# Structures 4 lecture notes

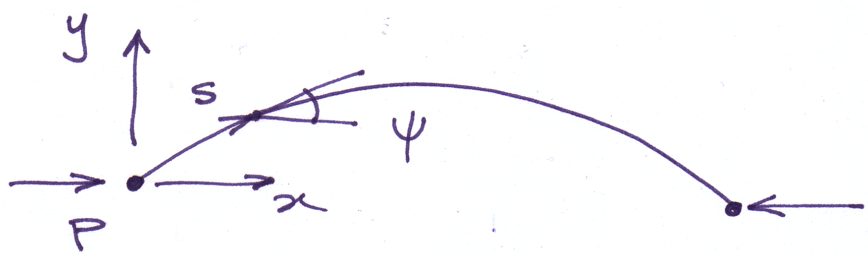
## Buckling

Buckling calculations are very difficult except for a few special cases, and so numerical methods on a computer are almost invariably used in practice for buckling modes involving plates, shells and assemblies of beams and columns. Single columns and beams aren't too bad.

The compression flange of a beam acts more or less like a column except that the torsional and warping resistance are involved.

However the simple special cases are still worth examining because they tell us what to look for in a numerical analysis.

### The elastica




The sagging curvature is $\frac{d\psi}{ds}$. This is the definition of curvature. The sagging moment is equal to $EI$ times the curvature. Sagging moment is also equal to $-Py$ and thus

$$M = EI\frac{d\psi}{ds} = -Py = -P\int_{s=0}^{s} \sin\psi \, ds$$

$$EI\frac{d^2\psi}{ds^2} = -P\sin\psi$$

$$EI\frac{d\psi}{ds}\frac{d^2\psi}{ds^2} = -P\sin\psi\frac{d\psi}{ds}$$

$$EI\frac{1}{2}\left(\frac{d\psi}{ds}\right)^2 = P\left(\cos\psi - \cos\psi_0\right)$$

where $\psi_0$ is the value of $\psi$ at $s=0$.

Hence we have

$$EI\frac{1}{2}\left(\frac{d\psi}{ds}\right)^2 = P\left(\cos\psi - \cos\psi_0\right)$$

$$s = \sqrt{\frac{EI}{2P}}\int_{\psi=\psi_0}^{\psi} \frac{d\psi}{\sqrt{\cos\psi - \cos\psi_0}}.$$

If we write $\psi = 2\theta$, $\cos\psi = \cos 2\theta = 1 - 2\sin^2\theta$ and

$$s = \sqrt{\frac{EI}{2P}}\int_{\psi=\psi_0}^{\psi} \frac{2d\theta}{\sqrt{2}\sqrt{\sin^2\theta_0 - \sin^2\theta}} = \sqrt{\frac{EI}{P\sin^2\theta_0}}\int_{\psi=\psi_0}^{\psi} \frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}}.$$

$$k = \frac{1}{\sin\theta_0}$$

This is an incomplete elliptic integral of the first kind, – see http://en.wikipedia.org/wiki/Elliptic_integral

This means that it cannot be evaluated using elementary functions – trigonometric functions, hyperbolic functions etc.

Alternative derivation:

$$\tan\psi = \frac{dy}{dx}$$

$$\sec^2\psi \frac{d\psi}{ds} = \frac{d^2y}{dx^2}\frac{dx}{ds} = \frac{d^2y}{dx^2}\cos\psi$$

$$\frac{d\psi}{ds} = \frac{\frac{d^2y}{dx^2}}{\sec^3\psi} = \frac{\frac{d^2y}{dx^2}}{\left(1+\tan^2\psi\right)^{\frac{3}{2}}} = \frac{\frac{d^2y}{dx^2}}{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}$$

Thus

$$\frac{EI\frac{d^2y}{dx^2}}{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}} + Py = 0$$

$$\frac{EI\frac{dy}{dx}\frac{d^2y}{dx^2}}{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}} + Py\frac{dy}{dx} = 0$$

$$\frac{EI}{\sqrt{1+\left(\frac{dy}{dx}\right)_0^2}} - \frac{EI}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} + \frac{1}{2}Py^2 = 0$$

We can carry on, but we will get stuck again with an elliptic integral. However, if we assume that $\frac{dy}{dx}$ is small, $\frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} \approx \frac{1}{1+\frac{1}{2}\left(\frac{dy}{dx}\right)^2} \approx 1-\frac{1}{2}\left(\frac{dy}{dx}\right)^2$ so that $EI\left(\left(\frac{dy}{dx}\right)^2 - \left(\frac{dy}{dx}\right)_0^2\right) + Py^2 = 0$ .

This is satisfied by

$$y = B\sin\left(\sqrt{\frac{P}{EI}}x\right)$$

$$\frac{dy}{dx} = B\sqrt{\frac{P}{EI}}\cos\left(\sqrt{\frac{P}{EI}}x\right).$$

$$\left(\frac{dy}{dx}\right)_0 = B\sqrt{\frac{P}{EI}}$$

This gives the Euler column formula,

$$P = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 EA}{\left(\frac{L}{r}\right)^2}$$

$$A = \text{cross-sectional area}$$

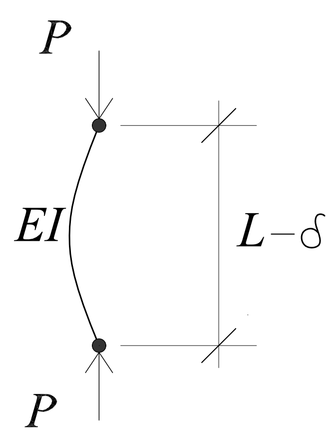
$$I = Ar^2 \qquad .$$

$$r = \text{radius of gyration}$$

$$\frac{L}{r} = \text{slenderness ratio}$$

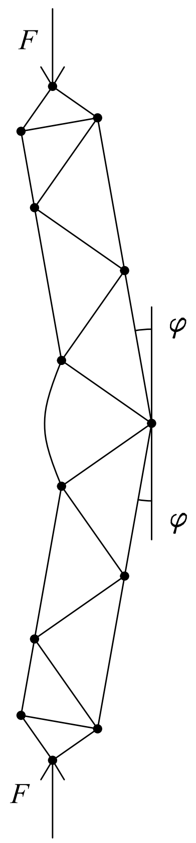
For columns other than pin-ended columns, $L$ is the **effective length.** For cantilever columns the effective length is more than twice the actual length, depending on how stiff the moment connection at the base is.

Analysis of the elastica shows that if a column remains elastic the load continues to increase as the column buckles.



The above figure shows a buckled pin ended column of length $L$ and bending stiffness $EI$. The column is initially perfectly straight. The relationship between the buckling load and the shortening *due to bending* is $\frac{P}{P_{\text{Euler}}} = 1 + \frac{\delta}{2L}$ where $P_{\text{Euler}} = \frac{\pi^2 EI}{L^2}$ is the Euler buckling load. This formula is obtained using complete elliptic integrals as described in §2.7 of Timoshenko and Gere, *Theory of Elastic Stability* and applies for small values of $\frac{\delta}{L}$.

However, it can be shown that for the truss column below that the load $F$ decreases with deflection (see question 3 in the 2013-14 Structures 4 exam). This is because as the truss deflects the buckled member attracts more than its fair share of the load.

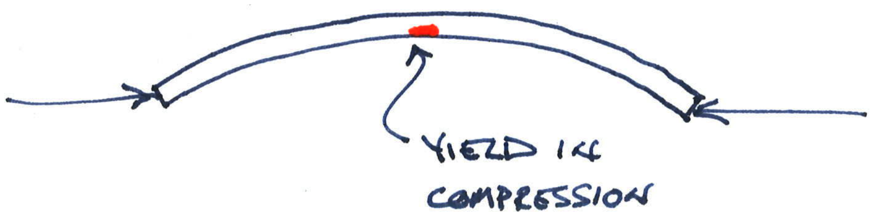


# Perry Robinson formula

See the A history of the safety factors by Alasdair N. Beal, The Structural Engineer 89 (20) 18 October 2011,

http://anbeal.co.uk/TSE2011HistoryofSafetyFactors.pdf

for a fascinating discussion of safety factors including the Perry Robertson formula.




Assume column has an initial bend, $y = \zeta L \sin\left(\frac{\pi x}{L}\right)$ (note that $\zeta$ is dimensionless) and that $\frac{dy}{dx}$ is **SMALL**. Then the sagging moment,

$$M = EI \times \text{change of curvature} = EI\left(\frac{d^2 y}{dx^2} - \frac{d^2}{dx^2}\left(\zeta L \sin\left(\frac{\pi x}{L}\right)\right)\right) = -Py$$

$$EI\frac{d^2 y}{dx^2} + Py = -EI\zeta L\frac{\pi^2}{L^2}\sin\left(\frac{\pi x}{L}\right)$$

Try solution $y = B\sin\left(\frac{\pi x}{L}\right)$ then

$$-EIB\frac{\pi^2}{L^2}\sin\left(\frac{\pi x}{L}\right) + PB\sin\left(\frac{\pi x}{L}\right) = -EI\zeta L\frac{\pi^2}{L^2}\sin\left(\frac{\pi x}{L}\right)$$

$$B = \frac{EI\zeta L\frac{\pi^2}{L^2}}{EI\frac{\pi^2}{L^2} - P} = \frac{\zeta L}{1 - \frac{P}{\frac{\pi^2 EI}{L^2}}}$$

The maximum stress is equal to

$$\sigma_{\max} = \frac{P}{A} + \frac{M}{Z} = \frac{P}{A} + \frac{PB}{Z} = \frac{P}{A} + \frac{\dfrac{PL\zeta}{Z}}{1 - \dfrac{P}{\dfrac{\pi^2 EI}{L^2}}} = \frac{P}{A} + \frac{\dfrac{P}{A}\dfrac{LA\zeta}{Z}}{1 - \dfrac{P}{P_{\text{Euler}}}}.$$

$Z = \frac{I}{c}$ is the section modulus.

If we set $I = Ar^2$, $Z = \frac{I}{c} = \frac{Ar^2}{c}$ $\sigma = \frac{P}{A}$, $\sigma_{\max} = \sigma_y$ and

$\sigma_{\text{Euler}} = \frac{P_{\text{Euler}}}{A} = \frac{\pi^2 EI}{AL^2} = \frac{\pi^2 E}{\left(\frac{L}{r}\right)^2}$, then we have

$$\sigma_y = \sigma + \frac{\theta\sigma}{1 - \dfrac{\sigma}{\sigma_{\text{Euler}}}}.$$

in which $\theta = \frac{LA\zeta}{Z} = \frac{Lc\zeta}{r^2}$.

Therefore

$$\begin{aligned} &(\sigma_{\text{Euler}} - \sigma)(\sigma_y - \sigma) + \theta\sigma_{\text{Euler}}\sigma = 0 \\ &\sigma^2 - \left(\sigma_y + (1+\theta)\sigma_{\text{Euler}}\right)\sigma + \sigma_y\sigma_{\text{Euler}} = 0 \end{aligned}.$$

$$\begin{aligned} \sigma &= \frac{1}{2}\left(\sigma_y + (1+\theta)\sigma_{\text{Euler}}\right) - \frac{1}{2}\sqrt{\left(\sigma_y + (1+\theta)\sigma_{\text{Euler}}\right)^2 - 4\sigma_y\sigma_{\text{Euler}}} \\ &= \frac{1}{2}\left(\sigma_y + (1+\theta)\sigma_{\text{Euler}}\right) - \frac{1}{2}\sqrt{\left(\sigma_y + (1+\theta)\sigma_{\text{Euler}}\right)^2 - 4(1+\theta)\sigma_y\sigma_{\text{Euler}} + 4\theta\sigma_y\sigma_{\text{Euler}}} \\ &= \frac{1}{2}\left(\sigma_y + (1+\theta)\sigma_{\text{Euler}}\right) - \frac{1}{2}\sqrt{\left(\sigma_y - (1+\theta)\sigma_{\text{Euler}}\right)^2 + 4\theta\sigma_y\sigma_{\text{Euler}}} \end{aligned}.$$

When $\theta = 0$,

$$\begin{aligned} \sigma &= \frac{1}{2}\left(\sigma_y + \sigma_{\text{Euler}}\right) - \frac{1}{2}\left|\sigma_y - \sigma_{\text{Euler}}\right| \\ &= \sigma_y \text{ or } \sigma_{\text{Euler}} \end{aligned}.$$

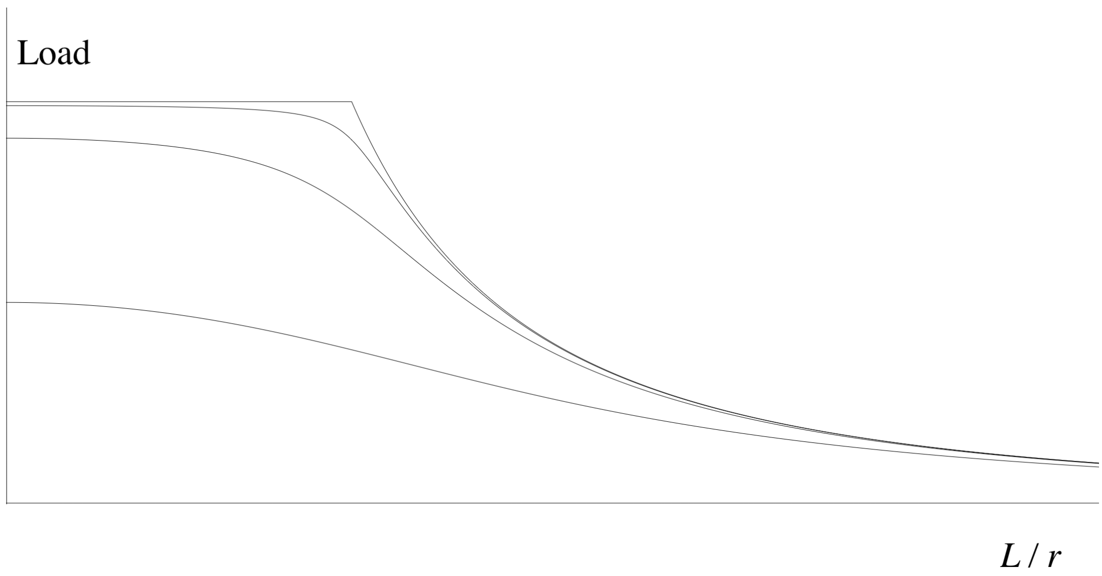
Note that $\sigma_{\text{Euler}} = \sigma_y$ when $\dfrac{\pi^2 E}{\left(\dfrac{L}{r}\right)^2} = \sigma_y$ so that $\dfrac{L}{r} = \pi\sqrt{\dfrac{E}{\sigma_y}}$.

When $\sigma_{\text{Euler}}$ is very small,

$$\begin{aligned}
\sigma &\simeq \frac{1}{2}\left(\sigma_y + (1+\theta)\sigma_{\text{Euler}}\right) - \frac{1}{2}\sqrt{\sigma_y^2 - (2-2\theta)\sigma_y\sigma_{\text{Euler}}} \\
&= \frac{1}{2}\left(\sigma_y + (1+\theta)\sigma_{\text{Euler}}\right) - \frac{1}{2}\sigma_y\sqrt{1 - 2(1-\theta)\frac{\sigma_{\text{Euler}}}{\sigma_y}} \\
&\simeq \frac{1}{2}\left(\sigma_y + (1+\theta)\sigma_{\text{Euler}}\right) - \frac{1}{2}\sigma_y\left[1 - (1-\theta)\frac{\sigma_{\text{Euler}}}{\sigma_y}\right] \\
&\simeq \sigma_{\text{Euler}}
\end{aligned}$$

and when $\sigma_{\text{Euler}}$ is very large,

$$\begin{aligned}
\sigma &\simeq \frac{1}{2}\left(\sigma_y + (1+\theta)\sigma_{\text{Euler}}\right) - \frac{1}{2}\sqrt{(1+\theta)^2\sigma_{\text{Euler}}^2 - (2-2\theta)\sigma_y\sigma_{\text{Euler}}} \\
&= \frac{1}{2}\left(\sigma_y + (1+\theta)\sigma_{\text{Euler}}\right) - \frac{1}{2}(1+\theta)\sigma_{\text{Euler}}\sqrt{1 - 2\frac{(1-\theta)}{(1+\theta)^2}\frac{\sigma_y}{\sigma_{\text{Euler}}}} \\
&\simeq \frac{1}{2}\left(\sigma_y + (1+\theta)\sigma_{\text{Euler}}\right) - \frac{1}{2}(1+\theta)\sigma_{\text{Euler}}\left(1 - \frac{(1-\theta)}{(1+\theta)^2}\frac{\sigma_y}{\sigma_{\text{Euler}}}\right) \\
&= \frac{1}{2}\sigma_y\left(1 + \frac{(1-\theta)}{(1+\theta)}\right) = \frac{\sigma_y}{1+\theta}
\end{aligned}$$

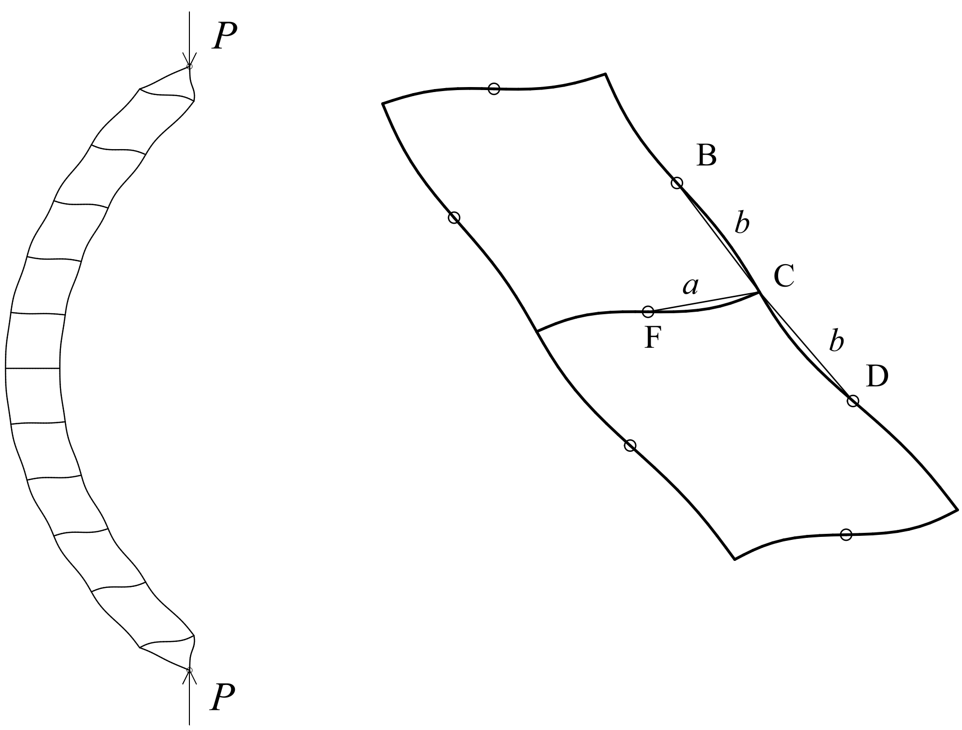



Perry-Robertson graphs with $\frac{E}{\sigma_y} = 1000$ and $\theta = 0,\ 0.01,\ 0.1 \text{ and } 1$.

See http://en.wikipedia.org/wiki/Perry_Robertson_formula

This is the basis for column design.

## Timoshenko or Cosserat column



The figure on the left a shows the buckling of a battened or Vierendeel column. The deformation has been exaggerated so that the deflected shape can be seen.

The figure on the right shows a detail of two bays in which the circles show the assumed points of contraflexure half way along the members.

If the column is treated as a Timoshenko or Cosserat beam, the differential equations describing deformation of the column are

$$
\begin{aligned}
&M = Py = -EI_{\text{composite}} \frac{d}{dx}\left(\frac{dy}{dx} - \theta\right) \\
&k\theta = F = P\frac{dy}{dx}
\end{aligned}
$$

in which $x$ is the vertical coordinate along the column, $y$ is the lateral displacement of the column, $M$ is the overall bending moment, $P$ is the overall axial load and $F$ is the overall shear force. $I_{\text{composite}}$ is the fully composite second moment of area and $k$ is the shear stiffness of the column.

Hence

$$
\begin{aligned}
EI_{\text{composite}} \frac{d}{dx}\left(\frac{dy}{dx} - \frac{P}{k}\frac{dy}{dx}\right) + Py = 0 \\
EI_{\text{composite}} \left(1 - \frac{P}{k}\right)\frac{d^2y}{dx^2} + Py = 0
\end{aligned}
.
$$

If the column is pin-ended and its length is length $L$, the differential equation is satisfied by

$$
y = B\sin\frac{\pi x}{L}
$$

if

$$P = \frac{\pi^2 EI_{\text{composite}}}{L^2}\left(1-\frac{P}{k}\right)$$
$$= P_{\text{Euler}}\left(1-\frac{P}{k}\right)$$
$$P\left(1+\frac{P_{\text{Euler}}}{k}\right) = P_{\text{Euler}}$$
$$P = \frac{kP_{\text{Euler}}}{k+P_{\text{Euler}}}$$

The shear stiffness $k$ depends upon the bending stiffness and length of the individual members. In the above column all the horizontal members all have length $2a$ so that the length $\text{CF}$ is $a$. The vertical members all have length $2b$ so that the lengths $\text{BC}$ and $\text{CD}$ are both $b$. The horizontal members all have second moment of area $I_{\text{horizontal}}$ and the vertical members all have second moment of area $I_{\text{vertical}}$ and cross-sectional area $A_{\text{vertical}}$. The members are all made from a material with Young's modulus $E$.

Shear deformation means that an angle such as $\text{D}\hat{\text{C}}\text{F}$ is deformed from a right angle to $\frac{\pi}{2}+\theta$ due to bending of the members. The connections between the members are assumed to be rigid.

If we ignore $I_{\text{vertical}}$, then the parallel axis theorem gives $I_{\text{composite}} = 2A_{\text{vertical}}a^2$.

To calculate the shear stiffness we first note that that the tip deflection of a cantilever of bending stiffness $EI$ and span $S$ loaded with a point load $W$ at its tip is $\frac{wS^3}{3EI}$. The shear force in each of the vertical members is $\frac{F}{2}$ and the shear force in the horizontal members is $\frac{2\frac{F}{2}b}{a} = \frac{Fb}{a}$. Thus the shear deformation is

$$\theta = \frac{\left(\frac{\frac{F}{2}b^3}{3EI_{\text{vertical}}}\right)}{b} + \frac{\left(\frac{\frac{Fb}{a}a^3}{3EI_{\text{horizontal}}}\right)}{a} = \frac{Fb^2}{6EI_{\text{vertical}}} + \frac{Fab}{3EI_{\text{horizontal}}}.$$

Hence $k = \dfrac{1}{\dfrac{b^2}{6EI_{\text{vertical}}} + \dfrac{ab}{3EI_{\text{horizontal}}}}$.

# Buckling of plates and shells

See, for example, Don O. Brush and Bo O. Almroth, *Buckling of Bars, Plates and Shells*, McGraw-Hill, New York 1975.

Plates behave fairly well when they buckle, the load usually does not drop off dramatically and may increase after buckling. On the other hand shells, including axially compressed cylinders, can be very imperfection sensitive so they collapse at a much smaller load than the eigenvalue buckling load – see

Hunt, G. W., 2011. Reflections and symmetries in space and time. *IMA Journal of Applied Mathematics*, 76 (1), pp. 2-26.

http://imamat.oxfordjournals.org/content/76/1/2

## Simply supported flat plate

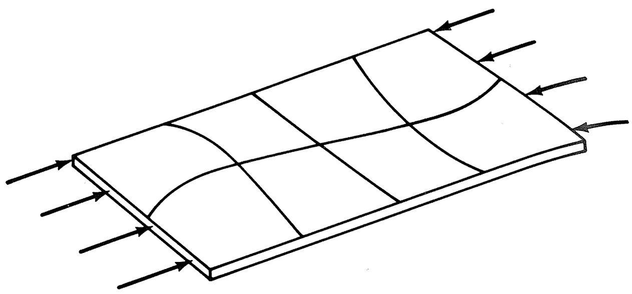


FIGURE 3.7
Buckled form of plate subjected to in-plane compressive loading.

Image from Brush and Almroth, *Buckling of Bars, Plates and Shells*

For a straight beam we have

$$EI\frac{d^2v}{dx^2} + Pv = 0$$

$$EI\frac{d^4v}{dx^4} + P\frac{d^2v}{dx^2} = 0$$

where $v$ is the displacement in the $y$ direction. The corresponding equation for a flat plate is

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) + \sigma_x \frac{\partial^2 w}{\partial x^2} = 0$$

$$D = \frac{Et^3}{12\left(1-\upsilon^2\right)}$$

in which it is assumed that that there is only a membrane stress $\sigma_x$ in the $x$ direction. Membrane stress has units force per unit width. $w$ is the displacement in the $z$ direction, $t$ is the plate thickness and $\upsilon$ is Poisson's ratio. $D$ is the bending stiffness per unit width and it replaces the $E\frac{bd^3}{12}$ for a rectangular beam.

Let us suppose that the plate is simply supported along $y=0$ and $y=b$ and that it is long in the $x$ direction. The differential equation is satisfied and the $y=0$ and $y=b$ boundary conditions are satisfied by

$$w = A\sin\frac{2\pi x}{\lambda}\sin\frac{\pi y}{b}$$

if

$$\sigma_x\left(\frac{2\pi}{\lambda}\right)^2 = D\left(\left(\frac{2\pi}{\lambda}\right)^4 + 2\left(\frac{2\pi}{\lambda}\right)^2\left(\frac{\pi}{b}\right)^2 + \left(\frac{\pi}{b}\right)^4\right) = D\left(\left(\frac{2\pi}{\lambda}\right)^2 + \left(\frac{\pi}{b}\right)^2\right)^2$$

so that

$$\sigma_x = D\left(\left(\frac{2\pi}{\lambda}\right) + \frac{\left(\frac{\pi}{b}\right)^2}{\left(\frac{2\pi}{\lambda}\right)}\right)^2$$

which is minimum when

$$\frac{d\sigma_x}{d\lambda} = -2D\left(\left(\frac{2\pi}{\lambda}\right) + \frac{\left(\frac{\pi}{b}\right)^2}{\left(\frac{2\pi}{\lambda}\right)}\right)\left(1 - \frac{\left(\frac{\pi}{b}\right)^2}{\left(\frac{2\pi}{\lambda}\right)^2}\right)\frac{2\pi}{\lambda^2} = 0 .$$

$$\lambda = 2b$$

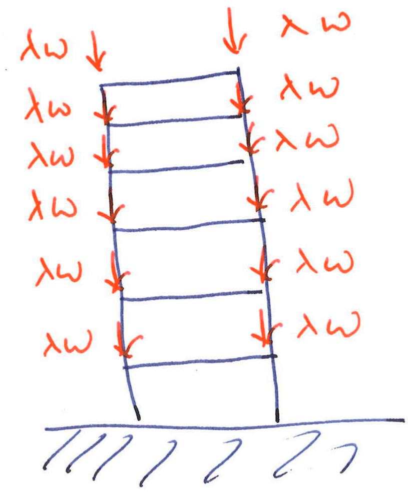
Thus the buckling membrane stress is

$$\sigma_x = 4D\left(\frac{\pi}{b}\right)^2 .$$

## Simple single degree of freedom models

These will be introduced in lectures, with particular reference to post-buckled stability and imperfection sensitivity. In particular the difference between non-linear and linear (eigenvalue) buckling will be emphasised.

## Linear or eigenvalue buckling




The above structure is loaded by a system of loads that are all multiplied by the same load factor $\lambda$. Imagine that it is analysed as a linear elastic structure with stiffness matrix $\mathbf{K}$ and that the axial forces in the members are found. These axial forces will all be proportional to $\lambda$ and from these forces we can find the **geometric stiffness matrix** $-\lambda\mathbf{G}$. The minus is put there because compressive forces produce a negative stiffness.

Linear buckling occurs when

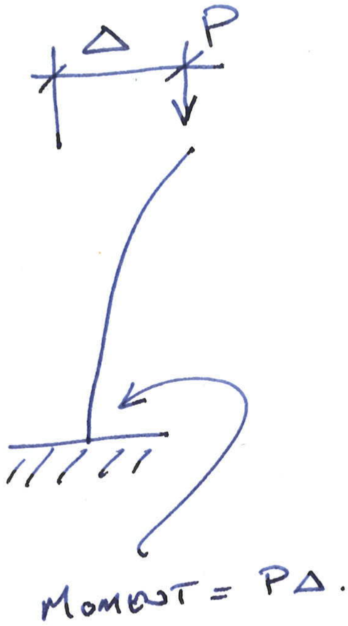
$$[\mathbf{K}-\lambda\mathbf{G}]\boldsymbol{\delta}=\mathbf{0}$$
$$\mathbf{K}^{-1}[\mathbf{K}-\lambda\mathbf{G}]\boldsymbol{\delta}=\mathbf{0}$$
$$\left[\mathbf{K}^{-1}\mathbf{K}-\lambda\mathbf{K}^{-1}\mathbf{G}\right]\boldsymbol{\delta}=\mathbf{0}.$$
$$\left[\mathbf{I}-\lambda\mathbf{K}^{-1}\mathbf{G}\right]\boldsymbol{\delta}=\mathbf{0}$$
$$\left[\mathbf{K}^{-1}\mathbf{G}-\frac{1}{\lambda}\mathbf{I}\right]\boldsymbol{\delta}=\mathbf{0}$$

We are only interested in the lowest buckling load for which the load factor $\lambda$ is equal to one over the highest eigenvalue of $\mathbf{K}^{-1}\mathbf{G}$. The corresponding eigenvector gives the mode shape.

Note the similarity to natural frequencies and mode shapes. Buckling corresponds to the natural frequency becoming zero.

Note that linear eigenvalue buckling analysis gives no information about imperfection sensitivity.

Therefore non-linear buckling analysis should always be done if there is any question of imperfection sensitivity.




The $P-\Delta$ effect refers to the moment caused by side-sway of a column. It is sometimes not clear what is meant by $P-\Delta$ analysis in a piece of software regarding linear or non-linear. Note that rotation is not a vector unless it is small and this assumption is often made even in so called non-linear analysis.

# Lagrange's Equations of Motion

Kinetic energy $= T = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} M_{ij}\dot{\delta}_i\dot{\delta}_j = \frac{1}{2}\dot{\boldsymbol{\delta}}^{\mathrm{T}}\mathbf{M}\dot{\boldsymbol{\delta}}$ where $n$ is the number of degrees of freedom. The $M_{ij}$ are functions of the $\delta$'s only.

$$\frac{dT}{dt} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left( M_{ij}\ddot{\delta}_i\dot{\delta}_j + M_{ij}\dot{\delta}_i\ddot{\delta}_j + \sum_{k=1}^{n}\frac{\partial M_{ij}}{\partial \delta_k}\dot{\delta}_i\dot{\delta}_j\dot{\delta}_k \right)$$

$$= \sum_{i=1}^{n}\left[ \dot{\delta}_i \sum_{j=1}^{n}\left( \frac{\left(M_{ij}+M_{ji}\right)}{2}\ddot{\delta}_j + \frac{1}{2}\sum_{k=1}^{n}\frac{\left(\frac{\partial M_{jk}}{\partial \delta_i}+\frac{\partial M_{kj}}{\partial \delta_i}\right)}{2}\dot{\delta}_j\dot{\delta}_k \right)\right]$$

$$= \sum_{i=1}^{n}\left[ \dot{\delta}_i \sum_{j=1}^{n}\left( \frac{\left(M_{ij}+M_{ji}\right)}{2}\ddot{\delta}_j + \sum_{k=1}^{n}\frac{\left(\frac{\partial M_{jk}}{\partial \delta_i}+\frac{\partial M_{kj}}{\partial \delta_i}\right)}{2}\dot{\delta}_j\dot{\delta}_k - \frac{1}{2}\sum_{k=1}^{n}\frac{\left(\frac{\partial M_{jk}}{\partial \delta_i}+\frac{\partial M_{kj}}{\partial \delta_i}\right)}{2}\dot{\delta}_j\dot{\delta}_k \right)\right]$$

$$= \sum_{i=1}^{n}\left[ \dot{\delta}_i \left( \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\delta}_i}\right) - \frac{\partial T}{\partial \delta_i} \right)\right]$$

If $U$ is the strain energy and $G$ is the gravitational potential energy, by conservation of energy,

$$0 = \frac{d}{dt}\left(T+U+G\right) = \sum_{i=1}^{n}\left[ \dot{\delta}_i \left( \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\delta}_i}\right) - \frac{\partial T}{\partial \delta_i} + \frac{\partial U}{\partial \delta_i} + \frac{\partial G}{\partial \delta_i} \right)\right]$$

This equation applies for arbitrary $\dot{\delta}_i$ and therefore

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\delta}_i}\right) - \frac{\partial T}{\partial \delta_i} + \frac{\partial U}{\partial \delta_i} + \frac{\partial G}{\partial \delta_i} = 0}.$$

These are Lagrange's equations of motion and apply for $i = 1 \text{ to } n$.

# Verlet integration

If the equations of motion can be rearranged so that:

$$\begin{aligned}
\ddot{\delta}_1 &= q_1\left(\delta_1,\delta_2,...,\delta_n,\dot{\delta}_1,\dot{\delta}_2,...,\dot{\delta}_n,p_1,p_2,...,p_n\right)\\
\ddot{\delta}_2 &= q_2\left(\delta_1,\delta_2,...,\delta_n,\dot{\delta}_1,\dot{\delta}_2,...,\dot{\delta}_n,p_1,p_2,...,p_n\right)\\
&.\\
&.\\
&.\\
&.\\
\ddot{\delta}_n &= q_n\left(\delta_1,\delta_2,...,\delta_n,\dot{\delta}_1,\dot{\delta}_2,...,\dot{\delta}_n,p_1,p_2,...,p_n\right)
\end{aligned}$$

in which the loads $p_1,p_2,...,p_n$ are known values of time and if we also know the initial values of $\delta_1,\delta_2,...,\delta_n,\dot{\delta}_1,\dot{\delta}_2,...,\dot{\delta}_n$ , then we can step through time updating values as follows:

$$\begin{aligned}
\dot{\delta}_1 &= \dot{\delta}_1 + \ddot{\delta}_1\Delta t & \qquad \delta_1 &= \delta_1 + \dot{\delta}_1\Delta t\\
\dot{\delta}_2 &= \dot{\delta}_2 + \ddot{\delta}_2\Delta t & \qquad \delta_2 &= \delta_2 + \dot{\delta}_2\Delta t\\
&. & &.\\
&. \qquad \text{and} & &. \qquad .\\
&. & &.\\
&. & &.\\
\dot{\delta}_n &= \dot{\delta}_n + \ddot{\delta}_n\Delta t & \qquad \delta_n &= \delta_n + \dot{\delta}_n\Delta t
\end{aligned}$$

There are a number of essentially similar ‘explicit’ methods like this with names such as Verlet Integration, Störmer's method, Gauss-Seidel and dynamic relaxation.

# Definitions of some terms used for non-aeroelastic vibrations

Note that single degree of freedom systems are very important because multidegree of freedom systems can be reduced to uncoupled single degree of freedom systems using modal analysis and the orthogonality conditions discussed in lectures – this is ignoring the coupling due to damping.

### Summary of results

In order to solve the equations

$$\mathbf{M}\ddot{\boldsymbol{\delta}}+\mathbf{D}\dot{\boldsymbol{\delta}}+\mathbf{K}\boldsymbol{\delta}=\mathbf{p}$$

for a system with $n$ degrees of freedom, we write $\boldsymbol{\delta}=\sum_{r=1}^{n} f_r(t)\boldsymbol{\Delta}_r$ where $\boldsymbol{\Delta}_r$ are the eigenvectors of $\mathbf{K}^{-1}\mathbf{M}$.

If we ignore coupling due to damping,

$$m_r\ddot{f}_r+\lambda_r\dot{f}_r+k_r f_r=p_r(t)$$
$$m_r=\boldsymbol{\Delta}_r^{\mathrm{T}}\mathbf{M}\boldsymbol{\Delta}_r$$
$$k_r=\boldsymbol{\Delta}_r^{\mathrm{T}}\mathbf{K}\boldsymbol{\Delta}_r$$
$$\lambda_r=2c\sqrt{k_r m_r}$$
$$p_r(t)=\boldsymbol{\Delta}_r^{\mathrm{T}}\mathbf{p}$$
$$c=\text{non-dimensional damping ratio}$$

which is the equation for a single degree of freedom system.

## Single degree of freedom system

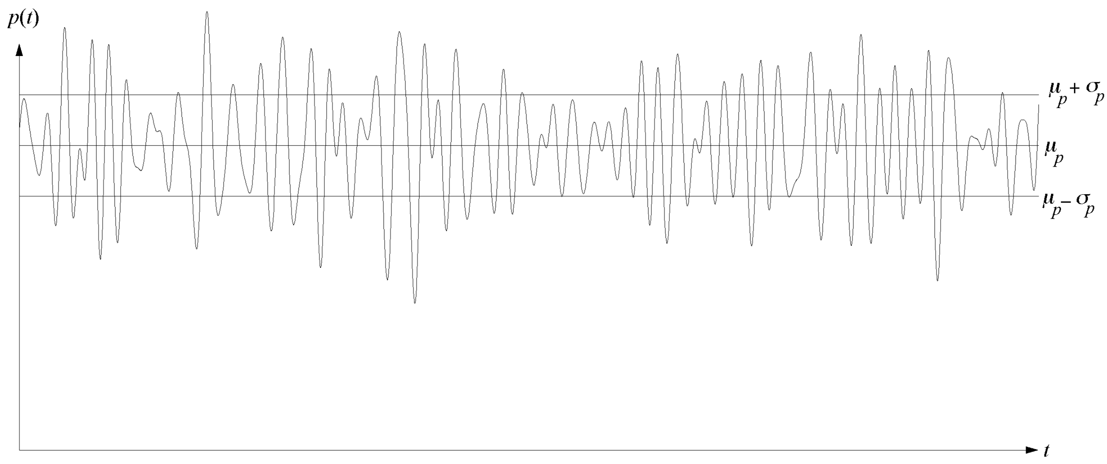



Figure 1

Figure 1 shows a typical 'random' load, $p(t)$.

The mean value of $p(t)$ is $\mu_p = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} p(t)\,dt \text{ as } T \to \infty$.

The auto-correlation function is $R_{pp}(\tau) = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} p(t)\,p(t+\tau)\,dt \text{ as } T \to \infty$.

The auto-correlation function of $p(t)$ minus its mean is

$$\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \left[p(t)-\mu_p\right]\left[p(t+\tau)-\mu_p\right]dt \text{ as } T \to \infty$$
$$= R_{pp}(\tau) - \mu_p^2.$$

The mean-square spectral density is $\phi_{pp}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[R_{pp}(\tau)-\mu_p^2\right]e^{-i\omega\tau}\,d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[R_{pp}(\tau)-\mu_p^2\right]\cos(\omega\tau)\,d\tau$ . Here $\omega = 2\pi \times \text{ frequency}$.

From the theory of Fourier transforms, $R_{pp}(\tau) = \mu_p^2 + \int_{-\infty}^{\infty}\phi_{pp}(\omega)e^{i\omega\tau}\,d\omega = \mu_p^2 + \int_{-\infty}^{\infty}\phi_{pp}(\omega)\cos(\omega\tau)\,d\omega$.

Note that $\mu_p$, $R_{pp}(\tau)$ and $\phi_{pp}(\omega)$ are all real, and $R_{pp}(-\tau) = R_{pp}(\tau)$ and $\phi_{pp}(-\omega) = \phi_{pp}(\omega)$.

$\sigma_p$ is the standard deviation of $p(t)$ which is defined as

$$\sigma_p^2 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \left[p(t) - \mu_p\right]\left[p(t) - \mu_p\right] dt \text{ as } T \to \infty$$

$$= R_{pp}(0) - \mu_p^2 = \int_{-\infty}^{\infty} \phi_{pp}(\omega) d\omega$$

The mean-square spectral density tells us about the amount of 'energy' at different frequencies, but gives no information about the relative phase.

$\psi_p$ is the root mean square (rms) value of $p(t)$ and $\psi_p^2 = \mu_p^2 + \sigma_p^2 = R_{pp}(0)$.

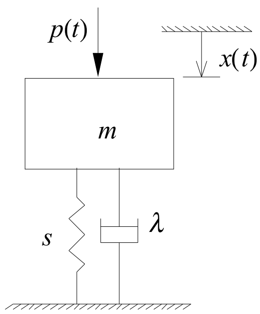



Figure 2

Figure 2 shows a mass – spring– damper. When the load $p(t)$ is applied,

$$m\frac{d^2x}{dt^2} + \lambda\frac{dx}{dt} + sx = p(t)$$

so that

$$\frac{1}{\Omega^2}\frac{d^2x}{dt^2} + \frac{2c}{\Omega}\frac{dx}{dt} + x = \frac{p(t)}{s}$$

where $\Omega = \sqrt{\frac{s}{m}} = 2\pi \times \text{natural frequency}$ and the viscous damping factor, $c = \frac{\lambda}{2\sqrt{sm}}$.

The mean value of $x(t)$ is $\mu_x = \frac{\mu_p}{s}$.

The mean-square spectral density of $x(t)$ is $\phi_{xx}(\omega) = \frac{\left(\frac{\phi_{pp}(\omega)}{s^2}\right)}{\left(1 - \frac{\omega^2}{\Omega^2}\right)^2 + \left(\frac{2c\omega}{\Omega}\right)^2}$.

Figure 3 shows plots of $\frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\Omega^2}\right)^2 + \left(\frac{2c\omega}{\Omega}\right)^2}}$.

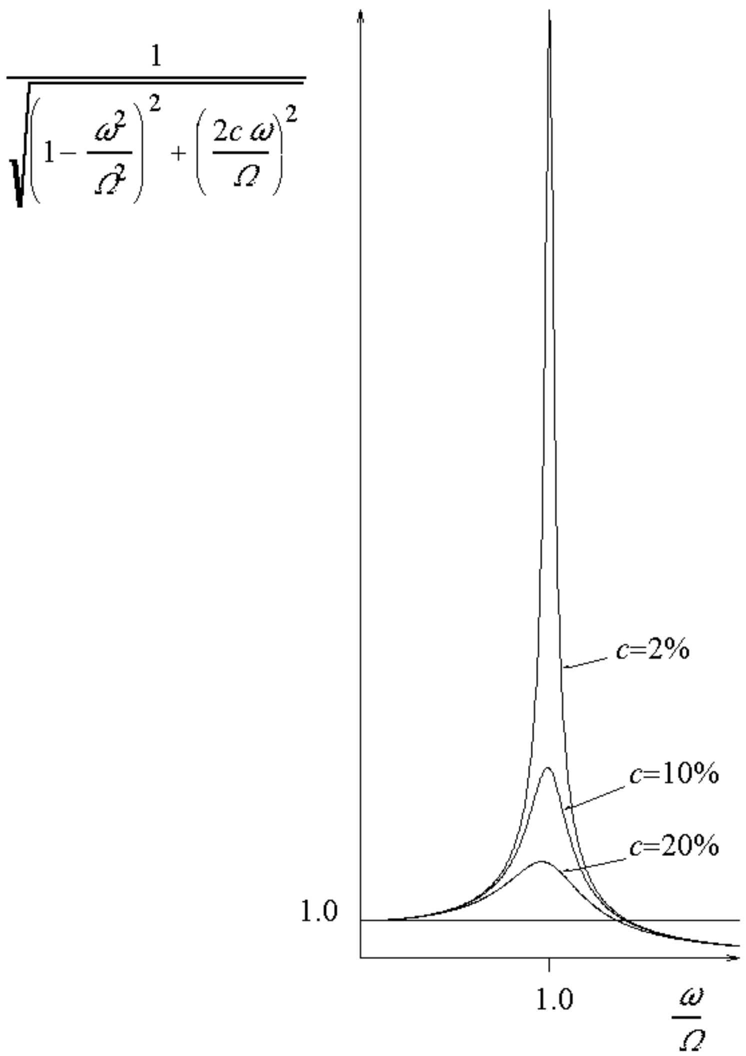



Figure 3

The standard deviation of $x(t)$ is $\sigma_x = \sqrt{\int_{-\infty}^{\infty} \phi_{xx}(\omega)d\omega} = \sqrt{R_{xx}(0)-\mu_x^2}$ and if $c$ is small,

$$\sigma_x = \frac{1}{s}\sqrt{\frac{\pi\Omega\phi_{pp}(\Omega)}{2c}} = \frac{1}{s}\sqrt{\frac{\pi\Omega\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[R_{pp}(\tau)-\mu_p^2\right]\cos(\Omega\tau)d\tau}{2c}}$$
$$= \frac{1}{s}\sqrt{\frac{\Omega\int_{-\infty}^{\infty}\left[R_{pp}(\tau)-\mu_p^2\right]\cos(\Omega\tau)d\tau}{4c}}$$

where $2\pi\Omega$ is the natural frequency.

The 'dynamic magnification factor' is $\sqrt{\frac{\pi\Omega\phi_{pp}(\Omega)}{2c\sigma_p^2}}$. **Note that this dynamic magnification factor applies only to the dynamic component of the load.**

Figure 4 shows the response to the load.

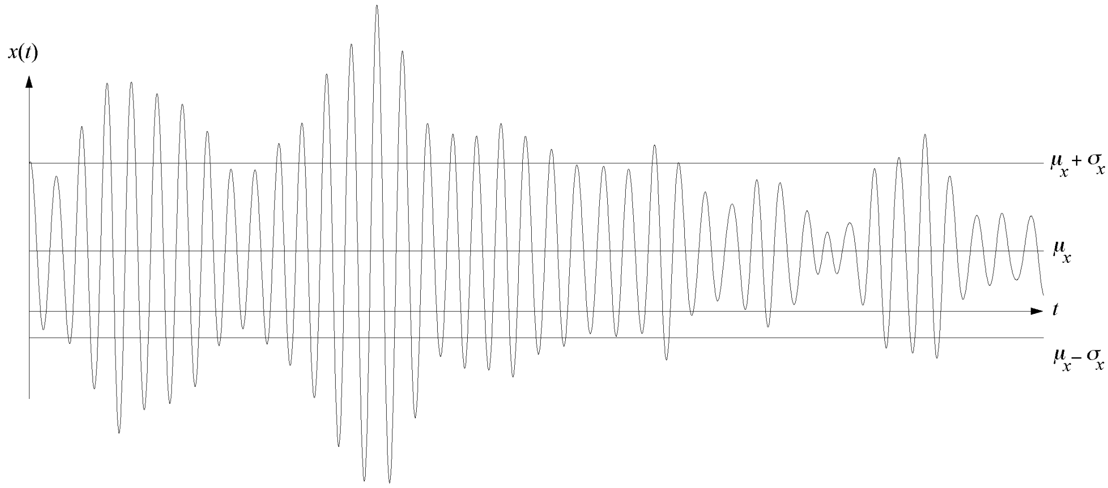



Figure 4

Note the dynamic magnification which can be seen by comparing the mean and standard deviation of the load and the response.

## Multi-degree of freedom systems

Let us suppose the 'load' exciting a particular mode of vibration is $p(t)=\sum_{i=1}^{N}A_i q_i(t)$ where the $q_i(t)$ are the pressures and the $A_i$ are the associated areas times displacement in the mode (which may be positive or negative).

The mean of the load exciting the mode is $\mu_p=\sum_{i=1}^{N}A_i\mu_{q_i}$ and the auto-correlation function is

$$R_{pp}(\tau) = \sum_{i=1}^{N}\sum_{j=1}^{N} A_i A_j R_{q_i q_j}(\tau)$$
$$= \mu_p^2 + \sum_{i=1}^{N}\sum_{j=1}^{N} A_i A_j \left[ R_{q_i q_j}(\tau) - \mu_{q_i}\mu_{q_j} \right]$$

where the cross-correlation,

$$R_{q_i q_j}(\tau) = R_{q_j q_i}(-\tau) = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} q_i(t) q_j(t+\tau) dt \text{ as } T \to \infty .$$

Again $\phi_{pp}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[R_{pp}(\tau) - \mu_p^2\right] e^{-i\omega\tau} d\tau$.

Note that in general $R_{q_i q_j}(-\tau) \neq R_{q_i q_j}(\tau)$ so that cross mean-square spectral density, $\phi_{q_i q_j}(\omega) = \overline{\phi_{q_i q_j}}(-\omega) = \phi_{q_j q_i}(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[R_{q_i q_j}(\tau) - \mu_{q_i}\mu_{q_j}\right] e^{-i\omega\tau} d\tau$ will be a complex function.

However in doing the summation $\phi_{pp}(\omega) = \sum_{i=1}^{N}\sum_{j=1}^{N} A_i A_j \phi_{q_i q_j}(\omega)$ , the imaginary parts will cancel out.

In terms of correlation functions, the response is given by

$$\sigma_x = \frac{1}{m}\sqrt{\frac{\int_{-\infty}^{\infty}\left(\sum_{i=1}^{N}\sum_{j=1}^{N} A_i A_j \left[R_{q_i q_j}(\tau) - \mu_{q_i}\mu_{q_j}\right]\right)\cos(\Omega\tau) d\tau}{4c}} .$$

## A note on Fourier series

For our purposes a stochastic random load can be approximated by the Fourier series,

$$p(t) = \mu_p + \sum_{n=1}^{\infty}\left(\sqrt{2\phi_{pp}(\omega_n)\Delta\omega}\sqrt{2}\cos\left(\omega_n t + \beta_p(\omega_n)\right)\right)$$

where $\omega_n = \frac{2\pi n}{T}$ and $\Delta\omega = \frac{2\pi}{T}$ if the period, $T$, is sufficiently large. Stochastic means 'governed by the laws of probability'.
The autocorrelation function,

$$R_{pp}(\tau) = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \left( \begin{array}{l} \left[ \mu_p + \sum_{n=1}^{\infty}\left( \sqrt{2\phi_{pp}(\omega_n)\Delta\omega}\sqrt{2}\cos\left(\omega_n t + \beta_p(\omega_n)\right)\right)\right] \\ \left[ \mu_p + \sum_{n=1}^{\infty}\left( \sqrt{2\phi_{pp}(\omega_n)\Delta\omega}\sqrt{2}\cos\left(\omega_n (t+\tau) + \beta_p(\omega_n)\right)\right)\right] \end{array}\right) dt \text{ as } T \to \infty$$

$$= \mu_p^2 + \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_{n=1}^{\infty}\left(2\phi_{pp}(\omega_n)\Delta\omega 2\cos\left(\omega_n(t+\tau)+\beta_p(\omega_n)\right)\cos\left(\omega_n t+\beta_p(\omega_n)\right)\right)dt$$

$$= \mu_p^2 + \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_{n=1}^{\infty}\left( 2\phi_{pp}(\omega_n)\Delta\omega 2 \left( \begin{array}{l} \cos(\omega_n\tau)\cos^2\left(\omega_n t+\beta_p(\omega_n)\right) \\ -\sin(\omega_n\tau)\sin\left(\omega_n t+\beta_p(\omega_n)\right)\cos\left(\omega_n t+\beta_p(\omega_n)\right) \end{array}\right)\right)dt$$

$$= \mu_p^2 + \sum_{n=1}^{\infty}\left(2\phi_{pp}(\omega_n)\Delta\omega\cos(\omega_n\tau)\right)$$

$$= \mu_p^2 + 2\sum_{n=1}^{\infty}\left(\phi_{pp}(\omega_n)\cos(\omega_n\tau)\right)\Delta\omega$$

You can also use the Fourier transform, but this seems to cause problems with spectral densities unless you limit the time to $-\frac{T}{2} \le t \le \frac{T}{2}$.

# Duhamel's integral

Duhamel's integral is a bit like Verlet integration, except that it only applies to linear systems.

The unloaded single degree of freedom system:

$m\frac{d^2x}{dt^2} + \lambda\frac{dx}{dt} + sx = 0$ or $\frac{1}{\Omega^2}\frac{d^2x}{dt^2} + \frac{2c}{\Omega}\frac{dx}{dt} + x = \frac{p(t)}{s}$ where $\Omega = \sqrt{\frac{s}{m}} = 2\pi \times \text{ natural frequency}$ and the viscous damping factor, $c = \frac{\lambda}{2\sqrt{sm}}$

is satisfied by

$$x = e^{-c\Omega t}\left(A\sin\left(\left(\sqrt{1-c^2}\right)\Omega t\right) + B\cos\left(\left(\sqrt{1-c^2}\right)\Omega t\right)\right).$$

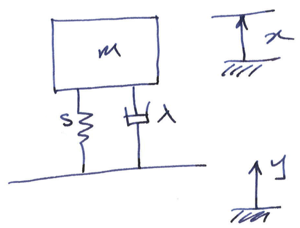
If the mass is stationary and receives an impulse $I$ at $t=0$, then when $t>0$,

$$x = \frac{I}{m}\frac{e^{-c\Omega t}\sin\left(\left(\sqrt{1-c^2}\right)\Omega t\right)}{\left(\sqrt{1-c^2}\right)\Omega}.$$

Duhamel's integral treats the load as lots of little impulses so that

$$x = \frac{1}{m\left(\sqrt{1-c^2}\right)\Omega}\int_0^t e^{-c\Omega(t-\tau)}\sin\left(\left(\sqrt{1-c^2}\right)\Omega(t-\tau)\right)p(\tau)\,d\tau.$$

# Seismic excitation




The ground motion in the above is $y$ (note that horizontal ground motion is more of a problem than vertical). The equation of motion is

$$m\ddot{x} + \lambda(\dot{x}-\dot{y}) + s(x-y) = 0.$$

We can rewrite this as

$$m\ddot{x} + \lambda\dot{x} + sx = \lambda\dot{y} + sy$$

so that the 'load' is $\lambda\dot{y} + sy$. However it is more usual to write

$$m\ddot{(x-y)} + \lambda\dot{(x-y)} + s(x-y) = -m\ddot{y}$$

in which $(x - y)$ is the motion relative to the ground and it is this motion which causes the stresses in the structure. The 'load' is now simply $-m\ddot{y}$. This is easy to implement in matrix notation for multi-degree of freedom systems.

# Aeroelasticity

This is discussed in lectures using the example of the Fokker E.V (later the D-VIII) monoplane (divergence) and the Tacoma Narrows bridge (single degree of freedom non-classical flutter) - see

http://books.google.co.uk/books?id=DnQOzYDJsm8C&dq=stall+flutter+tacoma&source=gbs_navlinks_s

Chris Williams