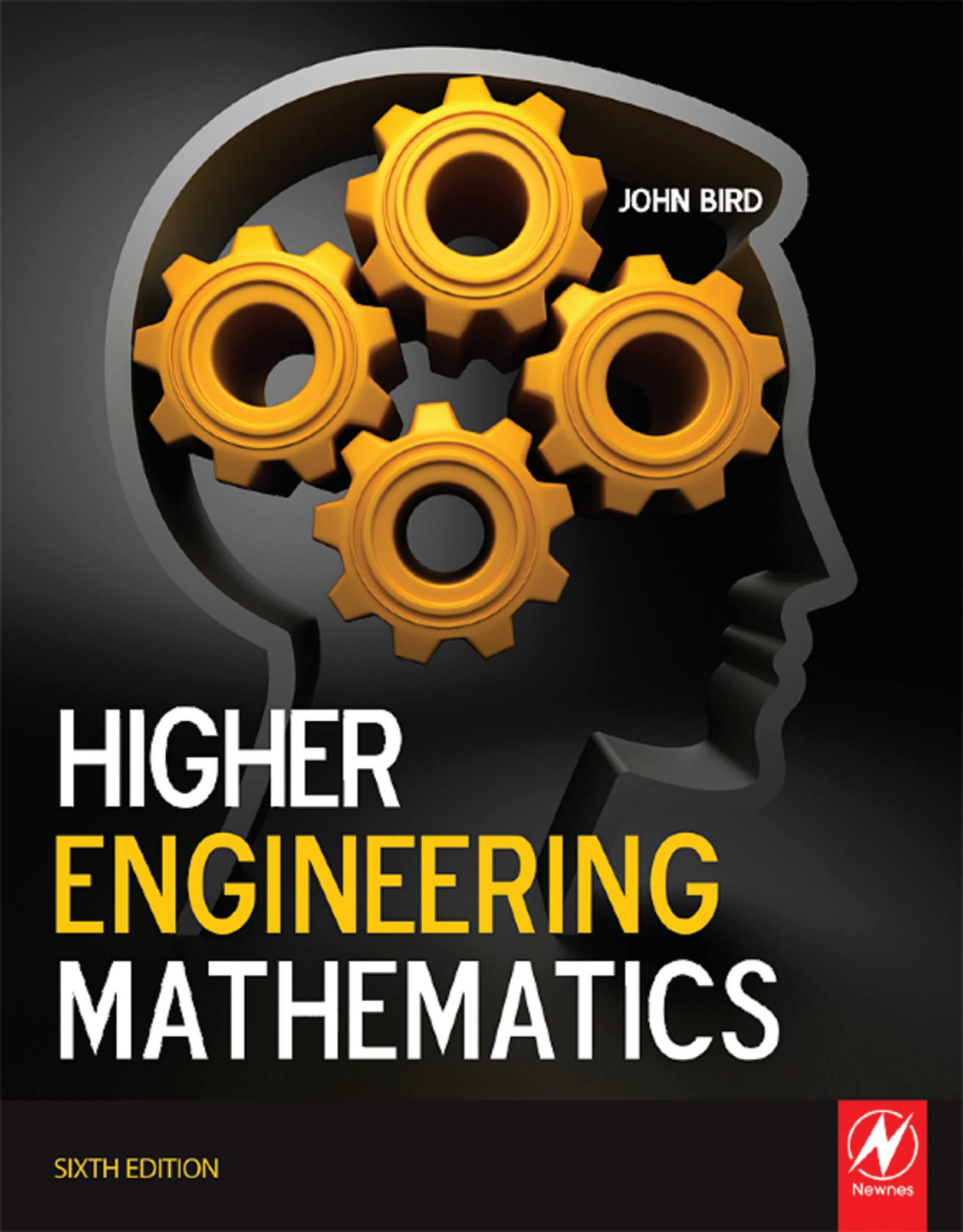
JOHN BIRD
HIGHER
ENGINEERING
MATHEMATICS
SIXTH EDITION
Newnes

Higher Engineering Mathematics

In memory of Elizabeth

Higher Engineering Mathematics

Sixth Edition

John Bird, BSc (Hons), CMath, CEng, CSci, FIMA, FIET, MIEE, FIIE, FCollT

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Preface

This **sixth edition of *'Higher Engineering Mathematics'*** covers essential mathematical material suitable for students studying **Degrees, Foundation Degrees, Higher National Certificate and Diploma courses in Engineering disciplines**.

In this edition the material has been ordered into the following **twelve convenient categories**: number and algebra, geometry and trigonometry, graphs, complex numbers, matrices and determinants, vector geometry, differential calculus, integral calculus, differential equations, statistics and probability, Laplace transforms and Fourier series. **New material** has been added on logarithms and exponential functions, binary, octal and hexadecimal, vectors and methods of adding alternating waveforms. Another feature is that a **free Internet download** is available of a sample (over 1100) of the further problems contained in the book.

The primary aim of the material in this text is to provide the fundamental analytical and underpinning knowledge and techniques needed to successfully complete scientific and engineering principles modules of Degree, Foundation Degree and Higher National Engineering programmes. The material has been designed to enable students to use techniques learned for the analysis, modelling and solution of realistic engineering problems at Degree and Higher National level. It also aims to provide some of the more advanced knowledge required for those wishing to pursue careers in mechanical engineering, aeronautical engineering, electronics, communications engineering, systems engineering and all variants of control engineering.

In ***Higher Engineering Mathematics 6th Edition***, theory is introduced in each chapter by a full outline of essential definitions, formulae, laws, procedures etc. The theory is kept to a minimum, for **problem solving** is extensively used to establish and exemplify the theory. It is intended that readers will gain real understanding through seeing problems solved and then through solving similar problems themselves.

Access to software packages such as Maple, Mathematica and Derive, or a graphics calculator, will enhance understanding of some of the topics in this text.

Each topic considered in the text is presented in a way that assumes in the reader only knowledge attained in BTEC National Certificate/Diploma, or similar, in an Engineering discipline.

'Higher Engineering Mathematics 6th Edition' provides a follow-up to *'Engineering Mathematics 6th Edition'*.

This textbook contains some **900 worked problems**, followed by over **1760 further problems (with answers)**, arranged within **238 Exercises**. Some **432 line diagrams** further enhance understanding.

A **sample of worked solutions** to over 1100 of the further problems has been prepared and can be **accessed free via the Internet** (see next page).

At the end of the text, a list of **Essential Formulae** is included for convenience of reference.

At intervals throughout the text are some **19 Revision Tests** (plus two more in the website chapters) to check understanding. For example, Revision Test 1 covers the material in Chapters 1 to 4, Revision Test 2 covers the material in Chapters 5 to 7, Revision Test 3 covers the material in Chapters 8 to 10, and so on. An **Instructor's Manual**, containing full solutions to the Revision Tests, is available free to lecturers adopting this text (see next page).

Due to restriction of extent, five chapters that appeared in the fifth edition have been removed from the text and placed on the website. For chapters on Inequalities, Boolean algebra and logic circuits, Sampling and estimation theories, Significance testing and Chi-square and distribution-free tests (see next page).

'Learning by example' is at the heart of *'Higher Engineering Mathematics 6th Edition'*.

JOHN BIRD
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HMS Sultan,
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It is recognised that the **level of understanding of algebra** on entry to higher courses is often inadequate. Since algebra provides the basis of so much of higher engineering studies, it is a situation that often needs urgent attention. Lack of space has prevented the inclusion of more basic algebra topics in this textbook; it is for this reason that some algebra topics – solution of simple, simultaneous and quadratic equations and transposition of formulae – have been made available to all via the Internet. Also included is a Remedial Algebra Revision Test to test understanding. To access the Algebra material visit the website.

Five extra chapters

Chapters on Inequalities, Boolean Algebra and logic circuits, Sampling and Estimation theories, Significance testing, and Chi-square and distribution-free tests are available to download at the website.

Sample of worked Solutions to Exercises

Within the text (plus the website chapters) are some 1900 further problems arranged within 260 Exercises. A sample of over 1100 worked solutions has been prepared and can be accessed free via the Internet. To access these worked solutions visit the website.

Instructor's manual

This provides fully worked solutions and mark scheme for all the Revision Tests in this book (plus 2 from the website chapters), together with solutions to the Remedial Algebra Revision Test mentioned above. The material is available to lecturers only. To obtain a password please visit the website with the following details: course title, number of students, your job title and work postal address.

To download the Instructor's Manual visit the website and enter the book title in the search box.

Syllabus Guidance

This textbook is written for **undergraduate engineering degree and foundation degree courses**; however, it is also most appropriate for **HNC/D studies** and three syllabuses are covered. The appropriate chapters for these three syllabuses are shown in the table below.

Chapter		Analytical Methods for Engineers	Further Analytical Methods for Engineers	Engineering Mathematics
1.	Algebra	×		
2.	Partial fractions	×		
3.	Logarithms	×		
4.	Exponential functions	×		
5.	Hyperbolic functions	×		
6.	Arithmetic and geometric progressions	×		
7.	The binomial series	×		
8.	Maclaurin's series	×		
9.	Solving equations by iterative methods		×	
10.	Binary, octal and hexadecimal		×	
11.	Introduction to trigonometry	×		
12.	Cartesian and polar co-ordinates	×		
13.	The circle and its properties	×		
14.	Trigonometric waveforms	×		
15.	Trigonometric identities and equations	×		
16.	The relationship between trigonometric and hyperbolic functions	×		
17.	Compound angles	×		
18.	Functions and their curves		×	
19.	Irregular areas, volumes and mean values of waveforms		×	
20.	Complex numbers		×	
21.	De Moivre's theorem		×	
22.	The theory of matrices and determinants		×	
23.	The solution of simultaneous equations by matrices and determinants		×	
24.	Vectors		×	
25.	Methods of adding alternating waveforms		×	

(*Continued*)

Chapter		Analytical Methods for Engineers	Further Analytical Methods for Engineers	Engineering Mathematics
26.	Scalar and vector products		×	
27.	Methods of differentiation	×		
28.	Some applications of differentiation	×		
29.	Differentiation of parametric equations			
30.	Differentiation of implicit functions	×		
31.	Logarithmic differentiation	×		
32.	Differentiation of hyperbolic functions	×		
33.	Differentiation of inverse trigonometric and hyperbolic functions	×		
34.	Partial differentiation			×
35.	Total differential, rates of change and small changes			×
36.	Maxima, minima and saddle points for functions of two variables			×
37.	Standard integration	×		
38.	Some applications of integration	×		
39.	Integration using algebraic substitutions	×		
40.	Integration using trigonometric and hyperbolic substitutions	×		
41.	Integration using partial fractions	×		
42.	The $t = \tan\theta/2$ substitution			
43.	Integration by parts	×		
44.	Reduction formulae	×		
45.	Numerical integration		×	
46.	Solution of first order differential equations by separation of variables		×	
47.	Homogeneous first order differential equations			
48.	Linear first order differential equations		×	
49.	Numerical methods for first order differential equations		×	×
50.	Second order differential equations of the form $a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0$		×	
51.	Second order differential equations of the form $a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = f(x)$		×	
52.	Power series methods of solving ordinary differential equations			×
53.	An introduction to partial differential equations			×
54.	Presentation of statistical data	×		

(*Continued*)

Chapter		Analytical Methods for Engineers	Further Analytical Methods for Engineers	Engineering Mathematics
55.	Measures of central tendency and dispersion	×		
56.	Probability	×		
57.	The binomial and Poisson distributions	×		
58.	The normal distribution	×		
59.	Linear correlation	×		
60.	Linear regression	×		
61.	Introduction to Laplace transforms			×
62.	Properties of Laplace transforms			×
63.	Inverse Laplace transforms			×
64.	Solution of differential equations using Laplace transforms			×
65.	The solution of simultaneous differential equations using Laplace transforms			×
66.	Fourier series for periodic functions of period 2π			×
67.	Fourier series for non-periodic functions over range 2π			×
68.	Even and odd functions and half-range Fourier series			×
69.	Fourier series over any range			×
70.	A numerical method of harmonic analysis			×
71.	The complex or exponential form of a Fourier series			×
Website Chapters				
72.	Inequalities			
73.	Boolean algebra and logic circuits		×	
74.	Sampling and estimation theories	×		
75.	Significance testing	×		
76.	Chi-square and distribution-free tests	×		

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Chapter 1

Algebra

1.1 Introduction

In this chapter, polynomial division and the factor and remainder theorems are explained (in Sections 1.4 to 1.6). However, before this, some essential algebra revision on basic laws and equations is included.

For further Algebra revision, go to website: http://books.elsevier.com/companions/0750681527

1.2 Revision of basic laws

(a) Basic operations and laws of indices

The **laws of indices** are:

(i) $a^m \times a^n = a^{m+n}$ (ii) $\dfrac{a^m}{a^n} = a^{m-n}$

(iii) $(a^m)^n = a^{m \times n}$ (iv) $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

(v) $a^{-n} = \dfrac{1}{a^n}$ (vi) $a^0 = 1$

Problem 1. Evaluate $4a^2bc^3 - 2ac$ when $a = 2$, $b = \frac{1}{2}$ and $c = 1\frac{1}{2}$

$$4a^2bc^3 - 2ac = 4(2)^2\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)^3 - 2(2)\left(\frac{3}{2}\right)$$

$$= \frac{4 \times 2 \times 2 \times 3 \times 3 \times 3}{2 \times 2 \times 2 \times 2} - \frac{12}{2}$$

$$= 27 - 6 = \mathbf{21}$$

Problem 2. Multiply $3x + 2y$ by $x - y$.

$$\begin{array}{rl} & 3x + 2y \\ & x - y \\ \hline \text{Multiply by } x \rightarrow & 3x^2 + 2xy \\ \text{Multiply by } -y \rightarrow & \quad -3xy - 2y^2 \\ \hline \text{Adding gives:} & \mathbf{3x^2 - xy - 2y^2} \\ \hline \end{array}$$

Alternatively,

$$(3x + 2y)(x - y) = 3x^2 - 3xy + 2xy - 2y^2$$

$$= \mathbf{3x^2 - xy - 2y^2}$$

Problem 3. Simplify $\dfrac{a^3b^2c^4}{abc^{-2}}$ and evaluate when $a = 3$, $b = \frac{1}{8}$ and $c = 2$.

$$\frac{a^3b^2c^4}{abc^{-2}} = a^{3-1}b^{2-1}c^{4-(-2)} = \mathbf{a^2bc^6}$$

When $a = 3$, $b = \frac{1}{8}$ and $c = 2$,

$$a^2bc^6 = (3)^2\left(\tfrac{1}{8}\right)(2)^6 = (9)\left(\tfrac{1}{8}\right)(64) = \mathbf{72}$$

Problem 4. Simplify $\dfrac{x^2y^3 + xy^2}{xy}$

$$\frac{x^2y^3 + xy^2}{xy} = \frac{x^2y^3}{xy} + \frac{xy^2}{xy}$$

$$= x^{2-1}y^{3-1} + x^{1-1}y^{2-1}$$

$$= \mathbf{xy^2 + y} \quad \text{or} \quad \mathbf{y(xy + 1)}$$

Problem 5. Simplify $\dfrac{(x^2\sqrt{y})(\sqrt{x}\sqrt[3]{y^2})}{(x^5y^3)^{\frac{1}{2}}}$

$$\frac{(x^2\sqrt{y})(\sqrt{x}\sqrt[3]{y^2})}{(x^5y^3)^{\frac{1}{2}}} = \frac{x^2y^{\frac{1}{2}}x^{\frac{1}{2}}y^{\frac{2}{3}}}{x^{\frac{5}{2}}y^{\frac{3}{2}}}$$

$$= x^{2+\frac{1}{2}-\frac{5}{2}}y^{\frac{1}{2}+\frac{2}{3}-\frac{3}{2}}$$

$$= x^0y^{-\frac{1}{3}}$$

$$= \boldsymbol{y^{-\frac{1}{3}}} \quad \text{or} \quad \boldsymbol{\frac{1}{y^{\frac{1}{3}}}} \quad \text{or} \quad \boldsymbol{\frac{1}{\sqrt[3]{y}}}$$

Now try the following exercise

Exercise 1 Revision of basic operations and laws of indices

1. Evaluate $2ab+3bc-abc$ when $a=2$, $b=-2$ and $c=4$. [-16]
2. Find the value of $5pq^2r^3$ when $p=\frac{2}{5}$, $q=-2$ and $r=-1$. [-8]
3. From $4x-3y+2z$ subtract $x+2y-3z$. [$3x-5y+5z$]
4. Multiply $2a-5b+c$ by $3a+b$. [$6a^2-13ab+3ac-5b^2+bc$]
5. Simplify $(x^2y^3z)(x^3yz^2)$ and evaluate when $x=\frac{1}{2}$, $y=2$ and $z=3$. [$x^5y^4z^3$, $13\frac{1}{2}$]
6. Evaluate $(a^{\frac{3}{2}}bc^{-3})(a^{\frac{1}{2}}b^{-\frac{1}{2}}c)$ when $a=3$, $b=4$ and $c=2$. [$\pm 4\frac{1}{2}$]
7. Simplify $\dfrac{a^2b+a^3b}{a^2b^2}$ $\left[\dfrac{1+a}{b}\right]$
8. Simplify $\dfrac{(a^3b^{\frac{1}{2}}c^{-\frac{1}{2}})(ab)^{\frac{1}{3}}}{(\sqrt{a^3}\sqrt{b}c)}$ $\left[a^{\frac{11}{6}}b^{\frac{1}{3}}c^{-\frac{3}{2}} \text{ or } \dfrac{\sqrt[6]{a^{11}}\sqrt[3]{b}}{\sqrt{c^3}}\right]$

(b) Brackets, factorization and precedence

Problem 6. Simplify $a^2-(2a-ab)-a(3b+a)$.

$$a^2-(2a-ab)-a(3b+a)$$
$$= a^2-2a+ab-3ab-a^2$$
$$= \boldsymbol{-2a-2ab} \quad \text{or} \quad \boldsymbol{-2a(1+b)}$$

Problem 7. Remove the brackets and simplify the expression:

$$2a-[3\{2(4a-b)-5(a+2b)\}+4a].$$

Removing the innermost brackets gives:

$$2a-[3\{8a-2b-5a-10b\}+4a]$$

Collecting together similar terms gives:

$$2a-[3\{3a-12b\}+4a]$$

Removing the 'curly' brackets gives:

$$2a-[9a-36b+4a]$$

Collecting together similar terms gives:

$$2a-[13a-36b]$$

Removing the square brackets gives:

$$2a-13a+36b = \boldsymbol{-11a+36b} \quad \text{or} \quad \boldsymbol{36b-11a}$$

Problem 8. Factorize (a) $xy-3xz$ (b) $4a^2+16ab^3$ (c) $3a^2b-6ab^2+15ab$.

(a) $xy-3xz = \boldsymbol{x(y-3z)}$

(b) $4a^2+16ab^3 = \boldsymbol{4a(a+4b^3)}$

(c) $3a^2b-6ab^2+15ab = \boldsymbol{3ab(a-2b+5)}$

Problem 9. Simplify $3c+2c\times 4c+c\div 5c-8c$.

The order of precedence is division, multiplication, addition and subtraction (sometimes remembered by BODMAS). Hence

$$3c + 2c \times 4c + c \div 5c - 8c$$
$$= 3c + 2c \times 4c + \left(\frac{c}{5c}\right) - 8c$$
$$= 3c + 8c^2 + \frac{1}{5} - 8c$$
$$= \mathbf{8c^2 - 5c + \frac{1}{5}} \quad \text{or} \quad \mathbf{c(8c - 5) + \frac{1}{5}}$$

Problem 10. Simplify $(2a - 3) \div 4a + 5 \times 6 - 3a$.

$$(2a - 3) \div 4a + 5 \times 6 - 3a$$
$$= \frac{2a - 3}{4a} + 5 \times 6 - 3a$$
$$= \frac{2a - 3}{4a} + 30 - 3a$$
$$= \frac{2a}{4a} - \frac{3}{4a} + 30 - 3a$$
$$= \frac{1}{2} - \frac{3}{4a} + 30 - 3a = \mathbf{30\frac{1}{2} - \frac{3}{4a} - 3a}$$

Now try the following exercise

Exercise 2 Further problems on brackets, factorization and precedence

1. Simplify $2(p + 3q - r) - 4(r - q + 2p) + p$. $[-5p + 10q - 6r]$
2. Expand and simplify $(x + y)(x - 2y)$. $[x^2 - xy - 2y^2]$
3. Remove the brackets and simplify: $24p - [2\{3(5p - q) - 2(p + 2q)\} + 3q]$. $[11q - 2p]$
4. Factorize $21a^2b^2 - 28ab$. $[7ab(3ab - 4)]$
5. Factorize $2xy^2 + 6x^2y + 8x^3y$. $[2xy(y + 3x + 4x^2)]$
6. Simplify $2y + 4 \div 6y + 3 \times 4 - 5y$. $\left[\frac{2}{3y} - 3y + 12\right]$
7. Simplify $3 \div y + 2 \div y - 1$. $\left[\frac{5}{y} - 1\right]$
8. Simplify $a^2 - 3ab \times 2a \div 6b + ab$. $[ab]$

1.3 Revision of equations

(a) Simple equations

Problem 11. Solve $4 - 3x = 2x - 11$.

Since $4 - 3x = 2x - 11$ then $4 + 11 = 2x + 3x$

i.e. $15 = 5x$ from which, $\boldsymbol{x = \frac{15}{5} = 3}$

Problem 12. Solve $4(2a - 3) - 2(a - 4) = 3(a - 3) - 1$.

Removing the brackets gives:

$$8a - 12 - 2a + 8 = 3a - 9 - 1$$

Rearranging gives:

$$8a - 2a - 3a = -9 - 1 + 12 - 8$$

i.e. $3a = -6$

and $\boldsymbol{a = \frac{-6}{3} = -2}$

Problem 13. Solve $\frac{3}{x - 2} = \frac{4}{3x + 4}$.

By 'cross-multiplying': $3(3x + 4) = 4(x - 2)$

Removing brackets gives: $9x + 12 = 4x - 8$

Rearranging gives: $9x - 4x = -8 - 12$

i.e. $5x = -20$

and $\boldsymbol{x = \frac{-20}{5} = -4}$

Problem 14. Solve $\left(\frac{\sqrt{t} + 3}{\sqrt{t}}\right) = 2$.

$$\sqrt{t}\left(\frac{\sqrt{t}+3}{\sqrt{t}}\right)=2\sqrt{t}$$

i.e. $\sqrt{t}+3=2\sqrt{t}$

and $3=2\sqrt{t}-\sqrt{t}$

i.e. $3=\sqrt{t}$

and $\mathbf{9=t}$

(b) Transposition of formulae

Problem 15. Transpose the formula $v=u+\dfrac{ft}{m}$ to make f the subject.

$u+\dfrac{ft}{m}=v$ from which, $\dfrac{ft}{m}=v-u$

and $m\left(\dfrac{ft}{m}\right)=m(v-u)$

i.e. $ft=m(v-u)$

and $\boldsymbol{f=\dfrac{m}{t}(v-u)}$

Problem 16. The impedance of an a.c. circuit is given by $Z=\sqrt{R^2+X^2}$. Make the reactance X the subject.

$\sqrt{R^2+X^2}=Z$ and squaring both sides gives

$R^2+X^2=Z^2$, from which,

$X^2=Z^2-R^2$ and **reactance** $\boldsymbol{X=\sqrt{Z^2-R^2}}$

Problem 17. Given that $\dfrac{D}{\mathrm{d}}=\sqrt{\left(\dfrac{f+p}{f-p}\right)}$, express p in terms of D, d and f.

Rearranging gives: $\sqrt{\left(\dfrac{f+p}{f-p}\right)}=\dfrac{D}{\mathrm{d}}$

Squaring both sides gives: $\dfrac{f+p}{f-p}=\dfrac{D^2}{\mathrm{d}^2}$

'Cross-multiplying' gives:

$$\mathrm{d}^2(f+p)=D^2(f-p)$$

Removing brackets gives:

$$\mathrm{d}^2f+\mathrm{d}^2p=D^2f-D^2p$$

Rearranging gives: $\mathrm{d}^2p+D^2p=D^2f-\mathrm{d}^2f$

Factorizing gives: $p(\mathrm{d}^2+D^2)=f(D^2-\mathrm{d}^2)$

and $$\boldsymbol{p=\frac{f(D^2-\mathrm{d}^2)}{(\mathrm{d}^2+D^2)}}$$

Now try the following exercise

Exercise 3 Further problems on simple equations and transposition of formulae

In problems 1 to 4 solve the equations

1. $3x-2-5x=2x-4$. $\left[\frac{1}{2}\right]$
2. $8+4(x-1)-5(x-3)=2(5-2x)$. $[-3]$
3. $\dfrac{1}{3a-2}+\dfrac{1}{5a+3}=0$. $\left[-\frac{1}{8}\right]$
4. $\dfrac{3\sqrt{t}}{1-\sqrt{t}}=-6$. $[4]$
5. Transpose $y=\dfrac{3(F-f)}{L}$ for f. $\left[f=\dfrac{3F-yL}{3}\quad\text{or}\quad f=F-\dfrac{yL}{3}\right]$
6. Make l the subject of $t=2\pi\sqrt{\dfrac{l}{g}}$. $\left[l=\dfrac{t^2g}{4\pi^2}\right]$
7. Transpose $m=\dfrac{\mu L}{L+rCR}$ for L. $\left[L=\dfrac{mrCR}{\mu-m}\right]$
8. Make r the subject of the formula $\dfrac{x}{y}=\dfrac{1+r^2}{1-r^2}$. $\left[r=\sqrt{\left(\dfrac{x-y}{x+y}\right)}\right]$

(c) Simultaneous equations

Problem 18. Solve the simultaneous equations:

$$7x-2y=26 \quad (1)$$

$$6x+5y=29. \quad (2)$$

$5 \times$ equation (1) gives:

$$35x - 10y = 130 \qquad (3)$$

$2 \times$ equation (2) gives:

$$12x + 10y = 58 \qquad (4)$$

equation (3) + equation (4) gives:

$$47x + 0 = 188$$

from which, $$x = \frac{188}{47} = \mathbf{4}$$

Substituting $x = 4$ in equation (1) gives:

$$28 - 2y = 26$$

from which, $28 - 26 = 2y$ and $\boldsymbol{y = 1}$

Problem 19. Solve

$$\frac{x}{8} + \frac{5}{2} = y \qquad (1)$$

$$11 + \frac{y}{3} = 3x. \qquad (2)$$

$8 \times$ equation (1) gives: $x + 20 = 8y$ (3)

$3 \times$ equation (2) gives: $33 + y = 9x$ (4)

i.e. $x - 8y = -20$ (5)

and $9x - y = 33$ (6)

$8 \times$ equation (6) gives: $72x - 8y = 264$ (7)

Equation (7) − equation (5) gives:

$$71x = 284$$

from which, $$x = \frac{284}{71} = \mathbf{4}$$

Substituting $x = 4$ in equation (5) gives:

$$4 - 8y = -20$$

from which, $4 + 20 = 8y$ and $\boldsymbol{y = 3}$

(d) Quadratic equations

Problem 20. Solve the following equations by factorization:

(a) $3x^2 - 11x - 4 = 0$

(b) $4x^2 + 8x + 3 = 0$.

(a) The factors of $3x^2$ are $3x$ and x and these are placed in brackets thus:

$(3x \quad)(x \quad)$

The factors of -4 are $+1$ and -4 or -1 and $+4$, or -2 and $+2$. Remembering that the product of the two inner terms added to the product of the two outer terms must equal $-11x$, the only combination to give this is $+1$ and -4, i.e.,

$$3x^2 - 11x - 4 = (3x + 1)(x - 4)$$

Thus $(3x + 1)(x - 4) = 0$ hence

either $(3x + 1) = 0$ i.e. $\boldsymbol{x = -\frac{1}{3}}$

or $(x - 4) = 0$ i.e. $\boldsymbol{x = 4}$

(b) $4x^2 + 8x + 3 = (2x + 3)(2x + 1)$

Thus $(2x + 3)(2x + 1) = 0$ hence

either $(2x + 3) = 0$ i.e. $\boldsymbol{x = -\frac{3}{2}}$

or $(2x + 1) = 0$ i.e. $\boldsymbol{x = -\frac{1}{2}}$

Problem 21. The roots of a quadratic equation are $\frac{1}{3}$ and -2. Determine the equation in x.

If $\frac{1}{3}$ and -2 are the roots of a quadratic equation then,

$$(x - \tfrac{1}{3})(x + 2) = 0$$

i.e. $x^2 + 2x - \frac{1}{3}x - \frac{2}{3} = 0$

i.e. $x^2 + \frac{5}{3}x - \frac{2}{3} = 0$

or $\boldsymbol{3x^2 + 5x - 2 = 0}$

Problem 22. Solve $4x^2 + 7x + 2 = 0$ giving the answer correct to 2 decimal places.

From the quadratic formula if $ax^2 + bx + c = 0$ then,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Hence if $4x^2 + 7x + 2 = 0$

then $$x = \frac{-7 \pm \sqrt{7^2 - 4(4)(2)}}{2(4)}$$

$$= \frac{-7 \pm \sqrt{17}}{8}$$

$$= \frac{-7 \pm 4.123}{8}$$

$$= \frac{-7 + 4.123}{8} \text{ or } \frac{-7 - 4.123}{8}$$

i.e. $\boldsymbol{x = -0.36}$ or $\boldsymbol{-1.39}$

Now try the following exercise

Exercise 4 Further problems on simultaneous and quadratic equations

In problems 1 to 3, solve the simultaneous equations

1. $8x - 3y = 51$
   $3x + 4y = 14.$ $[x = 6,\ y = -1]$

2. $5a = 1 - 3b$
   $2b + a + 4 = 0.$ $[a = 2,\ b = -3]$

3. $\frac{x}{5} + \frac{2y}{3} = \frac{49}{15}$
   $\frac{3x}{7} - \frac{y}{2} + \frac{5}{7} = 0.$ $[x = 3,\ y = 4]$

4. Solve the following quadratic equations by factorization:
   (a) $x^2 + 4x - 32 = 0$
   (b) $8x^2 + 2x - 15 = 0.$
   $[$(a) $4, -8$ (b) $\frac{5}{4}, -\frac{3}{2}]$

5. Determine the quadratic equation in x whose roots are 2 and -5.
   $[x^2 + 3x - 10 = 0]$

6. Solve the following quadratic equations, correct to 3 decimal places:
   (a) $2x^2 + 5x - 4 = 0$
   (b) $4t^2 - 11t + 3 = 0.$
   $\begin{bmatrix}$(a) 0.637, -3.137
   (b) 2.443, 0.307$\end{bmatrix}$

1.4 Polynomial division

Before looking at long division in algebra let us revise long division with numbers (we may have forgotten, since calculators do the job for us!)

For example, $\frac{208}{16}$ is achieved as follows:

$$\begin{array}{r} 13 \\ 16\overline{)\,208} \\ \underline{16} \\ 48 \\ \underline{48} \\ \cdot\,\cdot \end{array}$$

(1) 16 divided into 2 won't go

(2) 16 divided into 20 goes 1

(3) Put 1 above the zero

(4) Multiply 16 by 1 giving 16

(5) Subtract 16 from 20 giving 4

(6) Bring down the 8

(7) 16 divided into 48 goes 3 times

(8) Put the 3 above the 8

(9) $3 \times 16 = 48$

(10) $48 - 48 = 0$

Hence $\frac{208}{16} = \mathbf{13}$ exactly

Similarly, $\frac{172}{15}$ is laid out as follows:

$$\begin{array}{r} 11 \\ 15\overline{)\,172} \\ \underline{15} \\ 22 \\ \underline{15} \\ 7 \end{array}$$

Hence $\frac{172}{15} = 11$ remainder 7 or $11 + \frac{7}{15} = \mathbf{11\frac{7}{15}}$

Below are some examples of division in algebra, which in some respects, is similar to long division with numbers.

(Note that a **polynomial** is an expression of the form

$$f(x) = a + bx + cx^2 + dx^3 + \cdots$$

and **polynomial division** is sometimes required when resolving into partial fractions—see Chapter 2.)

Problem 23. Divide $2x^2+x-3$ by $x-1$.

$2x^2+x-3$ is called the **dividend** and $x-1$ the **divisor**. The usual layout is shown below with the dividend and divisor both arranged in descending powers of the symbols.

$$\begin{array}{r} 2x+3 \\ x-1\,\overline{)\,2x^2+x-3} \\ \underline{2x^2-2x} \\ 3x-3 \\ \underline{3x-3} \\ \cdots \end{array}$$

Dividing the first term of the dividend by the first term of the divisor, i.e. $\frac{2x^2}{x}$ gives $2x$, which is put above the first term of the dividend as shown. The divisor is then multiplied by $2x$, i.e. $2x(x-1)=2x^2-2x$, which is placed under the dividend as shown. Subtracting gives $3x-3$. The process is then repeated, i.e. the first term of the divisor, x, is divided into $3x$, giving $+3$, which is placed above the dividend as shown. Then $3(x-1)=3x-3$ which is placed under the $3x-3$. The remainder, on subtraction, is zero, which completes the process.

Thus $\mathbf{(2x^2+x-3)\div(x-1)=(2x+3)}$

[A check can be made on this answer by multiplying $(2x+3)$ by $(x-1)$ which equals $2x^2+x-3$]

Problem 24. Divide $3x^3+x^2+3x+5$ by $x+1$.

$$\begin{array}{r} \text{(1)}\quad\text{(4)}\quad\text{(7)} \\ 3x^2-2x+5 \\ x+1\,\overline{)\,3x^3+x^2+3x+5} \\ \underline{3x^3+3x^2} \\ -2x^2+3x+5 \\ \underline{-2x^2-2x} \\ 5x+5 \\ \underline{5x+5} \\ \cdots \end{array}$$

(1) x into $3x^3$ goes $3x^2$. Put $3x^2$ above $3x^3$

(2) $3x^2(x+1)=3x^3+3x^2$

(3) Subtract

(4) x into $-2x^2$ goes $-2x$. Put $-2x$ above the dividend

(5) $-2x(x+1)=-2x^2-2x$

(6) Subtract

(7) x into $5x$ goes 5. Put 5 above the dividend

(8) $5(x+1)=5x+5$

(9) Subtract

Thus $\dfrac{3x^3+x^2+3x+5}{x+1}=\mathbf{3x^2-2x+5}$

Problem 25. Simplify $\dfrac{x^3+y^3}{x+y}$.

$$\begin{array}{r} \text{(1)}\quad\text{(4)}\quad\text{(7)} \\ x^2-xy+y^2 \\ x+y\,\overline{)\,x^3+0+0+y^3} \\ \underline{x^3+x^2y} \\ -x^2y+y^3 \\ \underline{-x^2y-xy^2} \\ xy^2+y^3 \\ \underline{xy^2+y^3} \\ \cdots \end{array}$$

(1) x into x^3 goes x^2. Put x^2 above x^3 of dividend

(2) $x^2(x+y)=x^3+x^2y$

(3) Subtract

(4) x into $-x^2y$ goes $-xy$. Put $-xy$ above dividend

(5) $-xy(x+y)=-x^2y-xy^2$

(6) Subtract

(7) x into xy^2 goes y^2. Put y^2 above dividend

(8) $y^2(x+y)=xy^2+y^3$

(9) Subtract

Thus

$$\frac{x^3+y^3}{x+y}=\mathbf{x^2-xy+y^2}$$

The zero's shown in the dividend are not normally shown, but are included to clarify the subtraction process and to keep similar terms in their respective columns.

Problem 26. Divide (x^2+3x-2) by $(x-2)$.

$$
\begin{array}{r}
x\ +5 \\
x-2\ \overline{)\ x^2+3x-\ 2} \\
\underline{x^2-2x\ \ \ } \\
5x-\ 2 \\
\underline{5x-10} \\
8
\end{array}
$$

Hence

$$\frac{x^2+3x-2}{x-2} = \boldsymbol{x+5+\frac{8}{x-2}}$$

Problem 27. Divide $4a^3-6a^2b+5b^3$ by $2a-b$.

$$
\begin{array}{r}
2a^2-2ab\ -\ \ b^2 \\
2a-b\ \overline{)\ 4a^3-6a^2b\ \ \ \ \ \ +5b^3} \\
\underline{4a^3-2a^2b\ \ \ \ \ \ \ \ \ \ } \\
-4a^2b\ \ \ \ \ \ +5b^3 \\
\underline{-4a^2b+2ab^2\ \ \ \ } \\
-2ab^2+5b^3 \\
\underline{-2ab^2+\ b^3} \\
4b^3
\end{array}
$$

Thus

$$\frac{4a^3-6a^2b+5b^3}{2a-b} = \boldsymbol{2a^2-2ab-b^2+\frac{4b^3}{2a-b}}$$

Now try the following exercise

Exercise 5 Further problems on polynomial division

1. Divide $(2x^2+xy-y^2)$ by $(x+y)$. $[2x-y]$
2. Divide $(3x^2+5x-2)$ by $(x+2)$. $[3x-1]$
3. Determine $(10x^2+11x-6)\div(2x+3)$. $[5x-2]$
4. Find $\dfrac{14x^2-19x-3}{2x-3}$. $[7x+1]$
5. Divide $(x^3+3x^2y+3xy^2+y^3)$ by $(x+y)$. $[x^2+2xy+y^2]$
6. Find $(5x^2-x+4)\div(x-1)$. $\left[5x+4+\dfrac{8}{x-1}\right]$
7. Divide $(3x^3+2x^2-5x+4)$ by $(x+2)$. $\left[3x^2-4x+3-\dfrac{2}{x+2}\right]$
8. Determine $(5x^4+3x^3-2x+1)/(x-3)$. $\left[5x^3+18x^2+54x+160+\dfrac{481}{x-3}\right]$

1.5 The factor theorem

There is a simple relationship between the factors of a quadratic expression and the roots of the equation obtained by equating the expression to zero.

For example, consider the quadratic equation $x^2+2x-8=0$.

To solve this we may factorize the quadratic expression x^2+2x-8 giving $(x-2)(x+4)$.

Hence $(x-2)(x+4)=0$.

Then, if the product of two numbers is zero, one or both of those numbers must equal zero. Therefore,

either $(x-2)=0$, from which, $x=2$

or $(x+4)=0$, from which, $x=-4$

It is clear then that a factor of $(x-2)$ indicates a root of $+2$, while a factor of $(x+4)$ indicates a root of -4.

In general, we can therefore say that:

a factor of $(x-a)$ corresponds to a root of $x=a$

In practice, we always deduce the roots of a simple quadratic equation from the factors of the quadratic expression, as in the above example. However, we could reverse this process. If, by trial and error, we could determine that $x=2$ is a root of the equation $x^2+2x-8=0$ we could deduce at once that $(x-2)$ is a factor of the

expression x^2+2x-8. We wouldn't normally solve quadratic equations this way — but suppose we have to factorize a cubic expression (i.e. one in which the highest power of the variable is 3). A cubic equation might have three simple linear factors and the difficulty of discovering all these factors by trial and error would be considerable. It is to deal with this kind of case that we use the **factor theorem**. This is just a generalized version of what we established above for the quadratic expression. The factor theorem provides a method of factorizing any polynomial, $f(x)$, which has simple factors.

A statement of the **factor theorem** says:

'if $x = a$ is a root of the equation $f(x) = 0$, then $(x - a)$ is a factor of $f(x)$'

The following worked problems show the use of the factor theorem.

Problem 28. Factorize x^3-7x-6 and use it to solve the cubic equation $x^3-7x-6=0$.

Let $f(x)=x^3-7x-6$

If $x=1$, then $f(1)=1^3-7(1)-6=-12$

If $x=2$, then $f(2)=2^3-7(2)-6=-12$

If $x=3$, then $f(3)=3^3-7(3)-6=0$

If $f(3)=0$, then $(x-3)$ is a factor — from the factor theorem.

We have a choice now. We can divide x^3-7x-6 by $(x-3)$ or we could continue our 'trial and error' by substituting further values for x in the given expression — and hope to arrive at $f(x)=0$.

Let us do both ways. Firstly, dividing out gives:

$$
\begin{array}{r}
x^2+3x+2 \\
x-3\,\overline{)\,x^3-0\quad -7x\ -6} \\
\underline{x^3-3x^2} \\
3x^2-7x\ -6 \\
\underline{3x^2-9x} \\
2x-6 \\
\underline{2x-6} \\
\ldots
\end{array}
$$

Hence $\dfrac{x^3-7x-6}{x-3}=x^2+3x+2$

i.e. $x^3-7x-6=(x-3)(x^2+3x+2)$

x^2+3x+2 factorizes 'on sight' as $(x+1)(x+2)$. Therefore

$$\boldsymbol{x^3-7x-6=(x-3)(x+1)(x+2)}$$

A second method is to continue to substitute values of x into $f(x)$.

Our expression for $f(3)$ was $3^3-7(3)-6$. We can see that if we continue with positive values of x the first term will predominate such that $f(x)$ will not be zero.

Therefore let us try some negative values for x. Therefore $f(-1)=(-1)^3-7(-1)-6=0$; hence $(x+1)$ is a factor (as shown above). Also $f(-2)=(-2)^3-7(-2)-6=0$; hence $(x+2)$ is a factor (also as shown above).

To solve $x^3-7x-6=0$, we substitute the factors, i.e.,

$$(x-3)(x+1)(x+2)=0$$

from which, $\boldsymbol{x=3}$, $\boldsymbol{x=-1}$ **and** $\boldsymbol{x=-2}$.

Note that the values of x, i.e. 3, -1 and -2, are all factors of the constant term, i.e. the 6. This can give us a clue as to what values of x we should consider.

Problem 29. Solve the cubic equation $x^3-2x^2-5x+6=0$ by using the factor theorem.

Let $f(x)=x^3-2x^2-5x+6$ and let us substitute simple values of x like 1, 2, 3, -1, -2, and so on.

$f(1)=1^3-2(1)^2-5(1)+6=0$,

hence $(x-1)$ is a factor

$f(2)=2^3-2(2)^2-5(2)+6\neq 0$

$f(3)=3^3-2(3)^2-5(3)+6=0$,

hence $(x-3)$ is a factor

$f(-1)=(-1)^3-2(-1)^2-5(-1)+6\neq 0$

$f(-2)=(-2)^3-2(-2)^2-5(-2)+6=0$,

hence $(x+2)$ is a factor

Hence $x^3-2x^2-5x+6=(x-1)(x-3)(x+2)$

Therefore if $x^3-2x^2-5x+6=0$

then $(x-1)(x-3)(x+2)=0$

from which, $\boldsymbol{x=1}$, $\boldsymbol{x=3}$ **and** $\boldsymbol{x=-2}$

Alternatively, having obtained one factor, i.e. $(x-1)$ we could divide this into (x^3-2x^2-5x+6) as follows:

$$
\begin{array}{r}
x^2 - x - 6 \\
\hline
x-1 \;\big)\; x^3 - 2x^2 - 5x + 6 \\
x^3 - x^2 \\
\hline
-x^2 - 5x + 6 \\
-x^2 + x \\
\hline
-6x + 6 \\
-6x + 6 \\
\hline
\cdot \quad \cdot \\
\hline
\end{array}
$$

Hence $x^3 - 2x^2 - 5x + 6$

$= (x-1)(x^2 - x - 6)$

$= \mathbf{(x-1)(x-3)(x+2)}$

Summarizing, the factor theorem provides us with a method of factorizing simple expressions, and an alternative, in certain circumstances, to polynomial division.

Now try the following exercise

Exercise 6 Further problems on the factor theorem

Use the factor theorem to factorize the expressions given in problems 1 to 4.

1. $x^2 + 2x - 3$ $[(x-1)(x+3)]$

2. $x^3 + x^2 - 4x - 4$ $[(x+1)(x+2)(x-2)]$

3. $2x^3 + 5x^2 - 4x - 7$ $[(x+1)(2x^2 + 3x - 7)]$

4. $2x^3 - x^2 - 16x + 15$ $[(x-1)(x+3)(2x-5)]$

5. Use the factor theorem to factorize $x^3 + 4x^2 + x - 6$ and hence solve the cubic equation $x^3 + 4x^2 + x - 6 = 0$.

$$\left[\begin{array}{r} x^3 + 4x^2 + x - 6 \\ = (x-1)(x+3)(x+2) \\ x = 1, x = -3 \text{ and } x = -2 \end{array}\right]$$

6. Solve the equation $x^3 - 2x^2 - x + 2 = 0$. $[x = 1, x = 2 \text{ and } x = -1]$

1.6 The remainder theorem

Dividing a general quadratic expression $(ax^2 + bx + c)$ by $(x - p)$, where p is any whole number, by long division (see section 1.3) gives:

$$
\begin{array}{r}
ax + (b + ap) \\
\hline
x - p \;\big)\; ax^2 + bx + c \\
ax^2 - apx \\
\hline
(b + ap)x + c \\
(b + ap)x - (b + ap)p \\
\hline
c + (b + ap)p \\
\hline
\end{array}
$$

The remainder, $c + (b + ap)p = c + bp + ap^2$ or $ap^2 + bp + c$. This is, in fact, what the **remainder theorem** states, i.e.,

'if $(ax^2 + bx + c)$ is divided by $(x - p)$, the remainder will be $ap^2 + bp + c$'

If, in the dividend $(ax^2 + bx + c)$, we substitute p for x we get the remainder $ap^2 + bp + c$.

For example, when $(3x^2 - 4x + 5)$ is divided by $(x - 2)$ the remainder is $ap^2 + bp + c$ (where $a = 3$, $b = -4$, $c = 5$ and $p = 2$),
i.e. the remainder is

$$3(2)^2 + (-4)(2) + 5 = 12 - 8 + 5 = \mathbf{9}$$

We can check this by dividing $(3x^2 - 4x + 5)$ by $(x - 2)$ by long division:

$$
\begin{array}{r}
3x + 2 \\
\hline
x - 2 \;\big)\; 3x^2 - 4x + 5 \\
3x^2 - 6x \\
\hline
2x + 5 \\
2x - 4 \\
\hline
9 \\
\hline
\end{array}
$$

Similarly, when $(4x^2 - 7x + 9)$ is divided by $(x + 3)$, the remainder is $ap^2 + bp + c$, (where $a = 4$, $b = -7$, $c = 9$ and $p = -3$) i.e. the remainder is $4(-3)^2 + (-7)(-3) + 9 = 36 + 21 + 9 = \mathbf{66}$.

Also, when $(x^2 + 3x - 2)$ is divided by $(x - 1)$, the remainder is $1(1)^2 + 3(1) - 2 = \mathbf{2}$.

It is not particularly useful, on its own, to know the remainder of an algebraic division. However, if the remainder should be zero then $(x - p)$ is a factor. This is very useful therefore when factorizing expressions.

For example, when $(2x^2 + x - 3)$ is divided by $(x - 1)$, the remainder is $2(1)^2 + 1(1) - 3 = 0$, which means that $(x - 1)$ is a factor of $(2x^2 + x - 3)$.

In this case the other factor is $(2x+3)$, i.e.,

$$(2x^2+x-3)=(x-1)(2x-3)$$

The **remainder theorem** may also be stated for a **cubic equation** as:

'if (ax^3+bx^2+cx+d) is divided by $(x-p)$, the remainder will be ap^3+bp^2+cp+d'

As before, the remainder may be obtained by substituting p for x in the dividend.

For example, when $(3x^3+2x^2-x+4)$ is divided by $(x-1)$, the remainder is ap^3+bp^2+cp+d (where $a=3$, $b=2$, $c=-1$, $d=4$ and $p=1$), i.e. the remainder is $3(1)^3+2(1)^2+(-1)(1)+4=3+2-1+4=\mathbf{8}$.

Similarly, when (x^3-7x-6) is divided by $(x-3)$, the remainder is $1(3)^3+0(3)^2-7(3)-6=0$, which means that $(x-3)$ is a factor of (x^3-7x-6).

Here are some more examples on the remainder theorem.

Problem 30. Without dividing out, find the remainder when $2x^2-3x+4$ is divided by $(x-2)$.

By the remainder theorem, the remainder is given by ap^2+bp+c, where $a=2$, $b=-3$, $c=4$ and $p=2$.
Hence **the remainder is:**

$$2(2)^2+(-3)(2)+4=8-6+4=\mathbf{6}$$

Problem 31. Use the remainder theorem to determine the remainder when $(3x^3-2x^2+x-5)$ is divided by $(x+2)$.

By the remainder theorem, the remainder is given by ap^3+bp^2+cp+d, where $a=3$, $b=-2$, $c=1$, $d=-5$ and $p=-2$.
Hence **the remainder is:**

$$\begin{aligned}&3(-2)^3+(-2)(-2)^2+(1)(-2)+(-5)\\&=-24-8-2-5\\&=\mathbf{-39}\end{aligned}$$

Problem 32. Determine the remainder when (x^3-2x^2-5x+6) is divided by (a) $(x-1)$ and (b) $(x+2)$. Hence factorize the cubic expression.

(a) When (x^3-2x^2-5x+6) is divided by $(x-1)$, the remainder is given by ap^3+bp^2+cp+d, where $a=1$, $b=-2$, $c=-5$, $d=6$ and $p=1$,

i.e. **the remainder** $=(1)(1)^3+(-2)(1)^2+(-5)(1)+6$

$$=1-2-5+6=\mathbf{0}$$

Hence $(x-1)$ is a factor of (x^3-2x^2-5x+6).

(b) When (x^3-2x^2-5x+6) is divided by $(x+2)$, **the remainder is** given by

$$(1)(-2)^3+(-2)(-2)^2+(-5)(-2)+6$$

$$=-8-8+10+6=\mathbf{0}$$

Hence $(x+2)$ is also a factor of (x^3-2x^2-5x+6). Therefore $(x-1)(x+2)(x\ \ )=x^3-2x^2-5x+6$. To determine the third factor (shown blank) we could

(i) divide (x^3-2x^2-5x+6) by $(x-1)(x+2)$.

or (ii) use the factor theorem where $f(x)=x^3-2x^2-5x+6$ and hoping to choose a value of x which makes $f(x)=0$.

or (iii) use the remainder theorem, again hoping to choose a factor $(x-p)$ which makes the remainder zero.

(i) Dividing (x^3-2x^2-5x+6) by (x^2+x-2) gives:

$$\begin{array}{r} x\ -3 \\ \hline x^2+x-2\ \big)\ x^3-2x^2-5x+6 \\ \underline{x^3+\ x^2-2x} \\ -3x^2-3x+6 \\ \underline{-3x^2-3x+6} \\ \quad .\ \ .\ \ . \\ \hline \end{array}$$

Thus $\mathbf{(x^3-2x^2-5x+6)=(x-1)(x+2)(x-3)}$

(ii) Using the factor theorem, we let

$$f(x)=x^3-2x^2-5x+6$$

Then $f(3)=3^3-2(3)^2-5(3)+6$

$$=27-18-15+6=0$$

Hence $(x-3)$ is a factor.

(iii) Using the remainder theorem, when (x^3-2x^2-5x+6) is divided by $(x-3)$, the remainder is given by

$ap^3 + bp^2 + cp + d$, where $a = 1$, $b = -2$, $c = -5$, $d = 6$ and $p = 3$.

Hence the remainder is:

$$1(3)^3 + (-2)(3)^2 + (-5)(3) + 6$$
$$= 27 - 18 - 15 + 6 = 0$$

Hence $(x - 3)$ is a factor.

Thus $\mathbf{(x^3 - 2x^2 - 5x + 6)}$
$\mathbf{= (x - 1)(x + 2)(x - 3)}$

Now try the following exercise

Exercise 7 Further problems on the remainder theorem

1. Find the remainder when $3x^2 - 4x + 2$ is divided by

   (a) $(x - 2)$ (b) $(x + 1)$. [(a) 6 (b) 9]

2. Determine the remainder when $x^3 - 6x^2 + x - 5$ is divided by

   (a) $(x + 2)$ (b) $(x - 3)$. [(a) −39 (b) −29]

3. Use the remainder theorem to find the factors of $x^3 - 6x^2 + 11x - 6$.

   $[(x - 1)(x - 2)(x - 3)]$

4. Determine the factors of $x^3 + 7x^2 + 14x + 8$ and hence solve the cubic equation $x^3 + 7x^2 + 14x + 8 = 0$.

   $[x = -1, x = -2$ and $x = -4]$

5. Determine the value of 'a' if $(x + 2)$ is a factor of $(x^3 - ax^2 + 7x + 10)$.

   $[a = -3]$

6. Using the remainder theorem, solve the equation $2x^3 - x^2 - 7x + 6 = 0$.

   $[x = 1, x = -2$ and $x = 1.5]$

Chapter 2

Partial fractions

2.1 Introduction to partial fractions

By algebraic addition,

$$\frac{1}{x-2}+\frac{3}{x+1}=\frac{(x+1)+3(x-2)}{(x-2)(x+1)}$$

$$=\frac{4x-5}{x^2-x-2}$$

The reverse process of moving from $\dfrac{4x-5}{x^2-x-2}$ to $\dfrac{1}{x-2}+\dfrac{3}{x+1}$ is called resolving into **partial fractions**.

In order to resolve an algebraic expression into partial fractions:

(i) the denominator must factorize (in the above example, x^2-x-2 factorizes as $(x-2)\,(x+1)$), and

(ii) the numerator must be at least one degree less than the denominator (in the above example $(4x-5)$ is of degree 1 since the highest powered x term is x^1 and (x^2-x-2) is of degree 2).

When the degree of the numerator is equal to or higher than the degree of the denominator, the numerator must be divided by the denominator until the remainder is of less degree than the denominator (see Problems 3 and 4).

There are basically three types of partial fraction and the form of partial fraction used is summarized in Table 2.1, where $f(x)$ is assumed to be of less degree than the relevant denominator and A, B and C are constants to be determined.

(In the latter type in Table 2.1, ax^2+bx+c is a quadratic expression which does not factorize without containing surds or imaginary terms.)

Resolving an algebraic expression into partial fractions is used as a preliminary to integrating certain functions (see Chapter 41) and in determining inverse Laplace transforms (see Chapter 63).

2.2 Worked problems on partial fractions with linear factors

Problem 1. Resolve $\dfrac{11-3x}{x^2+2x-3}$ into partial fractions.

Table 2.1

Type	Denominator containing	Expression	Form of partial fraction
1	Linear factors (see Problems 1 to 4)	$\dfrac{f(x)}{(x+a)(x-b)(x+c)}$	$\dfrac{A}{(x+a)}+\dfrac{B}{(x-b)}+\dfrac{C}{(x+c)}$
2	Repeated linear factors (see Problems 5 to 7)	$\dfrac{f(x)}{(x+a)^3}$	$\dfrac{A}{(x+a)}+\dfrac{B}{(x+a)^2}+\dfrac{C}{(x+a)^3}$
3	Quadratic factors (see Problems 8 and 9)	$\dfrac{f(x)}{(ax^2+bx+c)(x+d)}$	$\dfrac{Ax+B}{(ax^2+bx+c)}+\dfrac{C}{(x+d)}$

The denominator factorizes as $(x-1)(x+3)$ and the numerator is of less degree than the denominator. Thus $\dfrac{11-3x}{x^2+2x-3}$ may be resolved into partial fractions.

Let $\dfrac{11-3x}{x^2+2x-3} \equiv \dfrac{11-3x}{(x-1)(x+3)}$

$$\equiv \frac{A}{(x-1)} + \frac{B}{(x+3)}$$

where A and B are constants to be determined,

i.e. $\dfrac{11-3x}{(x-1)(x+3)} \equiv \dfrac{A(x+3)+B(x-1)}{(x-1)(x+3)}$,

by algebraic addition.

Since the denominators are the same on each side of the identity then the numerators are equal to each other.

Thus, $11-3x \equiv A(x+3)+B(x-1)$

To determine constants A and B, values of x are chosen to make the term in A or B equal to zero.

When $x=1$, then

$$11-3(1) \equiv A(1+3)+B(0)$$

i.e. $\quad 8 = 4A$

i.e. $\quad \boldsymbol{A = 2}$

When $x=-3$, then

$$11-3(-3) \equiv A(0)+B(-3-1)$$

i.e. $\quad 20 = -4B$

i.e. $\quad \boldsymbol{B = -5}$

Thus $\dfrac{\mathbf{11-3x}}{\mathbf{x^2+2x-3}} \equiv \dfrac{2}{(x-1)} + \dfrac{-5}{(x+3)}$

$$\equiv \frac{\mathbf{2}}{\mathbf{(x-1)}} - \frac{\mathbf{5}}{\mathbf{(x+3)}}$$

$$\left[\text{Check: } \frac{2}{(x-1)} - \frac{5}{(x+3)} = \frac{2(x+3)-5(x-1)}{(x-1)(x+3)} = \frac{11-3x}{x^2+2x-3}\right]$$

Problem 2. Convert $\dfrac{2x^2-9x-35}{(x+1)(x-2)(x+3)}$ into the sum of three partial fractions.

Let $\dfrac{2x^2-9x-35}{(x+1)(x-2)(x+3)}$

$$\equiv \frac{A}{(x+1)} + \frac{B}{(x-2)} + \frac{C}{(x+3)}$$

$$\equiv \frac{\begin{pmatrix}A(x-2)(x+3)+B(x+1)(x+3)\\ +C(x+1)(x-2)\end{pmatrix}}{(x+1)(x-2)(x+3)}$$

by algebraic addition.

Equating the numerators gives:

$$2x^2-9x-35 \equiv A(x-2)(x+3) + B(x+1)(x+3) + C(x+1)(x-2)$$

Let $x=-1$. Then

$$2(-1)^2-9(-1)-35 \equiv A(-3)(2) + \mathrm{B}(0)(2) + C(0)(-3)$$

i.e. $\quad -24 = -6A$

i.e. $\quad \boldsymbol{A} = \dfrac{-24}{-6} = \mathbf{4}$

Let $x=2$. Then

$$2(2)^2-9(2)-35 \equiv A(0)(5)+B(3)(5)+C(3)(0)$$

i.e. $\quad -45 = 15B$

i.e. $\quad \boldsymbol{B} = \dfrac{-45}{15} = \mathbf{-3}$

Let $x=-3$. Then

$$2(-3)^2-9(-3)-35 \equiv A(-5)(0)+B(-2)(0) + C(-2)(-5)$$

i.e. $\quad 10 = 10C$

i.e. $\quad \boldsymbol{C = 1}$

Thus $\dfrac{2x^2-9x-35}{(x+1)(x-2)(x+3)}$

$$\equiv \frac{\mathbf{4}}{\mathbf{(x+1)}} - \frac{\mathbf{3}}{\mathbf{(x-2)}} + \frac{\mathbf{1}}{\mathbf{(x+3)}}$$

Problem 3. Resolve $\dfrac{x^2+1}{x^2-3x+2}$ into partial fractions.

The denominator is of the same degree as the numerator. Thus dividing out gives:

$$x^2-3x+2\overline{)\begin{array}{r}1\\ x^2+1\\ x^2-3x+2\\ \hline 3x-1\end{array}}$$

For more on polynomial division, see Section 1.4, page 6.

Hence $\dfrac{x^2+1}{x^2-3x+2} \equiv 1+\dfrac{3x-1}{x^2-3x+2}$

$$\equiv 1+\frac{3x-1}{(x-1)(x-2)}$$

Let $\dfrac{3x-1}{(x-1)(x-2)} \equiv \dfrac{A}{(x-1)}+\dfrac{B}{(x-2)}$

$$\equiv \frac{A(x-2)+B(x-1)}{(x-1)(x-2)}$$

Equating numerators gives:

$$3x-1 \equiv A(x-2)+B(x-1)$$

Let $x=1$. Then $2=-A$

i.e. $\boldsymbol{A=-2}$

Let $x=2$. Then $\boldsymbol{5=B}$

Hence $\dfrac{3x-1}{(x-1)(x-2)} \equiv \dfrac{-2}{(x-1)}+\dfrac{5}{(x-2)}$

Thus $\dfrac{\boldsymbol{x^2+1}}{\boldsymbol{x^2-3x+2}} \boldsymbol{\equiv 1-\dfrac{2}{(x-1)}+\dfrac{5}{(x-2)}}$

Problem 4. Express $\dfrac{x^3-2x^2-4x-4}{x^2+x-2}$ in partial fractions.

The numerator is of higher degree than the denominator. Thus dividing out gives:

$$x^2+x-2\overline{)\begin{array}{r}x-3\\ x^3-2x^2-4x-4\\ x^3+x^2-2x\\ \hline -3x^2-2x-4\\ -3x^2-3x+6\\ \hline x-10\end{array}}$$

Thus $\dfrac{x^3-2x^2-4x-4}{x^2+x-2} \equiv x-3+\dfrac{x-10}{x^2+x-2}$

$$\equiv x-3+\frac{x-10}{(x+2)(x-1)}$$

Let $\dfrac{x-10}{(x+2)(x-1)} \equiv \dfrac{A}{(x+2)}+\dfrac{B}{(x-1)}$

$$\equiv \frac{A(x-1)+B(x+2)}{(x+2)(x-1)}$$

Equating the numerators gives:

$$x-10 \equiv A(x-1)+B(x+2)$$

Let $x=-2$. Then $-12=-3A$

i.e. $\boldsymbol{A=4}$

Let $x=1$. Then $-9=3B$

i.e. $\boldsymbol{B=-3}$

Hence $\dfrac{x-10}{(x+2)(x-1)} \equiv \dfrac{4}{(x+2)}-\dfrac{3}{(x-1)}$

Thus $\dfrac{\boldsymbol{x^3-2x^2-4x-4}}{\boldsymbol{x^2+x-2}}$

$$\boldsymbol{\equiv x-3+\frac{4}{(x+2)}-\frac{3}{(x-1)}}$$

Now try the following exercise

Exercise 8 Further problems on partial fractions with linear factors

Resolve the following into partial fractions.

1. $\dfrac{12}{x^2-9}$ $\left[\dfrac{2}{(x-3)}-\dfrac{2}{(x+3)}\right]$

2. $\dfrac{4(x-4)}{x^2-2x-3}$ $\left[\dfrac{5}{(x+1)}-\dfrac{1}{(x-3)}\right]$

3. $\dfrac{x^2-3x+6}{x(x-2)(x-1)}$ $\left[\dfrac{3}{x}+\dfrac{2}{(x-2)}-\dfrac{4}{(x-1)}\right]$

4. $\dfrac{3(2x^2-8x-1)}{(x+4)(x+1)(2x-1)}$

$$\left[\frac{7}{(x+4)}-\frac{3}{(x+1)}-\frac{2}{(2x-1)}\right]$$

5. $\dfrac{x^2+9x+8}{x^2+x-6}$ $\left[1+\dfrac{2}{(x+3)}+\dfrac{6}{(x-2)}\right]$

6. $\dfrac{x^2-x-14}{x^2-2x-3}$ $\left[1-\dfrac{2}{(x-3)}+\dfrac{3}{(x+1)}\right]$

7. $\dfrac{3x^3-2x^2-16x+20}{(x-2)(x+2)}$ $\left[3x-2+\dfrac{1}{(x-2)}-\dfrac{5}{(x+2)}\right]$

2.3 Worked problems on partial fractions with repeated linear factors

Problem 5. Resolve $\dfrac{2x+3}{(x-2)^2}$ into partial fractions.

The denominator contains a repeated linear factor, $(x-2)^2$.

Let $\dfrac{2x+3}{(x-2)^2} \equiv \dfrac{A}{(x-2)}+\dfrac{B}{(x-2)^2}$

$$\equiv \frac{A(x-2)+B}{(x-2)^2}$$

Equating the numerators gives:

$$2x+3 \equiv A(x-2)+B$$

Let $x=2$. Then $\quad 7 = A(0)+B$

i.e. $\quad \boldsymbol{B=7}$

$$2x+3 \equiv A(x-2)+B \equiv Ax-2A+B$$

Since an identity is true for all values of the unknown, the coefficients of similar terms may be equated.

Hence, equating the coefficients of x gives: $\boldsymbol{2=A}$.

[Also, as a check, equating the constant terms gives:

$$3 = -2A+B$$

When $A=2$ and $B=7$,

$$\text{R.H.S.} = -2(2)+7 = 3 = \text{L.H.S.}]$$

Hence $\dfrac{\boldsymbol{2x+3}}{\boldsymbol{(x-2)^2}} \equiv \dfrac{\boldsymbol{2}}{\boldsymbol{(x-2)}}+\dfrac{\boldsymbol{7}}{\boldsymbol{(x-2)^2}}$

Problem 6. Express $\dfrac{5x^2-2x-19}{(x+3)(x-1)^2}$ as the sum of three partial fractions.

The denominator is a combination of a linear factor and a repeated linear factor.

Let $\dfrac{5x^2-2x-19}{(x+3)(x-1)^2}$

$$\equiv \frac{A}{(x+3)}+\frac{B}{(x-1)}+\frac{C}{(x-1)^2}$$

$$\equiv \frac{A(x-1)^2+B(x+3)(x-1)+C(x+3)}{(x+3)(x-1)^2}$$

by algebraic addition.

Equating the numerators gives:

$$5x^2-2x-19 \equiv A(x-1)^2+B(x+3)(x-1) + C(x+3) \qquad (1)$$

Let $x=-3$. Then

$$5(-3)^2-2(-3)-19 \equiv A(-4)^2+B(0)(-4)+C(0)$$

i.e. $\quad 32 = 16A$

i.e. $\quad \boldsymbol{A=2}$

Let $x=1$. Then

$$5(1)^2-2(1)-19 \equiv A(0)^2+B(4)(0)+C(4)$$

i.e. $\quad -16 = 4C$

i.e. $\quad \boldsymbol{C=-4}$

Without expanding the RHS of equation (1) it can be seen that equating the coefficients of x^2 gives: $5=A+B$, and since $A=2$, $\boldsymbol{B=3}$.

[Check: Identity (1) may be expressed as:

$$5x^2-2x-19 \equiv A(x^2-2x+1) + B(x^2+2x-3)+C(x+3)$$

i.e. $5x^2-2x-19 \equiv Ax^2-2Ax+A+Bx^2+2Bx -3B+Cx+3C$

Equating the x term coefficients gives:

$$-2 \equiv -2A + 2B + C$$

When $A = 2$, $B = 3$ and $C = -4$ then

$$\begin{aligned} -2A + 2B + C &= -2(2) + 2(3) - 4 \\ &= -2 = \text{LHS} \end{aligned}$$

Equating the constant term gives:

$$-19 \equiv A - 3B + 3C$$

$$\begin{aligned} \text{RHS} &= 2 - 3(3) + 3(-4) = 2 - 9 - 12 \\ &= -19 = \text{LHS}] \end{aligned}$$

Hence $\dfrac{\mathbf{5x^2 - 2x - 19}}{\mathbf{(x+3)(x-1)^2}}$

$$\equiv \frac{\mathbf{2}}{\mathbf{(x+3)}} + \frac{\mathbf{3}}{\mathbf{(x-1)}} - \frac{\mathbf{4}}{\mathbf{(x-1)^2}}$$

Problem 7. Resolve $\dfrac{3x^2 + 16x + 15}{(x+3)^3}$ into partial fractions.

Let $\dfrac{3x^2 + 16x + 15}{(x+3)^3}$

$$\equiv \frac{A}{(x+3)} + \frac{B}{(x+3)^2} + \frac{C}{(x+3)^3}$$

$$\equiv \frac{A(x+3)^2 + B(x+3) + C}{(x+3)^3}$$

Equating the numerators gives:

$$3x^2 + 16x + 15 \equiv A(x+3)^2 + B(x+3) + C \qquad (1)$$

Let $x = -3$. Then

$$3(-3)^2 + 16(-3) + 15 \equiv A(0)^2 + B(0) + C$$

i.e. $\mathbf{-6 = C}$

Identity (1) may be expanded as:

$$\begin{aligned} 3x^2 + 16x + 15 &\equiv A(x^2 + 6x + 9) \\ &\quad + B(x+3) + C \end{aligned}$$

$$\begin{aligned} \text{i.e. } 3x^2 + 16x + 15 &\equiv Ax^2 + 6Ax + 9A \\ &\quad + Bx + 3B + C \end{aligned}$$

Equating the coefficients of x^2 terms gives: $\mathbf{3 = A}$

Equating the coefficients of x terms gives:

$$16 = 6A + B$$

Since $A = 3$, $\boldsymbol{B = -2}$

[Check: equating the constant terms gives:

$$15 = 9A + 3B + C$$

When $A = 3$, $B = -2$ and $C = -6$,

$$\begin{aligned} 9A + 3B + C &= 9(3) + 3(-2) + (-6) \\ &= 27 - 6 - 6 = 15 = \text{LHS}] \end{aligned}$$

Thus $\dfrac{\mathbf{3x^2 + 16x + 15}}{\mathbf{(x+3)^3}}$

$$\equiv \frac{\mathbf{3}}{\mathbf{(x+3)}} - \frac{\mathbf{2}}{\mathbf{(x+3)^2}} - \frac{\mathbf{6}}{\mathbf{(x+3)^3}}$$

Now try the following exercise

Exercise 9 Further problems on partial fractions with linear factors

1. $\dfrac{4x-3}{(x+1)^2}$ $\left[\dfrac{4}{(x+1)} - \dfrac{7}{(x+1)^2}\right]$

2. $\dfrac{x^2+7x+3}{x^2(x+3)}$ $\left[\dfrac{1}{x^2} + \dfrac{2}{x} - \dfrac{1}{(x+3)}\right]$

3. $\dfrac{5x^2 - 30x + 44}{(x-2)^3}$ $\left[\dfrac{5}{(x-2)} - \dfrac{10}{(x-2)^2} + \dfrac{4}{(x-2)^3}\right]$

4. $\dfrac{18 + 21x - x^2}{(x-5)(x+2)^2}$ $\left[\dfrac{2}{(x-5)} - \dfrac{3}{(x+2)} + \dfrac{4}{(x+2)^2}\right]$

2.4 Worked problems on partial fractions with quadratic factors

Problem 8. Express $\dfrac{7x^2 + 5x + 13}{(x^2+2)(x+1)}$ in partial fractions.

The denominator is a combination of a quadratic factor, (x^2+2), which does not factorize without introducing imaginary surd terms, and a linear factor, $(x+1)$. Let,

$$\frac{7x^2+5x+13}{(x^2+2)(x+1)} \equiv \frac{Ax+B}{(x^2+2)} + \frac{C}{(x+1)}$$
$$\equiv \frac{(Ax+B)(x+1)+C(x^2+2)}{(x^2+2)(x+1)}$$

Equating numerators gives:

$$7x^2+5x+13 \equiv (Ax+B)(x+1)+C(x^2+2) \quad (1)$$

Let $x=-1$. Then

$7(-1)^2+5(-1)+13 \equiv (Ax+B)(0)+C(1+2)$

i.e. $15=3C$

i.e. $\boldsymbol{C=5}$

Identity (1) may be expanded as:

$$7x^2+5x+13 \equiv Ax^2+Ax+Bx+B+Cx^2+2C$$

Equating the coefficients of x^2 terms gives:

$7 = A+C$, and since $C=5$, $\boldsymbol{A=2}$

Equating the coefficients of x terms gives:

$5 = A+B$, and since $A=2$, $\boldsymbol{B=3}$

[Check: equating the constant terms gives:

$$13 = B+2C$$

When $B=3$ and $C=5$,

$B+2C = 3+10 = 13 = \text{LHS}$]

Hence $\dfrac{\mathbf{7x^2+5x+13}}{\mathbf{(x^2+2)(x+1)}} \equiv \dfrac{\mathbf{2x+3}}{\mathbf{(x^2+2)}} + \dfrac{\mathbf{5}}{\mathbf{(x+1)}}$

Problem 9. Resolve $\dfrac{3+6x+4x^2-2x^3}{x^2(x^2+3)}$ into partial fractions.

Terms such as x^2 may be treated as $(x+0)^2$, i.e. they are repeated linear factors.

Let $\dfrac{3+6x+4x^2-2x^3}{x^2(x^2+3)} \equiv \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{Cx+D}{(x^2+3)}$

$$\equiv \frac{Ax(x^2+3)+B(x^2+3)+(Cx+D)x^2}{x^2(x^2+3)}$$

Equating the numerators gives:

$$3+6x+4x^2-2x^3 \equiv Ax(x^2+3)+B(x^2+3)+(Cx+D)x^2$$
$$\equiv Ax^3+3Ax+Bx^2+3B+Cx^3+Dx^2$$

Let $x=0$. Then $3=3B$

i.e. $\boldsymbol{B=1}$

Equating the coefficients of x^3 terms gives:

$$-2 = A+C \quad (1)$$

Equating the coefficients of x^2 terms gives:

$$4 = B+D$$

Since $B=1$, $\boldsymbol{D=3}$

Equating the coefficients of x terms gives:

$$6 = 3A$$

i.e. $\boldsymbol{A=2}$

From equation (1), since $A=2$, $\boldsymbol{C=-4}$

Hence $\dfrac{\mathbf{3+6x+4x^2-2x^3}}{\mathbf{x^2(x^2+3)}} \equiv \dfrac{2}{x} + \dfrac{1}{x^2} + \dfrac{-4x+3}{x^2+3}$

$$\equiv \frac{\mathbf{2}}{\mathbf{x}} + \frac{\mathbf{1}}{\mathbf{x^2}} + \frac{\mathbf{3-4x}}{\mathbf{x^2+3}}$$

Now try the following exercise

Exercise 10 Further problems on partial fractions with quadratic factors

1. $\dfrac{x^2-x-13}{(x^2+7)(x-2)}$ $\left[\dfrac{2x+3}{(x^2+7)} - \dfrac{1}{(x-2)}\right]$

2. $\dfrac{6x-5}{(x-4)(x^2+3)}$ $\left[\dfrac{1}{(x-4)} + \dfrac{2-x}{(x^2+3)}\right]$

3. $\dfrac{15+5x+5x^2-4x^3}{x^2(x^2+5)}$ $\left[\dfrac{1}{x} + \dfrac{3}{x^2} + \dfrac{2-5x}{(x^2+5)}\right]$

4. $\dfrac{x^3+4x^2+20x-7}{(x-1)^2(x^2+8)}$

$$\left[\frac{3}{(x-1)}+\frac{2}{(x-1)^2}+\frac{1-2x}{(x^2+8)}\right]$$

5. When solving the differential equation $\dfrac{\mathrm{d}^2\theta}{\mathrm{d}t^2}-6\dfrac{\mathrm{d}\theta}{\mathrm{d}t}-10\theta=20-\mathrm{e}^{2t}$ by Laplace transforms, for given boundary conditions, the following expression for $\mathcal{L}\{\theta\}$ results:

$$\mathcal{L}\{\theta\}=\frac{4s^3-\dfrac{39}{2}s^2+42s-40}{s(s-2)(s^2-6s+10)}$$

Show that the expression can be resolved into partial fractions to give:

$$\mathcal{L}\{\theta\}=\frac{2}{s}-\frac{1}{2(s-2)}+\frac{5s-3}{2(s^2-6s+10)}$$

Chapter 3

Logarithms

3.1 Introduction to logarithms

With the use of calculators firmly established, logarithmic tables are now rarely used for calculation. However, the theory of logarithms is important, for there are several scientific and engineering laws that involve the rules of logarithms.

From the laws of indices: $16 = 2^4$

The number 4 is called the **power** or the **exponent** or the **index**. In the expression 2^4, the number 2 is called the **base**.

In another example: $64 = 8^2$

In this example, 2 is the power, or exponent, or index. The number 8 is the base.

What is a logarithm?

Consider the expression $16 = 2^4$.

An alternative, yet equivalent, way of writing this expression is: $\log_2 16 = 4$.

This is stated as 'log to the base 2 of 16 equals 4'.

We see that the logarithm is the same as the power or index in the original expression. It is the base in the original expression which becomes the base of the logarithm.

The two statements: $16 = 2^4$ and $\log_2 16 = 4$ are equivalent.

If we write either of them, we are automatically implying the other.

In general, if a number y can be written in the form a^x, then the index 'x' is called the 'logarithm of y to the base of a',

i.e. if $\boldsymbol{y = a^x}$ then $\boldsymbol{x = \log_a y}$

In another example, if we write down that $64 = 8^2$ then the equivalent statement using logarithms is:

$$\log_8 64 = 2$$

In another example, if we write down that: $\log_3 81 = 4$ then the equivalent statement using powers is:

$$3^4 = 81$$

So the two sets of statements, one involving powers and one involving logarithms, are equivalent.

Common logarithms

From above, if we write down that: $1000 = 10^3$, then $3 = \log_{10} 1000$

This may be checked using the 'log' button on your calculator.

Logarithms having a base of 10 are called **common logarithms** and $\log_{10}$ is often abbreviated to lg.

The following values may be checked by using a calculator:

$$\lg 27.5 = 1.4393\ldots, \quad \lg 378.1 = 2.5776\ldots$$
$$\text{and} \quad \lg 0.0204 = -1.6903\ldots$$

Napierian logarithms

Logarithms having a base of e (where 'e' is a mathematical constant approximately equal to 2.7183) are called **hyperbolic**, **Napierian** or **natural logarithms**, and $\log_e$ is usually abbreviated to ln.

The following values may be checked by using a calculator:

$$\ln 3.65 = 1.2947\ldots, \ln 417.3 = 6.0338\ldots$$
$$\text{and} \quad \ln 0.182 = -1.7037\ldots$$

More on Napierian logarithms is explained in Chapter 4 following.

Here are some worked problems to help understanding of logarithms.

Problem 1. Evaluate $\log_3 9$.

Let $x = \log_3 9$ then $3^x = 9$ from the definition of a logarithm,

i.e. $3^x = 3^2$ from which, $x = 2$

Hence, $\mathbf{\log_3 9 = 2}$

Problem 2. Evaluate $\log_{10} 10$.

Let $x = \log_{10} 10$ then $10^x = 10$ from the definition of a logarithm,

i.e. $10^x = 10^1$ from which, $x = 1$

Hence, $\mathbf{\log_{10} 10 = 1}$ (which may be checked by a calculator)

Problem 3. Evaluate $\log_{16} 8$.

Let $x = \log_{16} 8$ then $16^x = 8$ from the definition of a logarithm,

i.e. $(2^4)^x = 2^3$ i.e. $2^{4x} = 2^3$ from the laws of indices,

from which, $4x = 3$ and $x = \frac{3}{4}$

Hence, $\mathbf{\log_{16} 8 = \frac{3}{4}}$

Problem 4. Evaluate $\lg 0.001$.

Let $x = \lg 0.001 = \log_{10} 0.001$ then $10^x = 0.001$

i.e. $10^x = 10^{-3}$ from which, $x = -3$

Hence, $\mathbf{\lg 0.001 = -3}$ (which may be checked by a calculator)

Problem 5. Evaluate $\ln e$.

Let $x = \ln e = \log_e e$ then $e^x = e$

i.e. $e^x = e^1$

from which, $x = 1$

Hence, $\mathbf{\ln e = 1}$ (which may be checked by a calculator)

Problem 6. Evaluate $\log_3 \frac{1}{81}$.

Let $x = \log_3 \frac{1}{81}$ then $3^x = \frac{1}{81} = \frac{1}{3^4} = 3^{-4}$

from which, $x = -4$

Hence, $\mathbf{\log_3 \frac{1}{81} = -4}$

Problem 7. Solve the equation: $\lg x = 3$.

If $\lg x = 3$ then $\log_{10} x = 3$

and $x = 10^3$ i.e. $\boldsymbol{x = 1000}$

Problem 8. Solve the equation: $\log_2 x = 5$.

If $\log_2 x = 5$ then $\boldsymbol{x = 2^5 = 32}$

Problem 9. Solve the equation: $\log_5 x = -2$.

If $\log_5 x = -2$ then $x = 5^{-2} = \frac{1}{5^2} = \mathbf{\frac{1}{25}}$

Now try the following exercise

Exercise 11 Further problems on laws of logarithms

In Problems 1 to 11, evaluate the given expressions:

1. $\log_{10} 10000$ [4]
2. $\log_2 16$ [4]
3. $\log_5 125$ [3]
4. $\log_2 \frac{1}{8}$ [−3]
5. $\log_8 2$ $\left[\frac{1}{3}\right]$
6. $\log_7 343$ [3]
7. $\lg 100$ [2]
8. $\lg 0.01$ [−2]
9. $\log_4 8$ $\left[1\frac{1}{2}\right]$
10. $\log_{27} 3$ $\left[\frac{1}{3}\right]$
11. $\ln e^2$ [2]

In Problems 12 to 18 solve the equations:

12. $\log_{10} x = 4$ [10000]
13. $\lg x = 5$ [100000]
14. $\log_3 x = 2$ [9]
15. $\log_4 x = -2\frac{1}{2}$ $\left[\frac{1}{32}\right]$
16. $\lg x = -2$ [0.01]
17. $\log_8 x = -\frac{4}{3}$ $\left[\frac{1}{16}\right]$
18. $\ln x = 3$ [e^3]

3.2 Laws of logarithms

There are three laws of logarithms, which apply to any base:

(i) To multiply two numbers:

$$\log(A \times B) = \log A + \log B$$

The following may be checked by using a calculator:

$$\lg 10 = 1$$

Also, $\lg 5 + \lg 2 = 0.69897\ldots + 0.301029\ldots = 1$

Hence, $\lg(5 \times 2) = \lg 10 = \lg 5 + \lg 2$

(ii) To divide two numbers:

$$\log\left(\frac{A}{B}\right) = \log A - \log B$$

The following may be checked using a calculator:

$$\ln\left(\frac{5}{2}\right) = \ln 2.5 = 0.91629\ldots$$

Also, $\ln 5 - \ln 2 = 1.60943\ldots - 0.69314\ldots = 0.91629\ldots$

Hence, $\ln\left(\frac{5}{2}\right) = \ln 5 - \ln 2$

(iii) To raise a number to a power:

$$\log A^n = n \log A$$

The following may be checked using a calculator:

$$\lg 5^2 = \lg 25 = 1.39794\ldots$$

Also, $2\lg 5 = 2 \times 0.69897\ldots = 1.39794\ldots$

Hence, $\lg 5^2 = 2\lg 5$

Here are some worked problems to help understanding of the laws of logarithms.

Problem 10. Write $\log 4 + \log 7$ as the logarithm of a single number.

$\log 4 + \log 7 = \log(7 \times 4)$ by the first law of logarithms

$= \mathbf{\log 28}$

Problem 11. Write $\log 16 - \log 2$ as the logarithm of a single number.

$\log 16 - \log 2 = \log\left(\frac{16}{2}\right)$ by the second law of logarithms

$= \mathbf{\log 8}$

Problem 12. Write $2\log 3$ as the logarithm of a single number.

$2\log 3 = \log 3^2$ by the third law of logarithms

$= \mathbf{\log 9}$

Problem 13. Write $\frac{1}{2}\log 25$ as the logarithm of a single number.

$\frac{1}{2}\log 25 = \log 25^{\frac{1}{2}}$ by the third law of logarithms

$= \log\sqrt{25} = \mathbf{\log 5}$

Problem 14. Simplify: $\log 64 - \log 128 + \log 32$.

$64 = 2^6$, $128 = 2^7$ and $32 = 2^5$

Hence, $\log 64 - \log 128 + \log 32$

$= \log 2^6 - \log 2^7 + \log 2^5$

$= 6\log 2 - 7\log 2 + 5\log 2$

by the third law of logarithms

$= \mathbf{4\log 2}$

Problem 15. Write $\frac{1}{2}\log 16 + \frac{1}{3}\log 27 - 2\log 5$ as the logarithm of a single number.

$\frac{1}{2}\log 16 + \frac{1}{3}\log 27 - 2\log 5$

$= \log 16^{\frac{1}{2}} + \log 27^{\frac{1}{3}} - \log 5^2$

by the third law of logarithms

$= \log\sqrt{16} + \log\sqrt[3]{27} - \log 25$

by the laws of indices

$= \log 4 + \log 3 - \log 25$

$= \log\left(\frac{4\times 3}{25}\right)$

by the first and second laws of logarithms

$= \log\left(\frac{12}{25}\right) = \mathbf{\log 0.48}$

Problem 16. Write (a) $\log 30$ (b) $\log 450$ in terms of $\log 2$, $\log 3$ and $\log 5$ to any base.

(a) $\log 30 = \log(2\times 15) = \log(2\times 3\times 5)$

$= \mathbf{\log 2 + \log 3 + \log 5}$

by the first law of logarithms

(b) $\log 450 = \log(2\times 225) = \log(2\times 3\times 75)$

$= \log(2\times 3\times 3\times 25)$

$= \log(2\times 3^2\times 5^2)$

$= \log 2 + \log 3^2 + \log 5^2$

by the first law of logarithms

i.e. $\log 450 = \mathbf{\log 2 + 2\log 3 + 2\log 5}$

by the third law of logarithms

Problem 17. Write $\log\left(\frac{8\times\sqrt[4]{5}}{81}\right)$ in terms of $\log 2$, $\log 3$ and $\log 5$ to any base.

$\log\left(\frac{8\times\sqrt[4]{5}}{81}\right) = \log 8 + \log\sqrt[4]{5} - \log 81$

by the first and second laws of logarithms

$= \log 2^3 + \log 5^{\frac{1}{4}} - \log 3^4$

by the laws of indices

i.e. $\mathbf{\log\left(\frac{8\times\sqrt[4]{5}}{81}\right) = 3\log 2 + \frac{1}{4}\log 5 - 4\log 3}$

by the third law of logarithms

Problem 18. Evaluate: $\dfrac{\log 25 - \log 125 + \frac{1}{2}\log 625}{3\log 5}$.

$$\frac{\log 25 - \log 125 + \frac{1}{2}\log 625}{3\log 5} = \frac{\log 5^2 - \log 5^3 + \frac{1}{2}\log 5^4}{3\log 5}$$

$$= \frac{2\log 5 - 3\log 5 + \frac{4}{2}\log 5}{3\log 5} = \frac{1\log 5}{3\log 5} = \mathbf{\frac{1}{3}}$$

Problem 19. Solve the equation: $\log(x-1) + \log(x+8) = 2\log(x+2)$.

$\text{LHS} = \log(x-1) + \log(x+8)$

$= \log(x-1)(x+8)$

from the first law of logarithms

$= \log(x^2 + 7x - 8)$

$\text{RHS} = 2\log(x+2) = \log(x+2)^2$

from the third law of logarithms

$= \log(x^2 + 4x + 4)$

Hence, $\log(x^2+7x-8) = \log(x^2+4x+4)$

from which, $x^2 + 7x - 8 = x^2 + 4x + 4$

i.e. $7x - 8 = 4x + 4$

i.e. $3x = 12$

and $\boldsymbol{x = 4}$

Problem 20. Solve the equation: $\frac{1}{2}\log 4 = \log x$.

$\frac{1}{2}\log 4 = \log 4^{\frac{1}{2}}$ from the third law of logarithms

$= \log\sqrt{4}$ from the laws of indices

Hence, $\frac{1}{2}\log 4 = \log x$

becomes $\log\sqrt{4} = \log x$

i.e. $\log 2 = \log x$

from which, $2 = x$

i.e. the **solution of the equation is: $x = 2$**

Problem 21. Solve the equation: $\log(x^2-3) - \log x = \log 2$.

$$\log(x^2-3) - \log x = \log\left(\frac{x^2-3}{x}\right)$$

from the second law of logarithms

Hence, $\log\left(\frac{x^2-3}{x}\right) = \log 2$

from which, $\frac{x^2-3}{x} = 2$

Rearranging gives: $x^2 - 3 = 2x$

and $x^2 - 2x - 3 = 0$

Factorizing gives: $(x-3)(x+1) = 0$

from which, $x = 3$ or $x = -1$

$x = -1$ is not a valid solution since the logarithm of a negative number has no real root.

Hence, **the solution of the equation is: $x = 3$**

Now try the following exercise

Exercise 12 Further problems on laws of logarithms

In Problems 1 to 11, write as the logarithm of a single number:

1. $\log 2 + \log 3$ [$\log 6$]
2. $\log 3 + \log 5$ [$\log 15$]
3. $\log 3 + \log 4 - \log 6$ [$\log 2$]
4. $\log 7 + \log 21 - \log 49$ [$\log 3$]
5. $2\log 2 + \log 3$ [$\log 12$]
6. $2\log 2 + 3\log 5$ [$\log 500$]
7. $2\log 5 - \frac{1}{2}\log 81 + \log 36$ [$\log 100$]
8. $\frac{1}{3}\log 8 - \frac{1}{2}\log 81 + \log 27$ [$\log 6$]
9. $\frac{1}{2}\log 4 - 2\log 3 + \log 45$ [$\log 10$]
10. $\frac{1}{4}\log 16 + 2\log 3 - \log 18$ [$\log 1 = 0$]
11. $2\log 2 + \log 5 - \log 10$ [$\log 2$]

Simplify the expressions given in Problems 12 to 14:

12. $\log 27 - \log 9 + \log 81$ [$\log 243$ or $\log 3^5$ or $5\log 3$]
13. $\log 64 + \log 32 - \log 128$ [$\log 16$ or $\log 2^4$ or $4\log 2$]
14. $\log 8 - \log 4 + \log 32$ [$\log 64$ or $\log 2^6$ or $6\log 2$]

Evaluate the expressions given in Problems 15 and 16:

15. $\frac{\frac{1}{2}\log 16 - \frac{1}{3}\log 8}{\log 4}$ [0.5]
16. $\frac{\log 9 - \log 3 + \frac{1}{2}\log 81}{2\log 3}$ [1.5]

Solve the equations given in Problems 17 to 22:

17. $\log x^4 - \log x^3 = \log 5x - \log 2x$ [$x = 2.5$]
18. $\log 2t^3 - \log t = \log 16 + \log t$ [$t = 8$]
19. $2\log b^2 - 3\log b = \log 8b - \log 4b$ [$b = 2$]
20. $\log(x+1) + \log(x-1) = \log 3$ [$x = 2$]
21. $\frac{1}{3}\log 27 = \log(0.5a)$ [$a = 6$]
22. $\log(x^2-5) - \log x = \log 4$ [$x = 5$]

3.3 Indicial equations

The laws of logarithms may be used to solve certain equations involving powers—called **indicial equations**. For example, to solve, say, $3^x = 27$, logarithms to a base of 10 are taken of both sides,

i.e. $\log_{10} 3^x = \log_{10} 27$

and $x\log_{10} 3 = \log_{10} 27$, by the third law of logarithms

Rearranging gives

$$x = \frac{\log_{10} 27}{\log_{10} 3} = \frac{1.43136\ldots}{0.4771\ldots} = 3$$

which may be readily checked

$$\left(\text{Note, } \left(\frac{\log 8}{\log 2}\right) \text{ is } \textbf{not} \text{ equal to } \lg\left(\frac{8}{2}\right)\right)$$

Problem 22. Solve the equation $2^x = 3$, correct to 4 significant figures.

Taking logarithms to base 10 of both sides of $2^x = 3$ gives:

$$\log_{10} 2^x = \log_{10} 3$$

i.e. $$x \log_{10} 2 = \log_{10} 3$$

Rearranging gives:

$$x = \frac{\log_{10} 3}{\log_{10} 2} = \frac{0.47712125\ldots}{0.30102999\ldots}$$

$$= \textbf{1.585}, \text{ correct to 4 significant figures}$$

Problem 23. Solve the equation $2^{x+1} = 3^{2x-5}$ correct to 2 decimal places.

Taking logarithms to base 10 of both sides gives:

$$\log_{10} 2^{x+1} = \log_{10} 3^{2x-5}$$

i.e. $$(x+1)\log_{10} 2 = (2x-5)\log_{10} 3$$

$$x \log_{10} 2 + \log_{10} 2 = 2x \log_{10} 3 - 5 \log_{10} 3$$

$$x(0.3010) + (0.3010) = 2x(0.4771) - 5(0.4771)$$

i.e. $$0.3010x + 0.3010 = 0.9542x - 2.3855$$

Hence

$$2.3855 + 0.3010 = 0.9542x - 0.3010x$$

$$2.6865 = 0.6532x$$

from which $\boldsymbol{x} = \dfrac{2.6865}{0.6532} = \textbf{4.11}$, correct to 2 decimal places

Problem 24. Solve the equation $x^{3.2} = 41.15$, correct to 4 significant figures.

Taking logarithms to base 10 of both sides gives:

$$\log_{10} x^{3.2} = \log_{10} 41.15$$

$$3.2 \log_{10} x = \log_{10} 41.15$$

Hence $\log_{10} x = \dfrac{\log_{10} 41.15}{3.2} = 0.50449$

Thus $\boldsymbol{x} = \text{antilog } 0.50449 = 10^{0.50449} = \textbf{3.195}$ correct to 4 significant figures.

Now try the following exercise

Exercise 13 Indicial equations

Solve the following indicial equations for x, each correct to 4 significant figures:

1. $3^x = 6.4$ [1.690]
2. $2^x = 9$ [3.170]
3. $2^{x-1} = 3^{2x-1}$ [0.2696]
4. $x^{1.5} = 14.91$ [6.058]
5. $25.28 = 4.2^x$ [2.251]
6. $4^{2x-1} = 5^{x+2}$ [3.959]
7. $x^{-0.25} = 0.792$ [2.542]
8. $0.027^x = 3.26$ [−0.3272]
9. The decibel gain n of an amplifier is given by:

$$n = 10 \log_{10}\left(\frac{P_2}{P_1}\right)$$

where P_1 is the power input and P_2 is the power output. Find the power gain $\dfrac{P_2}{P_1}$ when $n = 25$ decibels.

[316.2]

3.4 Graphs of logarithmic functions

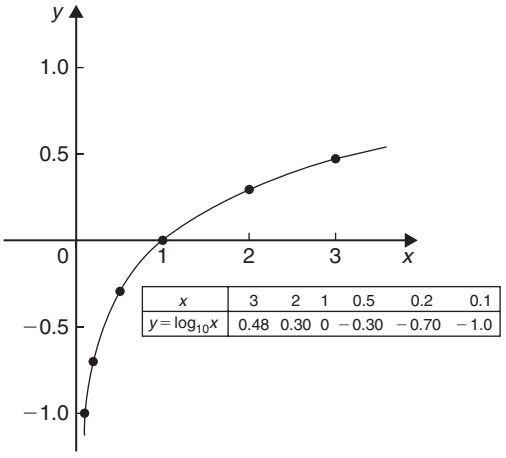
A graph of $y = \log_{10} x$ is shown in Fig. 3.1 and a graph of $y = \log_e x$ is shown in Fig. 3.2. Both are seen to be of similar shape; in fact, the same general shape occurs for a logarithm to any base.

In general, with a logarithm to any base a, it is noted that:

(i) $\boldsymbol{\log_a 1 = 0}$

Let $\log_a = x$, then $a^x = 1$ from the definition of the logarithm.

If $a^x = 1$ then $x = 0$ from the laws of indices. Hence $\log_a 1 = 0$. In the above graphs it is seen that $\log_{10} 1 = 0$ and $\log_e 1 = 0$



x	3	2	1	0.5	0.2	0.1
$y = \log_{10} x$	0.48	0.30	0	−0.30	−0.70	−1.0

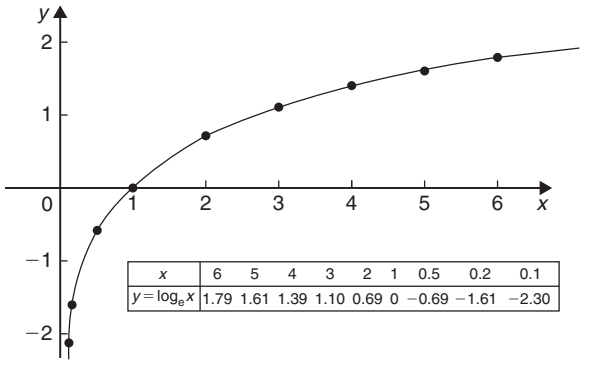
Figure 3.1

(ii) **$\log_a a = 1$**

Let $\log_a a = x$ then $a^x = a$ from the definition of a logarithm.

If $a^x = a$ then $x = 1$.

Hence $\log_a a = 1$. (Check with a calculator that $\log_{10} 10 = 1$ and $\log_e e = 1$)



x	6	5	4	3	2	1	0.5	0.2	0.1
$y = \log_e x$	1.79	1.61	1.39	1.10	0.69	0	−0.69	−1.61	−2.30

Figure 3.2

(iii) **$\log_a 0 \to -\infty$**

Let $\log_a 0 = x$ then $a^x = 0$ from the definition of a logarithm.

If $a^x = 0$, and a is a positive real number, then x must approach minus infinity. (For example, check with a calculator, $2^{-2} = 0.25$, $2^{-20} = 9.54 \times 10^{-7}$, $2^{-200} = 6.22 \times 10^{-61}$, and so on)

Hence $\log_a 0 \to -\infty$

Chapter 4

Exponential functions

4.1 Introduction to exponential functions

An exponential function is one which contains e^x, e being a constant called the exponent and having an approximate value of 2.7183. The exponent arises from the natural laws of growth and decay and is used as a base for natural or Napierian logarithms.

The most common method of evaluating an exponential function is by using a scientific notation **calculator**. Use your calculator to check the following values:

$e^{1} = 2.7182818$, correct to 8 significant figures,

$e^{-1.618} = 0.1982949$, each correct to 7 significant figures,

$e^{0.12} = 1.1275$, correct to 5 significant figures,

$e^{-1.47} = 0.22993$, correct to 5 decimal places,

$e^{-0.431} = 0.6499$, correct to 4 decimal places,

$e^{9.32} = 11159$, correct to 5 significant figures,

$e^{-2.785} = 0.0617291$, correct to 7 decimal places.

Problem 1. Evaluate the following correct to 4 decimal places, using a calculator:

$$0.0256\left(e^{5.21} - e^{2.49}\right)$$

$$\begin{aligned} 0.0256\left(e^{5.21} - e^{2.49}\right) &= 0.0256\,(183.094058\ldots \\ &\quad - 12.0612761\ldots) \\ &= \mathbf{4.3784,}\ \text{correct to 4 decimal places.} \end{aligned}$$

Problem 2. Evaluate the following correct to 4 decimal places, using a calculator:

$$5\left(\frac{e^{0.25} - e^{-0.25}}{e^{0.25} + e^{-0.25}}\right)$$

$$\begin{aligned} &5\left(\frac{e^{0.25} - e^{-0.25}}{e^{0.25} + e^{-0.25}}\right) \\ &= 5\left(\frac{1.28402541\ldots - 0.77880078\ldots}{1.28402541\ldots + 0.77880078\ldots}\right) \\ &= 5\left(\frac{0.5052246\ldots}{2.0628262\ldots}\right) \\ &= \mathbf{1.2246},\ \text{correct to 4 decimal places.} \end{aligned}$$

Problem 3. The instantaneous voltage v in a capacitive circuit is related to time t by the equation: $v = Ve^{-t/CR}$ where V, C and R are constants. Determine v, correct to 4 significant figures, when $t = 50\,\text{ms}$, $C = 10\,\mu\text{F}$, $R = 47\,\text{k}\Omega$ and $V = 300$ volts.

$$v = Ve^{-t/CR} = 300e^{(-50\times10^{-3})/(10\times10^{-6}\times47\times10^{3})}$$

Using a calculator,

$$v = 300e^{-0.1063829\ldots} = 300(0.89908025\ldots)$$
$$= \mathbf{269.7\ volts}$$

Now try the following exercise

Exercise 14 Further problems on evaluating exponential functions

1. Evaluate the following, correct to 4 significant figures: (a) $e^{-1.8}$ (b) $e^{-0.78}$ (c) e^{10}
   [(a) 0.1653 (b) 0.4584 (c) 22030]

2. Evaluate the following, correct to 5 significant figures:
   (a) $e^{1.629}$ (b) $e^{-2.7483}$ (c) $0.62e^{4.178}$
   [(a) 5.0988 (b) 0.064037 (c) 40.446]

In Problems 3 and 4, evaluate correct to 5 decimal places:

3. (a) $\frac{1}{7}e^{3.4629}$ (b) $8.52e^{-1.2651}$ (c) $\frac{5e^{2.6921}}{3e^{1.1171}}$
   [(a) 4.55848 (b) 2.40444 (c) 8.05124]

4. (a) $\frac{5.6823}{e^{-2.1347}}$ (b) $\frac{e^{2.1127} - e^{-2.1127}}{2}$
   (c) $\frac{4(e^{-1.7295} - 1)}{e^{3.6817}}$
   [(a) 48.04106 (b) 4.07482 (c) −0.08286]

5. The length of a bar, l, at a temperature θ is given by $l = l_0e^{\alpha\theta}$, where l_0 and α are constants. Evaluate 1, correct to 4 significant figures, where $l_0 = 2.587$, $\theta = 321.7$ and $\alpha = 1.771 \times 10^{-4}$. [2.739]

6. When a chain of length $2L$ is suspended from two points, $2D$ metres apart, on the same horizontal level: $D = k\left\{\ln\left(\frac{L+\sqrt{L^2+k^2}}{k}\right)\right\}$. Evaluate D when $k = 75$ m and $L = 180$ m.
   [120.7m]

4.2 The power series for e^x

The value of e^x can be calculated to any required degree of accuracy since it is defined in terms of the following **power series**:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

(where $3! = 3 \times 2 \times 1$ and is called 'factorial 3')
The series is valid for all values of x.

The series is said to **converge**, i.e. if all the terms are added, an actual value for e^x (where x is a real number) is obtained. The more terms that are taken, the closer will be the value of e^x to its actual value. The value of the exponent e, correct to say 4 decimal places, may be determined by substituting $x = 1$ in the power series of equation (1). Thus,

$$e^1 = 1 + 1 + \frac{(1)^2}{2!} + \frac{(1)^3}{3!} + \frac{(1)^4}{4!} + \frac{(1)^5}{5!} + \frac{(1)^6}{6!} + \frac{(1)^7}{7!} + \frac{(1)^8}{8!} + \cdots$$
$$= 1 + 1 + 0.5 + 0.16667 + 0.04167 + 0.00833 + 0.00139 + 0.00020 + 0.00002 + \cdots$$

i.e. $e = 2.71828 = 2.7183$, correct to 4 decimal places

The value of $e^{0.05}$, correct to say 8 significant figures, is found by substituting $x = 0.05$ in the power series for e^x. Thus

$$e^{0.05} = 1 + 0.05 + \frac{(0.05)^2}{2!} + \frac{(0.05)^3}{3!} + \frac{(0.05)^4}{4!} + \frac{(0.05)^5}{5!} + \cdots$$
$$= 1 + 0.05 + 0.00125 + 0.000020833 + 0.000000260 + 0.000000003$$

and by adding,

$e^{0.05} = 1.0512711$, correct to 8 significant figures

In this example, successive terms in the series grow smaller very rapidly and it is relatively easy to determine the value of $e^{0.05}$ to a high degree of accuracy. However, when x is nearer to unity or larger than unity, a very large number of terms are required for an accurate result.

If in the series of equation (1), x is replaced by $-x$, then,

$$e^{-x} = 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \cdots$$

i.e. $$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

In a similar manner the power series for e^x may be used to evaluate any exponential function of the form ae^{kx},

where a and k are constants. In the series of equation (1), let x be replaced by kx. Then,

$$a\,e^{kx} = a\left\{1 + (kx) + \frac{(kx)^2}{2!} + \frac{(kx)^3}{3!} + \cdots\right\}$$

Thus $5\,e^{2x} = 5\left\{1 + (2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \cdots\right\}$

$$= 5\left\{1 + 2x + \frac{4x^2}{2} + \frac{8x^3}{6} + \cdots\right\}$$

i.e. $5\,e^{2x} = 5\left\{1 + 2x + 2x^2 + \frac{4}{3}x^3 + \cdots\right\}$

Problem 4. Determine the value of $5\,e^{0.5}$, correct to 5 significant figures by using the power series for e^x.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Hence $e^{0.5} = 1 + 0.5 + \frac{(0.5)^2}{(2)(1)} + \frac{(0.5)^3}{(3)(2)(1)} + \frac{(0.5)^4}{(4)(3)(2)(1)} + \frac{(0.5)^5}{(5)(4)(3)(2)(1)} + \frac{(0.5)^6}{(6)(5)(4)(3)(2)(1)}$

$= 1 + 0.5 + 0.125 + 0.020833 + 0.0026042 + 0.0002604 + 0.0000217$

i.e. $e^{0.5} = 1.64872$, correct to 6 significant figures

Hence $\mathbf{5e^{0.5}} = 5(1.64872) = \mathbf{8.2436}$, correct to 5 significant figures

Problem 5. Expand $e^x(x^2 - 1)$ as far as the term in x^5.

The power series for e^x is,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

Hence $e^x(x^2 - 1)$

$$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\right)(x^2 - 1)$$

$$= \left(x^2 + x^3 + \frac{x^4}{2!} + \frac{x^5}{3!} + \cdots\right) - \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\right)$$

Grouping like terms gives:

$$e^x(x^2 - 1) = -1 - x + \left(x^2 - \frac{x^2}{2!}\right) + \left(x^3 - \frac{x^3}{3!}\right) + \left(\frac{x^4}{2!} - \frac{x^4}{4!}\right) + \left(\frac{x^5}{3!} - \frac{x^5}{5!}\right) + \cdots$$

$$= \mathbf{-1 - x + \frac{1}{2}x^2 + \frac{5}{6}x^3 + \frac{11}{24}x^4 + \frac{19}{120}x^5}$$

when expanded as far as the term in x^5.

Now try the following exercise

Exercise 15 Further problems on the power series for e^x

1. Evaluate $5.6\,e^{-1}$, correct to 4 decimal places, using the power series for e^x. [2.0601]
2. Use the power series for e^x to determine, correct to 4 significant figures, (a) e^2 (b) $e^{-0.3}$ and check your result by using a calculator. [(a) 7.389 (b) 0.7408]
3. Expand $(1 - 2x)\,e^{2x}$ as far as the term in x^4. $\left[1 - 2x^2 - \frac{8x^3}{3} - 2x^4\right]$
4. Expand $\left(2e^{x^2}\right)\left(x^{\frac{1}{2}}\right)$ to six terms. $\left[2x^{\frac{1}{2}} + 2x^{\frac{5}{2}} + x^{\frac{9}{2}} + \frac{1}{3}x^{\frac{13}{2}} + \frac{1}{12}x^{\frac{17}{2}} + \frac{1}{60}x^{\frac{21}{2}}\right]$

4.3 Graphs of exponential functions

Values of e^x and e^{-x} obtained from a calculator, correct to 2 decimal places, over a range $x = -3$ to $x = 3$, are shown in the following table.

x	−3.0	−2.5	−2.0	−1.5	−1.0	−0.5	0
e^{x}	0.05	0.08	0.14	0.22	0.37	0.61	1.00
e^{-x}	20.09	12.18	7.39	4.48	2.72	1.65	1.00

x	0.5	1.0	1.5	2.0	2.5	3.0
e^{x}	1.65	2.72	4.48	7.39	12.18	20.09
e^{-x}	0.61	0.37	0.22	0.14	0.08	0.05

Figure 4.1 shows graphs of $y = e^{x}$ and $y = e^{-x}$

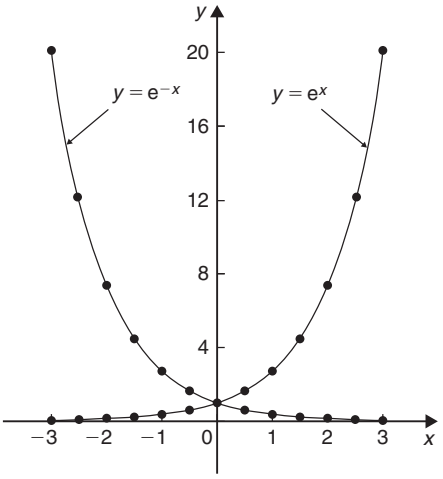


Figure 4.1

Problem 6. Plot a graph of $y = 2e^{0.3x}$ over a range of $x = -2$ to $x = 3$. Hence determine the value of y when $x = 2.2$ and the value of x when $y = 1.6$.

A table of values is drawn up as shown below.

x	−3	−2	−1	0	1	2	3
$0.3x$	−0.9	−0.6	−0.3	0	0.3	0.6	0.9
$e^{0.3x}$	0.407	0.549	0.741	1.000	1.350	1.822	2.460
$2e^{0.3x}$	0.81	1.10	1.48	2.00	2.70	3.64	4.92

A graph of $y = 2e^{0.3x}$ is shown plotted in Fig. 4.2. From the graph, **when $x = 2.2$, $y = 3.87$** and **when $y = 1.6$, $x = -0.74$**.

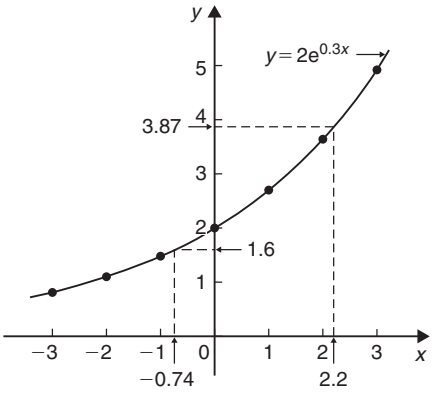


Figure 4.2

Problem 7. Plot a graph of $y = \frac{1}{3}e^{-2x}$ over the range $x = -1.5$ to $x = 1.5$. Determine from the graph the value of y when $x = -1.2$ and the value of x when $y = 1.4$.

A table of values is drawn up as shown below.

x	−1.5	−1.0	−0.5	0	0.5	1.0	1.5
$-2x$	3	2	1	0	−1	−2	−3
e^{-2x}	20.086	7.389	2.718	1.00	0.368	0.135	0.050
$\frac{1}{3}e^{-2x}$	6.70	2.46	0.91	0.33	0.12	0.05	0.02

A graph of $\frac{1}{3}e^{-2x}$ is shown in Fig. 4.3.

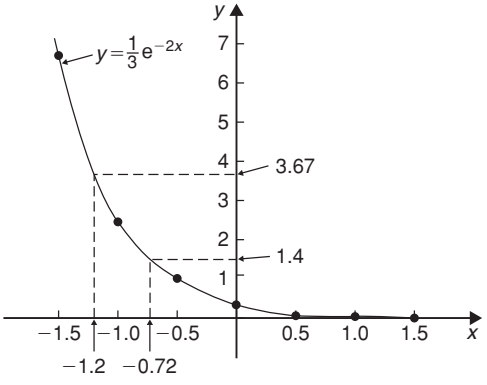


Figure 4.3

From the graph, **when $x = -1.2$, $y = 3.67$** and **when $y = 1.4$, $x = -0.72$.**

Problem 8. The decay of voltage, v volts, across a capacitor at time t seconds is given by $v = 250e^{\frac{-t}{3}}$. Draw a graph showing the natural

decay curve over the first 6 seconds. From the graph, find (a) the voltage after 3.4 s, and (b) the time when the voltage is 150 V.

A table of values is drawn up as shown below.

t	0	1	2	3
$e^{\frac{-t}{3}}$	1.00	0.7165	0.5134	0.3679
$v=250e^{\frac{-t}{3}}$	250.0	179.1	128.4	91.97

t	4	5	6
$e^{\frac{-t}{3}}$	0.2636	0.1889	0.1353
$v=250e^{\frac{-t}{3}}$	65.90	47.22	33.83

The natural decay curve of $v=250e^{\frac{-t}{3}}$ is shown in Fig. 4.4.

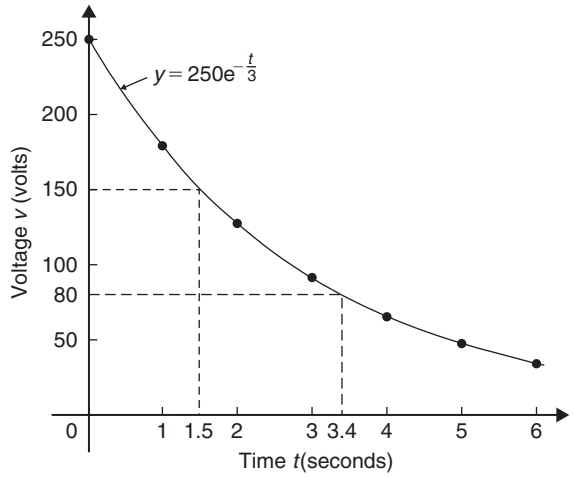


Figure 4.4

From the graph:

(a) **when time $t=3.4$ s, voltage $v=80$ V** and

(b) **when voltage $v=150$ V, time $t=1.5$ s**.

Now try the following exercise

Exercise 16 Further problems on exponential graphs

1. Plot a graph of $y=3e^{0.2x}$ over the range $x=-3$ to $x=3$. Hence determine the value of y when $x=1.4$ and the value of x when $y=4.5$. [3.95, 2.05]

2. Plot a graph of $y=\frac{1}{2}e^{-1.5x}$ over a range $x=-1.5$ to $x=1.5$ and hence determine the value of y when $x=-0.8$ and the value of x when $y=3.5$. [1.65, −1.30]

3. In a chemical reaction the amount of starting material C cm^3 left after t minutes is given by $C=40e^{-0.006t}$. Plot a graph of C against t and determine (a) the concentration C after 1 hour, and (b) the time taken for the concentration to decrease by half.
[(a) 28 cm^3 (b) 116 min]

4. The rate at which a body cools is given by $\theta=250e^{-0.05t}$ where the excess of temperature of a body above its surroundings at time t minutes is θ°C. Plot a graph showing the natural decay curve for the first hour of cooling. Hence determine (a) the temperature after 25 minutes, and (b) the time when the temperature is 195°C.
[(a) 70°C (b) 5 min]

4.4 Napierian logarithms

Logarithms having a base of 'e' are called **hyperbolic, Napierian** or **natural logarithms** and the Napierian logarithm of x is written as $\log_e x$, or more commonly as $\ln x$. Logarithms were invented by John Napier, a Scotsman (1550–1617).

The most common method of evaluating a Napierian logarithm is by a scientific notation **calculator**. Use your calculator to check the following values:

$\ln 4.328 = 1.46510554\ldots$
$= 1.4651$, correct to 4 decimal places

$\ln 1.812 = 0.59443$, correct to 5 significant figures

$\ln 1 = 0$

$\ln 527 = 6.2672$, correct to 5 significant figures

$\ln 0.17 = -1.772$, correct to 4 significant figures

$\ln 0.00042 = -7.77526$, correct to 6 significant figures

$\ln e^3 = 3$

$\ln e^1 = 1$

From the last two examples we can conclude that:

$$\log_e e^x = x$$

This is useful when **solving equations involving exponential functions**. For example, to solve $e^{3x} = 7$, take Napierian logarithms of both sides, which gives:

$$\ln e^{3x} = \ln 7$$

i.e. $$3x = \ln 7$$

from which $$x = \frac{1}{3}\ln 7 = \mathbf{0.6486},$$ correct to 4 decimal places.

Problem 9. Evaluate the following, each correct to 5 significant figures:

(a) $\frac{1}{2}\ln 4.7291$ (b) $\frac{\ln 7.8693}{7.8693}$ (c) $\frac{3.17\ln 24.07}{e^{-0.1762}}$.

(a) $\frac{1}{2}\ln 4.7291 = \frac{1}{2}(1.5537349\ldots) = \mathbf{0.77687}$, correct to 5 significant figures

(b) $\frac{\ln 7.8693}{7.8693} = \frac{2.06296911\ldots}{7.8693} = \mathbf{0.26215}$, correct to 5 significant figures

(c) $\frac{3.17\ln 24.07}{e^{-0.1762}} = \frac{3.17(3.18096625\ldots)}{0.83845027\ldots} = \mathbf{12.027}$, correct to 5 significant figures.

Problem 10. Evaluate the following:

(a) $\frac{\ln e^{2.5}}{\lg 10^{0.5}}$ (b) $\frac{5e^{2.23}\lg 2.23}{\ln 2.23}$ (correct to 3 decimal places).

(a) $\frac{\ln e^{2.5}}{\lg 10^{0.5}} = \frac{2.5}{0.5} = \mathbf{5}$

(b) $\frac{5e^{2.23}\lg 2.23}{\ln 2.23} = \frac{5(9.29986607\ldots)(0.34830486\ldots)}{0.80200158\ldots} = \mathbf{20.194}$, correct to 3 decimal places.

Problem 11. Solve the equation: $9 = 4e^{-3x}$ to find x, correct to 4 significant figures.

Rearranging $9 = 4e^{-3x}$ gives: $\frac{9}{4} = e^{-3x}$

Taking the reciprocal of both sides gives:

$$\frac{4}{9} = \frac{1}{e^{-3x}} = e^{3x}$$

Taking Napierian logarithms of both sides gives:

$$\ln\left(\frac{4}{9}\right) = \ln(e^{3x})$$

Since $\log_e e^{\alpha} = \alpha$, then $\ln\left(\frac{4}{9}\right) = 3x$

Hence, $x = \frac{1}{3}\ln\left(\frac{4}{9}\right) = \frac{1}{3}(-0.81093) = \mathbf{-0.2703}$, correct to 4 significant figures.

Problem 12. Given $32 = 70\left(1 - e^{-\frac{t}{2}}\right)$ determine the value of t, correct to 3 significant figures.

Rearranging $32 = 70(1 - e^{-\frac{t}{2}})$ gives:

$$\frac{32}{70} = 1 - e^{-\frac{t}{2}}$$

and $$e^{-\frac{t}{2}} = 1 - \frac{32}{70} = \frac{38}{70}$$

Taking the reciprocal of both sides gives:

$$e^{\frac{t}{2}} = \frac{70}{38}$$

Taking Napierian logarithms of both sides gives:

$$\ln e^{\frac{t}{2}} = \ln\left(\frac{70}{38}\right)$$

i.e. $$\frac{t}{2} = \ln\left(\frac{70}{38}\right)$$

from which, $t = 2\ln\left(\frac{70}{38}\right) = \mathbf{1.22}$, correct to 3 significant figures.

Problem 13. Solve the equation: $2.68 = \ln\left(\frac{4.87}{x}\right)$ to find x.

From the definition of a logarithm, since $2.68 = \ln\left(\frac{4.87}{x}\right)$ then $e^{2.68} = \frac{4.87}{x}$

Rearranging gives: $x = \frac{4.87}{e^{2.68}} = 4.87e^{-2.68}$

i.e. $x = \mathbf{0.3339}$, correct to 4 significant figures.

Problem 14. Solve $\frac{7}{4} = e^{3x}$ correct to 4 significant figures.

Taking natural logs of both sides gives:

$$\ln\frac{7}{4} = \ln e^{3x}$$

$$\ln\frac{7}{4} = 3x\ln e$$

Since $\ln e = 1$ $\quad \ln\frac{7}{4} = 3x$

i.e. $\quad 0.55962 = 3x$

i.e. $\quad x = \mathbf{0.1865}$, correct to 4 significant figures.

Problem 15. Solve: $e^{x-1} = 2e^{3x-4}$ correct to 4 significant figures.

Taking natural logarithms of both sides gives:

$$\ln\left(e^{x-1}\right) = \ln\left(2e^{3x-4}\right)$$

and by the first law of logarithms,

$$\ln\left(e^{x-1}\right) = \ln 2 + \ln\left(e^{3x-4}\right)$$

i.e. $\quad x - 1 = \ln 2 + 3x - 4$

Rearranging gives: $\quad 4 - 1 - \ln 2 = 3x - x$

i.e. $\quad 3 - \ln 2 = 2x$

from which, $\quad x = \dfrac{3 - \ln 2}{2}$

$$= \mathbf{1.153}$$

Problem 16. Solve, correct to 4 significant figures: $\ln(x-2)^2 = \ln(x-2) - \ln(x+3) + 1.6$

Rearranging gives:

$$\ln(x-2)^2 - \ln(x-2) + \ln(x+3) = 1.6$$

and by the laws of logarithms,

$$\ln\left\{\frac{(x-2)^2(x+3)}{(x-2)}\right\} = 1.6$$

Cancelling gives: $\quad \ln\{(x-2)(x+3)\} = 1.6$

and $\quad (x-2)(x+3) = e^{1.6}$

i.e. $\quad x^2 + x - 6 = e^{1.6}$

or $\quad x^2 + x - 6 - e^{1.6} = 0$

i.e. $\quad x^2 + x - 10.953 = 0$

Using the quadratic formula,

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-10.953)}}{2}$$

$$= \frac{-1 \pm \sqrt{44.812}}{2} = \frac{-1 \pm 6.6942}{2}$$

i.e. $\quad x = 2.847$ or -3.8471

$x = -3.8471$ is not valid since the logarithm of a negative number has no real root.

Hence, **the solution of the equation is: $x = 2.847$**

Now try the following exercise

Exercise 17 Further problems on evaluating Napierian logarithms

In Problems 1 and 2, evaluate correct to 5 significant figures:

1. (a) $\frac{1}{3}\ln 5.2932$ (b) $\dfrac{\ln 82.473}{4.829}$ (c) $\dfrac{5.62\ln 321.62}{e^{1.2942}}$

   [(a) 0.55547 (b) 0.91374 (c) 8.8941]

2. (a) $\dfrac{1.786\ln e^{1.76}}{\lg 10^{1.41}}$ (b) $\dfrac{5e^{-0.1629}}{2\ln 0.00165}$ (c) $\dfrac{\ln 4.8629 - \ln 2.4711}{5.173}$

   [(a) 2.2293 (b) −0.33154 (c) 0.13087]

In Problems 3 to 7 solve the given equations, each correct to 4 significant figures.

3. $\ln x = 2.10$ [8.166]
4. $24 + e^{2x} = 45$ [1.522]
5. $5 = e^{x+1} - 7$ [1.485]
6. $1.5 = 4e^{2t}$ [−0.4904]
7. $7.83 = 2.91e^{-1.7x}$ [−0.5822]
8. $16 = 24\left(1 - e^{-\frac{t}{2}}\right)$ [2.197]
9. $5.17 = \ln\left(\dfrac{x}{4.64}\right)$ [816.2]
10. $3.72\ln\left(\dfrac{1.59}{x}\right) = 2.43$ [0.8274]

11. $5 = 8\left(1 - e^{\frac{-x}{2}}\right)$ [1.962]

12. $\ln(x+3) - \ln x = \ln(x-1)$ [3]

13. $\ln(x-1)^2 - \ln 3 = \ln(x-1)$ [4]

14. $\ln(x+3) + 2 = 12 - \ln(x-2)$ [147.9]

15. $e^{(x+1)} = 3e^{(2x-5)}$ [4.901]

16. $\ln(x+1)^2 = 1.5 - \ln(x-2) + \ln(x+1)$ [3.095]

17. Transpose: $b = \ln t - a \ln D$ to make t the subject.
[$t = e^{b + a\ln D} = e^b e^{a \ln D} = e^b e^{\ln D^a}$ i.e. $t = e^b D^a$]

18. If $\frac{P}{Q} = 10 \log_{10}\left(\frac{R_1}{R_2}\right)$ find the value of R_1 when $P = 160$, $Q = 8$ and $R_2 = 5$. [500]

19. If $U_2 = U_1 e^{\left(\frac{W}{PV}\right)}$ make W the subject of the formula. $\left[W = PV \ln\left(\frac{U_2}{U_1}\right)\right]$

20. The work done in an isothermal expansion of a gas from pressure p_1 to p_2 is given by:

$$w = w_0 \ln\left(\frac{p_1}{p_2}\right)$$

If the initial pressure $p_1 = 7.0$ kPa, calculate the final pressure p_2 if $w = 3w_0$.
[$p_2 = 348.5$ Pa]

4.5 Laws of growth and decay

The laws of exponential growth and decay are of the form $y = Ae^{-kx}$ and $y = A(1 - e^{-kx})$, where A and k are constants. When plotted, the form of each of these equations is as shown in Fig. 4.5. The laws occur frequently in engineering and science and examples of quantities related by a natural law include.

(i) Linear expansion $l = l_0 e^{\alpha\theta}$

(ii) Change in electrical resistance with temperature $R_\theta = R_0 e^{\alpha\theta}$

(iii) Tension in belts $T_1 = T_0 e^{\mu\theta}$

(iv) Newton's law of cooling $\theta = \theta_0 e^{-kt}$

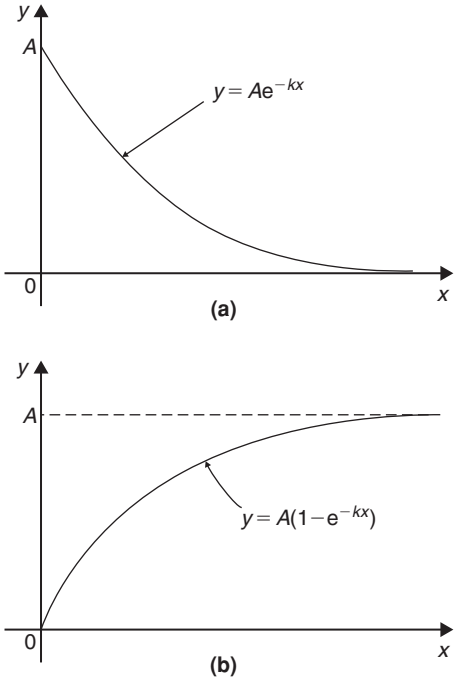


Figure 4.5

(v) Biological growth $y = y_0 e^{kt}$

(vi) Discharge of a capacitor $q = Qe^{-t/CR}$

(vii) Atmospheric pressure $p = p_0 e^{-h/c}$

(viii) Radioactive decay $N = N_0 e^{-\lambda t}$

(ix) Decay of current in an inductive circuit $i = Ie^{-Rt/L}$

(x) Growth of current in a capacitive circuit $i = I(1 - e^{-t/CR})$

Problem 17. The resistance R of an electrical conductor at temperature θ°C is given by $R = R_0 e^{\alpha\theta}$, where α is a constant and $R_0 = 5 \times 10^3$ ohms. Determine the value of α, correct to 4 significant figures, when $R = 6 \times 10^3$ ohms and $\theta = 1500$°C. Also, find the temperature, correct to the nearest degree, when the resistance R is 5.4×10^3 ohms.

Transposing $R = R_0 e^{\alpha\theta}$ gives $\frac{R}{R_0} = e^{\alpha\theta}$.

Taking Napierian logarithms of both sides gives:

$$\ln\frac{R}{R_0} = \ln e^{\alpha\theta} = \alpha\theta$$

Hence $\alpha = \frac{1}{\theta}\ln\frac{R}{R_0} = \frac{1}{1500}\ln\left(\frac{6\times10^3}{5\times10^3}\right)$

$= \frac{1}{1500}(0.1823215\ldots)$

$= 1.215477\cdots\times10^{-4}$

Hence $\boldsymbol{\alpha = 1.215\times10^{-4}}$,
correct to 4 significant figures.

From above, $\ln\frac{R}{R_0} = \alpha\theta$

hence $\theta = \frac{1}{\alpha}\ln\frac{R}{R_0}$

When $R = 5.4\times10^3$, $\alpha = 1.215477\ldots\times10^{-4}$ and $R_0 = 5\times10^3$

$$\theta = \frac{1}{1.215477\ldots\times10^{-4}}\ln\left(\frac{5.4\times10^3}{5\times10^3}\right)$$

$$= \frac{10^4}{1.215477\ldots}(7.696104\ldots\times10^{-2})$$

$= \mathbf{633°C}$, correct to the nearest degree.

Problem 18. In an experiment involving Newton's law of cooling, the temperature θ(°C) is given by $\theta = \theta_0 e^{-kt}$. Find the value of constant k when $\theta_0 = 56.6$°C, $\theta = 16.5$°C and $t = 83.0$ seconds.

Transposing $\theta = \theta_0 e^{-kt}$ gives

$$\frac{\theta}{\theta_0} = e^{-kt}$$

from which $\frac{\theta_0}{\theta} = \frac{1}{e^{-kt}} = e^{kt}$

Taking Napierian logarithms of both sides gives:

$$\ln\frac{\theta_0}{\theta} = kt$$

from which,

$$k = \frac{1}{t}\ln\frac{\theta_0}{\theta} = \frac{1}{83.0}\ln\left(\frac{56.6}{16.5}\right)$$

$$= \frac{1}{83.0}(1.2326486\ldots)$$

Hence $\boldsymbol{k = 1.485\times10^{-2}}$

Problem 19. The current i amperes flowing in a capacitor at time t seconds is given by $i = 8.0(1 - e^{\frac{-t}{CR}})$, where the circuit resistance R is 25×10^3 ohms and capacitance C is 16×10^{-6} farads. Determine (a) the current i after 0.5 seconds and (b) the time, to the nearest millisecond, for the current to reach 6.0 A. Sketch the graph of current against time.

(a) Current $i = 8.0(1 - e^{\frac{-t}{CR}})$

$$= 8.0[1 - e^{\frac{-0.5}{(16\times10^{-6})(25\times10^3)}}] = 8.0(1 - e^{-1.25})$$

$$= 8.0(1 - 0.2865047\ldots) = 8.0(0.7134952\ldots)$$

$= \mathbf{5.71\ amperes}$

(b) Transposing $i = 8.0(1 - e^{\frac{-t}{CR}})$

gives $\frac{i}{8.0} = 1 - e^{\frac{-t}{CR}}$

from which, $e^{\frac{-t}{CR}} = 1 - \frac{i}{8.0} = \frac{8.0 - i}{8.0}$

Taking the reciprocal of both sides gives:

$$e^{\frac{t}{CR}} = \frac{8.0}{8.0 - i}$$

Taking Napierian logarithms of both sides gives:

$$\frac{t}{CR} = \ln\left(\frac{8.0}{8.0 - i}\right)$$

Hence

$$t = CR\ln\left(\frac{8.0}{8.0 - i}\right)$$

$$= (16\times10^{-6})(25\times10^3)\ln\left(\frac{8.0}{8.0 - 6.0}\right)$$

when $i = 6.0$ amperes,

i.e. $t = \frac{400}{10^3}\ln\left(\frac{8.0}{2.0}\right) = 0.4\ln4.0$

$$= 0.4(1.3862943\ldots) = 0.5545\,\text{s}$$

$= \mathbf{555\ ms}$, to the nearest millisecond.

A graph of current against time is shown in Fig. 4.6.

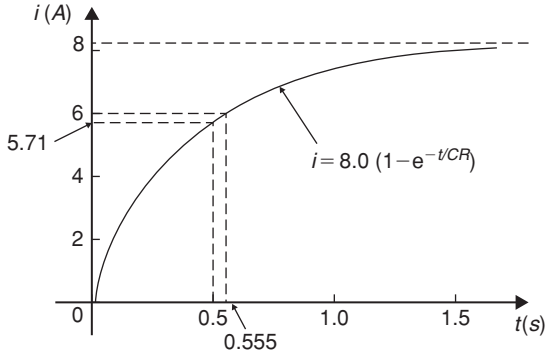


Figure 4.6

Problem 20. The temperature θ_2 of a winding which is being heated electrically at time t is given by: $\theta_2 = \theta_1(1 - e^{\frac{-t}{\tau}})$ where θ_1 is the temperature (in degrees Celsius) at time $t = 0$ and τ is a constant. Calculate,

(a) θ_1, correct to the nearest degree, when θ_2 is 50°C, t is 30 s and τ is 60 s

(b) the time t, correct to 1 decimal place, for θ_2 to be half the value of θ_1.

(a) Transposing the formula to make θ_1 the subject gives:

$$\theta_1 = \frac{\theta_2}{(1 - e^{\frac{-t}{T}})} = \frac{50}{1 - e^{\frac{-30}{60}}}$$

$$= \frac{50}{1 - e^{-0.5}} = \frac{50}{0.393469\ldots}$$

i.e. $\boldsymbol{\theta_1 = 127°C}$, correct to the nearest degree.

(b) Transposing to make t the subject of the formula gives:

$$\frac{\theta_2}{\theta_1} = 1 - e^{\frac{-t}{\tau}}$$

from which, $e^{\frac{-t}{\tau}} = 1 - \frac{\theta_2}{\theta_1}$

Hence $$-\frac{t}{\tau} = \ln\left(1 - \frac{\theta_2}{\theta_1}\right)$$

i.e. $$t = -\tau \ln\left(1 - \frac{\theta_2}{\theta_1}\right)$$

Since $$\theta_2 = \frac{1}{2}\theta_1$$

$$t = -60 \ln\left(1 - \frac{1}{2}\right)$$

$$= -60 \ln 0.5 = 41.59 \text{ s}$$

Hence the time for the temperature θ_2 to be one half of the value of θ_1 is 41.6 s, correct to 1 decimal place.

Now try the following exercise

Exercise 18 Further problems on the laws of growth and decay

1. The temperature, T°C, of a cooling object varies with time, t minutes, according to the equation: $T = 150e^{-0.04t}$. Determine the temperature when (a) $t = 0$, (b) $t = 10$ minutes.
[(a) 150°C (b) 100.5°C]

2. The pressure p pascals at height h metres above ground level is given by $p = p_0 e^{\frac{-h}{C}}$, where p_0 is the pressure at ground level and C is a constant. Find pressure p when $p_0 = 1.012 \times 10^5$ Pa, height $h = 1420$ m, and $C = 71500$. [99210]

3. The voltage drop, v volts, across an inductor L henrys at time t seconds is given by $v = 200e^{\frac{-Rt}{L}}$, where $R = 150\,\Omega$ and $L = 12.5 \times 10^{-3}$ H. Determine (a) the voltage when $t = 160 \times 10^{-6}$ s, and (b) the time for the voltage to reach 85 V.
[(a) 29.32 volts (b) 71.31×10^{-6} s]

4. The length l metres of a metal bar at temperature t°C is given by $l = l_0 e^{\alpha t}$, where l_0 and α are constants. Determine (a) the value of α when $l = 1.993$ m, $l_0 = 1.894$ m and $t = 250$°C, and (b) the value of l_0 when $l = 2.416$, $t = 310$°C and $\alpha = 1.682 \times 10^{-4}$.
[(a) 2.038×10^{-4} (b) 2.293 m]

5. The temperature θ_2°C of an electrical conductor at time t seconds is given by: $\theta_2 = \theta_1(1 - e^{-t/T})$, where θ_1 is the initial temperature and T seconds is a constant. Determine:
   (a) θ_2 when $\theta_1 = 159.9$°C, $t = 30$ s and $T = 80$ s, and
   (b) the time t for θ_2 to fall to half the value of θ_1 if T remains at 80 s.
[(a) 50°C (b) 55.45 s]

6. A belt is in contact with a pulley for a sector of $\theta = 1.12$ radians and the coefficient

of friction between these two surfaces is $\mu = 0.26$. Determine the tension on the taut side of the belt, T newtons, when tension on the slack side $T_0 = 22.7$ newtons, given that these quantities are related by the law $T = T_0 e^{\mu\theta}$. Determine also the value of θ when $T = 28.0$ newtons.

[30.4 N, 0.807 rad]

7. The instantaneous current i at time t is given by: $i = 10e^{\frac{-t}{CR}}$ when a capacitor is being charged. The capacitance C is 7×10^{-6} farads and the resistance R is 0.3×10^{6} ohms. Determine:
   (a) the instantaneous current when t is 2.5 seconds, and
   (b) the time for the instantaneous current to fall to 5 amperes

   Sketch a curve of current against time from $t = 0$ to $t = 6$ seconds.

   [(a) 3.04 A (b) 1.46 s]

8. The amount of product x (in mol/cm^3) found in a chemical reaction starting with 2.5 mol/cm^3 of reactant is given by $x = 2.5(1 - e^{-4t})$ where t is the time, in minutes, to form product x. Plot a graph at 30 second intervals up to 2.5 minutes and determine x after 1 minute.

   [2.45 mol/cm^3]

9. The current i flowing in a capacitor at time t is given by:

$$i = 12.5(1 - e^{\frac{-t}{CR}})$$

   where resistance R is 30 kilohms and the capacitance C is 20 micro-farads. Determine:
   (a) the current flowing after 0.5 seconds, and
   (b) the time for the current to reach 10 amperes. [(a) 7.07 A (b) 0.966 s]

4.6 Reduction of exponential laws to linear form

Frequently, the relationship between two variables, say x and y, is not a linear one, i.e. when x is plotted against y a curve results. In such cases the non-linear equation may be modified to the linear form, $y = mx + c$, so that the constants, and thus the law relating the variables can be determined. This technique is called **'determination of law'**.

Graph paper is available where the scale markings along the horizontal and vertical axes are proportional to the logarithms of the numbers. Such graph paper is called **log-log graph paper**.

A **logarithmic scale** is shown in Fig. 4.7 where the distance between, say 1 and 2, is proportional to lg 2 − lg 1, i.e. 0.3010 of the total distance from 1 to 10. Similarly, the distance between 7 and 8 is proportional to lg 8 − lg 7, i.e. 0.05799 of the total distance from 1 to 10. Thus the distance between markings progressively decreases as the numbers increase from 1 to 10.

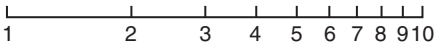


Figure 4.7

With log-log graph paper the scale markings are from 1 to 9, and this pattern can be repeated several times. The number of times the pattern of markings is repeated on an axis signifies the number of **cycles**. When the vertical axis has, say, 3 sets of values from 1 to 9, and the horizontal axis has, say, 2 sets of values from 1 to 9, then this log-log graph paper is called 'log 3 cycle × 2 cycle'. Many different arrangements are available ranging from 'log 1 cycle × 1 cycle' through to 'log 5 cycle × 5 cycle'.

To depict a set of values, say, from 0.4 to 161, on an axis of log-log graph paper, 4 cycles are required, from 0.1 to 1, 1 to 10, 10 to 100 and 100 to 1000.

Graphs of the form $y = a e^{kx}$

Taking logarithms to a base of e of both sides of $y = a e^{kx}$ gives:

$$\ln y = \ln(a e^{kx}) = \ln a + \ln e^{kx} = \ln a + kx \ln e$$

i.e. $\ln y = kx + \ln a$ (since $\ln e = 1$)

which compares with $Y = mX + c$

Thus, by plotting ln y vertically against x horizontally, a straight line results, i.e. the equation $y = a e^{kx}$ is reduced to linear form. In this case, graph paper having a linear horizontal scale and a logarithmic vertical scale may be used. This type of graph paper is called **log-linear graph paper**, and is specified by the number of cycles on the logarithmic scale.

Problem 21. The data given below is believed to be related by a law of the form $y = a e^{kx}$, where a and b are constants. Verify that the law is true and

determine approximate values of a and b. Also determine the value of y when x is 3.8 and the value of x when y is 85.

x	-1.2	0.38	1.2	2.5	3.4	4.2	5.3
y	9.3	22.2	34.8	71.2	117	181	332

Since $y = a\,e^{kx}$ then $\ln y = kx + \ln a$ (from above), which is of the form $Y = mX + c$, showing that to produce a straight line graph $\ln y$ is plotted vertically against x horizontally. The value of y ranges from 9.3 to 332 hence 'log3 cycle × linear' graph paper is used. The plotted co-ordinates are shown in Fig. 4.8 and since a straight line passes through the points the law $y = a\,e^{kx}$ is verified.

Gradient of straight line,

$$k = \frac{AB}{BC} = \frac{\ln 100 - \ln 10}{3.12 - (-1.08)} = \frac{2.3026}{4.20}$$

$= 0.55$, correct to 2 significant figures.

Since $\ln y = kx + \ln a$, when $x = 0$, $\ln y = \ln a$, i.e. $y = a$

The vertical axis intercept value at $x = 0$ is 18, hence $a = 18$

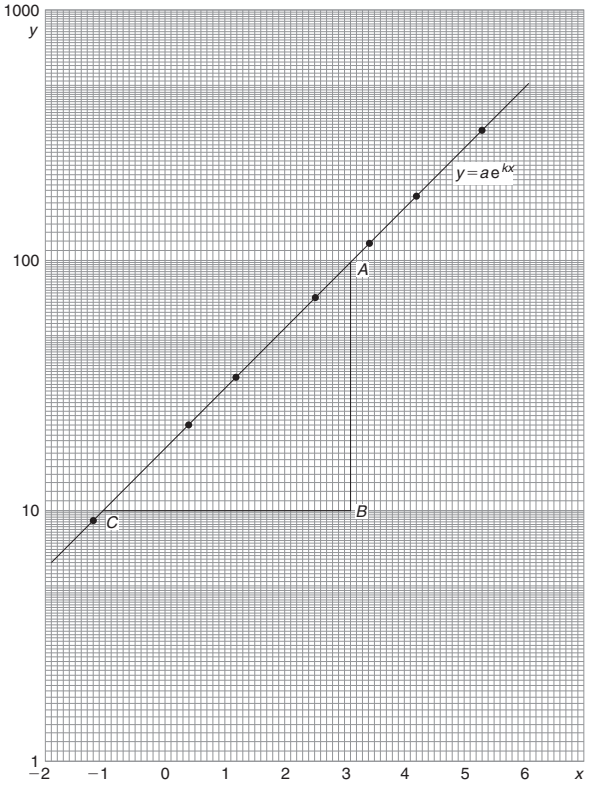


Figure 4.8

The law of the graph is thus $y = 18\,e^{0.55x}$

When x is 3.8, $y = 18\,e^{0.55(3.8)} = 18\,e^{2.09}$

$= 18(8.0849) = \mathbf{146}$

When y is 85, $85 = 18\,e^{0.55x}$

Hence, $e^{0.55x} = \dfrac{85}{18} = 4.7222$

and $0.55x = \ln 4.7222 = 1.5523$

Hence $x = \dfrac{1.5523}{0.55} = \mathbf{2.82}$

Problem 22. The voltage, v volts, across an inductor is believed to be related to time, t ms, by the law $v = V\,e^{\frac{t}{T}}$, where V and T are constants. Experimental results obtained are:

v volts	883	347	90	55.5	18.6	5.2
t ms	10.4	21.6	37.8	43.6	56.7	72.0

Show that the law relating voltage and time is as stated and determine the approximate values of V and T. Find also the value of voltage after 25 ms and the time when the voltage is 30.0 V.

Since $v = V\,e^{\frac{t}{T}}$ then $\ln v = \frac{1}{T}t + \ln V$ which is of the form $Y = mX + c$.

Using 'log3 cycle × linear' graph paper, the points are plotted as shown in Fig. 4.9.

Since the points are joined by a straight line the law $v = Ve^{\frac{t}{T}}$ is verified.

Gradient of straight line,

$$\frac{1}{T} = \frac{AB}{BC}$$

$$= \frac{\ln 100 - \ln 10}{36.5 - 64.2}$$

$$= \frac{2.3026}{-27.7}$$

Hence $\boldsymbol{T} = \dfrac{-27.7}{2.3026}$

$= \mathbf{-12.0}$, correct to 3 significant figures.

Since the straight line does not cross the vertical axis at $t = 0$ in Fig. 4.9, the value of V is determined by selecting any point, say A, having co-ordinates (36.5,100) and substituting these values into $v = V\,e^{\frac{t}{T}}$.

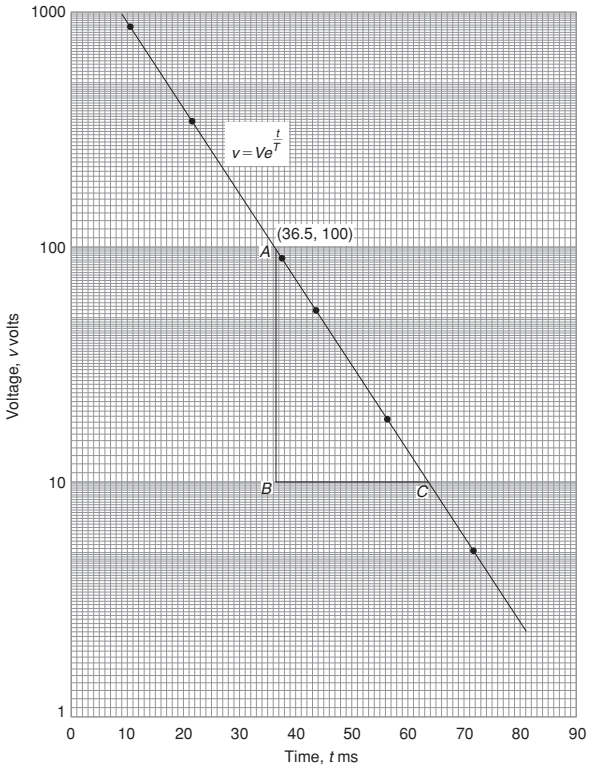


Figure 4.9

Thus $100 = V\,e^{\frac{36.5}{-12.0}}$

i.e. $$V = \frac{100}{e^{\frac{-36.5}{12.0}}}$$

$$= \mathbf{2090\ volts},$$

correct to 3 significant figures.

Hence the law of the graph is $v = 2090\,e^{\frac{-t}{12.0}}$.

When time $t = 25$ ms,

voltage $v = 2090\,e^{\frac{-25}{12.0}} = \mathbf{260\ V}$

When the voltage is 30.0 volts, $30.0 = 2090\,e^{\frac{-t}{12.0}}$,

hence $$e^{\frac{-t}{12.0}} = \frac{30.0}{2090}$$

and $$e^{\frac{t}{12.0}} = \frac{2090}{30.0} = 69.67$$

Taking Napierian logarithms gives:

$$\frac{t}{12.0} = \ln 69.67 = 4.2438$$

from which, time $\boldsymbol{t} = (12.0)(4.2438) = \mathbf{50.9\ ms}$

Now try the following exercise

Exercise 19 Further problems on reducing exponential laws to linear form

1. Atmospheric pressure p is measured at varying altitudes h and the results are as shown below:

Altitude, h m	pressure, p cm
500	73.39
1500	68.42
3000	61.60
5000	53.56
8000	43.41

Show that the quantities are related by the law $p = a\,e^{kh}$, where a and k are constants. Determine the values of a and k and state the law. Find also the atmospheric pressure at 10 000 m.

$$\left[\begin{array}{l} a = 76, k = -7 \times 10^{-5}, \\ p = 76\,e^{-7\times10^{-5}h}, 37.74\,\text{cm} \end{array}\right]$$

2. At particular times, t minutes, measurements are made of the temperature, θ°C, of a cooling liquid and the following results are obtained:

Temperature θ°C	Time t minutes
92.2	10
55.9	20
33.9	30
20.6	40
12.5	50

Prove that the quantities follow a law of the form $\theta = \theta_0\,e^{kt}$, where θ_0 and k are constants, and determine the approximate value of θ_0 and k.

$[\theta_0 = 152, k = -0.05]$

Revision Test 1

This Revision Test covers the material contained in Chapters 1 to 4. *The marks for each question are shown in brackets at the end of each question.*

1. Factorise x^3+4x^2+x-6 using the factor theorem. Hence solve the equation

$$x^3+4x^2+x-6=0 \qquad (6)$$

2. Use the remainder theorem to find the remainder when $2x^3+x^2-7x-6$ is divided by

(a) $(x-2)$ (b) $(x+1)$

Hence factorise the cubic expression (7)

3. Simplify $\dfrac{6x^2+7x-5}{2x-1}$ by dividing out (4)

4. Resolve the following into partial fractions

(a) $\dfrac{x-11}{x^2-x-2}$ (b) $\dfrac{3-x}{(x^2+3)(x+3)}$

(c) $\dfrac{x^3-6x+9}{x^2+x-2}$ (24)

5. Evaluate, correct to 3 decimal places,

$$\frac{5\,\mathrm{e}^{-0.982}}{3\ln 0.0173} \qquad (2)$$

6. Solve the following equations, each correct to 4 significant figures:

(a) $\ln x = 2.40$ (b) $3^{x-1}=5^{x-2}$

(c) $5=8(1-\mathrm{e}^{-\frac{x}{2}})$ (10)

7. (a) The pressure p at height h above ground level is given by: $p=p_0\mathrm{e}^{-kh}$ where p_0 is the pressure at ground level and k is a constant. When p_0 is 101 kilopascals and the pressure at a height of 1500 m is 100 kilopascals, determine the value of k.

(b) Sketch a graph of p against h (p the vertical axis and h the horizontal axis) for values of height from zero to 12000 m when p_0 is 101 kilopascals.

(c) If pressure $p=95$ kPa, ground level pressure $p_0=101$ kPa, constant $k=5\times10^{-6}$, determine the height above ground level, h, in kilometres correct to 2 decimal places. (13)

8. Solve the following equations:

(a) $\log\left(x^2+8\right)-\log(2x)=\log 3$

(b) $\ln x+\ln(x-3)=\ln 6x-\ln(x-2)$ (13)

9. If $\theta_f-\theta_i=\dfrac{R}{J}\ln\left(\dfrac{U_2}{U_1}\right)$ find the value of U_2 given that $\theta_f=3.5$, $\theta_i=2.5$, $R=0.315$, $J=0.4$, $U_1=50$ (6)

10. Solve, correct to 4 significant figures:

(a) $13\mathrm{e}^{2x-1}=7\mathrm{e}^{x}$

(b) $\ln(x+1)^2=\ln(x+1)-\ln(x+2)+2$ (15)

Chapter 5

Hyperbolic functions

5.1 Introduction to hyperbolic functions

Functions which are associated with the geometry of the conic section called a hyperbola are called **hyperbolic functions**. Applications of hyperbolic functions include transmission line theory and catenary problems. By definition:

(i) Hyperbolic sine of x,

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad (1)$$

'sinh x' is often abbreviated to 'sh x' and is pronounced as 'shine x'

(ii) Hyperbolic cosine of x,

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad (2)$$

'cosh x' is often abbreviated to 'ch x' and is pronounced as 'kosh x'

(iii) Hyperbolic tangent of x,

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad (3)$$

'tanh x' is often abbreviated to 'th x' and is pronounced as 'than x'

(iv) Hyperbolic cosecant of x,

$$\operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}} \qquad (4)$$

'cosech x' is pronounced as 'coshec x'

(v) Hyperbolic secant of x,

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} \qquad (5)$$

'sech x' is pronounced as 'shec x'

(vi) Hyperbolic cotangent of x,

$$\coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}} \qquad (6)$$

'coth x' is pronounced as 'koth x'

Some properties of hyperbolic functions

Replacing x by 0 in equation (1) gives:

$$\sinh 0 = \frac{e^0 - e^{-0}}{2} = \frac{1-1}{2} = 0$$

Replacing x by 0 in equation (2) gives:

$$\cosh 0 = \frac{e^0 + e^{-0}}{2} = \frac{1+1}{2} = 1$$

If a function of x, $f(-x) = -f(x)$, then $f(x)$ is called an **odd function** of x. Replacing x by $-x$ in equation (1) gives:

$$\sinh(-x) = \frac{e^{-x} - e^{-(-x)}}{2} = \frac{e^{-x} - e^{x}}{2}$$
$$= -\left(\frac{e^x - e^{-x}}{2}\right) = -\sinh x$$

Replacing x by $-x$ in equation (3) gives:

$$\tanh(-x) = \frac{e^{-x} - e^{-(-x)}}{e^{-x} + e^{-(-x)}} = \frac{e^{-x} - e^{x}}{e^{-x} + e^{x}}$$
$$= -\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) = -\tanh x$$

Hence **sinh x and tanh x are both odd functions** (see Section 5.1), as also are cosech x $\left(=\dfrac{1}{\sinh x}\right)$ and $\coth x\left(=\dfrac{1}{\tanh x}\right)$

If a function of x, $f(-x) = f(x)$, then $f(x)$ is called an **even function** of x. Replacing x by $-x$ in equation (2) gives:

$$\cosh(-x) = \frac{e^{-x} + e^{-(-x)}}{2} = \frac{e^{-x} + e^{x}}{2}$$
$$= \cosh x$$

Hence **cosh x is an even function** (see Section 5.2), as also is sech $x\left(=\dfrac{1}{\cosh x}\right)$

Hyperbolic functions may be evaluated easiest using a calculator. Many scientific notation calculators actually possess sinh and cosh functions; however, if a calculator does not contain these functions, then the definitions given above may be used.

Problem 1. Evaluate sinh 5.4, correct to 4 significant figures.

Using a calculator,

(i) press hyp
(ii) press 1 and sinh(appears
(iii) type in 5.4
(iv) press) to close the brackets
(v) press = and 110.7009498 appears

Hence, **sinh 5.4 = 110.7**, correct to 4 significant figures.

Alternatively, $\sinh 5.4 = \dfrac{1}{2}(e^{5.4} - e^{-5.4})$

$$= \frac{1}{2}(221.406416\ldots - 0.00451658\ldots)$$
$$= \frac{1}{2}(221.401899\ldots)$$
$$= \mathbf{110.7}, \text{ correct to 4 significant figures.}$$

Problem 2. Evaluate cosh 1.86, correct to 3 decimal places.

Using a calculator with the procedure similar to that used in Worked Problem 1,

cosh 1.86 = 3.290, correct to 3 decimal places.

Problem 3. Evaluate th 0.52, correct to 4 significant figures.

Using a calculator with the procedure similar to that used in Worked Problem 1,

th 0.52 = 0.4777, correct to 4 significant figures.

Problem 4. Evaluate cosech 1.4, correct to 4 significant figures.

$$\text{cosech } 1.4 = \frac{1}{\sinh 1.4}$$

Using a calculator,

(i) press hyp
(ii) press 1 and sinh(appears
(iii) type in 1.4
(iv) press) to close the brackets
(v) press = and 1.904301501 appears
(vi) press x^{-1}
(vii) press = and 0.5251269293 appears

Hence, **cosech 1.4 = 0.5251**, correct to 4 significant figures.

Problem 5. Evaluate sech 0.86, correct to 4 significant figures.

$$\text{sech } 0.86 = \frac{1}{\cosh 0.86}$$

Using a calculator with the procedure similar to that used in Worked Problem 4,

sech 0.86 = 0.7178, correct to 4 significant figures.

Problem 6. Evaluate coth 0.38, correct to 3 decimal places.

$$\coth 0.38 = \frac{1}{\tanh 0.38}$$

Using a calculator with the procedure similar to that used in Worked Problem 4,

coth 0.38 = 2.757, correct to 3 decimal places.

Now try the following exercise

Exercise 20 Further problems on evaluating hyperbolic functions

In Problems 1 to 6, evaluate correct to 4 significant figures.

1. (a) sh 0.64 (b) sh 2.182
   [(a) 0.6846 (b) 4.376]
2. (a) ch 0.72 (b) ch 2.4625
   [(a) 1.271 (b) 5.910]
3. (a) th 0.65 (b) th 1.81
   [(a) 0.5717 (b) 0.9478]
4. (a) cosech 0.543 (b) cosech 3.12
   [(a) 1.754 (b) 0.08849]
5. (a) sech 0.39 (b) sech 2.367
   [(a) 0.9285 (b) 0.1859]
6. (a) coth 0.444 (b) coth 1.843
   [(a) 2.398 (b) 1.051]
7. A telegraph wire hangs so that its shape is described by $y = 50 \operatorname{ch} \frac{x}{50}$. Evaluate, correct to 4 significant figures, the value of y when $x = 25$. [56.38]
8. The length l of a heavy cable hanging under gravity is given by $l = 2c \operatorname{sh}(L/2c)$. Find the value of l when $c = 40$ and $L = 30$.
   [30.71]
9. $V^2 = 0.55L \tanh(6.3\,\mathrm{d}/L)$ is a formula for velocity V of waves over the bottom of shallow water, where d is the depth and L is the wavelength. If $\mathrm{d} = 8.0$ and $L = 96$, calculate the value of V. [5.042]

5.2 Graphs of hyperbolic functions

A graph of $\boldsymbol{y = \sinh x}$ may be plotted using calculator values of hyperbolic functions. The curve is shown in Fig. 5.1. Since the graph is symmetrical about the origin, $\sinh x$ is an **odd function** (as stated in Section 5.1).

A graph of $\boldsymbol{y = \cosh x}$ may be plotted using calculator values of hyperbolic functions. The curve is shown in Fig. 5.2. Since the graph is symmetrical about the y-axis,

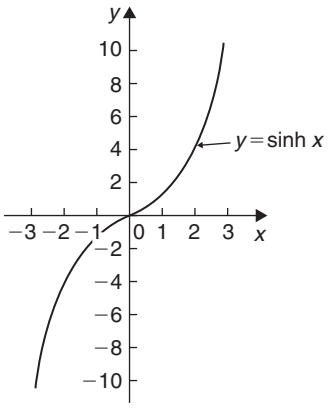


Figure 5.1

$\cosh x$ is an **even function** (as stated in Section 5.1). The shape of $y = \cosh x$ is that of a heavy rope or chain hanging freely under gravity and is called a **catenary**. Examples include transmission lines, a telegraph wire or a fisherman's line, and is used in the design of roofs and arches. Graphs of $y = \tanh x$, $y = \operatorname{cosech} x$, $y = \operatorname{sech} x$ and $y = \coth x$ are deduced in Problems 7 and 8.

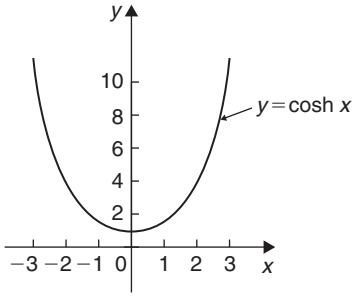


Figure 5.2

Problem 7. Sketch graphs of (a) $y = \tanh x$ and (b) $y = \coth x$ for values of x between -3 and 3.

A table of values is drawn up as shown below

x	-3	-2	-1
$\operatorname{sh} x$	-10.02	-3.63	-1.18
$\operatorname{ch} x$	10.07	3.76	1.54
$y = \operatorname{th} x = \frac{\operatorname{sh} x}{\operatorname{ch} x}$	-0.995	-0.97	-0.77
$y = \coth x = \frac{\operatorname{ch} x}{\operatorname{sh} x}$	-1.005	-1.04	-1.31

x	0	1	2	3
$\text{sh}\,x$	0	1.18	3.63	10.02
$\text{ch}\,x$	1	1.54	3.76	10.07
$y = \text{th}\,x = \dfrac{\text{sh}\,x}{\text{ch}\,x}$	0	0.77	0.97	0.995
$y = \coth x = \dfrac{\text{ch}\,x}{\text{sh}\,x}$	$\pm\infty$	1.31	1.04	1.005

(a) A graph of $y = \tanh x$ is shown in Fig. 5.3(a)

(b) A graph of $y = \coth x$ is shown in Fig. 5.3(b)

Both graphs are symmetrical about the origin thus $\tanh x$ and $\coth x$ are odd functions.

Problem 8. Sketch graphs of (a) $y = \text{cosech}\,x$ and (b) $y = \text{sech}\,x$ from $x = -4$ to $x = 4$, and, from the graphs, determine whether they are odd or even functions.

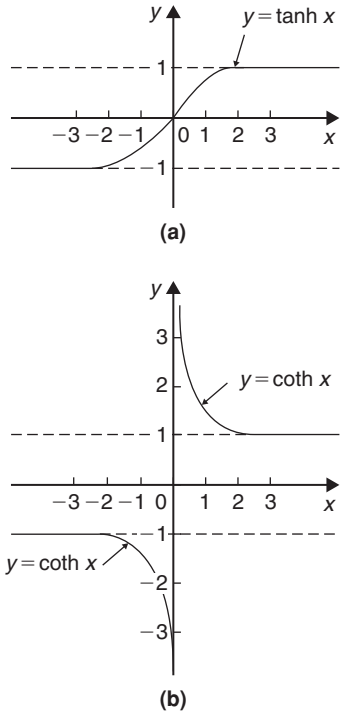


Figure 5.3

A table of values is drawn up as shown below

x	-4	-3	-2	-1
$\text{sh}\,x$	-22.29	-10.02	-3.63	-1.18
$\text{cosech}\,x = \dfrac{1}{\text{sh}\,x}$	-0.04	-0.10	-0.28	-0.85
$\text{ch}\,x$	27.31	10.07	3.76	1.54
$\text{sech}\,x = \dfrac{1}{\text{ch}\,x}$	0.04	0.10	0.27	0.65

x	0	1	2	3	4
$\text{sh}\,x$	0	1.18	3.63	10.02	27.29
$\text{cosech}\,x = \dfrac{1}{\text{sh}\,x}$	$\pm\infty$	0.85	0.28	0.10	0.04
$\text{ch}\,x$	1	1.54	3.76	10.07	27.31
$\text{sech}\,x = \dfrac{1}{\text{ch}\,x}$	1	0.65	0.27	0.10	0.04

(a) A graph of $y = \text{cosech}\,x$ is shown in Fig. 5.4(a). The graph is symmetrical about the origin and is thus an **odd function**.

(b) A graph of $y = \text{sech}\,x$ is shown in Fig. 5.4(b). The graph is symmetrical about the y-axis and is thus an **even function**.

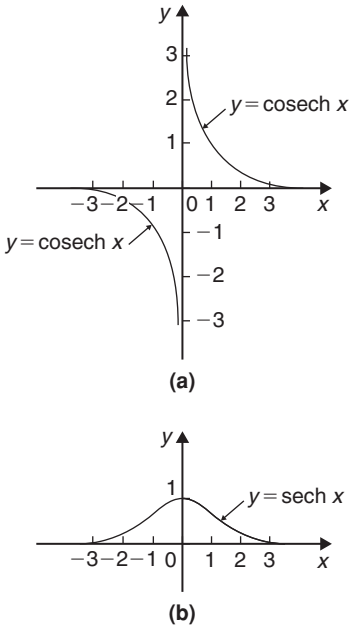


Figure 5.4

5.3 Hyperbolic identities

For every trigonometric identity there is a corresponding hyperbolic identity. **Hyperbolic identities** may be proved by either

(i) replacing $\sh x$ by $\dfrac{e^x - e^{-x}}{2}$ and $\ch x$ by $\dfrac{e^x + e^{-x}}{2}$, or

(ii) by using **Osborne's rule**, which states: *'the six trigonometric ratios used in trigonometrical identities relating general angles may be replaced by their corresponding hyperbolic functions, but the sign of any direct or implied product of two sines must be changed'.*

For example, since $\cos^2 x + \sin^2 x = 1$ then, by Osborne's rule, $\ch^2 x - \sh^2 x = 1$, i.e. the trigonometric functions have been changed to their corresponding hyperbolic functions and since $\sin^2 x$ is a product of two sines the sign is changed from + to −. Table 5.1 shows some trigonometric identities and their corresponding hyperbolic identities.

Problem 9. Prove the hyperbolic identities (a) $\ch^2 x - \sh^2 x = 1$ (b) $1 - \th^2 x = \sech^2 x$ (c) $\coth^2 x - 1 = \cosech^2 x$.

(a) $$\ch x + \sh x = \left(\frac{e^x + e^{-x}}{2}\right) + \left(\frac{e^x - e^{-x}}{2}\right) = e^x$$

$$\ch x - \sh x = \left(\frac{e^x + e^{-x}}{2}\right) - \left(\frac{e^x - e^{-x}}{2}\right)$$

$$= e^{+-x}$$

$$(\ch x + \sh x)(\ch x - \sh x) = (e^x)(e^{-x}) = e^0 = 1$$

i.e. $$\mathbf{ch^2 x - sh^2 x = 1} \qquad (1)$$

(b) Dividing each term in equation (1) by $\ch^2 x$ gives:

$$\frac{\ch^2 x}{\ch^2 x} - \frac{\sh^2 x}{\ch^2 x} = \frac{1}{\ch^2 x}$$

i.e. $$\mathbf{1 - th^2 x = sech^2 x}$$

Table 5.1

Trigonometric identity	Corresponding hyperbolic identity
$\cos^2 x + \sin^2 x = 1$	$\ch^2 x - \sh^2 x = 1$
$1 + \tan^2 x = \sec^2 x$	$1 - \th^2 x = \sech^2 x$
$\cot^2 x + 1 = \cosec^2 x$	$\coth^2 x - 1 = \cosech^2 x$
Compound angle formulae	
$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$	$\sh(A \pm B) = \sh A \ch B \pm \ch A \sh B$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$	$\ch(A \pm B) = \ch A \ch B \pm \sh A \sh B$
$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$	$\th(A \pm B) = \dfrac{\th A \pm \th B}{1 \pm \th A \th B}$
Double angles	
$\sin 2x = 2 \sin x \cos x$	$\sh 2x = 2 \sh x \ch x$
$\cos 2x = \cos^2 x - \sin^2 x$	$\ch 2x = \ch^2 x + \sh^2 x$
$= 2\cos^2 x - 1$	$= 2\ch^2 x - 1$
$= 1 - 2\sin^2 x$	$= 1 + 2\sh^2 x$
$\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$	$\th 2x = \dfrac{2 \th x}{1 + \th^2 x}$

(c) Dividing each term in equation (1) by $\text{sh}^2 x$ gives:

$$\frac{\text{ch}^2 x}{\text{sh}^2 x} - \frac{\text{sh}^2 x}{\text{sh}^2 x} = \frac{1}{\text{sh}^2 x}$$

i.e. $\mathbf{\coth^2 x - 1 = \text{cosech}^2 x}$

Problem 10. Prove, using Osborne's rule
(a) $\text{ch}\,2A = \text{ch}^2 A + \text{sh}^2 A$
(b) $1 - \text{th}^2 x = \text{sech}^2 x$.

(a) From trigonometric ratios,

$$\cos 2A = \cos^2 A - \sin^2 A \qquad (1)$$

Osborne's rule states that trigonometric ratios may be replaced by their corresponding hyperbolic functions but the sign of any product of two sines has to be changed. In this case, $\sin^2 A = (\sin A)(\sin A)$, i.e. a product of two sines, thus the sign of the corresponding hyperbolic function, $\text{sh}^2 A$, is changed from + to −. Hence, from (1), $\mathbf{\text{ch}\,2A = \text{ch}^2 A + \text{sh}^2 A}$

(b) From trigonometric ratios,

$$1 + \tan^2 x = \sec^2 x \qquad (2)$$

and $\tan^2 x = \dfrac{\sin^2 x}{\cos^2 x} = \dfrac{(\sin x)(\sin x)}{\cos^2 x}$

i.e. a product of two sines.

Hence, in equation (2), the trigonometric ratios are changed to their equivalent hyperbolic function and the sign of $\text{th}^2 x$ changed + to −, i.e. $\mathbf{1 - \text{th}^2 x = \text{sech}^2 x}$

Problem 11. Prove that $1 + 2\,\text{sh}^2 x = \text{ch}\,2x$.

Left hand side (L.H.S.)

$$= 1 + 2\,\text{sh}^2 x = 1 + 2\left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= 1 + 2\left(\frac{e^{2x} - 2e^x e^{-x} + e^{-2x}}{4}\right)$$

$$= 1 + \frac{e^{2x} - 2 + e^{-2x}}{2}$$

$$= 1 + \left(\frac{e^{2x} + e^{-2x}}{2}\right) - \frac{2}{2}$$

$$= \frac{e^{2x} + e^{-2x}}{2} = \text{ch}\,2x = \text{R.H.S.}$$

Problem 12. Show that $\text{th}^2 x + \text{sech}^2 x = 1$.

$$\text{L.H.S.} = \text{th}^2 x + \text{sech}^2 x = \frac{\text{sh}^2 x}{\text{ch}^2 x} + \frac{1}{\text{ch}^2 x}$$

$$= \frac{\text{sh}^2 x + 1}{\text{ch}^2 x}$$

Since $\text{ch}^2 x - \text{sh}^2 x = 1$ then $1 + \text{sh}^2 x = \text{ch}^2 x$

Thus $\dfrac{\text{sh}^2 x + 1}{\text{ch}^2 x} = \dfrac{\text{ch}^2 x}{\text{ch}^2 x} = 1 = \text{R.H.S.}$

Problem 13. Given $Ae^x + Be^{-x} \equiv 4\,\text{ch}\,x - 5\,\text{sh}\,x$, determine the values of A and B.

$$Ae^x + Be^{-x} \equiv 4\,\text{ch}\,x - 5\,\text{sh}\,x$$

$$= 4\left(\frac{e^x + e^{-x}}{2}\right) - 5\left(\frac{e^x - e^{-x}}{2}\right)$$

$$= 2e^x + 2e^{-x} - \frac{5}{2}e^x + \frac{5}{2}e^{-x}$$

$$= -\frac{1}{2}e^x + \frac{9}{2}e^{-x}$$

Equating coefficients gives: $\mathbf{A = -\frac{1}{2}}$ **and** $\mathbf{B = 4\frac{1}{2}}$

Problem 14. If $4e^x - 3e^{-x} \equiv P\,\text{sh}\,x + Q\,\text{ch}\,x$, determine the values of P and Q.

$$4e^x - 3e^{-x} \equiv P\,\text{sh}\,x + Q\,\text{ch}\,x$$

$$= P\left(\frac{e^x - e^{-x}}{2}\right) + Q\left(\frac{e^x + e^{-x}}{2}\right)$$

$$= \frac{P}{2}e^x - \frac{P}{2}e^{-x} + \frac{Q}{2}e^x + \frac{Q}{2}e^{-x}$$

$$= \left(\frac{P+Q}{2}\right)e^x + \left(\frac{Q-P}{2}\right)e^{-x}$$

Equating coefficients gives:

$$4 = \frac{P+Q}{2} \text{ and } -3 = \frac{Q-P}{2}$$

i.e. $P + Q = 8 \qquad (1)$

$$-P + Q = -6 \qquad (2)$$

Adding equations (1) and (2) gives: $2Q = 2$, i.e. $\mathbf{Q = 1}$
Substituting in equation (1) gives: $\mathbf{P = 7}$.

Now try the following exercise

Exercise 21 Further problems on hyperbolic identities

In Problems 1 to 4, prove the given identities.

1. (a) $\text{ch}(P-Q) \equiv \text{ch}\,P\,\text{ch}\,Q - \text{sh}\,P\,\text{sh}\,Q$
   (b) $\text{ch}\,2x \equiv \text{ch}^2 x + \text{sh}^2 x$
2. (a) $\coth x \equiv 2\,\text{cosech}\,2x + \text{th}\,x$
   (b) $\text{ch}\,2\theta - 1 \equiv 2\,\text{sh}^2\theta$
3. (a) $\text{th}(A-B) \equiv \dfrac{\text{th}\,A - \text{th}\,B}{1 - \text{th}\,A\,\text{th}\,B}$
   (b) $\text{sh}\,2A \equiv 2\,\text{sh}\,A\,\text{ch}\,A$
4. (a) $\text{sh}(A+B) \equiv \text{sh}\,A\,\text{ch}\,B + \text{ch}\,A\,\text{sh}\,B$
   (b) $\dfrac{\text{sh}^2 x + \text{ch}^2 x - 1}{2\text{ch}^2 x \coth^2 x} \equiv \tanh^4 x$
5. Given $P\text{e}^x - Q\text{e}^{-x} \equiv 6\,\text{ch}\,x - 2\,\text{sh}\,x$, find P and Q $[P=2,\ Q=-4]$
6. If $5\text{e}^x - 4\text{e}^{-x} \equiv A\,\text{sh}\,x + B\,\text{ch}\,x$, find A and B. $[A=9,\ B=1]$

5.4 Solving equations involving hyperbolic functions

Equations such as $\sinh x = 3.25$ or $\coth x = 3.478$ may be determined using a calculator. This is demonstrated in Worked Problems 15 to 21.

Problem 15. Solve the equation $\text{sh}\,x = 3$, correct to 4 significant figures.

If $\sinh x = 3$, then $x = \sinh^{-1} 3$
This can be determined by calculator.

(i) Press hyp
(ii) Choose 4, which is $\sinh^{-1}$
(iii) Type in 3
(iv) Close bracket)
(v) Press = and the answer is 1.818448459

i.e. the solution of $\text{sh}\,x = 3$ is: $\boldsymbol{x = 1.818}$, correct to 4 significant figures.

Problem 16. Solve the equation $\text{ch}\,x = 1.52$, correct to 3 decimal places.

Using a calculator with a similar procedure as in Worked Problem 15, check that:

$x = 0.980$, correct to 3 decimal places.

With reference to Fig. 5.2, it can be seen that there will be two values corresponding to $y = \cosh x = 1.52$. Hence, $\boldsymbol{x = \pm 0.980}$

Problem 17. Solve the equation $\tanh\theta = 0.256$, correct to 4 significant figures.

Using a calculator with a similar procedure as in Worked Problem 15, check that gives

$\boldsymbol{\theta = 0.2618}$, correct to 4 significant figures.

Problem 18. Solve the equation $\text{sech}\,x = 0.4562$, correct to 3 decimal places.

If $\text{sech}\,x = 0.4562$, then $x = \text{sech}^{-1} 0.4562 = \cosh^{-1}\left(\dfrac{1}{0.4562}\right)$ since $\cosh = \dfrac{1}{\text{sech}}$

i.e. $x = 1.421$, correct to 3 decimal places.

With reference to the graph of $y = \text{sech}\,x$ in Fig. 5.4, it can be seen that there will be two values corresponding to $y = \text{sech}\,x = 0.4562$
Hence, $\boldsymbol{x = \pm 1.421}$

Problem 19. Solve the equation $\text{cosech}\,y = -0.4458$, correct to 4 significant figures.

If $\text{cosech}\,y = -0.4458$, then $y = \text{cosech}^{-1}(-0.4458) = \sinh^{-1}\left(\dfrac{1}{-0.4458}\right)$ since $\sinh = \dfrac{1}{\text{cosech}}$
i.e. $\boldsymbol{y = -1.547}$, correct to 4 significant figures.

Problem 20. Solve the equation $\coth A = 2.431$, correct to 3 decimal places.

If $\coth A = 2.431$, then $A = \coth^{-1} 2.431 = \tanh^{-1}\left(\dfrac{1}{2.431}\right)$ since $\tanh = \dfrac{1}{\coth}$
i.e. $\boldsymbol{A = 0.437}$, correct to 3 decimal places.

Problem 21. A chain hangs in the form given by $y = 40\,\text{ch}\,\dfrac{x}{40}$. Determine, correct to 4 significant figures, (a) the value of y when x is 25, and (b) the value of x when $y = 54.30$

(a) $y = 40\,\text{ch}\dfrac{x}{40}$, and when $x = 25$,

$$\boldsymbol{y} = 40\,\text{ch}\,\frac{25}{40} = 40\,\text{ch}\,0.625$$

$$= 40(1.2017536\ldots) = \mathbf{48.07}$$

(b) When $y = 54.30$, $54.30 = 40\,\text{ch}\dfrac{x}{40}$, from which $\text{ch}\dfrac{x}{40} = \dfrac{54.30}{40} = 1.3575$

Hence, $\dfrac{x}{40} = \cosh^{-1} 1.3575 = \pm 0.822219\ldots$.

(see Fig. 5.2 for the reason as to why the answer is $\pm$) from which, $\boldsymbol{x} = 40(\pm 0.822219\ldots) = \mathbf{\pm 32.89}$

Equations of the form $\boldsymbol{a\,\text{ch}\,x + b\,\text{sh}\,x = c}$, where a, b and c are constants may be solved either by:

(a) plotting graphs of $y = a\,\text{ch}\,x + b\,\text{sh}\,x$ and $y = c$ and noting the points of intersection, or more accurately,

(b) by adopting the following procedure:

(i) Change $\text{sh}\,x$ to $\left(\dfrac{e^x - e^{-x}}{2}\right)$ and $\text{ch}\,x$ to $\left(\dfrac{e^x + e^{-x}}{2}\right)$

(ii) Rearrange the equation into the form $pe^x + qe^{-x} + r = 0$, where p, q and r are constants.

(iii) Multiply each term by e^x, which produces an equation of the form $p(e^x)^2 + re^x + q = 0$ (since $(e^{-x})(e^x) = e^0 = 1$)

(iv) Solve the quadratic equation $p(e^x)^2 + re^x + q = 0$ for e^x by factorising or by using the quadratic formula.

(v) Given e^x = a constant (obtained by solving the equation in (iv)), take Napierian logarithms of both sides to give $x = \ln(\text{constant})$

This procedure is demonstrated in Problem 22.

Problem 22. Solve the equation $2.6\,\text{ch}\,x + 5.1\,\text{sh}\,x = 8.73$, correct to 4 decimal places.

Following the above procedure:

(i) $2.6\,\text{ch}\,x + 5.1\,\text{sh}\,x = 8.73$

i.e. $2.6\left(\dfrac{e^x + e^{-x}}{2}\right) + 5.1\left(\dfrac{e^x - e^{-x}}{2}\right) = 8.73$

(ii) $1.3e^x + 1.3e^{-x} + 2.55e^x - 2.55e^{-x} = 8.73$

i.e. $3.85e^x - 1.25e^{-x} - 8.73 = 0$

(iii) $3.85(e^x)^2 - 8.73e^x - 1.25 = 0$

(iv) e^x

$$= \frac{-(-8.73) \pm \sqrt{[(-8.73)^2 - 4(3.85)(-1.25)]}}{2(3.85)}$$

$$= \frac{8.73 \pm \sqrt{95.463}}{7.70} = \frac{8.73 \pm 9.7705}{7.70}$$

Hence $e^x = 2.4027$ or $e^x = -0.1351$

(v) $x = \ln 2.4027$ or $x = \ln(-0.1351)$ which has no real solution.

Hence $\boldsymbol{x = 0.8766}$, correct to 4 decimal places.

Now try the following exercise

Exercise 22 Further problems on hyperbolic equations

In Problems 1 to 8, solve the given equations correct to 4 decimal places.

1. (a) $\sinh x = 1$ (b) $\text{sh}\,A = -2.43$
   [(a) 0.8814 (b) −1.6209]
2. (a) $\cosh B = 1.87$ (b) $2\,\text{ch}\,x = 3$
   [(a) ±1.2384 (b) ±0.9624]
3. (a) $\tanh y = -0.76$ (b) $3\,\text{th}\,x = 2.4$
   [(a) −0.9962 (b) 1.0986]
4. (a) $\text{sech}\,B = 0.235$ (b) $\text{sech}\,Z = 0.889$
   [(a) ±2.1272 (b) ±0.4947]
5. (a) $\text{cosech}\,\theta = 1.45$ (b) $5\,\text{cosech}\,x = 4.35$
   [(a) 0.6442 (b) 0.5401]
6. (a) $\coth x = 2.54$ (b) $2\coth y = -3.64$
   [(a) 0.4162 (b) −0.6176]
7. $3.5\,\text{sh}\,x + 2.5\,\text{ch}\,x = 0$ [−0.8959]

8. $2\,\text{sh}\,x + 3\,\text{ch}\,x = 5$ [0.6389 or −2.2484]
9. $4\,\text{th}\,x - 1 = 0$ [0.2554]
10. A chain hangs so that its shape is of the form $y = 56\cosh\left(\frac{x}{56}\right)$. Determine, correct to 4 significant figures, (a) the value of y when x is 35, and (b) the value of x when y is 62.35 [(a) 67.30 (b) ±26.42]

5.5 Series expansions for cosh *x* and sinh *x*

By definition,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

from Chapter 4.
Replacing x by $-x$ gives:

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots.$$

$$\cosh x = \frac{1}{2}(e^x + e^{-x})$$
$$= \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\right) + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots\right)\right]$$
$$= \frac{1}{2}\left[\left(2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \cdots\right)\right]$$

i.e. $\boldsymbol{\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots}$ (which is valid for all values of x). $\cosh x$ is an even function and contains only even powers of x in its expansion.

$$\sinh x = \frac{1}{2}(e^x - e^{-x})$$
$$= \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots\right)\right]$$
$$= \frac{1}{2}\left[2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \cdots\right]$$

i.e. $\boldsymbol{\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots}$ (which is valid for all values of x). $\sinh x$ is an odd function and contains only odd powers of x in its series expansion.

Problem 23. Using the series expansion for ch x evaluate ch 1 correct to 4 decimal places.

$\text{ch}\,x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$ from above

Let $x = 1$,

$$\text{then ch}\,1 = 1 + \frac{1^2}{2\times 1} + \frac{1^4}{4\times 3\times 2\times 1} + \frac{1^6}{6\times 5\times 4\times 3\times 2\times 1} + \cdots$$
$$= 1 + 0.5 + 0.04167 + 0.001389 + \cdots$$

i.e. $\boldsymbol{\text{ch}\,1 = 1.5431}$, correct to 4 decimal places, which may be checked by using a calculator.

Problem 24. Determine, correct to 3 decimal places, the value of sh 3 using the series expansion for sh x.

$$\text{sh}\,x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \text{ from above}$$

Let $x = 3$, then

$$\text{sh}\,3 = 3 + \frac{3^3}{3!} + \frac{3^5}{5!} + \frac{3^7}{7!} + \frac{3^9}{9!} + \frac{3^{11}}{11!} + \cdots$$
$$= 3 + 4.5 + 2.025 + 0.43393 + 0.05424 + 0.00444 + \cdots$$

i.e. $\boldsymbol{\text{sh}\,3 = 10.018}$, correct to 3 decimal places.

Problem 25. Determine the power series for $2\,\text{ch}\left(\frac{\theta}{2}\right) - \text{sh}\,2\theta$ as far as the term in θ^5.

In the series expansion for ch x, let $x = \frac{\theta}{2}$ then:

$$2\,\text{ch}\left(\frac{\theta}{2}\right) = 2\left[1 + \frac{(\theta/2)^2}{2!} + \frac{(\theta/2)^4}{4!} + \cdots\right]$$
$$= 2 + \frac{\theta^2}{4} + \frac{\theta^4}{192} + \cdots$$

In the series expansion for $\text{sh}\,x$, let $x = 2\theta$, then:

$$\text{sh}\,2\theta = 2\theta + \frac{(2\theta)^3}{3!} + \frac{(2\theta)^5}{5!} + \cdots$$
$$= 2\theta + \frac{4}{3}\theta^3 + \frac{4}{15}\theta^5 + \cdots$$

Hence

$$\text{ch}\left(\frac{\theta}{2}\right) - \text{sh}\,2\theta = \left(2 + \frac{\theta^2}{4} + \frac{\theta^4}{192} + \cdots\right) - \left(2\theta + \frac{4}{3}\theta^3 + \frac{4}{15}\theta^5 + \cdots\right)$$
$$= \mathbf{2 - 2\theta + \frac{\theta^2}{4} - \frac{4}{3}\theta^3 + \frac{\theta^4}{192} - \frac{4}{15}\theta^5 + \cdots}$$ as far as the term in $\boldsymbol{\theta^5}$

Now try the following exercise

Exercise 23 Further problems on series expansions for cosh *x* and sinh *x*

1. Use the series expansion for $\text{ch}\,x$ to evaluate, correct to 4 decimal places: (a) $\text{ch}\,1.5$ (b) $\text{ch}\,0.8$

[(a) 2.3524 (b) 1.3374]

2. Use the series expansion for $\text{sh}\,x$ to evaluate, correct to 4 decimal places: (a) $\text{sh}\,0.5$ (b) $\text{sh}\,2$

[(a) 0.5211 (b) 3.6269]

3. Expand the following as a power series as far as the term in x^5: (a) $\text{sh}\,3x$ (b) $\text{ch}\,2x$

$$\left[\begin{array}{l}\text{(a) } 3x + \frac{9}{2}x^3 + \frac{81}{40}x^5 \\ \text{(b) } 1 + 2x^2 + \frac{2}{3}x^4\end{array}\right]$$

In Problems 4 and 5, prove the given identities, the series being taken as far as the term in θ^5 only.

4. $\text{sh}\,2\theta - \text{sh}\,\theta \equiv \theta + \frac{7}{6}\theta^3 + \frac{31}{120}\theta^5$

5. $2\,\text{sh}\,\frac{\theta}{2} - \text{ch}\,\frac{\theta}{2} \equiv -1 + \theta - \frac{\theta^2}{8} + \frac{\theta^3}{24} - \frac{\theta^4}{384} + \frac{\theta^5}{1920}$

Chapter 6

Arithmetic and geometric progressions

6.1 Arithmetic progressions

When a sequence has a constant difference between successive terms it is called an **arithmetic progression** (often abbreviated to AP).
Examples include:

(i) 1, 4, 7, 10, 13,... where the **common difference** is 3 and

(ii) a, $a+d$, $a+2d$, $a+3d$,... where the common difference is d.

General expression for the *n*'th term of an AP

If the first term of an AP is 'a' and the common difference is 'd' then

the n'th term is: $a+(n-1)d$

In example (i) above, the 7th term is given by $1+(7-1)3=\mathbf{19}$, which may be readily checked.

Sum of *n* terms of an AP

The sum S of an AP can be obtained by multiplying the average of all the terms by the number of terms.

The average of all the terms $=\dfrac{a+l}{2}$, where 'a' is the first term and l is the last term, i.e. $l=a+(n-1)d$, for n terms.

Hence the sum of n terms,

$$S_n = n\left(\frac{a+l}{2}\right)$$

$$= \frac{n}{2}\{a+[a+(n-1)d]\}$$

i.e. $$S_n = \frac{n}{2}[2a+(n-1)d]$$

For example, the sum of the first 7 terms of the series 1, 4, 7, 10, 13,... is given by

$$S_7 = \frac{7}{2}[2(1)+(7-1)3], \text{ since } a=1 \text{ and } d=3$$

$$= \frac{7}{2}[2+18] = \frac{7}{2}[20] = \mathbf{70}$$

6.2 Worked problems on arithmetic progressions

Problem 1. Determine (a) the ninth, and (b) the sixteenth term of the series 2, 7, 12, 17, ...

2, 7, 12, 17,... is an arithmetic progression with a common difference, d, of 5.

(a) The n'th term of an AP is given by $a+(n-1)d$
Since the first term $a=2$, $d=5$ and $n=9$ then the 9th term is:
$2+(9-1)5=2+(8)(5)=2+40=\mathbf{42}$

(b) The 16th term is:
$2+(16-1)5=2+(15)(5)=2+75=\mathbf{77}$.

Problem 2. The 6th term of an AP is 17 and the 13th term is 38. Determine the 19th term.

The n'th term of an AP is $a+(n-1)d$

The 6th term is: $a+5d=17$ (1)

The 13th term is: $a+12d=38$ (2)

Equation (2) − equation (1) gives: $7d=21$, from which, $d=\frac{21}{7}=3$.

Substituting in equation (1) gives: $a+15=17$, from which, $a=2$.

Hence the 19th term is:
$a+(n-1)d=2+(19-1)3=2+(18)(3)=$
$2+54=\mathbf{56}$.

Problem 3. Determine the number of the term whose value is 22 in the series $2\frac{1}{2}, 4, 5\frac{1}{2}, 7, \ldots$

$2\frac{1}{2}, 4, 5\frac{1}{2}, 7, \ldots$ is an AP where $a=2\frac{1}{2}$ and $d=1\frac{1}{2}$.
Hence if the n'th term is 22 then: $a+(n-1)d=22$
i.e. $2\frac{1}{2}+(n-1)\left(1\frac{1}{2}\right)=22$
$(n-1)\left(1\frac{1}{2}\right)=22-2\frac{1}{2}=19\frac{1}{2}$.

$$n-1=\frac{19\frac{1}{2}}{1\frac{1}{2}}=13 \text{ and } n=13+1=14$$

i.e. **the 14th term of the AP is 22**.

Problem 4. Find the sum of the first 12 terms of the series 5, 9, 13, 17, ...

5, 9, 13, 17, ... is an AP where $a=5$ and $d=4$. The sum of n terms of an AP,

$$S_n=\frac{n}{2}[2a+(n-1)d]$$

Hence the sum of the first 12 terms,

$$S_{12}=\frac{12}{2}[2(5)+(12-1)4]$$
$$=6[10+44]=6(54)=\mathbf{324}$$

Problem 5. Find the sum of the first 21 terms of the series 3.5, 4.1, 4.7, 5.3, ...

3.5, 4.1, 4.7, 5.3, ... is an AP where $a=3.5$ and $d=0.6$

The sum of the first 21 terms,

$$S_{21}=\frac{21}{2}[2a+(n-1)d]$$
$$=\frac{21}{2}[2(3.5)+(21-1)0.6]=\frac{21}{2}[7+12]$$
$$=\frac{21}{2}(19)=\frac{399}{2}=\mathbf{199.5}$$

Now try the following exercise

Exercise 24 Further problems on arithmetic progressions

1. Find the 11th term of the series 8, 14, 20, 26, ... [68]
2. Find the 17th term of the series 11, 10.7, 10.4, 10.1, ... [6.2]
3. The seventh term of a series is 29 and the eleventh term is 54. Determine the sixteenth term. [85.25]
4. Find the 15th term of an arithmetic progression of which the first term is 2.5 and the tenth term is 16. [23.5]
5. Determine the number of the term which is 29 in the series 7, 9.2, 11.4, 13.6, ... [11^{th}]
6. Find the sum of the first 11 terms of the series 4, 7, 10, 13, ... [209]
7. Determine the sum of the series 6.5, 8.0, 9.5, 11.0, ..., 32 [346.5]

6.3 Further worked problems on arithmetic progressions

Problem 6. The sum of 7 terms of an AP is 35 and the common difference is 1.2. Determine the first term of the series.

$n=7$, $d=1.2$ and $S_7=35$
Since the sum of n terms of an AP is given by

$$S_n=\frac{n}{2}[2a+(n-1)d], \text{ then}$$

$$35=\frac{7}{2}[2a+(7-1)1.2]=\frac{7}{2}[2a+7.2]$$

Hence $\dfrac{35\times 2}{7} = 2a + 7.2$

$10 = 2a + 7.2$

Thus $2a = 10 - 7.2 = 2.8,$

from which $a = \dfrac{2.8}{2} = 1.4$

i.e. **the first term, $a = 1.4$**

Problem 7. Three numbers are in arithmetic progression. Their sum is 15 and their product is 80. Determine the three numbers.

Let the three numbers be $(a-d)$, a and $(a+d)$

Then $(a-d)+a+(a+d)=15$, i.e. $3a=15$, from which, $a=5$

Also, $a(a-d)(a+d)=80$, i.e. $a(a^2-d^2)=80$

Since $a=5, 5(5^2-d^2)=80$

$125-5d^2=80$

$125-80=5d^2$

$45=5d^2$

from which, $d^2=\dfrac{45}{5}=9$. Hence $d=\sqrt{9}=\pm 3$.
The three numbers are thus $(5-3)$, 5 and $(5+3)$, i.e. **2, 5 and 8**.

Problem 8. Find the sum of all the numbers between 0 and 207 which are exactly divisible by 3.

The series 3, 6, 9, 12, …, 207 is an AP whose first term $a=3$ and common difference $d=3$

The last term is $a+(n-1)d = 207$

i.e. $3+(n-1)3 = 207,$

from which $(n-1) = \dfrac{207-3}{3} = 68$

Hence $n = 68+1 = 69$

The sum of all 69 terms is given by

$$S_{69} = \frac{n}{2}[2a+(n-1)d]$$
$$= \frac{69}{2}[2(3)+(69-1)3]$$
$$= \frac{69}{2}[6+204] = \frac{69}{2}(210) = \mathbf{7245}$$

Problem 9. The first, twelfth and last term of an arithmetic progression are 4, $31\frac{1}{2}$, and $376\frac{1}{2}$ respectively. Determine (a) the number of terms in the series, (b) the sum of all the terms and (c) the '80'th term.

(a) Let the AP be $a, a+d, a+2d, \ldots, a+(n-1)d$, where $a=4$

The 12th term is: $a+(12-1)d = 31\frac{1}{2}$

i.e. $4+11d = 31\frac{1}{2},$

from which, $11d = 31\frac{1}{2} - 4 = 27\frac{1}{2}$

Hence $d = \dfrac{27\frac{1}{2}}{11} = 2\frac{1}{2}$

The last term is $a+(n-1)d$

i.e. $4+(n-1)\left(2\frac{1}{2}\right) = 376\frac{1}{2}$

$$(n-1) = \frac{376\frac{1}{2}-4}{2\frac{1}{2}}$$
$$= \frac{372\frac{1}{2}}{2\frac{1}{2}} = 149$$

Hence the number of terms in the series, $n = 149+1 = 150$

(b) Sum of all the terms,

$$S_{150} = \frac{n}{2}[2a+(n-1)d]$$
$$= \frac{150}{2}\left[2(4)+(150-1)\left(2\frac{1}{2}\right)\right]$$
$$= 75\left[8+(149)\left(2\frac{1}{2}\right)\right]$$
$$= 85[8+372.5]$$
$$= 75(380.5) = \mathbf{28537\frac{1}{2}}$$

(c) The 80th term is:

$$a+(n-1)d = 4+(80-1)\left(2\tfrac{1}{2}\right)$$
$$= 4+(79)\left(2\tfrac{1}{2}\right)$$
$$= 4+197.5 = \mathbf{201\frac{1}{2}}$$

Problem 10. An oil company bores a hole 80 m deep. Estimate the cost of boring if the cost is £30 for drilling the first metre with an increase in cost of £2 per metre for each succeeding metre.

The series is: 30, 32, 34, ... to 80 terms, i.e. $a = 30$, $d = 2$ and $n = 80$

Thus, **total cost**,

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$
$$= \frac{80}{2}[2(30) + (80-1)(2)]$$
$$= 40[60 + 158] = 40(218) = \mathbf{£8720}$$

Now try the following exercise

Exercise 25 Further problems on arithmetic progressions

1. The sum of 15 terms of an arithmetic progression is 202.5 and the common difference is 2. Find the first term of the series. [−0.5]
2. Three numbers are in arithmetic progression. Their sum is 9 and their product is 20.25. Determine the three numbers. [1.5, 3, 4.5]
3. Find the sum of all the numbers between 5 and 250 which are exactly divisible by 4. [7808]
4. Find the number of terms of the series 5, 8, 11, ... of which the sum is 1025. [25]
5. Insert four terms between 5 and 22.5 to form an arithmetic progression. [8.5, 12, 15.5, 19]
6. The first, tenth and last terms of an arithmetic progression are 9, 40.5, and 425.5 respectively. Find (a) the number of terms, (b) the sum of all the terms and (c) the 70th term. [(a) 120 (b) 26070 (c) 250.5]
7. On commencing employment a man is paid a salary of £16000 per annum and receives annual increments of £480. Determine his salary in the 9th year and calculate the total he will have received in the first 12 years. [£19840, £223,680]
8. An oil company bores a hole 120 m deep. Estimate the cost of boring if the cost is £70 for drilling the first metre with an increase in cost of £3 per metre for each succeeding metre. [£29820]

6.4 Geometric progressions

When a sequence has a constant ratio between successive terms it is called a **geometric progression** (often abbreviated to GP). The constant is called the **common ratio, r**.
Examples include

(i) 1, 2, 4, 8, ... where the common ratio is 2 and

(ii) $a, ar, ar^2, ar^3, \ldots$ where the common ratio is r.

General expression for the n'th term of a GP

If the first term of a GP is 'a' and the common ratio is r, then

the n'th term is: ar^{n-1}

which can be readily checked from the above examples.

For example, the 8th term of the GP 1, 2, 4, 8,... is $(1)(2)^7 = \mathbf{128}$, since $a = 1$ and $r = 2$.

Sum of n terms of a GP

Let a GP be $a, ar, ar^2, ar^3, \ldots, ar^{n-1}$
then the sum of n terms,

$$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} \cdots \quad (1)$$

Multiplying throughout by r gives:

$$rS_n = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} + ar^n + \cdots \quad (2)$$

Subtracting equation (2) from equation (1) gives:

$$S_n - rS_n = a - ar^n$$

i.e. $S_n(1-r) = a(1-r^n)$

Thus the sum of n terms, $S_n = \frac{a(1-r^n)}{(1-r)}$ which is valid when $r < 1$.

Subtracting equation (1) from equation (2) gives

$$S_n = \frac{a(r^n - 1)}{(r-1)} \text{ which is valid when } r > 1.$$

For example, the sum of the first 8 terms of the GP 1, 2, 4, 8, 16, ... is given by $S_8 = \frac{1(2^8 - 1)}{(2-1)}$, since $a = 1$ and $r = 2$

$$\text{i.e. } S_8 = \frac{1(256-1)}{1} = \mathbf{255}$$

Sum to infinity of a GP

When the common ratio r of a GP is less than unity, the sum of n terms, $S_n = \frac{a(1-r^n)}{(1-r)}$, which may be written as $S_n = \frac{a}{(1-r)} - \frac{ar^n}{(1-r)}$

Since $r < 1$, r^n becomes less as n increases, i.e. $r^n \to 0$ as $n \to \infty$.

Hence $\frac{ar^n}{(1-r)} \to 0$ as $n \to \infty$. Thus $S_n \to \frac{a}{(1-r)}$ as $n \to \infty$.

The quantity $\frac{a}{(1-r)}$ is called the **sum to infinity**, S_∞, and is the limiting value of the sum of an infinite number of terms,

$$\text{i.e. } S_\infty = \frac{a}{(1-r)} \text{ which is valid when } -1 < r < 1.$$

For example, the sum to infinity of the GP $1 + \frac{1}{2} + \frac{1}{4} + \cdots$ is

$S_\infty = \frac{1}{1 - \frac{1}{2}}$, since $a = 1$ and $r = \frac{1}{2}$, i.e. $\boldsymbol{S_\infty = 2}$.

6.5 Worked problems on geometric progressions

Problem 11. Determine the tenth term of the series 3, 6, 12, 24, ...

3, 6, 12, 24, ... is a geometric progression with a common ratio r of 2. The n'th term of a GP is ar^{n-1}, where a is the first term. Hence the 10th term is: $(3)(2)^{10-1} = (3)(2)^9 = 3(512) = \mathbf{1536}$.

Problem 12. Find the sum of the first 7 terms of the series, $\frac{1}{2}, 1\frac{1}{2}, 4\frac{1}{2}, 13\frac{1}{2}, \ldots$

$\frac{1}{2}, 1\frac{1}{2}, 4\frac{1}{2}, 13\frac{1}{2}, \ldots$ is a GP with a common ratio $r = 3$

The sum of n terms, $S_n = \frac{a(r^n - 1)}{(r-1)}$

Hence $S_7 = \frac{\frac{1}{2}(3^7 - 1)}{(3-1)} = \frac{\frac{1}{2}(2187-1)}{2} = \mathbf{546\frac{1}{2}}$

Problem 13. The first term of a geometric progression is 12 and the fifth term is 55. Determine the 8'th term and the 11'th term.

The 5th term is given by $ar^4 = 55$, where the first term $a = 12$

$$\text{Hence } r^4 = \frac{55}{a} = \frac{55}{12}$$

$$\text{and } r = \sqrt[4]{\left(\frac{55}{12}\right)} = 1.4631719\ldots$$

The 8th term is $ar^7 = (12)(1.4631719\ldots)^7 = \mathbf{172.3}$

The 11th term is $ar^{10} = (12)(1.4631719\ldots)^{10} = \mathbf{539.7}$

Problem 14. Which term of the series 2187, 729, 243, ... is $\frac{1}{9}$?

2187, 729, 243, ... is a GP with a common ratio $r = \frac{1}{3}$ and first term $a = 2187$

The n'th term of a GP is given by: ar^{n-1}

$$\text{Hence } \frac{1}{9} = (2187)\left(\tfrac{1}{3}\right)^{n-1}$$

$$\text{from which } \left(\frac{1}{3}\right)^{n-1} = \frac{1}{(9)(2187)} = \frac{1}{3^2 3^7} = \frac{1}{3^9} = \left(\frac{1}{3}\right)^9$$

Thus $(n-1) = 9$, from which, $n = 9 + 1 = 10$

i.e. $\frac{1}{9}$ **is the 10th term of the GP**.

Problem 15. Find the sum of the first 9 terms of the series 72.0, 57.6, 46.08, ...

$$\text{The common ratio, } r = \frac{ar}{a} = \frac{57.6}{72.0} = 0.8$$

$$\left(\text{also } \frac{ar^2}{ar} = \frac{46.08}{57.6} = 0.8\right)$$

The sum of 9 terms,

$$S_9 = \frac{a(1-r^n)}{(1-r)} = \frac{72.0(1-0.8^9)}{(1-0.8)}$$
$$= \frac{72.0(1-0.1342)}{0.2} = \mathbf{311.7}$$

Problem 16. Find the sum to infinity of the series 3, 1, $\frac{1}{3}$, ...

3, 1, $\frac{1}{3}$, ... is a GP of common ratio, $r = \frac{1}{3}$
The sum to infinity,

$$S_\infty = \frac{a}{1-r} = \frac{3}{1-\frac{1}{3}} = \frac{3}{\frac{2}{3}} = \frac{9}{2} = \mathbf{4\frac{1}{2}}$$

Now try the following exercise

Exercise 26 Further problems on geometric progressions

1. Find the 10th term of the series 5, 10, 20, 40, ... [2560]
2. Determine the sum of the first 7 terms of the series $\frac{1}{4}$, $\frac{3}{4}$, $2\frac{1}{4}$, $6\frac{3}{4}$, ... [273.25]
3. The first term of a geometric progression is 4 and the 6th term is 128. Determine the 8th and 11th terms. [512, 4096]
4. Find the sum of the first 7 terms of the series 2, 5, $12\frac{1}{2}$, ... (correct to 4 significant figures). [812.5]
5. Determine the sum to infinity of the series 4, 2, 1, ... [8]
6. Find the sum to infinity of the series $2\frac{1}{2}$, $-1\frac{1}{4}$, $\frac{5}{8}$, ... $[1\frac{2}{3}]$

6.6 Further worked problems on geometric progressions

Problem 17. In a geometric progression the sixth term is 8 times the third term and the sum of the seventh and eighth terms is 192. Determine (a) the common ratio, (b) the first term, and (c) the sum of the fifth to eleventh terms, inclusive.

(a) Let the GP be $a, ar, ar^2, ar^3, \ldots, ar^{n-1}$
The 3rd term $= ar^2$ and the sixth term $= ar^5$
The 6th term is 8 times the 3rd.
Hence $ar^5 = 8ar^2$ from which, $r^3 = 8$, $r = \sqrt[3]{8}$
i.e. **the common ratio $r = 2$**.

(b) The sum of the 7th and 8th terms is 192. Hence $ar^6 + ar^7 = 192$.

Since $r = 2$, then $64a + 128a = 192$

$$192a = 192,$$

from which, $\boldsymbol{a}$, the first term, $\mathbf{= 1}$.

(c) The sum of the 5th to 11th terms (inclusive) is given by:

$$S_{11} - S_4 = \frac{a(r^{11}-1)}{(r-1)} - \frac{a(r^4-1)}{(r-1)}$$
$$= \frac{1(2^{11}-1)}{(2-1)} - \frac{1(2^4-1)}{(2-1)}$$
$$= (2^{11}-1) - (2^4-1)$$
$$= 2^{11} - 2^4 = 2048 - 16 = \mathbf{2032}$$

Problem 18. A hire tool firm finds that their net return from hiring tools is decreasing by 10% per annum. If their net gain on a certain tool this year is £400, find the possible total of all future profits from this tool (assuming the tool lasts for ever).

The net gain forms a series:

$$£400 + £400 \times 0.9 + £400 \times 0.9^2 + \cdots,$$

which is a GP with $a = 400$ and $r = 0.9$.
The sum to infinity,

$$S_\infty = \frac{a}{(1-r)} = \frac{400}{(1-0.9)}$$

$$\mathbf{= £4000 = \text{total future profits}}$$

Problem 19. If £100 is invested at compound interest of 8% per annum, determine (a) the value after 10 years, (b) the time, correct to the nearest year, it takes to reach more than £300.

(a) Let the GP be $a, ar, ar^2, \ldots, ar^n$
The first term $a = £100$
The common ratio $r = 1.08$
Hence the second term is

$$ar = (100)(1.08) = £108,$$

which is the value after 1 year,
the third term is

$$ar^2 = (100)(1.08)^2 = £116.64,$$

which is the value after 2 years, and so on.
Thus the value after 10 years

$$= ar^{10} = (100)(1.08)^{10} = \mathbf{£215.89}$$

(b) When £300 has been reached, $300 = ar^n$

i.e. $300 = 100(1.08)^n$

and $3 = (1.08)^n$

Taking logarithms to base 10 of both sides gives:

$$\lg 3 = \lg(1.08)^n = n\lg(1.08),$$

by the laws of logarithms
from which, $n = \dfrac{\lg 3}{\lg 1.08} = 14.3$

Hence it will take 15 years to reach more than £300.

Problem 20. A drilling machine is to have 6 speeds ranging from 50 rev/min to 750 rev/ min. If the speeds form a geometric progression determine their values, each correct to the nearest whole number.

Let the GP of n terms be given by $a, ar, ar^2, \ldots, ar^{n-1}$.
The first term $a = 50$ rev/min
The 6th term is given by ar^{6-1}, which is 750 rev/min,

i.e. $ar^5 = 750$

from which $r^5 = \dfrac{750}{a} = \dfrac{750}{50} = 15$

Thus the common ratio, $r = \sqrt[5]{15} = 1.7188$

The first term is $a = 50$ rev/min

the second term is $ar = (50)(1.7188) = 85.94$,

the third term is $ar^2 = (50)(1.7188)^2 = 147.71$,

the fourth term is $ar^3 = (50)(1.7188)^3 = 253.89$,

the fifth term is $ar^4 = (50)(1.7188)^4 = 436.39$,

the sixth term is $ar^5 = (50)(1.7188)^5 = 750.06$

Hence, correct to the nearest whole number, the 6 speeds of the drilling machine are **50, 86, 148, 254, 436 and 750 rev/min**.

Now try the following exercise

Exercise 27 Further problems on geometric progressions

1. In a geometric progression the 5th term is 9 times the 3rd term and the sum of the 6th and 7th terms is 1944. Determine (a) the common ratio, (b) the first term and (c) the sum of the 4th to 10th terms inclusive.
[(a) 3 (b) 2 (c) 59022]

2. Which term of the series 3, 9, 27,... is 59049? [10^{th}]

3. The value of a lathe originally valued at £3000 depreciates 15% per annum. Calculate its value after 4 years. The machine is sold when its value is less than £550. After how many years is the lathe sold?
[£1566, 11 years]

4. If the population of Great Britain is 55 million and is decreasing at 2.4% per annum, what will be the population in 5 years time?
[48.71 M]

5. 100 g of a radioactive substance disintegrates at a rate of 3% per annum. How much of the substance is left after 11 years? [71.53 g]

6. If £250 is invested at compound interest of 6% per annum determine (a) the value after 15 years, (b) the time, correct to the nearest year, it takes to reach £750.
[(a) £599.14 (b) 19 years]

7. A drilling machine is to have 8 speeds ranging from 100 rev/min to 1000 rev/min. If the speeds form a geometric progression determine their values, each correct to the nearest whole number.
[100, 139, 193, 268, 373, 518, 720, 1000 rev/min]

Chapter 7

The binomial series

7.1 Pascal's triangle

A **binomial expression** is one which contains two terms connected by a plus or minus sign. Thus $(p+q)$, $(a+x)^2$, $(2x+y)^3$ are examples of binomial expressions. Expanding $(a+x)^n$ for integer values of n from 0 to 6 gives the results as shown at the bottom of the page.

From these results the following patterns emerge:

(i) 'a' decreases in power moving from left to right.

(ii) 'x' increases in power moving from left to right.

(iii) The coefficients of each term of the expansions are symmetrical about the middle coefficient when n is even and symmetrical about the two middle coefficients when n is odd.

(iv) The coefficients are shown separately in Table 7.1 and this arrangement is known as **Pascal's triangle**. A coefficient of a term may be obtained by adding the two adjacent coefficients immediately above in the previous row. This is shown by the triangles in Table 7.1, where, for example, $1+3=4$, $10+5=15$, and so on.

(v) Pascal's triangle method is used for expansions of the form $(a+x)^n$ for integer values of n less than about 8.

Table 7.1

$(a+x)^0$				1			
$(a+x)^1$			1		1		
$(a+x)^2$		1		2		1	
$(a+x)^3$	1		3		3		1
$(a+x)^4$		1		4		6		4		1
$(a+x)^5$	1		5		10		10		5		1
$(a+x)^6$	1		6		15		20		15		6		1

Problem 1. Use the Pascal's triangle method to determine the expansion of $(a+x)^7$.

From Table 7.1, the row of Pascal's triangle corresponding to $(a+x)^6$ is as shown in (1) below. Adding adjacent coefficients gives the coefficients of $(a+x)^7$ as shown in (2) below.

1 6 15 20 15 6 1 (1)

1 7 21 35 35 21 7 1 (2)

The first and last terms of the expansion of $(a+x)^7$ are a^7 and x^7 respectively. The powers of 'a' decrease and the powers of 'x' increase moving from left to right.

$$
\begin{aligned}
(a+x)^0 &= 1\\
(a+x)^1 &= a+x\\
(a+x)^2 &= (a+x)(a+x) = a^2+2ax+x^2\\
(a+x)^3 &= (a+x)^2(a+x) = a^3+3a^2x+3ax^2+x^3\\
(a+x)^4 &= (a+x)^3(a+x) = a^4+4a^3x+6a^2x^2+4ax^3+x^4\\
(a+x)^5 &= (a+x)^4(a+x) = a^5+5a^4x+10a^3x^2+10a^2x^3+5ax^4+x^5\\
(a+x)^6 &= (a+x)^5(a+x) = a^6+6a^5x+15a^4x^2+20a^3x^3+15a^2x^4+6ax^5+x^6
\end{aligned}
$$

Hence

$$(a+x)^7 = a^7 + 7a^6x + 21a^5x^2 + 35a^4x^3 + 35a^3x^4 + 21a^2x^5 + 7ax^6 + x^7$$

Problem 2. Determine, using Pascal's triangle method, the expansion of $(2p-3q)^5$.

Comparing $(2p-3q)^5$ with $(a+x)^5$ shows that $a=2p$ and $x=-3q$.
Using Pascal's triangle method:

$$(a+x)^5 = a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + \cdots$$

Hence

$$(2p-3q)^5 = (2p)^5 + 5(2p)^4(-3q) + 10(2p)^3(-3q)^2 + 10(2p)^2(-3q)^3 + 5(2p)(-3q)^4 + (-3q)^5$$

i.e. $$(2p-3q)^5 = 32p^5 - 240p^4q + 720p^3q^2 - 1080p^2q^3 + 810pq^4 - 243q^5$$

Now try the following exercise

Exercise 28 Further problems on Pascal's triangle

1. Use Pascal's triangle to expand $(x-y)^7$.

   $$\left[x^7 - 7x^6y + 21x^5y^2 - 35x^4y^3 + 35x^3y^4 - 21x^2y^5 + 7xy^6 - y^7\right]$$

2. Expand $(2a+3b)^5$ using Pascal's triangle.

   $$\left[32a^5 + 240a^4b + 720a^3b^2 + 1080a^2b^3 + 810ab^4 + 243b^5\right]$$

7.2 The binomial series

The **binomial series** or **binomial theorem** is a formula for raising a binomial expression to any power without lengthy multiplication. The general binomial expansion of $(a+x)^n$ is given by:

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \cdots$$

where 3! denotes $3\times2\times1$ and is termed 'factorial 3'.
With the binomial theorem n may be a fraction, a decimal fraction or a positive or negative integer.
When n is a positive integer, the series is finite, i.e., it comes to an end; when n is a negative integer, or a fraction, the series is infinite.
In the general expansion of $(a+x)^n$ it is noted that the 4th term is: $\frac{n(n-1)(n-2)}{3!}a^{n-3}x^3$. The number 3 is very evident in this expression.

For any term in a binomial expansion, say the r'th term, $(r-1)$ is very evident. It may therefore be reasoned that **the r'th term of the expansion $(a+x)^n$** is:

$$\frac{n(n-1)(n-2)\ldots\text{to }(r-1)\text{ terms}}{(r-1)!}a^{n-(r-1)}x^{r-1}$$

If $a=1$ in the binomial expansion of $(a+x)^n$ then:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

which is valid for $-1<x<1$.
When x is small compared with 1 then:

$$(1+x)^n \approx 1 + nx$$

7.3 Worked problems on the binomial series

Problem 3. Use the binomial series to determine the expansion of $(2+x)^7$.

The binomial expansion is given by:

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \cdots$$

When $a=2$ and $n=7$:

$$(2+x)^7 = 2^7 + 7(2)^6x + \frac{(7)(6)}{(2)(1)}(2)^5x^2 + \frac{(7)(6)(5)}{(3)(2)(1)}(2)^4x^3 + \frac{(7)(6)(5)(4)}{(4)(3)(2)(1)}(2)^3x^4 + \frac{(7)(6)(5)(4)(3)}{(5)(4)(3)(2)(1)}(2)^2x^5 + \frac{(7)(6)(5)(4)(3)(2)}{(6)(5)(4)(3)(2)(1)}(2)x^6 + \frac{(7)(6)(5)(4)(3)(2)(1)}{(7)(6)(5)(4)(3)(2)(1)}x^7$$

i.e. $\mathbf{(2+x)^7 = 128 + 448x + 672x^2 + 560x^3 + 280x^4 + 84x^5 + 14x^6 + x^7}$

Problem 4. Use the binomial series to determine the expansion of $(2a-3b)^5$.

From equation (1), the binomial expansion is given by:

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \cdots$$

When $a=2a$, $x=-3b$ and $n=5$:

$$(2a-3b)^5 = (2a)^5 + 5(2a)^4(-3b) + \frac{(5)(4)}{(2)(1)}(2a)^3(-3b)^2 + \frac{(5)(4)(3)}{(3)(2)(1)}(2a)^2(-3b)^3 + \frac{(5)(4)(3)(2)}{(4)(3)(2)(1)}(2a)(-3b)^4 + \frac{(5)(4)(3)(2)(1)}{(5)(4)(3)(2)(1)}(-3b)^5$$

i.e. $\mathbf{(2a-3b)^5 = 32a^5 - 240a^4b + 720a^3b^2 - 1080a^2b^3 + 810ab^4 - 243b^5}$

Problem 5. Expand $\left(c-\frac{1}{c}\right)^5$ using the binomial series.

$$\left(c-\frac{1}{c}\right)^5 = c^5 + 5c^4\left(-\frac{1}{c}\right) + \frac{(5)(4)}{(2)(1)}c^3\left(-\frac{1}{c}\right)^2 + \frac{(5)(4)(3)}{(3)(2)(1)}c^2\left(-\frac{1}{c}\right)^3 + \frac{(5)(4)(3)(2)}{(4)(3)(2)(1)}c\left(-\frac{1}{c}\right)^4 + \frac{(5)(4)(3)(2)(1)}{(5)(4)(3)(2)(1)}\left(-\frac{1}{c}\right)^5$$

i.e. $\mathbf{\left(c-\frac{1}{c}\right)^5 = c^5 - 5c^3 + 10c - \frac{10}{c} + \frac{5}{c^3} - \frac{1}{c^5}}$

Problem 6. Without fully expanding $(3+x)^7$, determine the fifth term.

The r'th term of the expansion $(a+x)^n$ is given by:

$$\frac{n(n-1)(n-2)\ldots\text{to }(r-1)\text{ terms}}{(r-1)!}a^{n-(r-1)}x^{r-1}$$

Substituting $n=7$, $a=3$ and $r-1=5-1=4$ gives:

$$\frac{(7)(6)(5)(4)}{(4)(3)(2)(1)}(3)^{7-4}x^4$$

i.e. the fifth term of $(3+x)^7 = 35(3)^3x^4 = \mathbf{945x^4}$

Problem 7. Find the middle term of $\left(2p-\frac{1}{2q}\right)^{10}$.

In the expansion of $(a+x)^{10}$ there are $10+1$, i.e. 11 terms. Hence the middle term is the sixth. Using the general expression for the r'th term where $a=2p$, $x=-\frac{1}{2q}$, $n=10$ and $r-1=5$ gives:

$$\frac{(10)(9)(8)(7)(6)}{(5)(4)(3)(2)(1)}(2p)^{10-5}\left(-\frac{1}{2q}\right)^5 = 252(32p^5)\left(-\frac{1}{32q^5}\right)$$

Hence the middle term of $\left(2p-\frac{1}{2q}\right)^{10}$ is $\mathbf{-252\frac{p^5}{q^5}}$

Problem 8. Evaluate $(1.002)^9$ using the binomial theorem correct to (a) 3 decimal places and (b) 7 significant figures.

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

$$(1.002)^9 = (1+0.002)^9$$

Substituting $x = 0.002$ and $n = 9$ in the general expansion for $(1+x)^n$ gives:

$$\begin{aligned}(1+0.002)^9 &= 1 + 9(0.002) + \frac{(9)(8)}{(2)(1)}(0.002)^2 + \frac{(9)(8)(7)}{(3)(2)(1)}(0.002)^3 + \cdots \\ &= 1 + 0.018 + 0.000144 + 0.000000672 + \cdots \\ &= 1.018144672\ldots\end{aligned}$$

Hence $(1.002)^9 =$ **1.018, correct to 3 decimal places**

$=$ **1.018145, correct to 7 significant figures**

Problem 9. Evaluate $(0.97)^6$ correct to 4 significant figures using the binomial expansion.

$(0.97)^6$ is written as $(1-0.03)^6$

Using the expansion of $(1+x)^n$ where $n = 6$ and $x = -0.03$ gives:

$$\begin{aligned}(1-0.03)^6 &= 1 + 6(-0.03) + \frac{(6)(5)}{(2)(1)}(-0.03)^2 + \frac{(6)(5)(4)}{(3)(2)(1)}(-0.03)^3 \\ &\quad + \frac{(6)(5)(4)(3)}{(4)(3)(2)(1)}(-0.03)^4 + \cdots \\ &= 1 - 0.18 + 0.0135 - 0.00054 + 0.00001215 - \cdots \\ &\approx 0.83297215\end{aligned}$$

i.e. $(0.97)^6 =$ **0.8330, correct to 4 significant figures**

Problem 10. Determine the value of $(3.039)^4$, correct to 6 significant figures using the binomial theorem.

$(3.039)^4$ may be written in the form $(1+x)^n$ as:

$$\begin{aligned}(3.039)^4 &= (3+0.039)^4 \\ &= \left[3\left(1 + \frac{0.039}{3}\right)\right]^4 \\ &= 3^4(1+0.013)^4\end{aligned}$$

$$\begin{aligned}(1+0.013)^4 &= 1 + 4(0.013) + \frac{(4)(3)}{(2)(1)}(0.013)^2 + \frac{(4)(3)(2)}{(3)(2)(1)}(0.013)^3 + \cdots \\ &= 1 + 0.052 + 0.001014 + 0.000008788 + \cdots \\ &= 1.0530228\end{aligned}$$

correct to 8 significant figures

Hence $(3.039)^4 = 3^4(1.0530228)$

$=$ **85.2948, correct to 6 significant figures**

Now try the following exercise

Exercise 29 Further problems on the binomial series

1. Use the binomial theorem to expand $(a+2x)^4$.

   $\left[a^4 + 8a^3x + 24a^2x^2 + 32ax^3 + 16x^4\right]$

2. Use the binomial theorem to expand $(2-x)^6$.

   $\left[64 - 192x + 240x^2 - 160x^3 + 60x^4 - 12x^5 + x^6\right]$

3. Expand $(2x-3y)^4$.

   $\left[16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4\right]$

4. Determine the expansion of $\left(2x + \frac{2}{x}\right)^5$.

   $\left[32x^5 + 160x^3 + 320x + \frac{320}{x} + \frac{160}{x^3} + \frac{32}{x^5}\right]$

5. Expand $(p+2q)^{11}$ as far as the fifth term.

$$\left[p^{11}+22p^{10}q+220p^9q^2+1320p^8q^3+5280p^7q^4\right]$$

6. Determine the sixth term of $\left(3p+\frac{q}{3}\right)^{13}$.

$$[34749\,p^8q^5]$$

7. Determine the middle term of $(2a-5b)^8$.

$$[700000\,a^4b^4]$$

8. Use the binomial theorem to determine, correct to 4 decimal places:

(a) $(1.003)^8$ (b) $(1.042)^7$

[(a) 1.0243 (b) 1.3337]

9. Use the binomial theorem to determine, correct to 5 significant figures:

(a) $(0.98)^7$ (b) $(2.01)^9$

[(a) 0.86813 (b) 535.51]

10. Evaluate $(4.044)^6$ correct to 3 decimal places.

[4373.880]

7.4 Further worked problems on the binomial series

Problem 11.

(a) Expand $\frac{1}{(1+2x)^3}$ in ascending powers of x as far as the term in x^3, using the binomial series.

(b) State the limits of x for which the expansion is valid.

(a) Using the binomial expansion of $(1+x)^n$, where $n=-3$ and x is replaced by $2x$ gives:

$$\frac{1}{(1+2x)^3}=(1+2x)^{-3}$$
$$=1+(-3)(2x)+\frac{(-3)(-4)}{2!}(2x)^2+\frac{(-3)(-4)(-5)}{3!}(2x)^3+\cdots$$
$$=\mathbf{1-6x+24x^2-80x^3+\cdots}$$

(b) The expansion is valid provided $|2x|<1$,

i.e. $\mathbf{|x|<\frac{1}{2}}$ or $\mathbf{-\frac{1}{2}<x<\frac{1}{2}}$

Problem 12.

(a) Expand $\frac{1}{(4-x)^2}$ in ascending powers of x as far as the term in x^3, using the binomial theorem.

(b) What are the limits of x for which the expansion in (a) is true?

(a) $$\frac{1}{(4-x)^2}=\frac{1}{\left[4\left(1-\frac{x}{4}\right)\right]^2}=\frac{1}{4^2\left(1-\frac{x}{4}\right)^2}$$
$$=\frac{1}{16}\left(1-\frac{x}{4}\right)^{-2}$$

Using the expansion of $(1+x)^n$

$$\frac{1}{(4-x)^2}=\frac{1}{16}\left(1-\frac{x}{4}\right)^{-2}$$
$$=\frac{1}{16}\left[1+(-2)\left(-\frac{x}{4}\right)+\frac{(-2)(-3)}{2!}\left(-\frac{x}{4}\right)^2+\frac{(-2)(-3)(-4)}{3!}\left(-\frac{x}{4}\right)^3+\cdots\right]$$
$$=\mathbf{\frac{1}{16}\left(1+\frac{x}{2}+\frac{3x^2}{16}+\frac{x^3}{16}+\cdots\right)}$$

(b) The expansion in (a) is true provided $\left|\frac{x}{4}\right|<1$,

i.e. $\mathbf{|x|<4}$ or $\mathbf{-4<x<4}$

Problem 13. Use the binomial theorem to expand $\sqrt{4+x}$ in ascending powers of x to four terms. Give the limits of x for which the expansion is valid.

$$\sqrt{4+x}=\sqrt{\left[4\left(1+\frac{x}{4}\right)\right]}$$
$$=\sqrt{4}\sqrt{\left(1+\frac{x}{4}\right)}=2\left(1+\frac{x}{4}\right)^{\frac{1}{2}}$$

Using the expansion of $(1+x)^n$,

$$2\left(1+\frac{x}{4}\right)^{\frac{1}{2}}$$
$$=2\left[1+\left(\frac{1}{2}\right)\left(\frac{x}{4}\right)+\frac{(1/2)(-1/2)}{2!}\left(\frac{x}{4}\right)^2\right.$$

$$+\frac{(1/2)(-1/2)(-3/2)}{3!}\left(\frac{x}{4}\right)^3+\cdots\Bigg]$$

$$=2\left(1+\frac{x}{8}-\frac{x^2}{128}+\frac{x^3}{1024}-\cdots\right)$$

$$=\mathbf{2+\frac{x}{4}-\frac{x^2}{64}+\frac{x^3}{512}-\cdots}$$

This is valid when $\left|\frac{x}{4}\right|<1$,

i.e. $\mathbf{|x|<4}$ or $\mathbf{-4<x<4}$

Problem 14. Expand $\dfrac{1}{\sqrt{(1-2t)}}$ in ascending powers of t as far as the term in t^3.

State the limits of t for which the expression is valid.

$$\frac{1}{\sqrt{(1-2t)}}$$

$$=(1-2t)^{-\frac{1}{2}}$$

$$=1+\left(-\frac{1}{2}\right)(-2t)+\frac{(-1/2)(-3/2)}{2!}(-2t)^2$$

$$+\frac{(-1/2)(-3/2)(-5/2)}{3!}(-2t)^3+\cdots,$$

using the expansion for $(1+x)^n$

$$=\mathbf{1+t+\frac{3}{2}t^2+\frac{5}{2}t^3+\cdots}$$

The expression is valid when $|2t|<1$,

i.e. $\mathbf{|t|<\frac{1}{2}}$ or $\mathbf{-\frac{1}{2}<t<\frac{1}{2}}$

Problem 15. Simplify $\dfrac{\sqrt[3]{(1-3x)}\sqrt{(1+x)}}{\left(1+\frac{x}{2}\right)^3}$ given that powers of x above the first may be neglected.

$$\frac{\sqrt[3]{(1-3x)}\sqrt{(1+x)}}{\left(1+\frac{x}{2}\right)^3}$$

$$=(1-3x)^{\frac{1}{3}}(1+x)^{\frac{1}{2}}\left(1+\frac{x}{2}\right)^{-3}$$

$$\approx\left[1+\left(\frac{1}{3}\right)(-3x)\right]\left[1+\left(\frac{1}{2}\right)(x)\right]\left[1+(-3)\left(\frac{x}{2}\right)\right]$$

when expanded by the binomial theorem as far as the x term only,

$$=(1-x)\left(1+\frac{x}{2}\right)\left(1-\frac{3x}{2}\right)$$

$$=\left(1-x+\frac{x}{2}-\frac{3x}{2}\right)\quad\text{when powers of } x \text{ higher than unity are neglected}$$

$$=\mathbf{(1-2x)}$$

Problem 16. Express $\dfrac{\sqrt{(1+2x)}}{\sqrt[3]{(1-3x)}}$ as a power series as far as the term in x^2. State the range of values of x for which the series is convergent.

$$\frac{\sqrt{(1+2x)}}{\sqrt[3]{(1-3x)}}=(1+2x)^{\frac{1}{2}}(1-3x)^{-\frac{1}{3}}$$

$$(1+2x)^{\frac{1}{2}}=1+\left(\frac{1}{2}\right)(2x)$$

$$+\frac{(1/2)(-1/2)}{2!}(2x)^2+\cdots$$

$$=1+x-\frac{x^2}{2}+\cdots\quad\text{which is valid for}$$

$$|2x|<1,\ \text{i.e. } |x|<\frac{1}{2}$$

$$(1-3x)^{-\frac{1}{3}}=1+(-1/3)(-3x)$$

$$+\frac{(-1/3)(-4/3)}{2!}(-3x)^2+\cdots$$

$$=1+x+2x^2+\cdots\quad\text{which is valid for}$$

$$|3x|<1,\ \text{i.e. } |x|<\frac{1}{3}$$

Hence

$$\frac{\sqrt{(1+2x)}}{\sqrt[3]{(1-3x)}}=(1+2x)^{\frac{1}{2}}(1-3x)^{-\frac{1}{3}}$$

$$=\left(1+x-\frac{x^2}{2}+\cdots\right)(1+x+2x^2+\cdots)$$

$$=1+x+2x^2+x+x^2-\frac{x^2}{2},$$

neglecting terms of higher power than 2,

$$=\mathbf{1+2x+\frac{5}{2}x^2}$$

The series is convergent if $\mathbf{-\frac{1}{3}<x<\frac{1}{3}}$

Now try the following exercise

Exercise 30 Further problems on the binomial series

In problems 1 to 5 expand in ascending powers of x as far as the term in x^3, using the binomial theorem. State in each case the limits of x for which the series is valid.

1. $\dfrac{1}{(1-x)}$

$[1+x+x^2+x^3+\cdots,\ |x|<1]$

2. $\dfrac{1}{(1+x)^2}$

$[1-2x+3x^2-4x^3+\cdots,\ |x|<1]$

3. $\dfrac{1}{(2+x)^3}$

$\left[\dfrac{1}{8}\left(1-\dfrac{3x}{2}+\dfrac{3x^2}{2}-\dfrac{5x^3}{4}+\cdots\right),\ |x|<2\right]$

4. $\sqrt{2+x}$

$\left[\sqrt{2}\left(1+\dfrac{x}{4}-\dfrac{x^2}{32}+\dfrac{x^3}{128}-\cdots\right),\ |x|<2\right]$

5. $\dfrac{1}{\sqrt{1+3x}}$

$\left[\left(1-\dfrac{3}{2}x+\dfrac{27}{8}x^2-\dfrac{135}{16}x^3+\cdots\right),\ |x|<\dfrac{1}{3}\right]$

6. Expand $(2+3x)^{-6}$ to three terms. For what values of x is the expansion valid?

$\left[\dfrac{1}{64}\left(1-9x+\dfrac{189}{4}x^2\right),\ |x|<\dfrac{2}{3}\right]$

7. When x is very small show that:

(a) $\dfrac{1}{(1-x)^2\sqrt{(1-x)}}\approx 1+\dfrac{5}{2}x$

(b) $\dfrac{(1-2x)}{(1-3x)^4}\approx 1+10x$

(c) $\dfrac{\sqrt{1+5x}}{\sqrt[3]{1-2x}}\approx 1+\dfrac{19}{6}x$

8. If x is very small such that x^2 and higher powers may be neglected, determine the power series for $\dfrac{\sqrt{x+4}\sqrt[3]{8-x}}{\sqrt[5]{(1+x)^3}}$

$\left[4-\dfrac{31}{15}x\right]$

9. Express the following as power series in ascending powers of x as far as the term in x^2. State in each case the range of x for which the series is valid.

(a) $\sqrt{\left(\dfrac{1-x}{1+x}\right)}$ (b) $\dfrac{(1+x)\sqrt[3]{(1-3x)^2}}{\sqrt{(1+x^2)}}$

$\left[\begin{array}{l}\text{(a) } 1-x+\dfrac{1}{2}x^2,\ |x|<1\\ \text{(b) } 1-x-\dfrac{7}{2}x^2,\ |x|<\dfrac{1}{3}\end{array}\right]$

7.5 Practical problems involving the binomial theorem

Binomial expansions may be used for numerical approximations, for calculations with small variations and in probability theory (see Chapter 57).

Problem 17. The radius of a cylinder is reduced by 4% and its height is increased by 2%. Determine the approximate percentage change in (a) its volume and (b) its curved surface area, (neglecting the products of small quantities).

Volume of cylinder $=\pi r^2 h$.

Let r and h be the original values of radius and height.

The new values are $0.96r$ or $(1-0.04)r$ and $1.02h$ or $(1+0.02)h$.

(a) New volume $=\pi[(1-0.04)r]^2[(1+0.02)h]$

$$=\pi r^2 h(1-0.04)^2(1+0.02)$$

Now $(1-0.04)^2=1-2(0.04)+(0.04)^2$

$$=(1-0.08),$$

neglecting powers of small terms.

Hence new volume

$$\approx \pi r^2 h(1-0.08)(1+0.02)$$

$$\approx \pi r^2 h(1-0.08+0.02), \text{ neglecting products of small terms}$$

$$\approx \pi r^2 h(1-0.06) \text{ or } 0.94\pi r^2 h, \text{ i.e. 94\% of the original volume}$$

Hence the volume is reduced by approximately 6%.

(b) Curved surface area of cylinder $= 2\pi rh$.

New surface area

$$= 2\pi[(1-0.04)r][(1+0.02)h]$$

$$= 2\pi rh(1-0.04)(1+0.02)$$

$$\approx 2\pi rh(1-0.04+0.02), \text{ neglecting products of small terms}$$

$$\approx 2\pi rh(1-0.02) \text{ or } 0.98(2\pi rh), \text{ i.e. 98\% of the original surface area}$$

Hence the curved surface area is reduced by approximately 2%.

Problem 18. The second moment of area of a rectangle through its centroid is given by $\frac{bl^3}{12}$. Determine the approximate change in the second moment of area if b is increased by 3.5% and l is reduced by 2.5%.

New values of b and l are $(1+0.035)b$ and $(1-0.025)l$ respectively.

New second moment of area

$$= \frac{1}{12}[(1+0.035)b][(1-0.025)l]^3$$

$$= \frac{bl^3}{12}(1+0.035)(1-0.025)^3$$

$$\approx \frac{bl^3}{12}(1+0.035)(1-0.075), \text{ neglecting powers of small terms}$$

$$\approx \frac{bl^3}{12}(1+0.035-0.075), \text{ neglecting products of small terms}$$

$$\approx \frac{bl^3}{12}(1-0.040) \text{ or } (0.96)\frac{bl^3}{12}, \text{ i.e. 96\% of the original second moment of area}$$

Hence the second moment of area is reduced by approximately 4%.

Problem 19. The resonant frequency of a vibrating shaft is given by: $f = \frac{1}{2\pi}\sqrt{\frac{k}{I}}$, where k is the stiffness and I is the inertia of the shaft. Use the binomial theorem to determine the approximate percentage error in determining the frequency using the measured values of k and I when the measured value of k is 4% too large and the measured value of I is 2% too small.

Let f, k and I be the true values of frequency, stiffness and inertia respectively. Since the measured value of stiffness, k_1, is 4% too large, then

$$k_1 = \frac{104}{100}k = (1+0.04)k$$

The measured value of inertia, I_1, is 2% too small, hence

$$I_1 = \frac{98}{100}I = (1-0.02)I$$

The measured value of frequency,

$$f_1 = \frac{1}{2\pi}\sqrt{\frac{k_1}{I_1}} = \frac{1}{2\pi}k_1^{\frac{1}{2}}I_1^{-\frac{1}{2}}$$

$$= \frac{1}{2\pi}[(1+0.04)k]^{\frac{1}{2}}[(1-0.02)I]^{-\frac{1}{2}}$$

$$= \frac{1}{2\pi}(1+0.04)^{\frac{1}{2}}k^{\frac{1}{2}}(1-0.02)^{-\frac{1}{2}}I^{-\frac{1}{2}}$$

$$= \frac{1}{2\pi}k^{\frac{1}{2}}I^{-\frac{1}{2}}(1+0.04)^{\frac{1}{2}}(1-0.02)^{-\frac{1}{2}}$$

i.e. $f_1 = f(1+0.04)^{\frac{1}{2}}(1-0.02)^{-\frac{1}{2}}$

$$\approx f\left[1+\left(\frac{1}{2}\right)(0.04)\right]\left[1+\left(-\frac{1}{2}\right)(-0.02)\right]$$

$$\approx f(1+0.02)(1+0.01)$$

Neglecting the products of small terms,

$$f_1 \approx (1+0.02+0.01)f \approx 1.03f$$

Thus the percentage error in f based on the measured values of k and I is approximately $[(1.03)(100)-100]$, i.e. **3% too large**.

Now try the following exercise

Exercise 31 Further practical problems involving the binomial theorem

1. Pressure p and volume v are related by $pv^3 = c$, where c is a constant. Determine the approximate percentage change in c when p is increased by 3% and v decreased by 1.2%. [0.6% decrease]

2. Kinetic energy is given by $\frac{1}{2}mv^2$. Determine the approximate change in the kinetic energy when mass m is increased by 2.5% and the velocity v is reduced by 3%. [3.5% decrease]

3. An error of +1.5% was made when measuring the radius of a sphere. Ignoring the products of small quantities determine the approximate error in calculating (a) the volume, and (b) the surface area.
$$\begin{bmatrix}\text{(a) 4.5\% increase}\\ \text{(b) 3.0\% increase}\end{bmatrix}$$

4. The power developed by an engine is given by $I = k$ PLAN, where k is a constant. Determine the approximate percentage change in the power when P and A are each increased by 2.5% and L and N are each decreased by 1.4%. [2.2% increase]

5. The radius of a cone is increased by 2.7% and its height reduced by 0.9%. Determine the approximate percentage change in its volume, neglecting the products of small terms. [4.5% increase]

6. The electric field strength H due to a magnet of length $2l$ and moment M at a point on its axis distance x from the centre is given by
$$H = \frac{M}{2l}\left\{\frac{1}{(x-l)^2} - \frac{1}{(x+l)^2}\right\}$$
Show that if l is very small compared with x, then $H \approx \dfrac{2M}{x^3}$.

7. The shear stress τ in a shaft of diameter D under a torque T is given by: $\tau = \dfrac{kT}{\pi D^3}$. Determine the approximate percentage error in calculating τ if T is measured 3% too small and D 1.5% too large. [7.5% decrease]

8. The energy W stored in a flywheel is given by: $W = kr^5N^2$, where k is a constant, r is the radius and N the number of revolutions. Determine the approximate percentage change in W when r is increased by 1.3% and N is decreased by 2%. [2.5% increase]

9. In a series electrical circuit containing inductance L and capacitance C the resonant frequency is given by: $f_r = \dfrac{1}{2\pi\sqrt{LC}}$. If the values of L and C used in the calculation are 2.6% too large and 0.8% too small respectively, determine the approximate percentage error in the frequency. [0.9% too small]

10. The viscosity η of a liquid is given by: $\eta = \dfrac{kr^4}{vl}$, where k is a constant. If there is an error in r of +2%, in v of +4% and l of −3%, what is the resultant error in η? [+7%]

11. A magnetic pole, distance x from the plane of a coil of radius r, and on the axis of the coil, is subject to a force F when a current flows in the coil. The force is given by: $F = \dfrac{kx}{\sqrt{(r^2+x^2)^5}}$, where k is a constant. Use the binomial theorem to show that when x is small compared to r, then $F \approx \dfrac{kx}{r^5} - \dfrac{5kx^3}{2r^7}$.

12. The flow of water through a pipe is given by: $G = \sqrt{\dfrac{(3d)^5H}{L}}$. If d decreases by 2% and H by 1%, use the binomial theorem to estimate the decrease in G. [5.5%]

Revision Test 2

This Revision Test covers the material contained in Chapters 5 to 7. *The marks for each question are shown in brackets at the end of each question.*

1. Evaluate correct to 4 significant figures:

   (a) $\sinh 2.47$ (b) $\tanh 0.6439$

   (c) $\operatorname{sech} 1.385$ (d) $\operatorname{cosech} 0.874$ (6)

2. The increase in resistance of strip conductors due to eddy currents at power frequencies is given by:

$$\lambda = \frac{\alpha t}{2}\left[\frac{\sinh \alpha t + \sin \alpha t}{\cosh \alpha t - \cos \alpha t}\right]$$

   Calculate λ, correct to 5 significant figures, when $\alpha = 1.08$ and $t = 1$. (5)

3. If $A \operatorname{ch} x - B \operatorname{sh} x \equiv 4e^{x} - 3e^{-x}$ determine the values of A and B. (6)

4. Solve the following equation:

$$3.52 \operatorname{ch} x + 8.42 \operatorname{sh} x = 5.32$$

   correct to 4 decimal places. (7)

5. Determine the 20th term of the series 15.6, 15, 14.4, 13.8, ... (3)

6. The sum of 13 terms of an arithmetic progression is 286 and the common difference is 3. Determine the first term of the series. (4)

7. An engineer earns £21000 per annum and receives annual increments of £600. Determine the salary in the 9th year and calculate the total earnings in the first 11 years. (5)

8. Determine the 11th term of the series 1.5, 3, 6, 12, ... (2)

9. Find the sum of the first eight terms of the series 1, 2.5, 6.25, ..., correct to 1 decimal place. (4)

10. Determine the sum to infinity of the series $5, 1, \frac{1}{5}, \ldots$ (3)

11. A machine is to have seven speeds ranging from 25 rev/min to 500 rev/min. If the speeds form a geometric progression, determine their value, each correct to the nearest whole number. (8)

12. Use the binomial series to expand $(2a - 3b)^6$. (7)

13. Determine the middle term of $\left(3x - \frac{1}{3y}\right)^{18}$. (6)

14. Expand the following in ascending powers of t as far as the term in t^3

    (a) $\frac{1}{1+t}$ (b) $\frac{1}{\sqrt{(1-3t)}}$

    For each case, state the limits for which the expansion is valid. (12)

15. When x is very small show that:

$$\frac{1}{(1+x)^2\sqrt{(1-x)}} \approx 1 - \frac{3}{2}x$$ (5)

16. The modulus of rigidity G is given by $G = \frac{R^4\theta}{L}$ where R is the radius, θ the angle of twist and L the length. Find the approximate percentage error in G when R is measured 1.5% too large, θ is measured 3% too small and L is measured 1% too small. (7)

Chapter 8

Maclaurin's series

8.1 Introduction

Some mathematical functions may be represented as power series, containing terms in ascending powers of the variable. For example,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\text{and } \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

(as introduced in Chapter 5)

Using a series, called **Maclaurin's series**, mixed functions containing, say, algebraic, trigonometric and exponential functions, may be expressed solely as algebraic functions, and differentiation and integration can often be more readily performed.

To expand a function using Maclaurin's theorem, some knowledge of differentiation is needed (More on differentiation is given in Chapter 27). Here is a revision of derivatives of the main functions needed in this chapter.

y or $f(x)$	$\frac{dy}{dx}$ or $f'(x)$
ax^n	anx^{n-1}
$\sin ax$	$a\cos ax$
$\cos ax$	$-a\sin ax$
e^{ax}	ae^{ax}
$\ln ax$	$\frac{1}{x}$
$\sinh ax$	$a\cosh ax$
$\cosh ax$	$a\sinh ax$

Given a general function $f(x)$, then $f'(x)$ is the first derivative, $f''(x)$ is the second derivative, and so on. Also, $f(0)$ means the value of the function when $x = 0$, $f'(0)$ means the value of the first derivative when $x = 0$, and so on.

8.2 Derivation of Maclaurin's theorem

Let the power series for $f(x)$ be

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots \quad (1)$$

where $a_0, a_1, a_2, \ldots$ are constants.

When $x = 0$, $\boldsymbol{f(0) = a_0}$.

Differentiating equation (1) with respect to x gives:

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \cdots \quad (2)$$

When $x = 0$, $\boldsymbol{f'(0) = a_1}$.

Differentiating equation (2) with respect to x gives:

$$f''(x) = 2a_2 + (3)(2)a_3x + (4)(3)a_4x^2 + (5)(4)a_5x^3 + \cdots \quad (3)$$

When $x = 0$, $f''(0) = 2a_2 = 2!a_2$, i.e. $\boldsymbol{a_2 = \dfrac{f''(0)}{2!}}$

Differentiating equation (3) with respect to x gives:

$$f'''(x) = (3)(2)a_3 + (4)(3)(2)a_4x + (5)(4)(3)a_5x^2 + \cdots \quad (4)$$

When $x = 0$, $f'''(0) = (3)(2)a_3 = 3!a_3$, i.e. $\boldsymbol{a_3 = \dfrac{f'''(0)}{3!}}$

Continuing the same procedure gives $a_4 = \dfrac{f^{iv}(0)}{4!}$, $a_5 = \dfrac{f^{v}(0)}{5!}$, and so on.

Substituting for $a_0, a_1, a_2, \ldots$ in equation (1) gives:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

i.e.
$$\boldsymbol{f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots} \qquad (5)$$

Equation (5) is a mathematical statement called **Maclaurin's theorem** or **Maclaurin's series**.

8.3 Conditions of Maclaurin's series

Maclaurin's series may be used to represent any function, say $f(x)$, as a power series provided that at $x = 0$ the following three conditions are met:

(a) $\boldsymbol{f(0) \neq \infty}$

For example, for the function $f(x) = \cos x$, $f(0) = \cos 0 = 1$, thus $\cos x$ meets the condition. However, if $f(x) = \ln x$, $f(0) = \ln 0 = -\infty$, thus $\ln x$ does not meet this condition.

(b) $\boldsymbol{f'(0), f''(0), f'''(0), \ldots \neq \infty}$

For example, for the function $f(x) = \cos x$, $f'(0) = -\sin 0 = 0$, $f''(0) = -\cos 0 = -1$, and so on; thus $\cos x$ meets this condition. However, if $f(x) = \ln x$, $f'(0) = \frac{1}{0} = \infty$, thus $\ln x$ does not meet this condition.

(c) **The resultant Maclaurin's series must be convergent**

In general, this means that the values of the terms, or groups of terms, must get progressively smaller and the sum of the terms must reach a limiting value.

For example, the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ is convergent since the value of the terms is getting smaller and the sum of the terms is approaching a limiting value of 2.

8.4 Worked problems on Maclaurin's series

Problem 1. Determine the first four terms of the power series for $\cos x$.

The values of $f(0)$, $f'(0)$, $f''(0)$, ... in the Maclaurin's series are obtained as follows:

$f(x) = \cos x$ $\quad f(0) = \cos 0 = 1$

$f'(x) = -\sin x$ $\quad f'(0) = -\sin 0 = 0$

$f''(x) = -\cos x$ $\quad f''(0) = -\cos 0 = -1$

$f'''(x) = \sin x$ $\quad f'''(0) = \sin 0 = 0$

$f^{iv}(x) = \cos x$ $\quad f^{iv}(0) = \cos 0 = 1$

$f^{v}(x) = -\sin x$ $\quad f^{v}(0) = -\sin 0 = 0$

$f^{vi}(x) = -\cos x$ $\quad f^{vi}(0) = -\cos 0 = -1$

Substituting these values into equation (5) gives:

$$f(x) = \cos x = 1 + x(0) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(1) + \frac{x^5}{5!}(0) + \frac{x^6}{6!}(-1) + \cdots$$

i.e. $$\boldsymbol{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots}$$

Problem 2. Determine the power series for $\cos 2\theta$.

Replacing x with 2θ in the series obtained in Problem 1 gives:

$$\cos 2\theta = 1 - \frac{(2\theta)^2}{2!} + \frac{(2\theta)^4}{4!} - \frac{(2\theta)^6}{6!} + \cdots$$

$$= 1 - \frac{4\theta^2}{2} + \frac{16\theta^4}{24} - \frac{64\theta^6}{720} + \cdots$$

i.e. $$\boldsymbol{\cos 2\theta = 1 - 2\theta^2 + \frac{2}{3}\theta^4 - \frac{4}{45}\theta^6 + \cdots}$$

Problem 3. Using Maclaurin's series, find the first 5 (non zero) terms for the function $f(x) = \sin x$.

$f(x) = \sin x \qquad f(0) = \sin 0 = 0$

$f'(x) = \cos x \qquad f'(0) = \cos 0 = 1$

$f''(x) = -\sin x \qquad f''(0) = -\sin 0 = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -\cos 0 = -1$

$f^{\mathrm{iv}}(x) = \sin x \qquad f^{\mathrm{iv}}(0) = \sin 0 = 0$

$f^{\mathrm{v}}(x) = \cos x \qquad f^{\mathrm{v}}(0) = \cos 0 = 1$

$f^{\mathrm{vi}}(x) = -\sin x \qquad f^{\mathrm{vi}}(0) = -\sin 0 = 0$

$f^{\mathrm{vii}}(x) = -\cos x \qquad f^{\mathrm{vii}}(0) = -\cos 0 = -1$

Substituting the above values into Maclaurin's series of equation (5) gives:

$$\sin x = 0 + x\,(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(1) + \frac{x^6}{6!}(0) + \frac{x^7}{7!}(-1) + \cdots\cdots$$

i.e. $\boldsymbol{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}$

Problem 4. Using Maclaurin's series, find the first five terms for the expansion of the function $f(x) = \mathrm{e}^{3x}$.

$f(x) = \mathrm{e}^{3x} \qquad f(0) = \mathrm{e}^0 = 1$

$f'(x) = 3\mathrm{e}^{3x} \qquad f'(0) = 3\mathrm{e}^0 = 3$

$f''(x) = 9\mathrm{e}^{3x} \qquad f''(0) = 9\mathrm{e}^0 = 9$

$f'''(x) = 27\mathrm{e}^{3x} \qquad f'''(0) = 27\mathrm{e}^0 = 27$

$f^{\mathrm{iv}}(x) = 81\mathrm{e}^{3x} \qquad f^{\mathrm{iv}}(0) = 81\mathrm{e}^0 = 81$

Substituting the above values into Maclaurin's series of equation (5) gives:

$$\mathrm{e}^{3x} = 1 + x\,(3) + \frac{x^2}{2!}(9) + \frac{x^3}{3!}(27) + \frac{x^4}{4!}(81) + \cdots\cdots$$

$$\mathrm{e}^{3x} = 1 + 3x + \frac{9x^2}{2!} + \frac{27x^3}{3!} + \frac{81x^4}{4!} + \cdots$$

i.e. $\boldsymbol{\mathrm{e}^{3x} = 1 + 3x + \frac{9x^2}{2} + \frac{9x^3}{2} + \frac{27x^4}{8} + \cdots}$

Problem 5. Determine the power series for $\tan x$ as far as the term in x^3.

$f(x) = \tan x$

$f(0) = \tan 0 = 0$

$f'(x) = \sec^2 x$

$f'(0) = \sec^2 0 = \dfrac{1}{\cos^2 0} = 1$

$f''(x) = (2\sec x)(\sec x \tan x)$

$= 2\sec^2 x \tan x$

$f''(0) = 2\sec^2 0 \tan 0 = 0$

$f'''(x) = (2\sec^2 x)(\sec^2 x) + (\tan x)(4\sec x \sec x \tan x)$, by the product rule,

$= 2\sec^4 x + 4\sec^2 x \tan^2 x$

$f'''(0) = 2\sec^4 0 + 4\sec^2 0 \tan^2 0 = 2$

Substituting these values into equation (5) gives:

$$f(x) = \tan x = 0 + (x)(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(2)$$

i.e. $\boldsymbol{\tan x = x + \frac{1}{3}x^3}$

Problem 6. Expand $\ln(1+x)$ to five terms.

$f(x) = \ln(1+x) \qquad f(0) = \ln(1+0) = 0$

$f'(x) = \dfrac{1}{(1+x)} \qquad f'(0) = \dfrac{1}{1+0} = 1$

$f''(x) = \dfrac{-1}{(1+x)^2} \qquad f''(0) = \dfrac{-1}{(1+0)^2} = -1$

$f'''(x) = \dfrac{2}{(1+x)^3} \qquad f'''(0) = \dfrac{2}{(1+0)^3} = 2$

$f^{\mathrm{iv}}(x) = \dfrac{-6}{(1+x)^4} \qquad f^{\mathrm{iv}}(0) = \dfrac{-6}{(1+0)^4} = -6$

$f^{\mathrm{v}}(x) = \dfrac{24}{(1+x)^5} \qquad f^{\mathrm{v}}(0) = \dfrac{24}{(1+0)^5} = 24$

Substituting these values into equation (5) gives:

$$f(x) = \ln(1+x) = 0 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \frac{x^5}{5!}(24)$$

i.e. $\mathbf{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots}$

Problem 7. Expand $\ln(1-x)$ to five terms.

Replacing x by $-x$ in the series for $\ln(1+x)$ in Problem 6 gives:

$$\ln(1-x) = (-x) - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \frac{(-x)^4}{4} + \frac{(-x)^5}{5} - \cdots$$

i.e. $\mathbf{\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots}$

Problem 8. Determine the power series for $\ln\left(\frac{1+x}{1-x}\right)$.

$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$ by the laws of logarithms, and from Problems 6 and 7,

$$\ln\left(\frac{1+x}{1-x}\right) = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots\right)$$

$$= 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \cdots$$

i.e. $\mathbf{\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)}$

Problem 9. Use Maclaurin's series to find the expansion of $(2+x)^4$.

$f(x) = (2+x)^4 \qquad f(0) = 2^4 = 16$

$f'(x) = 4(2+x)^3 \qquad f'(0) = 4(2)^3 = 32$

$f''(x) = 12(2+x)^2 \qquad f''(0) = 12(2)^2 = 48$

$f'''(x) = 24(2+x)^1 \qquad f'''(0) = 24(2) = 48$

$f^{\mathrm{iv}}(x) = 24 \qquad f^{\mathrm{iv}}(0) = 24$

Substituting in equation (5) gives:

$$\mathbf{(2+x)^4}$$

$$= f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{\mathrm{iv}}(0)$$

$$= 16 + (x)(32) + \frac{x^2}{2!}(48) + \frac{x^3}{3!}(48) + \frac{x^4}{4!}(24)$$

$$\mathbf{= 16 + 32x + 24x^2 + 8x^3 + x^4}$$

(This expression could have been obtained by applying the binomial theorem.)

Problem 10. Expand $e^{\frac{x}{2}}$ as far as the term in x^4.

$f(x) = e^{\frac{x}{2}} \qquad f(0) = e^0 = 1$

$f'(x) = \frac{1}{2}e^{\frac{x}{2}} \qquad f'(0) = \frac{1}{2}e^0 = \frac{1}{2}$

$f''(x) = \frac{1}{4}e^{\frac{x}{2}} \qquad f''(0) = \frac{1}{4}e^0 = \frac{1}{4}$

$f'''(x) = \frac{1}{8}e^{\frac{x}{2}} \qquad f'''(0) = \frac{1}{8}e^0 = \frac{1}{8}$

$f^{\mathrm{iv}}(x) = \frac{1}{16}e^{\frac{x}{2}} \qquad f^{\mathrm{iv}}(0) = \frac{1}{16}e^0 = \frac{1}{16}$

Substituting in equation (5) gives:

$$e^{\frac{x}{2}} = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{\mathrm{iv}}(0) + \cdots$$

$$= 1 + (x)\left(\frac{1}{2}\right) + \frac{x^2}{2!}\left(\frac{1}{4}\right) + \frac{x^3}{3!}\left(\frac{1}{8}\right) + \frac{x^4}{4!}\left(\frac{1}{16}\right) + \cdots$$

i.e. $\mathbf{e^{\frac{x}{2}} = 1 + \frac{1}{2}x + \frac{1}{8}x^2 + \frac{1}{48}x^3 + \frac{1}{384}x^4 + \cdots}$

Problem 11. Develop a series for $\sinh x$ using Maclaurin's series.

$f(x) = \sinh x \qquad f(0) = \sinh 0 = \dfrac{e^0 - e^{-0}}{2} = 0$

$f'(x) = \cosh x \qquad f'(0) = \cosh 0 = \dfrac{e^0 + e^{-0}}{2} = 1$

$f''(x) = \sinh x \qquad f''(0) = \sinh 0 = 0$

$f'''(x) = \cosh x \qquad f'''(0) = \cosh 0 = 1$

$f^{\mathrm{iv}}(x) = \sinh x \qquad f^{\mathrm{iv}}(0) = \sinh 0 = 0$

$f^{\mathrm{v}}(x) = \cosh x \qquad f^{\mathrm{v}}(0) = \cosh 0 = 1$

Substituting in equation (5) gives:

$$\sinh x = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{\mathrm{iv}}(0) + \frac{x^5}{5!}f^{\mathrm{v}}(0) + \cdots$$

$$= 0 + (x)(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(1) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(1) + \cdots$$

i.e. $\boldsymbol{\sinh x = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \cdots}$

(as obtained in Section 5.5, page 49)

Problem 12. Produce a power series for $\cos^2 2x$ as far as the term in x^6.

From double angle formulae, $\cos 2A = 2\cos^2 A - 1$ (see Chapter 17).

from which, $\cos^2 A = \dfrac{1}{2}(1 + \cos 2A)$

and $\cos^2 2x = \dfrac{1}{2}(1 + \cos 4x)$

From Problem 1,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

hence $\cos 4x = 1 - \dfrac{(4x)^2}{2!} + \dfrac{(4x)^4}{4!} - \dfrac{(4x)^6}{6!} + \cdots$

$$= 1 - 8x^2 + \frac{32}{3}x^4 - \frac{256}{45}x^6 + \cdots$$

Thus $\cos^2 2x = \dfrac{1}{2}(1 + \cos 4x)$

$$= \frac{1}{2}\left(1 + 1 - 8x^2 + \frac{32}{3}x^4 - \frac{256}{45}x^6 + \cdots\right)$$

i.e. $\boldsymbol{\cos^2 2x = 1 - 4x^2 + \dfrac{16}{3}x^4 - \dfrac{128}{45}x^6 + \cdots}$

Now try the following exercise

Exercise 32 Further problems on Maclaurin's series

1. Determine the first four terms of the power series for $\sin 2x$ using Maclaurin's series.
$\left[\sin 2x = 2x - \dfrac{4}{3}x^3 + \dfrac{4}{15}x^5 - \dfrac{8}{315}x^7 + \cdots\right]$

2. Use Maclaurin's series to produce a power series for $\cosh 3x$ as far as the term in x^6.
$\left[1 + \dfrac{9}{2}x^2 + \dfrac{27}{8}x^4 + \dfrac{81}{80}x^6\right]$

3. Use Maclaurin's theorem to determine the first three terms of the power series for $\ln(1 + e^x)$.
$\left[\ln 2 + \dfrac{x}{2} + \dfrac{x^2}{8}\right]$

4. Determine the power series for $\cos 4t$ as far as the term in t^6.
$\left[1 - 8t^2 + \dfrac{32}{3}t^4 - \dfrac{256}{45}t^6\right]$

5. Expand $e^{\frac{3}{2}x}$ in a power series as far as the term in x^3.
$\left[1 + \dfrac{3}{2}x + \dfrac{9}{8}x^2 + \dfrac{9}{16}x^3\right]$

6. Develop, as far as the term in x^4, the power series for $\sec 2x$.
$\left[1 + 2x^2 + \dfrac{10}{3}x^4\right]$

7. Expand $e^{2\theta}\cos 3\theta$ as far as the term in θ^2 using Maclaurin's series.
$\left[1 + 2\theta - \dfrac{5}{2}\theta^2\right]$

8. Determine the first three terms of the series for $\sin^2 x$ by applying Maclaurin's theorem.
$\left[x^2 - \dfrac{1}{3}x^4 + \dfrac{2}{45}x^6 \cdots\right]$

9. Use Maclaurin's series to determine the expansion of $(3 + 2t)^4$.
$[81 + 216t + 216t^2 + 96t^3 + 16t^4]$

8.5 Numerical integration using Maclaurin's series

The value of many integrals cannot be determined using the various analytical methods. In Chapter 45, the trapezoidal, mid-ordinate and Simpson's rules are used to numerically evaluate such integrals. Another method of finding the approximate value of a definite integral is to express the function as a power series using Maclaurin's series, and then integrating each algebraic term in turn. This is demonstrated in the following worked problems.

As a reminder, the general solution of integrals of the form $\int ax^n \mathrm{d}x$, where a and n are constants, is given by:

$$\int ax^n \mathrm{d}x = \frac{ax^{n+1}}{n+1} + c$$

Problem 13. Evaluate $\int_{0.1}^{0.4} 2\mathrm{e}^{\sin\theta}\,\mathrm{d}\theta$, correct to 3 significant figures.

A power series for $\mathrm{e}^{\sin\theta}$ is firstly obtained using Maclaurin's series.

$f(\theta) = \mathrm{e}^{\sin\theta}$ $\quad f(0) = \mathrm{e}^{\sin 0} = \mathrm{e}^0 = 1$

$f'(\theta) = \cos\theta\,\mathrm{e}^{\sin\theta}$ $\quad f'(0) = \cos 0\,\mathrm{e}^{\sin 0} = (1)\mathrm{e}^0 = 1$

$f''(\theta) = (\cos\theta)(\cos\theta\,\mathrm{e}^{\sin\theta}) + (\mathrm{e}^{\sin\theta})(-\sin\theta)$, by the product rule,

$= \mathrm{e}^{\sin\theta}(\cos^2\theta - \sin\theta)$;

$f''(0) = \mathrm{e}^0(\cos^2 0 - \sin 0) = 1$

$f'''(\theta) = (\mathrm{e}^{\sin\theta})[(2\cos\theta(-\sin\theta) - \cos\theta)] + (\cos^2\theta - \sin\theta)(\cos\theta\,\mathrm{e}^{\sin\theta})$

$= \mathrm{e}^{\sin\theta}\cos\theta[-2\sin\theta - 1 + \cos^2\theta - \sin\theta]$

$f'''(0) = \mathrm{e}^0\cos 0[(0 - 1 + 1 - 0)] = 0$

Hence from equation (5):

$$\mathrm{e}^{\sin\theta} = f(0) + \theta f'(0) + \frac{\theta^2}{2!}f''(0) + \frac{\theta^3}{3!}f'''(0) + \cdots$$

$$= 1 + \theta + \frac{\theta^2}{2} + 0$$

Thus $\displaystyle\int_{0.1}^{0.4} 2\mathrm{e}^{\sin\theta}\,\mathrm{d}\theta = \int_{0.1}^{0.4} 2\left(1 + \theta + \frac{\theta^2}{2}\right)\mathrm{d}\theta$

$$= \int_{0.1}^{0.4} (2 + 2\theta + \theta^2)\,\mathrm{d}\theta$$

$$= \left[2\theta + \frac{2\theta^2}{2} + \frac{\theta^3}{3}\right]_{0.1}^{0.4}$$

$$= \left(0.8 + (0.4)^2 + \frac{(0.4)^3}{3}\right) - \left(0.2 + (0.1)^2 + \frac{(0.1)^3}{3}\right)$$

$= 0.98133 - 0.21033$

$= \mathbf{0.771}$, correct to 3 significant figures.

Problem 14. Evaluate $\displaystyle\int_0^1 \frac{\sin\theta}{\theta}\,\mathrm{d}\theta$ using Maclaurin's series, correct to 3 significant figures.

Let $f(\theta) = \sin\theta$ $\quad f(0) = 0$

$f'(\theta) = \cos\theta$ $\quad f'(0) = 1$

$f''(\theta) = -\sin\theta$ $\quad f''(0) = 0$

$f'''(\theta) = -\cos\theta$ $\quad f'''(0) = -1$

$f^{\mathrm{iv}}(\theta) = \sin\theta$ $\quad f^{\mathrm{iv}}(0) = 0$

$f^{\mathrm{v}}(\theta) = \cos\theta$ $\quad f^{\mathrm{v}}(0) = 1$

Hence from equation (5):

$$\sin\theta = f(0) + \theta f'(0) + \frac{\theta^2}{2!}f''(0) + \frac{\theta^3}{3!}f'''(0) + \frac{\theta^4}{4!}f^{\mathrm{iv}}(0) + \frac{\theta^5}{5!}f^{\mathrm{v}}(0) + \cdots$$

$$= 0 + \theta(1) + \frac{\theta^2}{2!}(0) + \frac{\theta^3}{3!}(-1) + \frac{\theta^4}{4!}(0) + \frac{\theta^5}{5!}(1) + \cdots$$

i.e. $\displaystyle\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$

Hence

$$\int_0^1 \frac{\sin\theta}{\theta}\mathrm{d}\theta = \int_0^1 \frac{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right)}{\theta}\mathrm{d}\theta$$

$$= \int_0^1 \left(1 - \frac{\theta^2}{6} + \frac{\theta^4}{120} - \frac{\theta^6}{5040} + \cdots\right)\mathrm{d}\theta$$

$$= \left[\theta - \frac{\theta^3}{18} + \frac{\theta^5}{600} - \frac{\theta^7}{7(5040)} + \cdots\right]_0^1$$

$$= 1 - \frac{1}{18} + \frac{1}{600} - \frac{1}{7(5040)} + \cdots$$

$$= \mathbf{0.946}, \text{ correct to 3 significant figures.}$$

Problem 15. Evaluate $\int_0^{0.4} x\ln(1+x)\,dx$ using Maclaurin's theorem, correct to 3 decimal places.

From Problem 6,

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$$

Hence $\int_0^{0.4} x\ln(1+x)\,dx$

$$= \int_0^{0.4} x\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right)dx$$

$$= \int_0^{0.4} \left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \frac{x^6}{5} - \cdots\right)dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{8} + \frac{x^5}{15} - \frac{x^6}{24} + \frac{x^7}{35} - \cdots\right]_0^{0.4}$$

$$= \left(\frac{(0.4)^3}{3} - \frac{(0.4)^4}{8} + \frac{(0.4)^5}{15} - \frac{(0.4)^6}{24} + \frac{(0.4)^7}{35} - \cdots\right) - (0)$$

$$= 0.02133 - 0.0032 + 0.0006827 - \cdots$$

$$= \mathbf{0.019}, \text{ correct to 3 decimal places.}$$

Now try the following exercise

Exercise 33 Further problems on numerical integration using Maclaurin's series

1. Evaluate $\int_{0.2}^{0.6} 3e^{\sin\theta}\,d\theta$, correct to 3 decimal places, using Maclaurin's series. [1.784]
2. Use Maclaurin's theorem to expand $\cos 2\theta$ and hence evaluate, correct to 2 decimal places, $\int_0^1 \frac{\cos 2\theta}{\theta^{\frac{1}{3}}}\,d\theta$. [0.88]
3. Determine the value of $\int_0^1 \sqrt{\theta}\cos\theta\,d\theta$, correct to 2 significant figures, using Maclaurin's series. [0.53]
4. Use Maclaurin's theorem to expand $\sqrt{x}\ln(x+1)$ as a power series. Hence evaluate, correct to 3 decimal places, $\int_0^{0.5} \sqrt{x}\ln(x+1)\,dx$. [0.061]

8.6 Limiting values

It is sometimes necessary to find limits of the form $\lim_{x\to a}\left\{\frac{f(x)}{g(x)}\right\}$, where $f(a)=0$ and $g(a)=0$.
For example,

$$\lim_{x\to 1}\left\{\frac{x^2+3x-4}{x^2-7x+6}\right\} = \frac{1+3-4}{1-7+6} = \frac{0}{0}$$

and $\frac{0}{0}$ is generally referred to as indeterminate.
For certain limits a knowledge of series can sometimes help.

For example,

$$\lim_{x\to 0}\left\{\frac{\tan x - x}{x^3}\right\}$$

$$\equiv \lim_{x\to 0}\left\{\frac{x + \frac{1}{3}x^3 + \cdots - x}{x^3}\right\} \text{ from Problem 5}$$

$$= \lim_{x\to 0}\left\{\frac{\frac{1}{3}x^3 + \cdots}{x^3}\right\} = \lim_{x\to 0}\left\{\frac{1}{3}\right\} = \mathbf{\frac{1}{3}}$$

Similarly,

$$\lim_{x\to 0}\left\{\frac{\sinh x}{x}\right\}$$

$$\equiv \lim_{x\to 0}\left\{\frac{x + \frac{x^3}{3!} + \frac{x^5}{5!} + }{x}\right\} \text{ from Problem 11}$$

$$= \lim_{x\to 0}\left\{1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right\} = \mathbf{1}$$

However, a knowledge of series does not help with examples such as $\lim_{x\to 1}\left\{\dfrac{x^2+3x-4}{x^2-7x+6}\right\}$

L'Hopital's rule will enable us to determine such limits when the differential coefficients of the numerator and denominator can be found.

L'Hopital's rule states:

$$\lim_{x\to a}\left\{\frac{f(x)}{g(x)}\right\} = \lim_{x\to a}\left\{\frac{f'(x)}{g'(x)}\right\}$$

provided $g'(a)\neq 0$

It can happen that $\lim_{x\to a}\left\{\dfrac{f'(x)}{g'(x)}\right\}$ is still $\frac{0}{0}$; if so, the numerator and denominator are differentiated again (and again) until a non-zero value is obtained for the denominator.

The following worked problems demonstrate how L'Hopital's rule is used. Refer to Chapter 27 for methods of differentiation.

Problem 16. Determine $\lim_{x\to 1}\left\{\dfrac{x^2+3x-4}{x^2-7x+6}\right\}$

The first step is to substitute $x=1$ into both numerator and denominator. In this case we obtain $\frac{0}{0}$. It is only when we obtain such a result that we then use L'Hopital's rule. Hence applying L'Hopital's rule,

$$\lim_{x\to 1}\left\{\frac{x^2+3x-4}{x^2-7x+6}\right\} = \lim_{x\to 1}\left\{\frac{2x+3}{2x-7}\right\}$$

i.e. both numerator and denominator have been differentiated

$$= \frac{5}{-5} = -\mathbf{1}$$

Problem 17. Determine $\lim_{x\to 0}\left\{\dfrac{\sin x - x}{x^2}\right\}$

Substituting $x=0$ gives

$$\lim_{x\to 0}\left\{\frac{\sin x - x}{x^2}\right\} = \frac{\sin 0 - 0}{0} = \frac{0}{0}$$

Applying L'Hopital's rule gives

$$\lim_{x\to 0}\left\{\frac{\sin x - x}{x^2}\right\} = \lim_{x\to 0}\left\{\frac{\cos x - 1}{2x}\right\}$$

Substituting $x=0$ gives

$$\frac{\cos 0 - 1}{0} = \frac{1-1}{0} = \frac{0}{0} \quad \text{again}$$

Applying L'Hopital's rule again gives

$$\lim_{x\to 0}\left\{\frac{\cos x - 1}{2x}\right\} = \lim_{x\to 0}\left\{\frac{-\sin x}{2}\right\} = \mathbf{0}$$

Problem 18. Determine $\lim_{x\to 0}\left\{\dfrac{x-\sin x}{x-\tan x}\right\}$

Substituting $x=0$ gives

$$\lim_{x\to 0}\left\{\frac{x-\sin x}{x-\tan x}\right\} = \frac{0-\sin 0}{0-\tan 0} = \frac{0}{0}$$

Applying L'Hopital's rule gives

$$\lim_{x\to 0}\left\{\frac{x-\sin x}{x-\tan x}\right\} = \lim_{x\to 0}\left\{\frac{1-\cos x}{1-\sec^2 x}\right\}$$

Substituting $x=0$ gives

$$\lim_{x\to 0}\left\{\frac{1-\cos x}{1-\sec^2 x}\right\} = \frac{1-\cos 0}{1-\sec^2 0} = \frac{1-1}{1-1} = \frac{0}{0} \text{ again}$$

Applying L'Hopital's rule gives

$$\lim_{x\to 0}\left\{\frac{1-\cos x}{1-\sec^2 x}\right\} = \lim_{x\to 0}\left\{\frac{\sin x}{(-2\sec x)(\sec x\tan x)}\right\}$$

$$= \lim_{x\to 0}\left\{\frac{\sin x}{-2\sec^2 x\tan x}\right\}$$

Substituting $x=0$ gives

$$\frac{\sin 0}{-2\sec^2 0\tan 0} = \frac{0}{0} \quad \text{again}$$

Applying L'Hopital's rule gives

$$\lim_{x\to 0}\left\{\frac{\sin x}{-2\sec^2 x\tan x}\right\}$$

$$= \lim_{x\to 0}\left\{\frac{\cos x}{(-2\sec^2 x)(\sec^2 x) + (\tan x)(-4\sec^2 x\tan x)}\right\}$$

using the product rule

Substituting $x=0$ gives

$$\frac{\cos 0}{-2\sec^4 0 - 4\sec^2 0\tan^2 0} = \frac{1}{-2-0}$$

$$= -\frac{1}{2}$$

Hence $\lim_{x\to 0}\left\{\dfrac{x-\sin x}{x-\tan x}\right\} = -\dfrac{\mathbf{1}}{\mathbf{2}}$

Now try the following exercise

Exercise 34 Further problems on limiting values

Determine the following limiting values

1. $\lim_{x\to 1}\left\{\dfrac{x^3-2x+1}{2x^3+3x-5}\right\}$ $\left[\dfrac{1}{9}\right]$
2. $\lim_{x\to 0}\left\{\dfrac{\sin x}{x}\right\}$ [1]
3. $\lim_{x\to 0}\left\{\dfrac{\ln(1+x)}{x}\right\}$ [1]
4. $\lim_{x\to 0}\left\{\dfrac{x^2-\sin 3x}{3x+x^2}\right\}$ [−1]
5. $\lim_{\theta\to 0}\left\{\dfrac{\sin\theta-\theta\cos\theta}{\theta^3}\right\}$ $\left[\dfrac{1}{3}\right]$
6. $\lim_{t\to 1}\left\{\dfrac{\ln t}{t^2-1}\right\}$ $\left[\dfrac{1}{2}\right]$
7. $\lim_{x\to 0}\left\{\dfrac{\sinh x-\sin x}{x^3}\right\}$ $\left[\dfrac{1}{3}\right]$
8. $\lim_{\theta\to\frac{\pi}{2}}\left\{\dfrac{\sin\theta-1}{\ln\sin\theta}\right\}$ [1]
9. $\lim_{t\to 0}\left\{\dfrac{\sec t-1}{t\sin t}\right\}$ $\left[\dfrac{1}{2}\right]$

Chapter 9

Solving equations by iterative methods

9.1 Introduction to iterative methods

Many equations can only be solved graphically or by methods of successive approximations to the roots, called **iterative methods**. Three methods of successive approximations are (i) bisection method, introduced in Section 9.2, (ii) an algebraic method, introduction in Section 9.3, and (iii) by using the Newton-Raphson formula, given in Section 9.4.

Each successive approximation method relies on a reasonably good first estimate of the value of a root being made. One way of determining this is to sketch a graph of the function, say $y = f(x)$, and determine the approximate values of roots from the points where the graph cuts the x-axis. Another way is by using a functional notation method. This method uses the property that the value of the graph of $f(x) = 0$ changes sign for values of x just before and just after the value of a root.

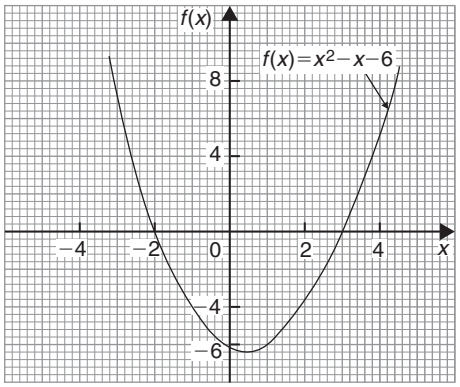


Figure 9.1

For example, one root of the equation $x^2 - x - 6 = 0$ is $x = 3$. Using functional notation:

$$f(x) = x^2 - x - 6$$
$$f(2) = 2^2 - 2 - 6 = -4$$
$$f(4) = 4^2 - 4 - 6 = +6$$

It can be seen from these results that the value of $f(x)$ changes from -4 at $f(2)$ to $+6$ at $f(4)$, indicating that a root lies between 2 and 4. This is shown more clearly in Fig. 9.1.

9.2 The bisection method

As shown above, by using functional notation it is possible to determine the vicinity of a root of an equation by the occurrence of a change of sign, i.e. if x_1 and x_2 are such that $f(x_1)$ and $f(x_2)$ have opposite signs, there is at least one root of the equation $f(x) = 0$ in the interval between x_1 and x_2 (provided $f(x)$ is a continuous function). In the **method of bisection** the mid-point of the interval, i.e. $x_3 = \dfrac{x_1 + x_2}{2}$, is taken, and from the sign of $f(x_3)$ it can be deduced whether a root lies in the half interval to the left or right of x_3. Whichever half interval is indicated, its mid-point is then taken and the procedure repeated. The method often requires many iterations and is therefore slow, but never fails to eventually produce the root. The procedure stops when two successive values of x are equal—to the required degree of accuracy.

The method of bisection is demonstrated in Problems 1 to 3 following.

Problem 1. Use the method of bisection to find the positive root of the equation $5x^2+11x-17=0$ correct to 3 significant figures.

Let $f(x)=5x^2+11x-17$

then, using functional notation:

$$f(0) = -17$$

$$\boldsymbol{f(1)} = 5(1)^2+11(1)-17 = \mathbf{-1}$$

$$\boldsymbol{f(2)} = 5(2)^2+11(2)-17 = \mathbf{+25}$$

Since there is a change of sign from negative to positive there must be a root of the equation between $x=1$ and $x=2$. This is shown graphically in Fig. 9.2.

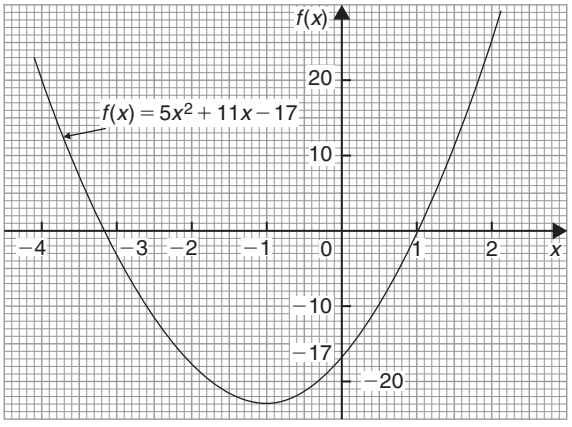


Figure 9.2

The method of bisection suggests that the root is at $\frac{1+2}{2}=1.5$, i.e. the interval between 1 and 2 has been bisected.

Hence

$$\boldsymbol{f(1.5)} = 5(1.5)^2+11(1.5)-17$$

$$= \mathbf{+10.75}$$

Since $f(1)$ is negative, $f(1.5)$ is positive, and $f(2)$ is also positive, a root of the equation must lie between $x=1$ and $x=1.5$, since a **sign change** has occurred between $f(1)$ and $f(1.5)$.

Bisecting this interval gives $\frac{1+1.5}{2}$ i.e. 1.25 as the next root.

Hence

$$\boldsymbol{f(1.25)} = 5(1.25)^2+11x-17$$

$$= \mathbf{+4.5625}$$

Since $f(1)$ is negative and $f(1.25)$ is positive, a root lies between $x=1$ and $x=1.25$.

Bisecting this interval gives $\frac{1+1.25}{2}$ i.e. 1.125

Hence

$$\boldsymbol{f(1.125)} = 5(1.125)^2+11(1.125)-17$$

$$= \mathbf{+1.703125}$$

Since $f(1)$ is negative and $f(1.125)$ is positive, a root lies between $x=1$ and $x=1.125$.

Bisecting this interval gives $\frac{1+1.125}{2}$ i.e. 1.0625.

Hence

$$\boldsymbol{f(1.0625)} = 5(1.0625)^2+11(1.0625)-17$$

$$= \mathbf{+0.33203125}$$

Since $f(1)$ is negative and $f(1.0625)$ is positive, a root lies between $x=1$ and $x=1.0625$.

Bisecting this interval gives $\frac{1+1.0625}{2}$ i.e. 1.03125.

Hence

$$\boldsymbol{f(1.03125)} = 5(1.03125)^2+11(1.03125)-17$$

$$= \mathbf{-0.338867\ldots}$$

Since $f(1.03125)$ is negative and $f(1.0625)$ is positive, a root lies between $x=1.03125$ and $x=1.0625$.

Bisecting this interval gives

$$\frac{1.03125+1.0625}{2} \text{ i.e. } 1.046875.$$

Hence

$$\boldsymbol{f(1.046875)} = 5(1.046875)^2+11(1.046875)-17$$

$$= \mathbf{-0.0046386\ldots}$$

Since $f(1.046875)$ is negative and $f(1.0625)$ is positive, a root lies between $x=1.046875$ and $x=1.0625$.

Bisecting this interval gives

$$\frac{1.046875+1.0625}{2} \text{ i.e. } \mathbf{1.0546875}.$$

The last three values obtained for the root are 1.03125, 1.046875 and 1.0546875. The last two values are both

1.05, correct to 3 significant figure. We therefore stop the iterations here.

Thus, correct to 3 significant figures, the positive root of $5x^2 + 11x - 17 = 0$ is 1.05

Problem 2. Use the bisection method to determine the positive root of the equation $x + 3 = e^x$, correct to 3 decimal places.

Let $f(x) = x + 3 - e^x$

then, using functional notation:

$$\boldsymbol{f(0) = 0 + 3 - e^0 = +2}$$
$$\boldsymbol{f(1) = 1 + 3 - e^1 = +1.2817\ldots}$$
$$\boldsymbol{f(2) = 2 + 3 - e^2 = -2.3890\ldots}$$

Since $f(1)$ is positive and $f(2)$ is negative, a root lies between $x = 1$ and $x = 2$. A sketch of $f(x) = x + 3 - e^x$, i.e. $x + 3 = e^x$ is shown in Fig. 9.3.

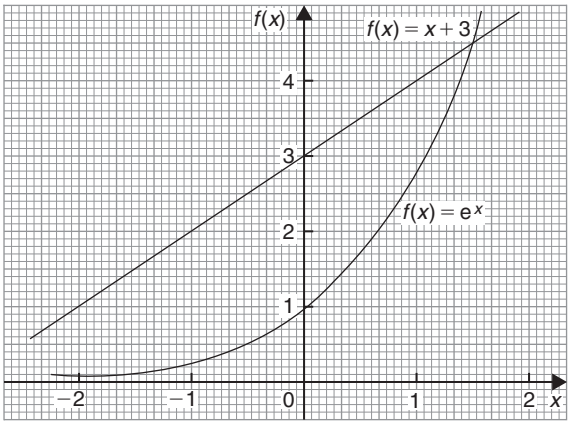


Figure 9.3

Bisecting the interval between $x = 1$ and $x = 2$ gives $\frac{1+2}{2}$ i.e. 1.5.

Hence

$$\boldsymbol{f(1.5) = 1.5 + 3 - e^{1.5}}$$
$$\boldsymbol{= +0.01831\ldots}$$

Since $f(1.5)$ is positive and $f(2)$ is negative, a root lies between $x = 1.5$ and $x = 2$.

Bisecting this interval gives $\frac{1.5+2}{2}$ i.e. 1.75.

Hence

$$\boldsymbol{f(1.75) = 1.75 + 3 - e^{1.75}}$$
$$\boldsymbol{= -1.00460\ldots}$$

Since $f(1.75)$ is negative and $f(1.5)$ is positive, a root lies between $x = 1.75$ and $x = 1.5$.

Bisecting this interval gives $\frac{1.75+1.5}{2}$ i.e. 1.625.

Hence

$$\boldsymbol{f(1.625) = 1.625 + 3 - e^{1.625}}$$
$$\boldsymbol{= -0.45341\ldots}$$

Since $f(1.625)$ is negative and $f(1.5)$ is positive, a root lies between $x = 1.625$ and $x = 1.5$.

Bisecting this interval gives $\frac{1.625+1.5}{2}$ i.e. 1.5625.

Hence

$$\boldsymbol{f(1.5625) = 1.5625 + 3 - e^{1.5625}}$$
$$\boldsymbol{= -0.20823\ldots}$$

Since $f(1.5625)$ is negative and $f(1.5)$ is positive, a root lies between $x = 1.5625$ and $x = 1.5$.

Bisecting this interval gives

$$\frac{1.5625 + 1.5}{2} \text{ i.e. } 1.53125.$$

Hence

$$\boldsymbol{f(1.53125) = 1.53125 + 3 - e^{1.53125}}$$
$$\boldsymbol{= -0.09270\ldots}$$

Since $f(1.53125)$ is negative and $f(1.5)$ is positive, a root lies between $x = 1.53125$ and $x = 1.5$.

Bisecting this interval gives

$$\frac{1.53125+1.5}{2} \text{ i.e. } 1.515625.$$

Hence

$$\boldsymbol{f(1.515625) = 1.515625 + 3 - e^{1.515625}}$$
$$\boldsymbol{= -0.03664\ldots}$$

Since $f(1.515625)$ is negative and $f(1.5)$ is positive, a root lies between $x = 1.515625$ and $x = 1.5$.

Bisecting this interval gives

$$\frac{1.515625 + 1.5}{2} \text{ i.e. } 1.5078125.$$

Hence

$$\boldsymbol{f(1.5078125) = 1.5078125 + 3 - e^{1.5078125}}$$
$$\boldsymbol{= -0.009026\ldots}$$

Since $f(1.5078125)$ is negative and $f(1.5)$ is positive, a root lies between $x = 1.5078125$ and $x = 1.5$.

Bisecting this interval gives

$$\frac{1.5078125 + 1.5}{2} \text{ i.e. } 1.50390625.$$

Hence

$$\boldsymbol{f(1.50390625)} = 1.50390625 + 3 - e^{1.50390625}$$
$$= \mathbf{+0.004676\ldots}$$

Since $f(1.50390625)$ is positive and $f(1.5078125)$ is negative, a root lies between $x = 1.50390625$ and $x = 1.5078125$.
Bisecting this interval gives

$$\frac{1.50390625 + 1.5078125}{2} \text{ i.e. } 1.505859375.$$

Hence

$$\boldsymbol{f(1.505859375)} = 1.505859375 + 3 - e^{1.505859375}$$
$$= \mathbf{-0.0021666\ldots}$$

Since $f(1.50589375)$ is negative and $f(1.50390625)$ is positive, a root lies between $x = 1.50589375$ and $x = 1.50390625$.

Bisecting this interval gives

$$\frac{1.505859375 + 1.50390625}{2} \text{ i.e. } 1.504882813.$$

Hence

$$\boldsymbol{f(1.504882813)} = 1.504882813 + 3 - e^{1.504882813}$$
$$= \mathbf{+0.001256\ldots}$$

Since $f(1.504882813)$ is positive and $f(1.505859375)$ is negative,

a root lies between $x = 1.504882813$ and $x = 1.505859375$.

Bisecting this interval gives

$$\frac{1.504882813 + 1.50589375}{2} \text{ i.e. } \mathbf{1.505388282}.$$

The last two values of x are 1.504882813 and 1.505388282, i.e. both are equal to 1.505, correct to 3 decimal places.

Hence the root of $x + 3 = e^x$ is $x = 1.505$, correct to 3 decimal places.

The above is a lengthy procedure and it is probably easier to present the data in a table as shown in the table.

x_1	x_2	$x_3 = \frac{x_1 + x_2}{2}$	$f(x_3)$
		0	+2
		1	+1.2817…
		2	−2.3890…
1	2	1.5	+0.0183…
1.5	2	1.75	−1.0046…
1.5	1.75	1.625	−0.4534…
1.5	1.625	1.5625	−0.2082…
1.5	1.5625	1.53125	−0.0927…
1.5	1.53125	1.515625	−0.0366…
1.5	1.515625	1.5078125	−0.0090…
1.5	1.5078125	1.50390625	+0.0046…
1.50390625	1.5078125	1.505859375	−0.0021…
1.50390625	1.505859375	**1.504882813**	+0.0012…
1.504882813	1.505859375	**1.505388282**	

Problem 3. Solve, correct to 2 decimal places, the equation $2\ln x + x = 2$ using the method of bisection.

Let $f(x) = 2\ln x + x - 2$

$f(0.1) = 2\ln(0.1) + 0.1 - 2 = -6.5051\ldots$

(Note that $\ln 0$ is infinite that is why $x = 0$ was not chosen)

$f(1) = 2\ln 1 + 1 - 2 = -1$

$f(2) = 2\ln 2 + 2 - 2 = +1.3862\ldots$

A change of sign indicates a root lies between $x = 1$ and $x = 2$.

Since $2\ln x + x = 2$ then $2\ln x = -x + 2$; sketches of $2\ln x$ and $-x + 2$ are shown in Fig. 9.4.

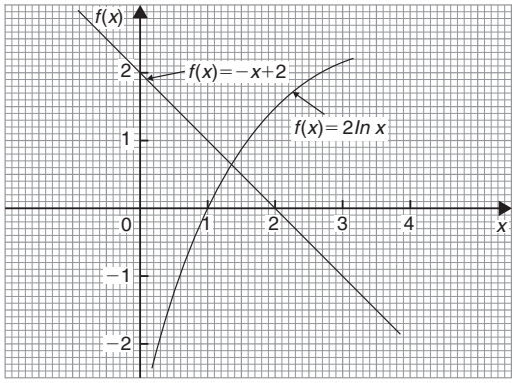


Figure 9.4

As shown in Problem 2, a table of values is produced to reduce space.

x_1	x_2	$x_3 = \frac{x_1 + x_2}{2}$	$f(x_3)$
		0.1	−6.6051...
		1	−1
		2	+1.3862...
1	2	1.5	+0.3109...
1	1.5	1.25	−0.3037...
1.25	1.5	1.375	+0.0119...
1.25	1.375	1.3125	−0.1436...
1.3125	1.375	1.34375	−0.0653...
1.34375	1.375	1.359375	−0.0265...
1.359375	1.375	**1.3671875**	−0.0073...
1.3671875	1.375	**1.37109375**	+0.0023...

The last two values of x_3 are both equal to 1.37 when expressed to 2 decimal places. We therefore stop the iterations.

Hence, the solution of $2\ln x + x = 2$ is $x = 1.37$, correct to 2 decimal places.

Now try the following exercise

Exercise 35 Further problems on the bisection method

Use the method of bisection to solve the following equations to the accuracy stated.

1. Find the positive root of the equation $x^2 + 3x - 5 = 0$, correct to 3 significant figures, using the method of bisection. [1.19]
2. Using the bisection method solve $e^x - x = 2$, correct to 4 significant figures. [1.146]
3. Determine the positive root of $x^2 = 4\cos x$, correct to 2 decimal places using the method of bisection. [1.20]
4. Solve $x - 2 - \ln x = 0$ for the root near to 3, correct to 3 decimal places using the bisection method. [3.146]
5. Solve, correct to 4 significant figures, $x - 2\sin^2 x = 0$ using the bisection method. [1.849]

9.3 An algebraic method of successive approximations

This method can be used to solve equations of the form:

$$a + bx + cx^2 + dx^3 + \cdots = 0,$$

where $a, b, c, d, \ldots$ are constants.

Procedure:

First approximation

(a) Using a graphical or the functional notation method (see Section 9.1) determine an approximate value of the root required, say x_1.

Second approximation

(b) Let the true value of the root be $(x_1 + \delta_1)$.

(c) Determine x_2 the approximate value of $(x_1 + \delta_1)$ by determining the value of $f(x_1 + \delta_1) = 0$, but neglecting terms containing products of δ_1.

Third approximation

(d) Let the true value of the root be $(x_2 + \delta_2)$.

(e) Determine x_3, the approximate value of $(x_2 + \delta_2)$ by determining the value of $f(x_2 + \delta_2) = 0$, but neglecting terms containing products of δ_2.

(f) The fourth and higher approximations are obtained in a similar way.

Using the techniques given in paragraphs (b) to (f), it is possible to continue getting values nearer and

nearer to the required root. The procedure is repeated until the value of the required root does not change on two consecutive approximations, when expressed to the required degree of accuracy.

Problem 4. Use an algebraic method of successive approximations to determine the value of the negative root of the quadratic equation: $4x^2 - 6x - 7 = 0$ correct to 3 significant figures. Check the value of the root by using the quadratic formula.

A first estimate of the values of the roots is made by using the functional notation method

$$f(x) = 4x^2 - 6x - 7$$
$$f(0) = 4(0)^2 - 6(0) - 7 = -7$$
$$f(-1) = 4(-1)^2 - 6(-1) - 7 = 3$$

These results show that the negative root lies between 0 and -1, since the value of $f(x)$ changes sign between $f(0)$ and $f(-1)$ (see Section 9.1). The procedure given above for the root lying between 0 and -1 is followed.

First approximation

(a) Let a first approximation be such that it divides the interval 0 to -1 in the ratio of -7 to 3, i.e. let $x_1 = -0.7$

Second approximation

(b) Let the true value of the root, x_2, be $(x_1 + \delta_1)$.

(c) Let $f(x_1 + \delta_1) = 0$, then, since $x_1 = -0.7$,

$$4(-0.7 + \delta_1)^2 - 6(-0.7 + \delta_1) - 7 = 0$$

Hence, $4[(-0.7)^2 + (2)(-0.7)(\delta_1) + \delta_1^2]$
$$- (6)(-0.7) - 6\delta_1 - 7 = 0$$

Neglecting terms containing products of δ_1 gives:

$$1.96 - 5.6\delta_1 + 4.2 - 6\delta_1 - 7 \approx 0$$

i.e. $$-5.6\delta_1 - 6\delta_1 = -1.96 - 4.2 + 7$$

i.e. $$\delta_1 \approx \frac{-1.96 - 4.2 + 7}{-5.6 - 6}$$
$$\approx \frac{0.84}{-11.6}$$
$$\approx -0.0724$$

Thus, x_2, a second approximation to the root is $[-0.7 + (-0.0724)]$,

i.e. $x_2 = -0.7724$, correct to 4 significant figures. (Since the question asked for 3 significant figure accuracy, it is usual to work to one figure greater than this).

The procedure given in (b) and (c) is now repeated for $x_2 = -0.7724$.

Third approximation

(d) Let the true value of the root, x_3, be $(x_2 + \delta_2)$.

(e) Let $f(x_2 + \delta_2) = 0$, then, since $x_2 = -0.7724$,

$$4(-0.7724 + \delta_2)^2 - 6(-0.7724 + \delta_2) - 7 = 0$$

$$4[(-0.7724)^2 + (2)(-0.7724)(\delta_2) + \delta_2^2]$$
$$- (6)(-0.7724) - 6\delta_2 - 7 = 0$$

Neglecting terms containing products of δ_2 gives:

$$2.3864 - 6.1792\delta_2 + 4.6344 - 6\delta_2 - 7 \approx 0$$

i.e. $$\delta_2 \approx \frac{-2.3864 - 4.6344 + 7}{-6.1792 - 6}$$

$$\approx \frac{-0.0208}{-12.1792}$$

$$\approx +0.001708$$

Thus x_3, the third approximation to the root is $(-0.7724 + 0.001708)$,

i.e. $x_3 = -0.7707$, correct to 4 significant figures (or -0.771 correct to 3 significant figures).

Fourth approximation

(f) The procedure given for the second and third approximations is now repeated for

$$x_3 = -0.7707$$

Let the true value of the root, x_4, be $(x_3 + \delta_3)$.

Let $f(x_3 + \delta_3) = 0$, then since $x_3 = -0.7707$,

$$4(-0.7707 + \delta_3)^2 - 6(-0.7707$$
$$+ \delta_3) - 7 = 0$$

$$4[(-0.7707)^2 + (2)(-0.7707)\delta_3 + \delta_3^2]$$
$$- 6(-0.7707) - 6\delta_3 - 7 = 0$$

Neglecting terms containing products of δ_3 gives:

$$2.3759 - 6.1656\delta_3 + 4.6242 - 6\delta_3 - 7 \approx 0$$

$$\text{i.e. } \delta_3 \approx \frac{-2.3759 - 4.6242 + 7}{-6.1656 - 6}$$

$$\approx \frac{-0.0001}{-12.156}$$

$$\approx +0.00000822$$

Thus, x_4, the fourth approximation to the root is $(-0.7707 + 0.00000822)$, i.e. $x_4 = -0.7707$, correct to 4 significant figures, and -0.771, correct to 3 significant figures.

Since the values of the roots are the same on two consecutive approximations, when stated to the required degree of accuracy, then the negative root of $4x^2 - 6x - 7 = 0$ is **-0.771, correct to 3 significant figures**.

[Checking, using the quadratic formula:

$$x = \frac{-(-6) \pm \sqrt{[(-6)^2 - (4)(4)(-7)]}}{(2)(4)}$$

$$= \frac{6 \pm 12.166}{8} = \mathbf{-0.771 \text{ and } 2.27},$$

correct to 3 significant figures]

[**Note on accuracy and errors.** Depending on the accuracy of evaluating the $f(x+\delta)$ terms, one or two iterations (i.e. successive approximations) might be saved. However, it is not usual to work to more than about 4 significant figures accuracy in this type of calculation. If a small error is made in calculations, the only likely effect is to increase the number of iterations.]

Problem 5. Determine the value of the smallest positive root of the equation $3x^3 - 10x^2 + 4x + 7 = 0$, correct to 3 significant figures, using an algebraic method of successive approximations.

The functional notation method is used to find the value of the first approximation.

$$f(x) = 3x^3 - 10x^2 + 4x + 7$$

$$f(0) = 3(0)^3 - 10(0)^2 + 4(0) + 7 = 7$$

$$f(1) = 3(1)^3 - 10(1)^2 + 4(1) + 7 = 4$$

$$f(2) = 3(2)^3 - 10(2)^2 + 4(2) + 7 = -1$$

Following the above procedure:

First approximation

(a) Let the first approximation be such that it divides the interval 1 to 2 in the ratio of 4 to -1, i.e. let x_1 be 1.8.

Second approximation

(b) Let the true value of the root, x_2, be $(x_1 + \delta_1)$.

(c) Let $f(x_1 + \delta_1) = 0$, then since $x_1 = 1.8$,

$$3(1.8 + \delta_1)^3 - 10(1.8 + \delta_1)^2 + 4(1.8 + \delta_1) + 7 = 0$$

Neglecting terms containing products of δ_1 and using the binomial series gives:

$$3[1.8^3 + 3(1.8)^2\delta_1] - 10[1.8^2 + (2)(1.8)\delta_1] + 4(1.8 + \delta_1) + 7 \approx 0$$

$$3(5.832 + 9.720\delta_1) - 32.4 - 36\delta_1 + 7.2 + 4\delta_1 + 7 \approx 0$$

$$17.496 + 29.16\delta_1 - 32.4 - 36\delta_1 + 7.2 + 4\delta_1 + 7 \approx 0$$

$$\delta_1 \approx \frac{-17.496 + 32.4 - 7.2 - 7}{29.16 - 36 + 4}$$

$$\approx -\frac{0.704}{2.84} \approx -0.2479$$

Thus $x_2 \approx 1.8 - 0.2479 = 1.5521$

Third approximation

(d) Let the true value of the root, x_3, be $(x_2 + \delta_2)$.

(e) Let $f(x_2 + \delta_2) = 0$, then since $x_2 = 1.5521$,

$$3(1.5521 + \delta_2)^3 - 10(1.5521 + \delta_2)^2 + 4(1.5521 + \delta_2) + 7 = 0$$

Neglecting terms containing products of δ_2 gives:

$$11.217 + 21.681\delta_2 - 24.090 - 31.042\delta_2 + 6.2084 + 4\delta_2 + 7 \approx 0$$

$$\delta_2 \approx \frac{-11.217 + 24.090 - 6.2084 - 7}{21.681 - 31.042 + 4}$$

$$\approx \frac{-0.3354}{-5.361}$$

$$\approx 0.06256$$

Thus $x_3 \approx 1.5521 + 0.06256 \approx 1.6147$

(f) Values of x_4 and x_5 are found in a similar way.

$$f(x_3 + \delta_3) = 3(1.6147 + \delta_3)^3 - 10(1.6147 + \delta_3)^2 + 4(1.6147 + \delta_3) + 7 = 0$$

giving $\delta_3 \approx 0.003175$ and $x_4 \approx 1.618$, i.e. 1.62 correct to 3 significant figures.

$$f(x_4 + \delta_4) = 3(1.618 + \delta_4)^3 - 10(1.618 + \delta_4)^2 + 4(1.618 + \delta_4) + 7 = 0$$

giving $\delta_4 \approx 0.0000417$, and $x_5 \approx 1.62$, correct to 3 significant figures.
Since x_4 and x_5 are the same when expressed to the required degree of accuracy, then the required root is **1.62**, correct to 3 significant figures.

Now try the following exercise

Exercise 36 Further problems on solving equations by an algebraic method of successive approximations

Use an algebraic method of successive approximation to solve the following equations to the accuracy stated.

1. $3x^2 + 5x - 17 = 0$, correct to 3 significant figures. [−3.36, 1.69]
2. $x^3 - 2x + 14 = 0$, correct to 3 decimal places. [−2.686]
3. $x^4 - 3x^3 + 7x - 5.5 = 0$, correct to 3 significant figures. [−1.53, 1.68]
4. $x^4 + 12x^3 - 13 = 0$, correct to 4 significant figures. [−12.01, 1.000]

9.4 The Newton-Raphson method

The Newton-Raphson formula, often just referred to as **Newton's method**, may be stated as follows:

If r_1 is the approximate value of a real root of the equation $f(x) = 0$, then a closer approximation to the root r_2 is given by:

$$r_2 = r_1 - \frac{f(r_1)}{f'(r_1)}$$

The advantages of Newton's method over the algebraic method of successive approximations is that it can be used for any type of mathematical equation (i.e. ones containing trigonometric, exponential, logarithmic, hyperbolic and algebraic functions), and it is usually easier to apply than the algebraic method.

Problem 6. Use Newton's method to determine the positive root of the quadratic equation $5x^2 + 11x - 17 = 0$, correct to 3 significant figures. Check the value of the root by using the quadratic formula.

The functional notation method is used to determine the first approximation to the root.

$$f(x) = 5x^2 + 11x - 17$$

$$f(0) = 5(0)^2 + 11(0) - 17 = -17$$

$$f(1) = 5(1)^2 + 11(1) - 17 = -1$$

$$f(2) = 5(2)^2 + 11(2) - 17 = 25$$

This shows that the value of the root is close to $x = 1$.

Let the first approximation to the root, r_1, be 1.

Newton's formula states that a closer approximation,

$$r_2 = r_1 - \frac{f(r_1)}{f'(r_1)}$$

$$f(x) = 5x^2 + 11x - 17,$$

thus, $f(r_1) = 5(r_1)^2 + 11(r_1) - 17$

$$= 5(1)^2 + 11(1) - 17 = -1$$

$f'(x)$ is the differential coefficient of $f(x)$,

i.e. $f'(x) = 10x + 11$.

Thus $f'(r_1) = 10(r_1) + 11$

$$= 10(1) + 11 = 21$$

By Newton's formula, a better approximation to the root is:

$$r_2 = 1 - \frac{-1}{21} = 1 - (-0.048) = 1.05,$$

correct to 3 significant figures.

A still better approximation to the root, r_3, is given by:

$$r_3 = r_2 - \frac{f(r_2)}{f'(r_2)}$$

$$= 1.05 - \frac{[5(1.05)^2 + 11(1.05) - 17]}{[10(1.05) + 11]}$$

$$= 1.05 - \frac{0.0625}{21.5}$$

$$= 1.05 - 0.003 = 1.047,$$

i.e. 1.05, correct to 3 significant figures.

Since the values of r_2 and r_3 are the same when expressed to the required degree of accuracy, the required root is **1.05**, correct to 3 significant figures. Checking, using the quadratic equation formula,

$$x = \frac{-11 \pm \sqrt{[121 - 4(5)(-17)]}}{(2)(5)}$$

$$= \frac{-11 \pm 21.47}{10}$$

The positive root is 1.047, i.e. **1.05**, correct to 3 significant figures (This root was determined in Problem 1 using the bisection method; Newton's method is clearly quicker).

Problem 7. Taking the first approximation as 2, determine the root of the equation $x^2 - 3\sin x + 2\ln(x+1) = 3.5$, correct to 3 significant figures, by using Newton's method.

Newton's formula states that $r_2 = r_1 - \dfrac{f(r_1)}{f'(r_1)}$, where r_1 is a first approximation to the root and r_2 is a better approximation to the root.

Since $f(x) = x^2 - 3\sin x + 2\ln(x+1) - 3.5$

$$f(r_1) = f(2) = 2^2 - 3\sin 2 + 2\ln 3 - 3.5,$$

where sin2 means the sine of 2 radians

$$= 4 - 2.7279 + 2.1972 - 3.5$$

$$= -0.0307$$

$$f'(x) = 2x - 3\cos x + \frac{2}{x+1}$$

$$f'(r_1) = f'(2) = 2(2) - 3\cos 2 + \frac{2}{3}$$

$$= 4 + 1.2484 + 0.6667$$

$$= 5.9151$$

Hence, $r_2 = r_1 - \dfrac{f(r_1)}{f'(r_1)}$

$$= 2 - \frac{-0.0307}{5.9151}$$

$$= 2.005 \text{ or } 2.01, \text{ correct to 3 significant figures.}$$

A still better approximation to the root, r_3, is given by:

$$r_3 = r_2 - \frac{f(r_2)}{f'(r_2)}$$

$$= 2.005 - \frac{[(2.005)^2 - 3\sin 2.005 + 2\ln 3.005 - 3.5]}{\left[2(2.005) - 3\cos 2.005 + \dfrac{2}{2.005+1}\right]}$$

$$= 2.005 - \frac{(-0.00104)}{5.9376} = 2.005 + 0.000175$$

i.e. $r_3 = 2.01$, correct to 3 significant figures.

Since the values of r_2 and r_3 are the same when expressed to the required degree of accuracy, then the required root is **2.01**, correct to 3 significant figures.

Problem 8. Use Newton's method to find the positive root of:

$$(x+4)^3 - e^{1.92x} + 5\cos\frac{x}{3} = 9,$$

correct to 3 significant figures.

The functional notational method is used to determine the approximate value of the root.

$$f(x) = (x+4)^3 - e^{1.92x} + 5\cos\frac{x}{3} - 9$$

$$f(0) = (0+4)^3 - e^0 + 5\cos 0 - 9 = 59$$

$$f(1) = 5^3 - e^{1.92} + 5\cos\frac{1}{3} - 9 \approx 114$$
$$f(2) = 6^3 - e^{3.84} + 5\cos\frac{2}{3} - 9 \approx 164$$
$$f(3) = 7^3 - e^{5.76} + 5\cos 1 - 9 \approx 19$$
$$f(4) = 8^3 - e^{7.68} + 5\cos\frac{4}{3} - 9 \approx -1660$$

From these results, let a first approximation to the root be $r_1 = 3$.

Newton's formula states that a better approximation to the root,

$$r_2 = r_1 - \frac{f(r_1)}{f'(r_1)}$$
$$f(r_1) = f(3) = 7^3 - e^{5.76} + 5\cos 1 - 9 = 19.35$$
$$f'(x) = 3(x+4)^2 - 1.92e^{1.92x} - \frac{5}{3}\sin\frac{x}{3}$$
$$f'(r_1) = f'(3) = 3(7)^2 - 1.92e^{5.76} - \frac{5}{3}\sin 1 = -463.7$$

Thus, $r_2 = 3 - \dfrac{19.35}{-463.7} = 3 + 0.042 = 3.042 = 3.04$, correct to 3 significant figures.

Similarly, $r_3 = 3.042 - \dfrac{f(3.042)}{f'(3.042)} = 3.042 - \dfrac{(-1.146)}{(-513.1)} = 3.042 - 0.0022 = 3.0398 = 3.04$, correct to 3 significant figures.

Since r_2 and r_3 are the same when expressed to the required degree of accuracy, then the required root is **3.04**, correct to 3 significant figures.

Now try the following exercise

Exercise 37 Further problems on Newton's method

In Problems 1 to 7, use **Newton's method** to solve the equations given to the accuracy stated.

1. $x^2 - 2x - 13 = 0$, correct to 3 decimal places. [−2.742, 4.742]
2. $3x^3 - 10x = 14$, correct to 4 significant figures. [2.313]
3. $x^4 - 3x^3 + 7x = 12$, correct to 3 decimal places. [−1.721, 2.648]
4. $3x^4 - 4x^3 + 7x - 12 = 0$, correct to 3 decimal places. [−1.386, 1.491]
5. $3\ln x + 4x = 5$, correct to 3 decimal places. [1.147]
6. $x^3 = 5\cos 2x$, correct to 3 significant figures. [−1.693, −0.846, 0.744]
7. $300e^{-2\theta} + \dfrac{\theta}{2} = 6$, correct to 3 significant figures. [2.05]
8. Solve the equations in Problems 1 to 5, Exercise 35, page 81 and Problems 1 to 4, Exercise 36, page 84 using Newton's method.
9. A Fourier analysis of the instantaneous value of a waveform can be represented by:
$$y = \left(t + \frac{\pi}{4}\right) + \sin t + \frac{1}{8}\sin 3t$$
Use Newton's method to determine the value of t near to 0.04, correct to 4 decimal places, when the amplitude, y, is 0.880. [0.0399]
10. A damped oscillation of a system is given by the equation:
$$y = -7.4e^{0.5t}\sin 3t.$$
Determine the value of t near to 4.2, correct to 3 significant figures, when the magnitude y of the oscillation is zero. [4.19]
11. The critical speeds of oscillation, λ, of a loaded beam are given by the equation:
$$\lambda^3 - 3.250\lambda^2 + \lambda - 0.063 = 0$$
Determine the value of λ which is approximately equal to 3.0 by Newton's method, correct to 4 decimal places. [2.9143]

Chapter 10

Binary, octal and hexadecimal

10.1 Introduction

All data in modern computers is stored as series of **bits**, a bit being a **b**inary dig**it**, and can have one of two values, the numbers 0 and 1. The most basic form of representing computer data is to represent a piece of data as a string of 1's and 0's, one for each bit. This is called a **binary** or base-2 number.

Because binary notation requires so many bits to represent relatively small numbers, two further compact notations are often used, called **octal** and **hexadecimal**. Computer programmers who design sequences of number codes instructing a computer what to do would have a very difficult task if they were forced to work with nothing but long strings of 1's and 0's, the 'native language' of any digital circuit.

Octal notation represents data as base-8 numbers with each digit in an octal number representing three bits. Similarly, hexadecimal notation uses base-16 numbers, representing four bits with each digit. Octal numbers use only the digits 0–7, while hexadecimal numbers use all ten base-10 digits (0–9) and the letters A–F (representing the numbers 10–15).

This chapter explains how to convert between the decimal, binary, octal and hexadecimal systems.

10.2 Binary numbers

The system of numbers in everyday use is the **denary** or **decimal** system of numbers, using the digits 0 to 9. It has ten different digits (0, 1, 2, 3, 4, 5, 6, 7, 8 and 9) and is said to have a **radix** or **base** of 10.

The **binary** system of numbers has a radix of 2 and uses only the digits 0 and 1.

(a) Conversion of binary to decimal

The decimal number 234.5 is equivalent to

$$2 \times 10^2 + 3 \times 10^1 + 4 \times 10^0 + 5 \times 10^{-1}$$

i.e. is the sum of terms comprising: (a digit) multiplied by (the base raised to some power).

In the binary system of numbers, the base is 2, so 1101.1 is equivalent to:

$$1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 + 1 \times 2^{-1}$$

Thus the decimal number equivalent to the binary number 1101.1 is $8+4+0+1+\frac{1}{2}$, that is 13.5 i.e. $\mathbf{1101.1_2 = 13.5_{10}}$, the suffixes 2 and 10 denoting binary and decimal systems of numbers respectively.

Problem 1. Convert 11011_2 to a decimal number.

$$\begin{aligned} \text{From above: } 11011_2 &= 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 \\ &\quad + 1 \times 2^1 + 1 \times 2^0 \\ &= 16 + 8 + 0 + 2 + 1 \\ &= \mathbf{27_{10}} \end{aligned}$$

Problem 2. Convert 0.1011_2 to a decimal fraction.

$$\begin{aligned} 0.1011_2 &= 1 \times 2^{-1} + 0 \times 2^{-2} + 1 \times 2^{-3} + 1 \times 2^{-4} \\ &= 1 \times \frac{1}{2} + 0 \times \frac{1}{2^2} + 1 \times \frac{1}{2^3} + 1 \times \frac{1}{2^4} \end{aligned}$$

$$= \frac{1}{2} + \frac{1}{8} + \frac{1}{16}$$
$$= 0.5 + 0.125 + 0.0625$$
$$= \mathbf{0.6875_{10}}$$

Problem 3. Convert 101.0101_2 to a decimal number.

$$101.0101_2 = 1\times2^2 + 0\times2^1 + 1\times2^0 + 0\times2^{-1} + 1\times2^{-2} + 0\times2^{-3} + 1\times2^{-4}$$
$$= 4 + 0 + 1 + 0 + 0.25 + 0 + 0.0625$$
$$= \mathbf{5.3125_{10}}$$

Now try the following exercise

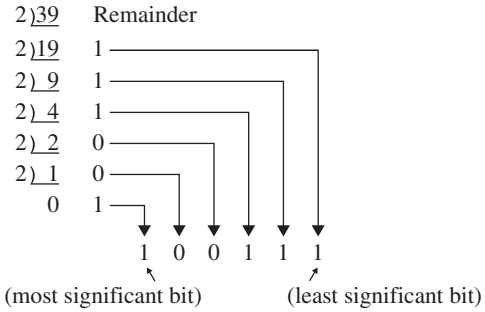
Exercise 38 Further problems on conversion of binary to decimal numbers

In Problems 1 to 5, convert the binary numbers given to decimal numbers.

1. (a) 110 (b) 1011 (c) 1110 (d) 1001
   [(a) 6_{10} (b) 11_{10} (c) 14_{10} (d) 9_{10}]
2. (a) 10101 (b) 11001 (c) 101101 (d) 110011
   [(a) 21_{10} (b) 25_{10} (c) 45_{10} (d) 51_{10}]
3. (a) 101010 (b) 111000 (c) 1000001 (d) 10111000
   [(a) 42_{10} (b) 56_{10} (c) 65_{10} (d) 184_{10}]
4. (a) 0.1101 (b) 0.11001 (c) 0.00111 (d) 0.01011
   [(a) 0.8125_{10} (b) 0.78125_{10} (c) 0.21875_{10} (d) 0.34375_{10}]
5. (a) 11010.11 (b) 10111.011 (c) 110101.0111 (d) 11010101.10111
   [(a) 26.75_{10} (b) 23.375_{10} (c) 53.4375_{10} (d) 213.71875_{10}]

(b) Conversion of decimal to binary

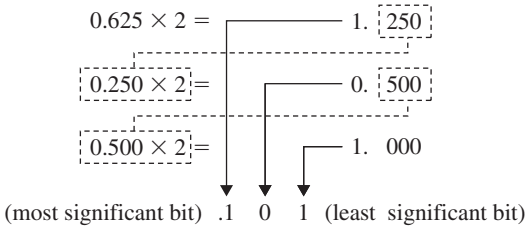
An integer decimal number can be converted to a corresponding binary number by repeatedly dividing by 2 and noting the remainder at each stage, as shown below for 39_{10}.



The result is obtained by writing the top digit of the remainder as the least significant bit, (a bit is a **b**inary dig**it** and the least significant bit is the one on the right). The bottom bit of the remainder is the most significant bit, i.e. the bit on the left.

Thus $\mathbf{39_{10} = 100111_2}$

The fractional part of a decimal number can be converted to a binary number by repeatedly multiplying by 2, as shown below for the fraction 0.625

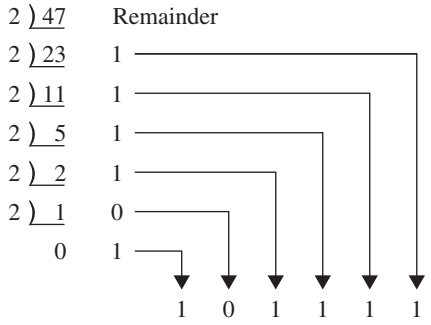


For fractions, the most significant bit of the result is the top bit obtained from the integer part of multiplication by 2. The least significant bit of the result is the bottom bit obtained from the integer part of multiplication by 2.

Thus $\mathbf{0.625_{10} = 0.101_2}$

Problem 4. Convert 47_{10} to a binary number.

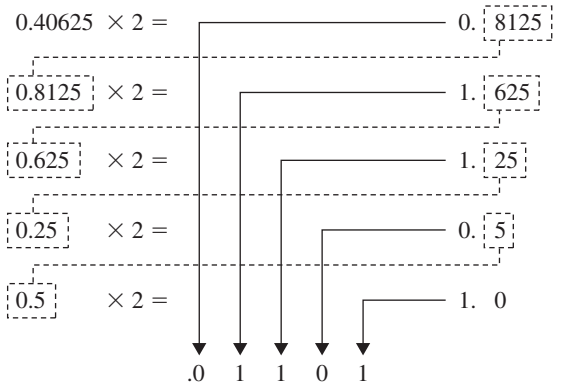
From above, repeatedly dividing by 2 and noting the remainder gives:



Thus $\mathbf{47_{10} = 101111_2}$

Problem 5. Convert 0.40625_{10} to a binary number.

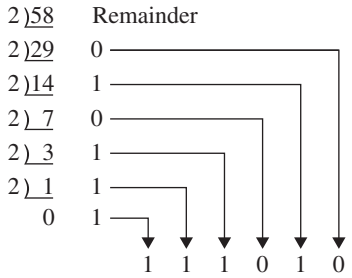
From above, repeatedly multiplying by 2 gives:



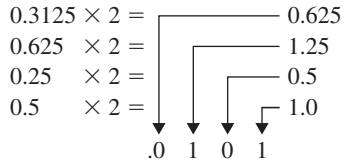
i.e. $\mathbf{0.40625_{10} = 0.01101_2}$

Problem 6. Convert 58.3125_{10} to a binary number.

The integer part is repeatedly divided by 2, giving:



The fractional part is repeatedly multiplied by 2 giving:



Thus $\mathbf{58.3125_{10} = 111010.0101_2}$

Now try the following exercise

Exercise 39 Further problems on conversion of decimal to binary numbers

In Problems 1 to 5, convert the decimal numbers given to binary numbers.

1. (a) 5 (b) 15 (c) 19 (d) 29

$$\begin{bmatrix} \text{(a) } 101_2 & \text{(b) } 1111_2 \\ \text{(c) } 10011_2 & \text{(d) } 11101_2 \end{bmatrix}$$

2. (a) 31 (b) 42 (c) 57 (d) 63

$$\begin{bmatrix} \text{(a) } 11111_2 & \text{(b) } 101010_2 \\ \text{(c) } 111001_2 & \text{(d) } 111111_2 \end{bmatrix}$$

3. (a) 47 (b) 60 (c) 73 (d) 84

$$\begin{bmatrix} \text{(a) } 101111_2 & \text{(b) } 111100_2 \\ \text{(c) } 1001001_2 & \text{(d) } 1010100_2 \end{bmatrix}$$

4. (a) 0.25 (b) 0.21875 (c) 0.28125 (d) 0.59375

$$\begin{bmatrix} \text{(a) } 0.01_2 & \text{(b) } 0.00111_2 \\ \text{(c) } 0.01001_2 & \text{(d) } 0.10011_2 \end{bmatrix}$$

5. (a) 47.40625 (b) 30.8125 (c) 53.90625 (d) 61.65625

$$\begin{bmatrix} \text{(a) } 101111.01101_2 \\ \text{(b) } 11110.1101_2 \\ \text{(c) } 110101.11101_2 \\ \text{(d) } 111101.10101_2 \end{bmatrix}$$

(c) Binary addition

Binary addition of two/three bits is achieved according to the following rules:

sum	carry	sum	carry
$0+0=0$	0	$0+0+0=0$	0
$0+1=1$	0	$0+0+1=1$	0
$1+0=1$	0	$0+1+0=1$	0
$1+1=0$	1	$0+1+1=0$	1
		$1+0+0=1$	0
		$1+0+1=0$	1
		$1+1+0=0$	1
		$1+1+1=1$	1

These rules are demonstrated in the following worked problems.

Problem 7. Perform the binary addition: $1001 + 10110$

$$\begin{array}{r} 1001 \\ +10110 \\ \hline \mathbf{11111} \end{array}$$

Problem 8. Perform the binary addition: $11111 + 10101$

$$\begin{array}{rr} & 11111 \\ & +10101 \\ \text{sum} & \mathbf{110100} \\ \text{carry} & 11111 \end{array}$$

Problem 9. Perform the binary addition: $1101001 + 1110101$

$$\begin{array}{rr} & 1101001 \\ & +1110101 \\ \text{sum} & \mathbf{11011110} \\ \text{carry} & 11\quad 1 \end{array}$$

Problem 10. Perform the binary addition: $1011101 + 1100001 + 110101$

$$\begin{array}{rr} & 1011101 \\ & 1100001 \\ + & 110101 \\ \text{sum} & \mathbf{11110011} \\ \text{carry} & 11111\ 1 \end{array}$$

Now try the following exercise

Exercise 40 Further problems on binary addition

Perform the following binary additions:

1. $10 + 11$ [101]
2. $101 + 110$ [1011]
3. $1101 + 111$ [10100]
4. $1111 + 11101$ [101100]
5. $110111 + 10001$ [1001000]
6. $10000101 + 10000101$ [100001010]
7. $11101100 + 111001011$ [1010110111]
8. $110011010 + 11100011$ [1001111101]
9. $10110 + 1011 + 11011$ [111100]
10. $111 + 10101 + 11011$ [110111]
11. $1101 + 1001 + 11101$ [110011]
12. $100011 + 11101 + 101110$ [1101110]

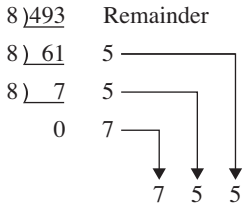
10.3 Octal numbers

For decimal integers containing several digits, repeatedly dividing by 2 can be a lengthy process. In this case, it is usually easier to convert a decimal number to a binary number via the octal system of numbers. This system has a radix of 8, using the digits 0, 1, 2, 3, 4, 5, 6 and 7. The decimal number equivalent to the octal number 4317_8 is:

$$4 \times 8^3 + 3 \times 8^2 + 1 \times 8^1 + 7 \times 8^0$$

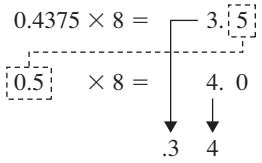
i.e. $4 \times 512 + 3 \times 64 + 1 \times 8 + 7 \times 1$ or 2255_{10}

An integer decimal number can be converted to a corresponding octal number by repeatedly dividing by 8 and noting the remainder at each stage, as shown below for 493_{10}.



Thus $\mathbf{493_{10} = 755_8}$

The fractional part of a decimal number can be converted to an octal number by repeatedly multiplying by 8, as shown below for the fraction 0.4375_{10}



For fractions, the most significant bit is the top integer obtained by multiplication of the decimal fraction by 8, thus,

$$0.4375_{10} = 0.34_8$$

The natural binary code for digits 0 to 7 is shown in Table 10.1, and an octal number can be converted to a binary number by writing down the three bits corresponding to the octal digit.

Thus $437_8 = 100\ 011\ 111_2$

and $26.35_8 = 010\ 110.011\ 101_2$

Table 10.1

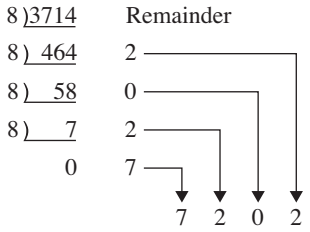
Octal digit	Natural binary number
0	000
1	001
2	010
3	011
4	100
5	101
6	110
7	111

The '0' on the extreme left does not signify anything, thus $26.35_8 = 10\ 110.011\ 101_2$

Conversion of decimal to binary via octal is demonstrated in the following worked problems.

Problem 11. Convert 3714_{10} to a binary number, via octal.

Dividing repeatedly by 8, and noting the remainder gives:

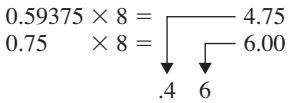


From Table 10.1, $7202_8 = 111\ 010\ 000\ 010_2$

i.e. $\mathbf{3714_{10} = 111\ 010\ 000\ 010_2}$

Problem 12. Convert 0.59375_{10} to a binary number, via octal.

Multiplying repeatedly by 8, and noting the integer values, gives:



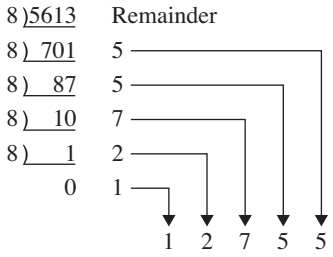
Thus $0.59375_{10} = 0.46_8$

From Table 10.1, $0.46_8 = 0.100\ 110_2$

i.e. $\mathbf{0.59375_{10} = 0.100\ 11_2}$

Problem 13. Convert 5613.90625_{10} to a binary number, via octal.

The integer part is repeatedly divided by 8, noting the remainder, giving:

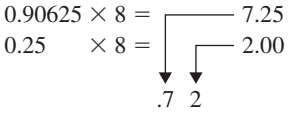


This octal number is converted to a binary number, (see Table 10.1).

$$12755_8 = 001\ 010\ 111\ 101\ 101_2$$

i.e. $5613_{10} = 1\ 010\ 111\ 101\ 101_2$

The fractional part is repeatedly multiplied by 8, and noting the integer part, giving:



This octal fraction is converted to a binary number, (see Table 10.1).

$$0.72_8 = 0.111\ 010_2$$

i.e. $0.90625_{10} = 0.111\ 01_2$

Thus, $\mathbf{5613.90625_{10} = 1\ 010\ 111\ 101\ 101.111\ 01_2}$

Problem 14. Convert $11\ 110\ 011.100\ 01_2$ to a decimal number via octal.

Grouping the binary number in three's from the binary point gives: $011\ 110\ 011.100\ 010_2$

Using Table 10.1 to convert this binary number to an octal number gives 363.42_8 and 363.42_8

$= 3 \times 8^2 + 6 \times 8^1 + 3 \times 8^0 + 4 \times 8^{-1} + 2 \times 8^{-2}$

$= 192 + 48 + 3 + 0.5 + 0.03125$

$= \mathbf{243.53125_{10}}$

Now try the following exercise

Exercise 41 Further problems on conversion between decimal and binary numbers via octal

In Problems 1 to 3, convert the decimal numbers given to binary numbers, via octal.

1. (a) 343 (b) 572 (c) 1265

   [(a) 101010111_2 (b) 1000111100_2 (c) 100111110001_2]

2. (a) 0.46875 (b) 0.6875 (c) 0.71875

   [(a) 0.01111_2 (b) 0.1011_2 (c) 0.10111_2]

3. (a) 247.09375 (b) 514.4375 (c) 1716.78125

   [(a) 11110111.00011_2 (b) 1000000010.0111_2 (c) 11010110100.11001_2]

4. Convert the binary numbers given to decimal numbers via octal.
   (a) 111.011 1 (b) 101 001.01
   (c) 1 110 011 011 010.001 1

   [(a) 7.4375_{10} (b) 41.25_{10} (c) 7386.1875_{10}]

10.4 Hexadecimal numbers

The hexadecimal system is particularly important in computer programming, since four bits (each consisting of a one or zero) can be succinctly expressed using a single hexadecimal digit. Two hexadecimal digits represent numbers from 0 to 255, a common range used, for example, to specify colours. Thus, in the HTML language of the web, colours are specified using three pairs of hexadecimal digits RRGGBB, where RR is the amount of red, GG the amount of green, and BB the amount of blue.

A **hexadecimal numbering system** has a radix of 16 and uses the following 16 distinct digits:

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E and F

'A' corresponds to 10 in the decimal system, B to 11, C to 12, and so on.

(a) Converting from hexadecimal to decimal:

For example

$$1A_{16} = 1 \times 16^1 + A \times 16^0$$
$$= 1 \times 16^1 + 10 \times 1$$
$$= 16 + 10 = 26$$

i.e. $\mathbf{1A_{16} = 26_{10}}$

Similarly, $\mathbf{2E_{16}} = 2 \times 16^1 + E \times 16^0$

$$= 2 \times 16^1 + 14 \times 16^0$$
$$= 32 + 14 = \mathbf{46_{10}}$$

and $\mathbf{1BF_{16}} = 1 \times 16^2 + B \times 16^1 + F \times 16^0$

$$= 1 \times 16^2 + 11 \times 16^1 + 15 \times 16^0$$
$$= 256 + 176 + 15 = \mathbf{447_{10}}$$

Table 10.2 compares decimal, binary, octal and hexadecimal numbers and shows, for example, that $23_{10} = 10111_2 = 27_8 = 17_{16}$

Problem 15. Convert the following hexadecimal numbers into their decimal equivalents:
(a) $7A_{16}$ (b) $3F_{16}$

(a) $7A_{16} = 7 \times 16^1 + A \times 16^0 = 7 \times 16 + 10 \times 1$

$= 112 + 10 = 122$

Thus $\mathbf{7A_{16} = 122_{10}}$

(b) $3F_{16} = 3 \times 16^1 + F \times 16^0 = 3 \times 16 + 15 \times 1$

$= 48 + 15 = 63$

Thus $\mathbf{3F_{16} = 63_{10}}$

Table 10.2

Decimal	Binary	Octal	Hexadecimal
0	0000	0	0
1	0001	1	1
2	0010	2	2
3	0011	3	3
4	0100	4	4
5	0101	5	5
6	0110	6	6
7	0111	7	7
8	1000	10	8
9	1001	11	9
10	1010	12	A
11	1011	13	B
12	1100	14	C
13	1101	15	D
14	1110	16	E
15	1111	17	F
16	10000	20	10
17	10001	21	11
18	10010	22	12
19	10011	23	13
20	10100	24	14
21	10101	25	15
22	10110	26	16
23	10111	27	17
24	11000	30	18
25	11001	31	19
26	11010	32	1A
27	11011	33	1B
28	11100	34	1C
29	11101	35	1D
30	11110	36	1E
31	11111	37	1F
32	100000	40	20

Problem 16. Convert the following hexadecimal numbers into their decimal equivalents:
(a) $C9_{16}$ (b) BD_{16}

(a) $C9_{16} = C \times 16^1 + 9 \times 16^0 = 12 \times 16 + 9 \times 1$
$= 192 + 9 = 201$

Thus $\mathbf{C9_{16} = 201_{10}}$

(b) $BD_{16} = B \times 16^1 + D \times 16^0$
$= 11 \times 16 + 13 \times 1 = 176 + 13 = 189$

Thus $\mathbf{BD_{16} = 189_{10}}$

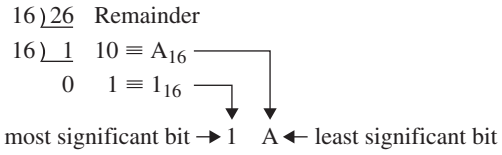
Problem 17. Convert $1A4E_{16}$ into a decimal number.

$1A4E_{16} = 1 \times 16^3 + A \times 16^2 + 4 \times 16^1 + E \times 16^0$
$= 1 \times 16^3 + 10 \times 16^2 + 4 \times 16^1$
$+ 14 \times 16^0$
$= 1 \times 4096 + 10 \times 256 + 4 \times 16 + 14 \times 1$
$= 4096 + 2560 + 64 + 14 = 6734$

Thus $\mathbf{1A4E_{16} = 6734_{10}}$

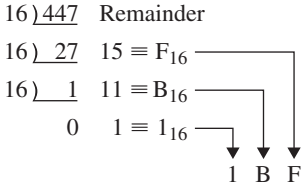
(b) Converting from decimal to hexadecimal

This is achieved by repeatedly dividing by 16 and noting the remainder at each stage, as shown below for 26_{10}.



Hence $\mathbf{26_{10} = 1A_{16}}$

Similarly, for 447_{10}



Thus $\mathbf{447_{10} = 1BF_{16}}$

Problem 18. Convert the following decimal numbers into their hexadecimal equivalents:
(a) 37_{10} (b) 108_{10}

(a) 16)37 Remainder
16) 2 $5 = 5_{16}$
0 $2 = 2_{16}$
2 5
most significant bit — least significant bit

Hence $\mathbf{37_{10} = 25_{16}}$

(b) 16)108 Remainder
16) 6 $12 = C_{16}$
0 $6 = 6_{16}$
6 C

Hence $\mathbf{108_{10} = 6C_{16}}$

Problem 19. Convert the following decimal numbers into their hexadecimal equivalents:
(a) 162_{10} (b) 239_{10}

(a) 16)162 Remainder
16) 10 $2 = 2_{16}$
0 $10 = A_{16}$
A 2

Hence $\mathbf{162_{10} = A2_{16}}$

(b) 16)239 Remainder
16) 14 $15 = F_{16}$
0 $14 = E_{16}$
E F

Hence $\mathbf{239_{10} = EF_{16}}$

Now try the following exercise

Exercise 42 Further problems on hexadecimal numbers

In Problems 1 to 4, convert the given hexadecimal numbers into their decimal equivalents.

1. $E7_{16}$ [231_{10}] 2. $2C_{16}$ [44_{10}]
3. 98_{16} [152_{10}] 4. $2F1_{16}$ [753_{10}]

In Problems 5 to 8, convert the given decimal numbers into their hexadecimal equivalents.

5. 54_{10} [36_{16}] 6. 200_{10} [$C8_{16}$]
7. 91_{10} [$5B_{16}$] 8. 238_{10} [EE_{16}]

(c) Converting from binary to hexadecimal:

The binary bits are arranged in groups of four, starting from right to left, and a hexadecimal symbol is assigned to each group. For example, the binary number 1110011110101001 is initially grouped in fours as: $\underbrace{1110}_{E}\ \underbrace{0111}_{7}\ \underbrace{1010}_{A}\ \underbrace{1001}_{9}$ and a hexadecimal symbol assigned to each group as above from Table 10.2.

Hence $\mathbf{1110011110101001_2 = E7A9_{16}}$

Problem 20. Convert the following binary numbers into their hexadecimal equivalents:
(a) 11010110_2 (b) 1100111_2

(a) Grouping bits in fours from the right gives: $\underbrace{1101}_{D}\ \underbrace{0110}_{6}$ and assigning hexadecimal symbols to each group gives as above from Table 10.2.

Thus, $\mathbf{11010110_2 = D6_{16}}$

(b) Grouping bits in fours from the right gives: $\underbrace{0110}_{6}\ \underbrace{0111}_{7}$ and assigning hexadecimal symbols to each group gives as above from Table 10.2.

Thus, $\mathbf{1100111_2 = 67_{16}}$

Problem 21. Convert the following binary numbers into their hexadecimal equivalents:
(a) 11001111_2 (b) 110011110_2

(a) Grouping bits in fours from the right gives: $\underbrace{1100}_{C}\ \underbrace{1111}_{F}$ and assigning hexadecimal symbols to each group gives as above from Table 10.2.

Thus, $\mathbf{11001111_2 = CF_{16}}$

(b) Grouping bits in fours from the right gives: $\underbrace{0001}_{1}\ \underbrace{1001}_{9}\ \underbrace{1110}_{E}$ and assigning hexadecimal

symbols to each group gives as above from Table 10.2.

Thus, $\mathbf{110011110_2 = 19E_{16}}$

(d) Converting from hexadecimal to binary:

The above procedure is reversed, thus, for example,

$6CF3_{16} = 0110\ 1100\ 1111\ 0011$ from Table 10.2

i.e. $\mathbf{6CF3_{16} = 110110011110011_2}$

Problem 22. Convert the following hexadecimal numbers into their binary equivalents:
(a) $3F_{16}$ (b) $A6_{16}$

(a) Spacing out hexadecimal digits gives: $\overbrace{3}^{0011}\ \overbrace{F}^{1111}$ and converting each into binary gives as above from Table 10.2.

Thus, $\mathbf{3F_{16} = 111111_2}$

(b) Spacing out hexadecimal digits gives: $\overbrace{A}^{1010}\ \overbrace{6}^{0110}$ and converting each into binary gives as above from Table 10.2.

Thus, $\mathbf{A6_{16} = 10100110_2}$

Problem 23. Convert the following hexadecimal numbers into their binary equivalents:
(a) $7B_{16}$ (b) $17D_{16}$

(a) Spacing out hexadecimal digits gives: $\overbrace{7}^{0111}\ \overbrace{B}^{1011}$ and converting each into binary gives as above from Table 10.2.

Thus, $\mathbf{7B_{16} = 1111011_2}$

(b) Spacing out hexadecimal digits gives: $\overbrace{1}^{0001}\ \overbrace{7}^{0111}\ \overbrace{D}^{1101}$ and converting each into binary gives as above from Table 10.2.

Thus, $\mathbf{17D_{16} = 101111101_2}$

Now try the following exercise

Exercise 43 Further problems on hexadecimal numbers

In Problems 1 to 4, convert the given binary numbers into their hexadecimal equivalents.

1. 11010111_2 [$D7_{16}$]
2. 11101010_2 [EA_{16}]
3. 10001011_2 [$8B_{16}$]
4. 10100101_2 [$A5_{16}$]

In Problems 5 to 8, convert the given hexadecimal numbers into their binary equivalents.

5. 37_{16} [110111_2]
6. ED_{16} [11101101_2]
7. $9F_{16}$ [10011111_2]
8. $A21_{16}$ [101000100001_2]

Revision Test 3

This Revision Test covers the material contained in Chapters 8 to 10. *The marks for each question are shown in brackets at the end of each question.*

1. Use Maclaurin's series to determine a power series for $e^{2x}\cos 3x$ as far as the term in x^2. (9)

2. Show, using Maclaurin's series, that the first four terms of the power series for $\cosh 2x$ is given by:

$$\cosh 2x = 1 + 2x^2 + \frac{2}{3}x^4 + \frac{4}{45}x^6.$$ (10)

3. Expand the function $x^2 \ln(1+\sin x)$ using Maclaurin's series and hence evaluate:

$\int_0^{\frac{1}{2}} x^2 \ln(1+\sin x)\,dx$ correct to 2 significant figures. (13)

4. Use the method of bisection to evaluate the root of the equation: $x^3 + 5x = 11$ in the range $x = 1$ to $x = 2$, correct to 3 significant figures. (11)

5. Repeat question 4 using an algebraic method of successive approximations. (16)

6. The solution to a differential equation associated with the path taken by a projectile for which the resistance to motion is proportional to the velocity is given by:

$$y = 2.5(e^x - e^{-x}) + x - 25$$

Use Newton's method to determine the value of x, correct to 2 decimal places, for which the value of y is zero. (10)

7. Convert the following binary numbers to decimal form:

(a) 1101 (b) 101101.0101 (5)

8. Convert the following decimal number to binary form:

(a) 27 (b) 44.1875 (9)

9. Convert the following decimal numbers to binary, via octal:

(a) 479 (b) 185.2890625 (9)

10. Convert

(a) $5F_{16}$ into its decimal equivalent

(b) 132_{10} into its hexadecimal equivalent

(c) 110101011_2 into its hexadecimal equivalent

(8)

Chapter 11

Introduction to trigonometry

11.1 Trigonometry

Trigonometry is the branch of mathematics which deals with the measurement of sides and angles of triangles, and their relationship with each other. There are many applications in engineering where a knowledge of trigonometry is needed.

11.2 The theorem of Pythagoras

With reference to Fig. 11.1, the side opposite the right angle (i.e. side b) is called the **hypotenuse**. The **theorem of Pythagoras** states:

'In any right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides.'

Hence $\boldsymbol{b^2 = a^2 + c^2}$

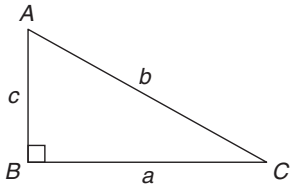


Figure 11.1

Problem 1. In Fig. 11.2, find the length of EF.

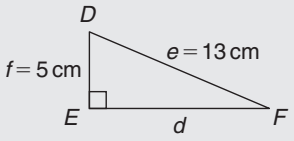


Figure 11.2

By Pythagoras' theorem:

$$e^2 = d^2 + f^2$$

Hence $13^2 = d^2 + 5^2$

$169 = d^2 + 25$

$d^2 = 169 - 25 = 144$

Thus $d = \sqrt{144} = 12\,\text{cm}$

i.e. $\boldsymbol{EF = 12\,cm}$

Problem 2. Two aircraft leave an airfield at the same time. One travels due north at an average speed of 300 km/h and the other due west at an average speed of 220 km/h. Calculate their distance apart after 4 hours.

After 4 hours, the first aircraft has travelled $4 \times 300 = 1200$ km, due north, and the second aircraft has travelled $4 \times 220 = 880$ km due west, as shown in Fig. 11.3. Distance apart after 4 hours $= BC$.

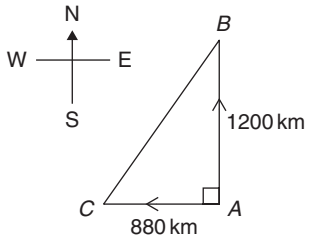


Figure 11.3

From Pythagoras' theorem:

$$BC^2 = 1200^2 + 880^2 = 1\,440\,000 + 774\,400$$

and $BC = \sqrt{(2\,214\,400)}$

Hence distance apart after 4 hours = 1488 km.

Now try the following exercise

Exercise 44 Further problems on the theorem of Pythagoras

1. In a triangle CDE, $D = 90°$, $CD = 14.83$ mm and $CE = 28.31$ mm. Determine the length of DE. [24.11 mm]
2. Triangle PQR is isosceles, Q being a right angle. If the hypotenuse is 38.47 cm find (a) the lengths of sides PQ and QR, and (b) the value of $\angle QPR$. [(a) 27.20 cm each (b) 45°]
3. A man cycles 24 km due south and then 20 km due east. Another man, starting at the same time as the first man, cycles 32 km due east and then 7 km due south. Find the distance between the two men. [20.81 km]
4. A ladder 3.5 m long is placed against a perpendicular wall with its foot 1.0 m from the wall. How far up the wall (to the nearest centimetre) does the ladder reach? If the foot of the ladder is now moved 30 cm further away from the wall, how far does the top of the ladder fall? [3.35 m, 10 cm]
5. Two ships leave a port at the same time. One travels due west at 18.4 km/h and the other due south at 27.6 km/h. Calculate how far apart the two ships are after 4 hours. [132.7 km]
6. Figure 11.4 shows a bolt rounded off at one end. Determine the dimension h. [2.94 mm]

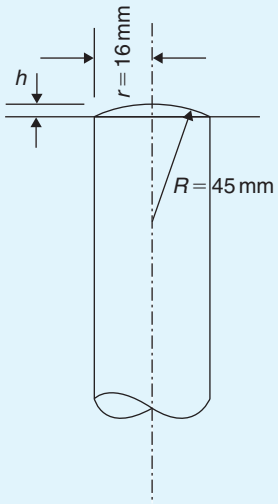


Figure 11.4

7. Figure 11.5 shows a cross-section of a component that is to be made from a round bar. If the diameter of the bar is 74 mm, calculate the dimension x. [24 mm]

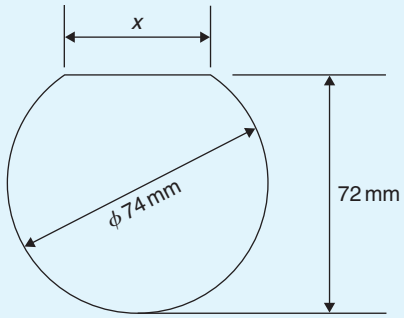


Figure 11.5

11.3 Trigonometric ratios of acute angles

(a) With reference to the right-angled triangle shown in Fig. 11.6:

(i) $$\text{sine}\,\theta = \frac{\text{opposite side}}{\text{hypotenuse}}$$

i.e. $$\boldsymbol{\sin\theta = \frac{b}{c}}$$

(ii) $$\text{cosine}\,\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

i.e. $$\boldsymbol{\cos\theta = \frac{a}{c}}$$

(iii) $$\text{tangent}\,\theta = \frac{\text{opposite side}}{\text{adjacent side}}$$

i.e. $$\boldsymbol{\tan\theta = \frac{b}{a}}$$

(iv) $$\text{secant}\,\theta = \frac{\text{hypotenuse}}{\text{adjacent side}}$$

i.e. $$\boldsymbol{\sec\theta = \frac{c}{a}}$$

(v) $$\text{cosecant}\,\theta = \frac{\text{hypotenuse}}{\text{opposite side}}$$

i.e. $$\boldsymbol{\text{cosec}\,\theta = \frac{c}{b}}$$

(vi) $\text{cotangent}\,\theta = \dfrac{\text{adjacent side}}{\text{opposite side}}$

i.e. $\boldsymbol{\cot\theta = \dfrac{a}{b}}$

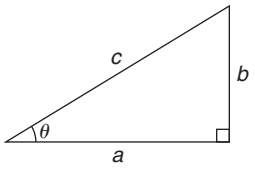


Figure 11.6

(b) From above,

(i) $\dfrac{\sin\theta}{\cos\theta} = \dfrac{\frac{b}{c}}{\frac{a}{c}} = \dfrac{b}{a} = \tan\theta,$

i.e. $\boldsymbol{\tan\theta = \dfrac{\sin\theta}{\cos\theta}}$

(ii) $\dfrac{\cos\theta}{\sin\theta} = \dfrac{\frac{a}{c}}{\frac{b}{c}} = \dfrac{a}{b} = \cot\theta,$

i.e. $\boldsymbol{\cot\theta = \dfrac{\cos\theta}{\sin\theta}}$

(iii) $\boldsymbol{\sec\theta = \dfrac{1}{\cos\theta}}$

(iv) $\boldsymbol{\text{cosec}\,\theta = \dfrac{1}{\sin\theta}}$

(Note 's' and 'c' go together)

(v) $\boldsymbol{\cot\theta = \dfrac{1}{\tan\theta}}$

Secants, cosecants and cotangents are called the **reciprocal ratios**.

Problem 3. If $\cos X = \dfrac{9}{41}$ determine the value of the other five trigonometry ratios.

Fig. 11.7 shows a right-angled triangle XYZ.

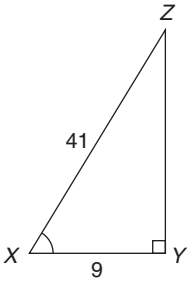


Figure 11.7

Since $\cos X = \dfrac{9}{41}$, then $XY = 9$ units and $XZ = 41$ units.

Using Pythagoras' theorem: $41^2 = 9^2 + YZ^2$ from which $YZ = \sqrt{(41^2 - 9^2)} = 40$ units.

Thus

$$\boldsymbol{\sin X = \frac{40}{41}, \tan X = \frac{40}{9} = 4\frac{4}{9},}$$

$$\boldsymbol{\text{cosec}\,X = \frac{41}{40} = 1\frac{1}{40},}$$

$$\boldsymbol{\sec X = \frac{41}{9} = 4\frac{5}{9}} \text{ and } \boldsymbol{\cot X = \frac{9}{40}}$$

Problem 4. If $\sin\theta = 0.625$ and $\cos\theta = 0.500$ determine, without using trigonometric tables or calculators, the values of $\text{cosec}\,\theta$, $\sec\theta$, $\tan\theta$ and $\cot\theta$.

$$\text{cosec}\,\theta = \frac{1}{\sin\theta} = \frac{1}{0.625} = \mathbf{1.60}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{0.500} = \mathbf{2.00}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{0.625}{0.500} = \mathbf{1.25}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{0.500}{0.625} = \mathbf{0.80}$$

Problem 5. Point A lies at co-ordinate (2, 3) and point B at (8, 7). Determine (a) the distance AB, (b) the gradient of the straight line AB, and (c) the angle AB makes with the horizontal.

(a) Points A and B are shown in Fig. 11.8(a).

In Fig. 11.8(b), the horizontal and vertical lines AC and BC are constructed.

Since ABC is a right-angled triangle, and $AC = (8-2) = 6$ and $BC = (7-3) = 4$, then by Pythagoras' theorem

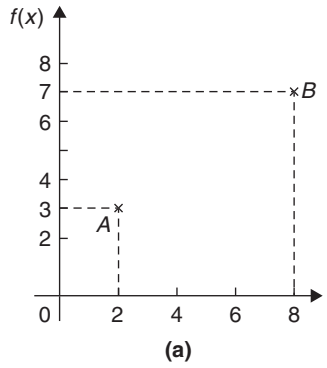
$$AB^2 = AC^2 + BC^2 = 6^2 + 4^2$$

and $\boldsymbol{AB = \sqrt{(6^2 + 4^2)} = \sqrt{52} = 7.211,}$ correct to 3 decimal places.

(b) The gradient of AB is given by $\tan A$,

$$\text{i.e. } \textbf{gradient} = \tan A = \frac{BC}{AC} = \frac{4}{6} = \frac{\mathbf{2}}{\mathbf{3}}$$

(c) **The angle AB makes with the horizontal** is given by $\tan^{-1}\frac{2}{3} = \mathbf{33.69°}$.



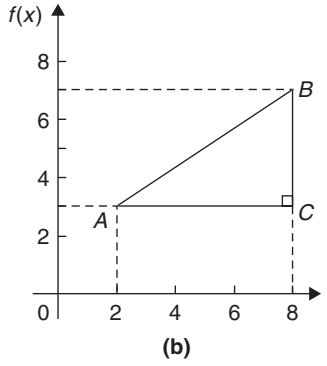


Figure 11.8

Now try the following exercise

Exercise 45 Further problems on trigonometric ratios of acute angles

1. In triangle ABC shown in Fig. 11.9, find $\sin A$, $\cos A$, $\tan A$, $\sin B$, $\cos B$ and $\tan B$.

$$\left[\begin{array}{l}\sin A=\frac{3}{5},\ \cos A=\frac{4}{5},\ \tan A=\frac{3}{4}\\ \sin B=\frac{4}{5},\ \cos B=\frac{3}{5},\ \tan B=\frac{4}{3}\end{array}\right]$$

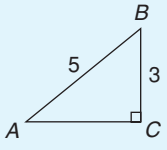


Figure 11.9

2. If $\cos A=\frac{15}{17}$ find $\sin A$ and $\tan A$, in fraction form. $\left[\sin A=\frac{8}{17},\ \tan A=\frac{8}{15}\right]$

3. For the right-angled triangle shown in Fig. 11.10, find:

(a) $\sin\alpha$ (b) $\cos\theta$ (c) $\tan\theta$

$$\left[\text{(a) }\frac{15}{17}\quad\text{(b) }\frac{15}{17}\quad\text{(c) }\frac{8}{15}\right]$$

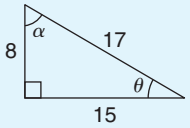


Figure 11.10

4. Point P lies at co-ordinate $(-3, 1)$ and point Q at $(5, -4)$. Determine
   (a) the distance PQ
   (b) the gradient of the straight line PQ and
   (c) the angle PQ makes with the horizontal.

[(a) 9.434 (b) −0.625 (c) 32°]

11.4 Evaluating trigonometric ratios

The easiest method of evaluating trigonometric functions of any angle is by using a **calculator**.

The following values, correct to 4 decimal places, may be checked:

sine 18° = 0.3090, cosine 56° = 0.5592
sine 172° = 0.1392 cosine 115° = –0.4226,
sine 241.63° = –0.8799, cosine 331.78° = 0.8811

tangent 29° = 0.5543,
tangent 178° = –0.0349
tangent 296.42° = –2.0127

To evaluate, say, sine 42°23′ using a calculator means finding sine $42\frac{23^\circ}{60}$ since there are 60 minutes in 1 degree.

$$\frac{23}{60}=0.383\dot{3}\text{ thus }42^\circ23'=42.38\dot{3}^\circ$$

Thus sine 42°23′ = sine 42.38$\dot{3}$° = 0.6741, correct to 4 decimal places.

Similarly, cosine $72°38' = \text{cosine } 72\frac{38^\circ}{60} = 0.2985$, correct to 4 decimal places.

Most calculators contain only sine, cosine and tangent functions. Thus to evaluate secants, cosecants and cotangents, reciprocals need to be used. The following

values, correct to 4 decimal places, may be checked:

$$\text{secant } 32° = \frac{1}{\cos 32°} = 1.1792$$

$$\text{cosecant } 75° = \frac{1}{\sin 75°} = 1.0353$$

$$\text{cotangent } 41° = \frac{1}{\tan 41°} = 1.1504$$

$$\text{secant } 215.12° = \frac{1}{\cos 215.12°} = -1.2226$$

$$\text{cosecant } 321.62° = \frac{1}{\sin 321.62°} = -1.6106$$

$$\text{cotangent } 263.59° = \frac{1}{\tan 263.59°} = 0.1123$$

If we know the value of a trigonometric ratio and need to find the angle we use the **inverse function** on our calculators.
For example, using shift and sin on our calculator gives $\sin^{-1}$ (
If, for example, we know the sine of an angle is 0.5 then the value of the angle is given by:

$$\sin^{-1} 0.5 = \mathbf{30°} \quad (\text{Check that } \sin 30° = 0.5)$$

(Note that $\sin^{-1} x$ does not mean $\frac{1}{\sin x}$; also, $\sin^{-1} x$ may also be written as $\arcsin x$)
Similarly, if $\cos\theta = 0.4371$ then
$\theta = \cos^{-1} 0.4371 = \mathbf{64.08°}$

and if $\tan A = 3.5984$ then $A = \tan^{-1} 3.5984 = \mathbf{74.47°}$

each correct to 2 decimal places.
Use your calculator to check the following worked examples.

Problem 6. Determine, correct to 4 decimal places, $\sin 43°39'$

$$\sin 43°39' = \sin 43\frac{39°}{60} = \sin 43.65° = \mathbf{0.6903}$$

This answer can be obtained using the **calculator** as follows:

1. Press sin
2. Enter 43
3. Press ° ’’’
4. Enter 39
5. Press ° ’’’
6. Press)
7. Press = Answer = **0.6902512**….

Problem 7. Determine, correct to 3 decimal places, $6\cos 62°12'$

$$6\cos 62°12' = 6\cos 62\frac{12°}{60} = 6\cos 62.20° = \mathbf{2.798}$$

This answer can be obtained using the **calculator** as follows:

1. Enter 6
2. Press cos
3. Enter 62
4. Press ° ’’’
5. Enter 12
6. Press ° ’’’
7. Press)
8. Press = Answer = **2.798319**….

Problem 8. Evaluate correct to 4 decimal places:
(a) sine $168°14'$ (b) cosine $271.41°$
(c) tangent $98°4'$

(a) $\text{sine } 168°14' = \text{sine } 168\frac{14°}{60} = \mathbf{0.2039}$

(b) $\text{cosine } 271.41° = \mathbf{0.0246}$

(c) $\text{tangent } 98°4' = \tan 98\frac{4°}{60} = \mathbf{-7.0558}$

Problem 9. Evaluate, correct to 4 decimal places:
(a) secant $161°$ (b) secant $302°29'$

(a) $\sec 161° = \frac{1}{\cos 161°} = \mathbf{-1.0576}$

(b) $\sec 302°29' = \frac{1}{\cos 302°29'} = \frac{1}{\cos 302\frac{29°}{60}} = \mathbf{1.8620}$

Problem 10. Evaluate, correct to 4 significant figures:
(a) cosecant $279.16°$ (b) cosecant $49°7'$

(a) $\text{cosec } 279.16° = \frac{1}{\sin 279.16°} = \mathbf{-1.013}$

(b) $\text{cosec } 49°7' = \frac{1}{\sin 49°7'} = \frac{1}{\sin 49\frac{7°}{60}} = \mathbf{1.323}$

Problem 11. Evaluate, correct to 4 decimal places:
(a) cotangent $17.49°$ (b) cotangent $163°52'$

(a) $\cot 17.49° = \dfrac{1}{\tan 17.49°} = \mathbf{3.1735}$

(b) $\cot 163°52' = \dfrac{1}{\tan 163°52'} = \dfrac{1}{\tan 163\dfrac{52°}{60}}$

$= \mathbf{-3.4570}$

Problem 12. Evaluate, correct to 4 significant figures:
(a) $\sin 1.481$ (b) $\cos(3\pi/5)$ (c) $\tan 2.93$

(a) $\sin 1.481$ means the sine of 1.481 radians. Hence a calculator needs to be on the radian function. Hence $\sin 1.481 = \mathbf{0.9960}$

(b) $\cos(3\pi/5) = \cos 1.884955\cdots = \mathbf{-0.3090}$

(c) $\tan 2.93 = \mathbf{-0.2148}$

Problem 13. Evaluate, correct to 4 decimal places:
(a) secant 5.37 (b) cosecant $\pi/4$
(c) cotangent $\pi/24$

(a) Again, with no degrees sign, it is assumed that 5.37 means 5.37 radians.

Hence $\sec 5.37 = \dfrac{1}{\cos 5.37} = \mathbf{1.6361}$

(b) $\operatorname{cosec}(\pi/4) = \dfrac{1}{\sin(\pi/4)} = \dfrac{1}{\sin 0.785398\ldots}$

$= \mathbf{1.4142}$

(c) $\cot(5\pi/24) = \dfrac{1}{\tan(5\pi/24)} = \dfrac{1}{\tan 0.654498\ldots}$

$= \mathbf{1.3032}$

Problem 14. Find, in degrees, the acute angle $\sin^{-1} 0.4128$ correct to 2 decimal places.

$\sin^{-1} 0.4128$ means 'the angle whose sine is 0.4128'

Using a calculator:

1. Press shift 2. Press sin 3. Enter 0.4128
4. Press) 5. Press = The answer 24.380848…… is displayed

Hence, $\mathbf{\sin^{-1} 0.4128 = 24.38°}$

Problem 15. Find the acute angle $\cos^{-1} 0.2437$ in degrees and minutes

$\cos^{-1} 0.2437$ means 'the angle whose cosine is 0.2437'

Using a calculator:

1. Press shift 2. Press cos 3. Enter 0.2437
4. Press) 5. Press = The answer 75.894979… is displayed
6. Press ° ''' and 75°53′41.93″ is displayed

Hence, $\mathbf{\cos^{-1} 0.2437 = 75.89° = 77°54'}$ correct to the nearest minute.

Problem 16. Find the acute angle $\tan^{-1} 7.4523$ in degrees and minutes

$\tan^{-1} 7.4523$ means 'the angle whose tangent is 7.4523'

Using a calculator:

1. Press shift 2. Press tan 3. Enter 7.4523
4. Press) 5. Press = The answer 82.357318… is displayed
6. Press ° ''' and 82°21′26.35″ is displayed

Hence, $\mathbf{\tan^{-1} 7.4523 = 82.36° = 82°21'}$ correct to the nearest minute.

Problem 17. Determine the acute angles:
(a) $\sec^{-1} 2.3164$ (b) $\operatorname{cosec}^{-1} 1.1784$
(c) $\cot^{-1} 2.1273$

(a) $\sec^{-1} 2.3164 = \cos^{-1}\left(\dfrac{1}{2.3164}\right)$

$= \cos^{-1} 0.4317\ldots$

$= \mathbf{64.42°}$ or $\mathbf{64°25'}$

or **1.124 radians**

(b) $\operatorname{cosec}^{-1} 1.1784 = \sin^{-1}\left(\dfrac{1}{1.1784}\right)$

$= \sin^{-1} 0.8486\ldots$

$= \mathbf{58.06°}$ or $\mathbf{58°4'}$

or **1.013 radians**

(c) $\cot^{-1} 2.1273 = \tan^{-1}\left(\dfrac{1}{2.1273}\right)$

$= \tan^{-1} 0.4700\ldots$

$= \mathbf{25.18°}$ or $\mathbf{25°11'}$

or **0.439 radians**

Problem 18. Evaluate the following expression, correct to 4 significant figures:

$$\frac{4\sec 32°10' - 2\cot 15°19'}{3\,\text{cosec}\,63°8'\tan 14°57'}$$

By calculator:

$\sec 32°10' = 1.1813$, $\cot 15°19' = 3.6512$

$\text{cosec}\,63°8' = 1.1210$, $\tan 14°57' = 0.2670$

Hence $$\frac{4\sec 32°10' - 2\cot 15°19'}{3\,\text{cosec}\,63°8'\tan 14°57'}$$

$$= \frac{4(1.1813) - 2(3.6512)}{3(1.1210)(0.2670)}$$

$$= \frac{4.7252 - 7.3024}{0.8979}$$

$$= \frac{-2.5772}{0.8979} = \mathbf{-2.870},$$

correct to 4 significant figures.

Problem 19. Evaluate correct to 4 decimal places: (a) sec(–115°) (b) cosec (–95°47′)

(a) Positive angles are considered by convention to be anticlockwise and negative angles as clockwise.

Hence –115° is actually the same as 245° (i.e. 360° – 115°)

Hence $\sec(-115°) = \sec 245° = \dfrac{1}{\cos 245°}$

$= \mathbf{-2.3662}$

(b) $\text{cosec}\,(-95°47') = \dfrac{1}{\sin\left(-95\frac{47°}{60}\right)} = \mathbf{-1.0051}$

Problem 20. In triangle EFG in Fig. 11.11, calculate angle G.

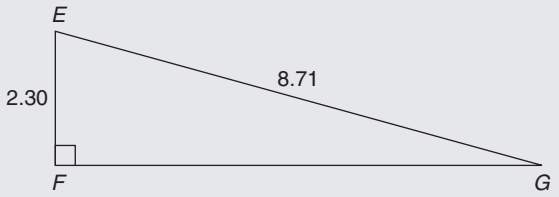


Figure 11.11

With reference to $\angle G$, the two sides of the triangle given are the opposite side EF and the hypotenuse EG; hence, sine is used,

i.e. $\sin G = \dfrac{2.30}{8.71} = 0.26406429\ldots$

from which, $G = \sin^{-1} 0.26406429\ldots$

i.e. $G = 15.311360\ldots$

Hence, $\angle \mathbf{G = 15.31°}$ or $\mathbf{15°19'}$

Now try the following exercise

Exercise 46 Further problems on evaluating trigonometric ratios

In Problems 1 to 8, evaluate correct to 4 decimal places:

1. (a) sine 27° (b) sine 172.41° (c) sine 302°52′

   [(a) 0.4540 (b) 0.1321 (c) −0.8399]

2. (a) cosine 124° (b) cosine 21.46° (c) cosine 284°10′

   [(a) −0.5592 (b) 0.9307 (c) 0.2447]

3. (a) tangent 145° (b) tangent 310.59° (c) tangent 49°16′

   [(a) −0.7002 (b) −1.1671 (c) 1.1612]

4. (a) secant 73° (b) secant 286.45° (c) secant 155°41′

   [(a) 3.4203 (b) 3.5313 (c) −1.0974]

5. (a) cosecant 213° (b) cosecant 15.62° (c) cosecant 311°50′

[(a) −1.8361 (b) 3.7139 (c) −1.3421]

6. (a) cotangent 71° (b) cotangent 151.62° (c) cotangent 321°23′

[(a) 0.3443 (b) −1.8510 (c) −1.2519]

7. (a) sine $\frac{2\pi}{3}$ (b) cos 1.681 (c) tan 3.672

[(a) 0.8660 (b) −0.1010 (c) 0.5865]

8. (a) sec $\frac{\pi}{8}$ (b) cosec 2.961 (c) cot 2.612

[(a) 1.0824 (b) 5.5675 (c) −1.7083]

In Problems 9 to 14, determine the acute angle in degrees (correct to 2 decimal places), degrees and minutes, and in radians (correct to 3 decimal places).

9. $\sin^{-1} 0.2341$ [13.54°, 13°32′, 0.236 rad]

10. $\cos^{-1} 0.8271$ [34.20°, 34°12′, 0.597 rad]

11. $\tan^{-1} 0.8106$ [39.03°, 39°2′, 0.681 rad]

12. $\sec^{-1} 1.6214$ [51.92°, 51°55′, 0.906 rad]

13. $\text{cosec}^{-1} 2.4891$ [23.69°, 23°41′, 0.413 rad]

14. $\cot^{-1} 1.9614$ [27.01°, 27°1′, 0.471 rad]

15. In the triangle shown in Fig. 11.12, determine angle θ, correct to 2 decimal places. [29.05°]

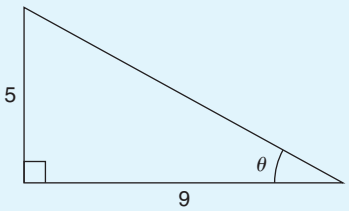


Figure 11.12

16. In the triangle shown in Fig. 11.13, determine angle θ in degrees and minutes. [20°21′]

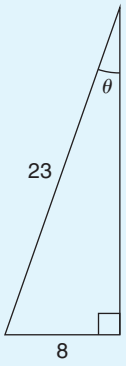


Figure 11.13

In Problems 17 to 20, evaluate correct to 4 significant figures.

17. $4\cos 56°19' - 3\sin 21°57'$ [1.097]

18. $\dfrac{11.5\tan 49°11' - \sin 90°}{3\cos 45°}$ [5.805]

19. $\dfrac{5\sin 86°3'}{3\tan 14°29' - 2\cos 31°9'}$ [−5.325]

20. $\dfrac{6.4\,\text{cosec}\, 29°5' - \sec 81°}{2\cot 12°}$ [0.7199]

21. Determine the acute angle, in degrees and minutes, correct to the nearest minute, given by $\sin^{-1}\left(\dfrac{4.32\sin 42°16'}{7.86}\right)$ [21°42′]

22. If $\tan x = 1.5276$, determine $\sec x$, $\text{cosec}\, x$, and $\cot x$. (Assume x is an acute angle) [1.8258, 1.1952, 0.6546]

In Problems 23 to 25 evaluate correct to 4 significant figures

23. $\dfrac{(\sin 34°27')(\cos 69°2')}{(2\tan 53°39')}$ [0.07448]

24. $3\cot 14°15'\sec 23°9'$ [12.85]

25. $\dfrac{\text{cosec}\, 27°19' + \sec 45°29'}{1 - \text{cosec}\, 27°19'\sec 45°29'}$ [−1.710]

26. Evaluate correct to 4 decimal places:
(a) sine (−125°) (b) tan(−241°)
(c) cos(−49°15′)

[(a) −0.8192 (b) −1.8040 (c) 0.6528]

27. Evaluate correct to 5 significant figures:
(a) cosec(−143°) (b) cot(−252°)
(c) sec(−67°22′)

[(a) −1.6616 (b) −0.32492
(c) 2.5985]

11.5 Solution of right-angled triangles

To 'solve a right-angled triangle' means 'to find the unknown sides and angles'. This is achieved by using (i) the theorem of Pythagoras, and/or (ii) trigonometric ratios. This is demonstrated in the following problems.

Problem 21. In triangle PQR shown in Fig. 11.14, find the lengths of PQ and PR.

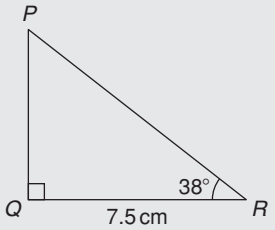


Figure 11.14

$$\tan 38° = \frac{PQ}{QR} = \frac{PQ}{7.5}$$

hence $PQ = 7.5 \tan 38° = 7.5(0.7813)$

$= \mathbf{5.860\ cm}$

$$\cos 38° = \frac{QR}{PR} = \frac{7.5}{PR}$$

hence $PR = \dfrac{7.5}{\cos 38°} = \dfrac{7.5}{0.7880} = \mathbf{9.518\ cm}$

[Check: Using Pythagoras' theorem

$(7.5)^2 + (5.860)^2 = 90.59 = (9.518)^2$]

Problem 22. Solve the triangle ABC shown in Fig. 11.15.

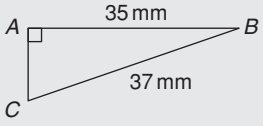


Figure 11.15

To 'solve triangle ABC' means 'to find the length AC and angles B and C'

$$\sin C = \frac{35}{37} = 0.94595$$

hence $\angle \boldsymbol{C} = \sin^{-1} 0.94595 = 71.08° = \mathbf{71°5'}$.

$\angle \boldsymbol{B} = 180° - 90° - 71°5' = \mathbf{18°55'}$ (since angles in a triangle add up to 180°)

$$\sin B = \frac{AC}{37}$$

hence $\boldsymbol{AC} = 37 \sin 18°55' = 37(0.3242)$

$= \mathbf{12.0\ mm}$

or, using Pythagoras' theorem, $37^2 = 35^2 + AC^2$, from which, $\boldsymbol{AC} = \sqrt{(37^2 - 35^2)} = \mathbf{12.0\ mm}$.

Problem 23. Solve triangle XYZ given $\angle X = 90°$, $\angle Y = 23°17'$ and $YZ = 20.0$ mm. Determine also its area.

It is always advisable to make a reasonably accurate sketch so as to visualize the expected magnitudes of unknown sides and angles. Such a sketch is shown in Fig. 11.16.

$$\angle Z = 180° - 90° - 23°17' = \mathbf{66°43'}$$

$$\sin 23°17' = \frac{XZ}{20.0}$$

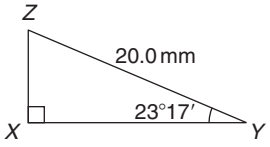


Figure 11.16

hence $\boldsymbol{XZ} = 20.0 \sin 23°17'$

$= 20.0(0.3953) = \mathbf{7.906\ mm}$

$$\cos 23°17' = \frac{XY}{20.0}$$

hence $\boldsymbol{XY} = 20.0 \cos 23°17'$

$= 20.0(0.9186) = \mathbf{18.37\ mm}$

[Check: Using Pythagoras' theorem

$(18.37)^2 + (7.906)^2 = 400.0 = (20.0)^2$]

Area of triangle XYZ

$= \frac{1}{2}$ (base) (perpendicular height)

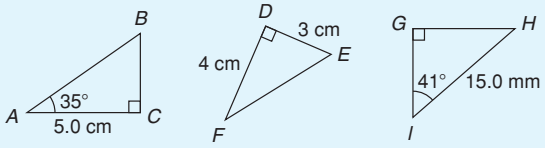
$= \frac{1}{2}(XY)(XZ) = \frac{1}{2}(18.37)(7.906)$

$= \mathbf{72.62\,mm^2}$

Now try the following exercise

Exercise 47 Further problems on the solution of right-angled triangles

1. Solve triangle ABC in Fig. 11.17(i). [$BC = 3.50\,\text{cm}$, $AB = 6.10\,\text{cm}$, $\angle B = 55°$]



(i) (ii) (iii)

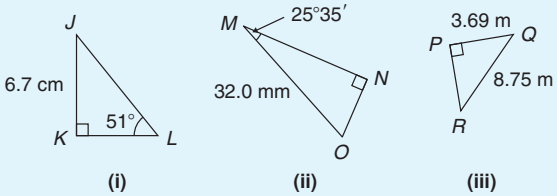
Figure 11.17

2. Solve triangle DEF in Fig. 11.17(ii). [$FE = 5\,\text{cm}$, $\angle E = 53°8'$, $\angle F = 36°52'$]

3. Solve triangle GHI in Fig. 11.17(iii). [$GH = 9.841\,\text{mm}$, $GI = 11.32\,\text{mm}$, $\angle H = 49°$]

4. Solve the triangle JKL in Fig. 11.18(i) and find its area. [$KL = 5.43\,\text{cm}$, $JL = 8.62\,\text{cm}$, $\angle J = 39°$, area $= 18.19\,\text{cm}^2$]

5. Solve the triangle MNO in Fig. 11.18(ii) and find its area. [$MN = 28.86\,\text{mm}$, $NO = 13.82\,\text{mm}$, $\angle O = 64°25'$, area $= 199.4\,\text{mm}^2$]



(i) (ii) (iii)

Figure 11.18

6. Solve the triangle PQR in Fig. 11.18(iii) and find its area. [$PR = 7.934\,\text{m}$, $\angle Q = 65°3'$, $\angle R = 24°57'$, area $= 14.64\,\text{m}^2$]

7. A ladder rests against the top of the perpendicular wall of a building and makes an angle of 73° with the ground. If the foot of the ladder is 2 m from the wall, calculate the height of the building. [6.54 m]

11.6 Angles of elevation and depression

(a) If, in Fig. 11.19, BC represents horizontal ground and AB a vertical flagpole, then the **angle of elevation** of the top of the flagpole, A, from the point C is the angle that the imaginary straight line AC must be raised (or elevated) from the horizontal CB, i.e. angle θ.

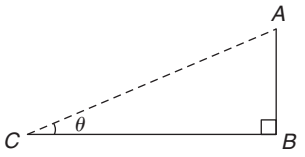


Figure 11.19

(b) If, in Fig. 11.20, PQ represents a vertical cliff and R a ship at sea, then the **angle of depression** of the ship from point P is the angle through which the imaginary straight line PR must be lowered (or depressed) from the horizontal to the ship, i.e. angle ϕ.

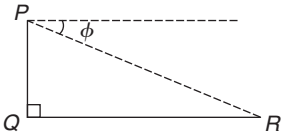


Figure 11.20

(Note, $\angle PRQ$ is also ϕ—alternate angles between parallel lines.)

Problem 24. An electricity pylon stands on horizontal ground. At a point 80 m from the base of the pylon, the angle of elevation of the top of the pylon is 23°. Calculate the height of the pylon to the nearest metre.

Figure 11.21 shows the pylon AB and the angle of elevation of A from point C is 23°

$$\tan 23° = \frac{AB}{BC} = \frac{AB}{80}$$

Hence height of pylon AB

$= 80\tan 23° = 80(0.4245) = 33.96\,\text{m}$

$= \mathbf{34\,m\ to\ the\ nearest\ metre}$.

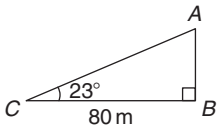


Figure 11.21

Problem 25. A surveyor measures the angle of elevation of the top of a perpendicular building as 19°. He moves 120 m nearer the building and finds the angle of elevation is now 47°. Determine the height of the building.

The building PQ and the angles of elevation are shown in Fig. 11.22.

In triangle PQS,

$$\tan 19° = \frac{h}{x+120}$$

hence $h = \tan 19°(x+120)$,

i.e. $h = 0.3443(x+120)$ (1)

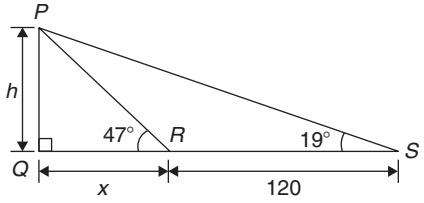


Figure 11.22

In triangle PQR, $\tan 47° = \dfrac{h}{x}$

hence $h = \tan 47°(x)$, i.e. $h = 1.0724x$ (2)

Equating equations (1) and (2) gives:

$$0.3443(x+120) = 1.0724x$$

$$0.3443x + (0.3443)(120) = 1.0724x$$

$$(0.3443)(120) = (1.0724 - 0.3443)x$$

$$41.316 = 0.7281x$$

$$x = \frac{41.316}{0.7281} = 56.74\,\text{m}$$

From equation (2), **height of building**,

$\boldsymbol{h} = 1.0724x = 1.0724(56.74) = \mathbf{60.85\,m}$.

Problem 26. The angle of depression of a ship viewed at a particular instant from the top of a 75 m vertical cliff is 30°. Find the distance of the ship from the base of the cliff at this instant. The ship is sailing away from the cliff at constant speed and 1 minute later its angle of depression from the top of the cliff is 20°. Determine the speed of the ship in km/h.

Figure 11.23 shows the cliff AB, the initial position of the ship at C and the final position at D. Since the angle of depression is initially 30° then $\angle ACB = 30°$ (alternate angles between parallel lines).

$$\tan 30° = \frac{AB}{BC} = \frac{75}{BC}$$

hence $BC = \dfrac{75}{\tan 30°} = \dfrac{75}{0.5774} = \mathbf{129.9\,m}$

$= \mathbf{initial\ position\ of\ ship\ from\ base\ of\ cliff}$

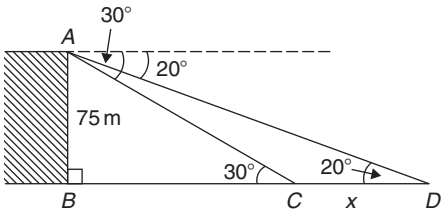


Figure 11.23

In triangle ABD,

$$\tan 20° = \frac{AB}{BD} = \frac{75}{BC+CD}$$

$$= \frac{75}{129.9+x}$$

Hence $129.9 + x = \dfrac{75}{\tan 20°} = \dfrac{75}{0.3640}$

$= 206.0\,\text{m}$

from which $x = 206.0 - 129.9 = 76.1\,\text{m}$

Thus the ship sails 76.1 m in 1 minute, i.e. 60 s, hence speed of ship

$$= \frac{\text{distance}}{\text{time}} = \frac{76.1}{60}\ \text{m/s}$$

$$= \frac{76.1 \times 60 \times 60}{60 \times 1000}\ \text{km/h} = \mathbf{4.57\,km/h}$$

Now try the following exercise

Exercise 48 Further problems on angles of elevation and depression

1. If the angle of elevation of the top of a vertical 30 m high aerial is 32°, how far is it to the aerial? [48 m]
2. From the top of a vertical cliff 80.0 m high the angles of depression of two buoys lying due west of the cliff are 23° and 15°, respectively. How far are the buoys apart? [110.1 m]
3. From a point on horizontal ground a surveyor measures the angle of elevation of the top of a flagpole as 18°40′. He moves 50 m nearer to the flagpole and measures the angle of elevation as 26°22′. Determine the height of the flagpole. [53.0 m]
4. A flagpole stands on the edge of the top of a building. At a point 200 m from the building the angles of elevation of the top and bottom of the pole are 32° and 30° respectively. Calculate the height of the flagpole. [9.50 m]
5. From a ship at sea, the angles of elevation of the top and bottom of a vertical lighthouse standing on the edge of a vertical cliff are 31° and 26°, respectively. If the lighthouse is 25.0 m high, calculate the height of the cliff. [107.8 m]
6. From a window 4.2 m above horizontal ground the angle of depression of the foot of a building across the road is 24° and the angle of elevation of the top of the building is 34°. Determine, correct to the nearest centimetre, the width of the road and the height of the building. [9.43 m, 10.56 m]
7. The elevation of a tower from two points, one due east of the tower and the other due west of it are 20° and 24°, respectively, and the two points of observation are 300 m apart. Find the height of the tower to the nearest metre. [60 m]

11.7 Sine and cosine rules

To **'solve a triangle'** means 'to find the values of unknown sides and angles'. If a triangle is **right angled**, trigonometric ratios and the theorem of Pythagoras may be used for its solution, as shown in Section 11.5. However, for a **non-right-angled triangle**, trigonometric ratios and Pythagoras' theorem **cannot** be used. Instead, two rules, called the **sine rule** and the **cosine rule**, are used.

Sine rule

With reference to triangle ABC of Fig. 11.24, the **sine rule** states:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

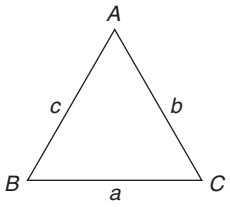


Figure 11.24

The rule may be used only when:

(i) 1 side and any 2 angles are initially given, or

(ii) 2 sides and an angle (not the included angle) are initially given.

Cosine rule

With reference to triangle ABC of Fig. 11.24, the **cosine rule** states:

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$\text{or } b^2 = a^2 + c^2 - 2ac\cos B$$
$$\text{or } c^2 = a^2 + b^2 - 2ab\cos C$$

The rule may be used only when:

(i) 2 sides and the included angle are initially given, or

(ii) 3 sides are initially given.

11.8 Area of any triangle

The **area of any triangle** such as ABC of Fig. 11.24 is given by:

(i) $\frac{1}{2}$ × base × perpendicular height, or

(ii) $\frac{1}{2}ab\sin C$ or $\frac{1}{2}ac\sin B$ or $\frac{1}{2}bc\sin A$, or

(iii) $\sqrt{[s(s-a)(s-b)(s-c)]}$, where $s=\frac{a+b+c}{2}$

11.9 Worked problems on the solution of triangles and finding their areas

Problem 27. In a triangle XYZ, $\angle X=51°$, $\angle Y=67°$ and $YZ=15.2$ cm. Solve the triangle and find its area.

The triangle XYZ is shown in Fig. 11.25. Since the angles in a triangle add up to 180°, then $Z=180°-51°-67°=\mathbf{62°}$. Applying the sine rule:

$$\frac{15.2}{\sin 51°}=\frac{y}{\sin 67°}=\frac{z}{\sin 62°}$$

Using $\frac{15.2}{\sin 51°}=\frac{y}{\sin 67°}$ and transposing gives:

$$y=\frac{15.2\ \sin 67°}{\sin 51°}=\mathbf{18.00\,cm}=\boldsymbol{XZ}$$

Using $\frac{15.2}{\sin 51°}=\frac{z}{\sin 62°}$ and transposing gives:

$$z=\frac{15.2\sin 62°}{\sin 51°}=\mathbf{17.27\,cm}=\boldsymbol{XY}$$

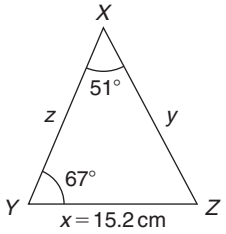


Figure 11.25

Area of triangle XYZ $=\frac{1}{2}xy\sin Z$
$=\frac{1}{2}(15.2)(18.00)\sin 62°=\mathbf{120.8\,cm^2}$ (or area
$=\frac{1}{2}xz\sin Y=\frac{1}{2}(15.2)(17.27)\sin 67°=\mathbf{120.8\,cm^2}$).

It is always worth checking with triangle problems that the longest side is opposite the largest angle, and vice-versa. In this problem, Y is the largest angle and XZ is the longest of the three sides.

Problem 28. Solve the triangle PQR and find its area given that $QR=36.5$ mm, $PR=29.6$ mm and $\angle Q=36°$.

Triangle PQR is shown in Fig. 11.26.

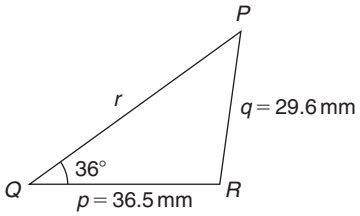


Figure 11.26

Applying the sine rule:

$$\frac{29.6}{\sin 36°}=\frac{36.5}{\sin P}$$

from which,

$$\sin P=\frac{36.5\sin 36°}{29.6}=0.7248$$

Hence $P=\sin^{-1}0.7248=46°27'$ or $133°33'$.
When $P=46°27'$ and $Q=36°$ then
$R=180°-46°27'-36°=97°33'$.
When $P=133°33'$ and $Q=36°$ then
$R=180°-133°33'-36°=10°27'$.

Thus, in this problem, there are **two** separate sets of results and both are feasible solutions. Such a situation is called the **ambiguous case**.

Case 1. $P=46°27'$, $Q=36°$, $R=97°33'$, $p=36.5$ mm and $q=29.6$ mm.
From the sine rule:

$$\frac{r}{\sin 97°33'}=\frac{29.6}{\sin 36°}$$

from which,

$$r=\frac{29.6\sin 97°33'}{\sin 36°}=\mathbf{49.92\ mm}$$

$$\textbf{Area}=\tfrac{1}{2}pq\sin R=\tfrac{1}{2}(36.5)(29.6)\sin 97°33'$$
$$=\mathbf{535.5\,mm^2}$$

Case 2. $P=133°33'$, $Q=36°$, $R=10°27'$, $p=36.5$ mm and $q=29.6$ mm.

From the sine rule:

$$\frac{r}{\sin 10°27'} = \frac{29.6}{\sin 36°}$$

from which,

$$r = \frac{29.6 \sin 10°27'}{\sin 36°} = \mathbf{9.134\,mm}$$

$$\textbf{Area} = \tfrac{1}{2}pq \sin R = \tfrac{1}{2}(36.5)(29.6)\sin 10°27'$$
$$= \mathbf{97.98\,mm^2}.$$

Triangle PQR for case 2 is shown in Fig. 11.27.

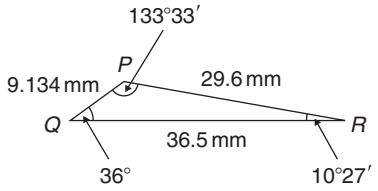


Figure 11.27

Now try the following exercise

Exercise 49 Further problems on solving triangles and finding their areas

In Problems 1 and 2, use the sine rule to solve the triangles ABC and find their areas.

1. $A = 29°$, $B = 68°$, $b = 27\,mm$.
$$\left[\begin{array}{l} C = 83°, a = 14.1\,mm, \\ c = 28.9\,mm, \text{area} = 189\,mm^2 \end{array}\right]$$

2. $B = 71°26'$, $C = 56°32'$, $b = 8.60\,cm$.
$$\left[\begin{array}{l} A = 52°2', c = 7.568\,cm, \\ a = 7.152\,cm, \text{area} = 25.65\,cm^2 \end{array}\right]$$

In Problems 3 and 4, use the sine rule to solve the triangles DEF and find their areas.

3. $d = 17\,cm$, $f = 22\,cm$, $F = 26°$.
$$\left[\begin{array}{l} D = 19°48', E = 134°12', \\ e = 36.0\,cm, \text{area} = 134\,cm^2 \end{array}\right]$$

4. $d = 32.6\,mm$, $e = 25.4\,mm$, $D = 104°22'$.
$$\left[\begin{array}{l} E = 49°0', F = 26°38', \\ f = 15.09\,mm, \text{area} = 185.6\,mm^2 \end{array}\right]$$

In Problems 5 and 6, use the sine rule to solve the triangles JKL and find their areas.

5. $j = 3.85\,cm$, $k = 3.23\,cm$, $K = 36°$.
$$\left[\begin{array}{l} J = 44°29', L = 99°31', \\ l = 5.420\,cm, \text{area} = 6.132\,cm^2 \text{ or} \\ J = 135°31', L = 8°29', \\ l = 0.811\,cm, \text{area} = 0.917\,cm^2 \end{array}\right]$$

6. $k = 46\,mm$, $l = 36\,mm$, $L = 35°$.
$$\left[\begin{array}{l} K = 47°8', J = 97°52', \\ j = 62.2\,mm, \text{area} = 820.2\,mm^2 \text{ or} \\ K = 132°52', J = 12°8', \\ j = 13.19\,mm, \text{area} = 174.0\,mm^2 \end{array}\right]$$

11.10 Further worked problems on solving triangles and finding their areas

Problem 29. Solve triangle DEF and find its area given that $EF = 35.0\,mm$, $DE = 25.0\,mm$ and $\angle E = 64°$.

Triangle DEF is shown in Fig. 11.28.

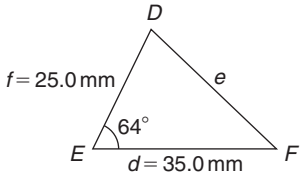


Figure 11.28

Applying the cosine rule:

$$e^2 = d^2 + f^2 - 2d\,f\cos E$$

i.e. $e^2 = (35.0)^2 + (25.0)^2$
$$- [2(35.0)(25.0)\cos 64°]$$
$$= 1225 + 625 - 767.1 = 1083$$

from which, $\boldsymbol{e} = \sqrt{1083} = \mathbf{32.91\,mm}$

Applying the sine rule:

$$\frac{32.91}{\sin 64°} = \frac{25.0}{\sin F}$$

from which, $\sin F = \dfrac{25.0 \sin 64°}{32.91} = 0.6828$

Thus $\angle F = \sin^{-1} 0.6828$
$$= 43°4' \text{ or } 136°56'$$

$F = 136°56'$ is not possible in this case since $136°56' + 64°$ is greater than $180°$. Thus only $\boldsymbol{F = 43°4'}$ is valid

$$\angle \boldsymbol{D} = 180° - 64° - 43°4' = \boldsymbol{72°56'}$$

Area of triangle $DEF = \frac{1}{2}\,d\,f \sin E$

$$= \tfrac{1}{2}(35.0)(25.0)\sin 64° = \mathbf{393.2\,mm^2}.$$

Problem 30. A triangle ABC has sides $a = 9.0\,\text{cm}$, $b = 7.5\,\text{cm}$ and $c = 6.5\,\text{cm}$. Determine its three angles and its area.

Triangle ABC is shown in Fig. 11.29. It is usual first to calculate the largest angle to determine whether the triangle is acute or obtuse. In this case the largest angle is A (i.e. opposite the longest side).

Applying the cosine rule:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

from which, $2bc\cos A = b^2 + c^2 - a^2$

and $$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{7.5^2 + 6.5^2 - 9.0^2}{2(7.5)(6.5)}$$

$$= 0.1795$$

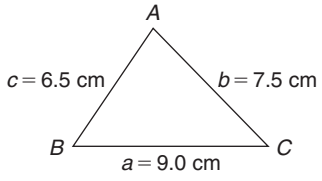


Figure 11.29

Hence $A = \cos^{-1} 0.1795 = \boldsymbol{79°40'}$ (or $280°20'$, which is obviously impossible). The triangle is thus acute angled since $\cos A$ is positive. (If $\cos A$ had been negative, angle A would be obtuse, i.e. lie between $90°$ and $180°$).

Applying the sine rule:

$$\frac{9.0}{\sin 79°40'} = \frac{7.5}{\sin B}$$

from which,

$$\sin B = \frac{7.5\sin 79°40'}{9.0} = 0.8198$$

Hence $\boldsymbol{B} = \sin^{-1} 0.8198 = \boldsymbol{55°4'}$

and $\boldsymbol{C} = 180° - 79°40' - 55°4' = \boldsymbol{45°16'}$

Area $= \sqrt{[s(s-a)(s-b)(s-c)]}$,

where $$s = \frac{a+b+c}{2} = \frac{9.0 + 7.5 + 6.5}{2}$$

$$= 11.5\,\text{cm}$$

Hence **area**

$$= \sqrt{[11.5(11.5-9.0)(11.5-7.5)(11.5-6.5)]}$$

$$= \sqrt{[11.5(2.5)(4.0)(5.0)]} = \mathbf{23.98\,cm^2}$$

Alternatively, area $= \frac{1}{2}ab\sin C$

$= \frac{1}{2}(9.0)(7.5)\sin 45°16' = \mathbf{23.98\,cm^2}$.

Now try the following exercise

Exercise 50 Further problems on solving triangles and finding their areas

In Problems 1 and 2, use the cosine and sine rules to solve the triangles PQR and find their areas.

1. $q = 12\,\text{cm}, r = 16\,\text{cm}, P = 54°$.
   $[\,p = 13.2\,\text{cm}, Q = 47°21',$ $R = 78°39',$ area $= 77.7\,\text{cm}^2\,]$
2. $q = 3.25\,\text{m}, r = 4.42\,\text{m}, P = 105°$.
   $[\,p = 6.127\,\text{m}, Q = 30°50',$ $R = 44°10',$ area $= 6.938\,\text{m}^2\,]$

In problems 3 and 4, use the cosine and sine rules to solve the triangles XYZ and find their areas.

3. $x = 10.0\,\text{cm}, y = 8.0\,\text{cm}, z = 7.0\,\text{cm}$.
   $[\,X = 83°20', Y = 52°37',$ $Z = 44°3',$ area $= 27.8\,\text{cm}^2\,]$
4. $x = 21\,\text{mm}, y = 34\,\text{mm}, z = 42\,\text{mm}$.
   $[\,X = 29°46', Y = 53°30',$ $Z = 96°44',$ area $= 355\,\text{mm}^2\,]$

11.11 Practical situations involving trigonometry

There are a number of **practical situations** where the use of trigonometry is needed to find unknown sides and angles of triangles. This is demonstrated in the following problems.

Problem 31. A room 8.0 m wide has a span roof which slopes at 33° on one side and 40° on the other. Find the length of the roof slopes, correct to the nearest centimetre.

A section of the roof is shown in Fig. 11.30.

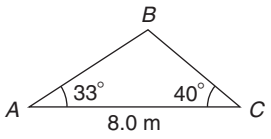


Figure 11.30

Angle at ridge, $B = 180° - 33° - 40° = 107°$
From the sine rule:

$$\frac{8.0}{\sin 107°} = \frac{a}{\sin 33°}$$

from which,

$$a = \frac{8.0 \sin 33°}{\sin 107°} = 4.556\,\text{m}$$

Also from the sine rule:

$$\frac{8.0}{\sin 107°} = \frac{c}{\sin 40°}$$

from which,

$$c = \frac{8.0 \sin 40°}{\sin 107°} = 5.377\,\text{m}$$

Hence the roof slopes are 4.56 m and 5.38 m, correct to the nearest centimetre.

Problem 32. Two voltage phasors are shown in Fig. 11.31. If $V_1 = 40\,\text{V}$ and $V_2 = 100\,\text{V}$ determine the value of their resultant (i.e. length OA) and the angle the resultant makes with V_1.

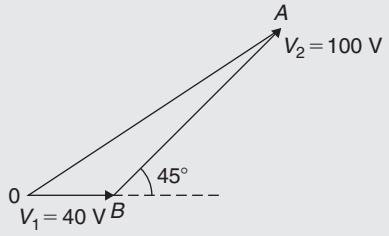


Figure 11.31

Angle $OBA = 180° - 45° = 135°$

Applying the cosine rule:

$$\begin{aligned} OA^2 &= V_1^2 + V_2^2 - 2V_1V_2\cos OBA \\ &= 40^2 + 100^2 - \{2(40)(100)\cos 135°\} \\ &= 1600 + 10000 - \{-5657\} \\ &= 1600 + 10000 + 5657 = 17257 \end{aligned}$$

The resultant

$$OA = \sqrt{(17257)} = 131.4\,\text{V}$$

Applying the sine rule:

$$\frac{131.4}{\sin 135°} = \frac{100}{\sin AOB}$$

from which, $\sin AOB = \dfrac{100 \sin 135°}{131.4}$

$$= 0.5381$$

Hence angle $AOB = \sin^{-1} 0.5381 = 32°33'$ (or $147°27'$, which is impossible in this case).

Hence the resultant voltage is 131.4 volts at 32°33′ to V_1.

Problem 33. In Fig. 11.32, PR represents the inclined jib of a crane and is 10.0 long. PQ is 4.0 m long. Determine the inclination of the jib to the vertical and the length of tie QR.

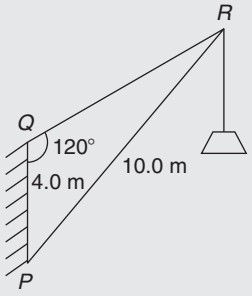


Figure 11.32

Applying the sine rule:

$$\frac{PR}{\sin 120°} = \frac{PQ}{\sin R}$$

from which,

$$\sin R = \frac{PQ \sin 120°}{PR} = \frac{(4.0)\sin 120°}{10.0}$$
$$= 0.3464$$

Hence $\angle R = \sin^{-1} 0.3464 = 20°16'$ (or $159°44'$, which is impossible in this case).
$\angle \boldsymbol{P} = 180° - 120° - 20°16' = \mathbf{39°44'}$, **which is the inclination of the jib to the vertical**.

Applying the sine rule:

$$\frac{10.0}{\sin 120°} = \frac{QR}{\sin 39°44'}$$

from which, **length of tie**,

$$QR = \frac{10.0 \sin 39°44'}{\sin 120°} = \mathbf{7.38\,m}$$

Now try the following exercise

Exercise 51 Further problems on practical situations involving trigonometry

1. A ship P sails at a steady speed of 45 km/h in a direction of W 32° N (i.e. a bearing of 302°) from a port. At the same time another ship Q leaves the port at a steady speed of 35 km/h in a direction N 15° E (i.e. a bearing of 015°). Determine their distance apart after 4 hours.

   [193 km]

2. Two sides of a triangular plot of land are 52.0 m and 34.0 m, respectively. If the area of the plot is 620 m^2 find (a) the length of fencing required to enclose the plot and (b) the angles of the triangular plot.

   [(a) 122.6 m (b) 94°49′, 40°39′, 44°32′]

3. A jib crane is shown in Fig. 11.33. If the tie rod PR is 8.0 long and PQ is 4.5 m long determine (a) the length of jib RQ and (b) the angle between the jib and the tie rod.

   [(a) 11.4 m (b) 17°33′]

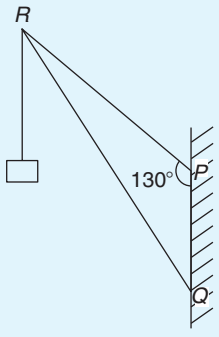


Figure 11.33

4. A building site is in the form of a quadrilateral as shown in Fig. 11.34, and its area is 1510 m^2. Determine the length of the perimeter of the site. [163.4 m]

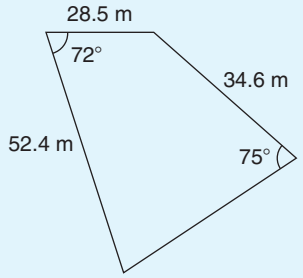


Figure 11.34

5. Determine the length of members BF and EB in the roof truss shown in Fig. 11.35.

   [$BF = 3.9$ m, $EB = 4.0$ m]

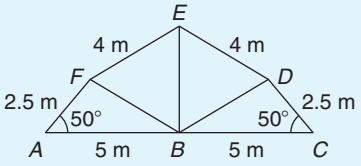


Figure 11.35

6. A laboratory 9.0 m wide has a span roof which slopes at 36° on one side and 44° on the other. Determine the lengths of the roof slopes. [6.35 m, 5.37 m]

11.12 Further practical situations involving trigonometry

Problem 34. A vertical aerial stands on horizontal ground. A surveyor positioned due east of the aerial measures the elevation of the top as 48°. He moves due south 30.0 m and measures the elevation as 44°. Determine the height of the aerial.

In Fig. 11.36, DC represents the aerial, A is the initial position of the surveyor and B his final position.

From triangle ACD, $\tan 48° = \dfrac{DC}{AC}$,

from which $$AC = \frac{DC}{\tan 48°}$$

Similarly, from triangle BCD,

$$BC = \frac{DC}{\tan 44°}$$

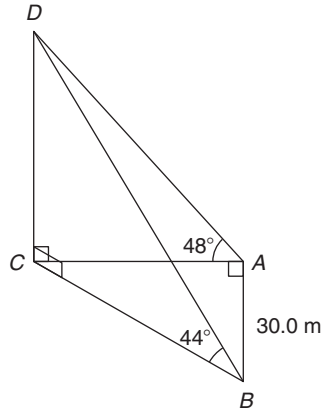


Figure 11.36

For triangle ABC, using Pythagoras' theorem:

$$BC^2 = AB^2 + AC^2$$

$$\left(\frac{DC}{\tan 44°}\right)^2 = (30.0)^2 + \left(\frac{DC}{\tan 48°}\right)^2$$

$$DC^2\left(\frac{1}{\tan^2 44°} - \frac{1}{\tan^2 48°}\right) = 30.0^2$$

$$DC^2(1.072323 - 0.810727) = 30.0^2$$

$$DC^2 = \frac{30.0^2}{0.261596} = 3440.4$$

Hence, height of aerial,

$$\boldsymbol{DC = \sqrt{3440.4} = 58.65\,\text{m}}$$

Problem 35. A crank mechanism of a petrol engine is shown in Fig. 11.37. Arm OA is 10.0 cm long and rotates clockwise about O. The connecting rod AB is 30.0 cm long and end B is constrained to move horizontally.

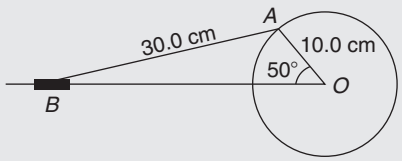


Figure 11.37

(a) For the position shown in Fig. 11.37 determine the angle between the connecting rod AB and the horizontal and the length of OB.

(b) How far does B move when angle AOB changes from 50° to 120°?

(a) Applying the sine rule:

$$\frac{AB}{\sin 50°} = \frac{AO}{\sin B}$$

from which,

$$\sin B = \frac{AO \sin 50°}{AB} = \frac{10.0 \sin 50°}{30.0}$$
$$= 0.2553$$

Hence $B = \sin^{-1} 0.2553 = 14°47'$ (or $165°13'$, which is impossible in this case).

Hence the connecting rod AB makes an angle of 14°47′ with the horizontal.

Angle $OAB = 180° - 50° - 14°47' = 115°13'$.

Applying the sine rule:

$$\frac{30.0}{\sin 50°} = \frac{OB}{\sin 115°13'}$$

from which,

$$\boldsymbol{OB = \frac{30.0 \sin 115°13'}{\sin 50°} = 35.43\,\text{cm}}$$

(b) Figure 11.38 shows the initial and final positions of the crank mechanism. In triangle $OA'B'$, applying the sine rule:

$$\frac{30.0}{\sin 120°} = \frac{10.0}{\sin A'B'O}$$

from which,

$$\sin A'B'O = \frac{10.0 \sin 120°}{30.0} = 0.2887$$

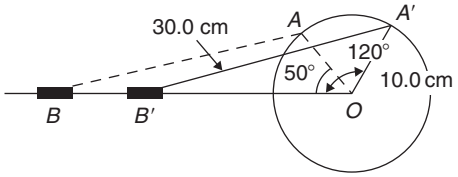


Figure 11.38

Hence $A'B'O = \sin^{-1} 0.2887 = 16°47'$ (or $163°13'$ which is impossible in this case).

Angle $OA'B' = 180° - 120° - 16°47' = 43°13'$.

Applying the sine rule:

$$\frac{30.0}{\sin 120°} = \frac{OB'}{\sin 43°13'}$$

from which,

$$OB' = \frac{30.0 \sin 43°13'}{\sin 120°} = 23.72\,\text{cm}$$

Since $OB = 35.43$ cm and $OB' = 23.72$ cm then $BB' = 35.43 - 23.72 = 11.71$ cm.

Hence B moves 11.71 cm when angle AOB changes from 50° to 120°.

Problem 36. The area of a field is in the form of a quadrilateral $ABCD$ as shown in Fig. 11.39. Determine its area.

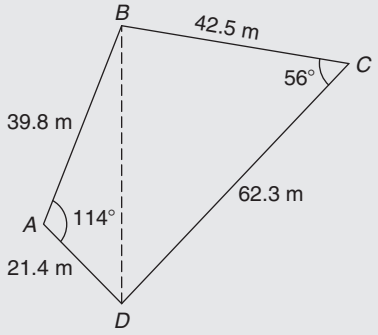


Figure 11.39

A diagonal drawn from B to D divides the quadrilateral into two triangles.

Area of quadrilateral ABCD

$$= \text{area of triangle } ABD + \text{area of triangle } BCD$$

$$= \tfrac{1}{2}(39.8)(21.4)\sin 114° + \tfrac{1}{2}(42.5)(62.3)\sin 56°$$

$$= 389.04 + 1097.5 = \mathbf{1487\,m^2}$$

Now try the following exercise

Exercise 52 Further problems on practical situations involving trigonometry

1. PQ and QR are the phasors representing the alternating currents in two branches of a circuit. Phasor PQ is 20.0 A and is horizontal. Phasor QR (which is joined to the end of PQ to form triangle PQR) is 14.0 A and is at an angle of 35° to the horizontal. Determine the resultant phasor PR and the angle it makes with phasor PQ. [32.48 A, 14°19′]

2. Three forces acting on a fixed point are represented by the sides of a triangle of dimensions 7.2 cm, 9.6 cm and 11.0 cm. Determine the angles between the lines of action and the three forces. [80°25′, 59°23′, 40°12′]

3. Calculate, correct to 3 significant figures, the co-ordinates x and y to locate the hole centre at P shown in Fig. 11.40. [$x = 69.3$ mm, $y = 142$ mm]

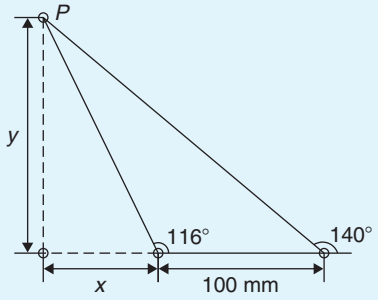


Figure 11.40

4. An idler gear, 30 mm in diameter, has to be fitted between a 70 mm diameter driving gear and a 90 mm diameter driven gear as shown in Fig. 11.41. Determine the value of angle θ between the center lines. [130°]

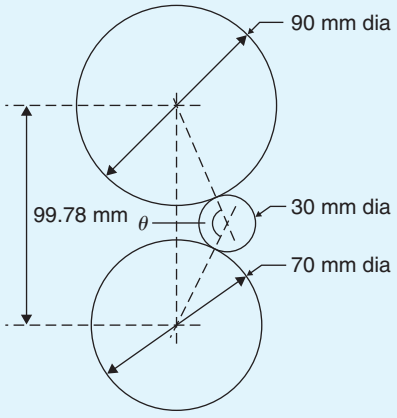


Figure 11.41

5. A reciprocating engine mechanism is shown in Fig. 11.42. The crank AB is 12.0 cm long and the connecting rod BC is 32.0 cm long.

For the position shown determine the length of AC and the angle between the crank and the connecting rod. [40.25 cm, 126°3′]

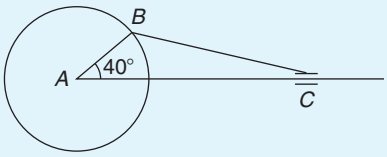


Figure 11.42

6. From Fig. 11.42, determine how far C moves, correct to the nearest millimetre when angle CAB changes from 40° to 160°, B moving in an anticlockwise direction. [19.8 cm]

7. A surveyor, standing W 25° S of a tower measures the angle of elevation of the top of the tower as 46°30′. From a position E 23° S from the tower the elevation of the top is 37°15′. Determine the height of the tower if the distance between the two observations is 75 m. [36.2 m]

8. An aeroplane is sighted due east from a radar station at an elevation of 40° and a height of 8000 m and later at an elevation of 35° and height 5500 m in a direction E 70° S. If it is descending uniformly, find the angle of descent. Determine also the speed of the aeroplane in km/h if the time between the two observations is 45 s. [13°57′, 829.9 km/h]

9. Sixteen holes are equally spaced on a pitch circle of 70 mm diameter. Determine the length of the chord joining the centres of two adjacent holes. [13.66 mm]

Chapter 12

Cartesian and polar co-ordinates

12.1 Introduction

There are two ways in which the position of a point in a plane can be represented. These are

(a) by **Cartesian co-ordinates**, i.e. (x, y), and

(b) by **polar co-ordinates**, i.e. (r, θ), where r is a 'radius' from a fixed point and θ is an angle from a fixed point.

12.2 Changing from Cartesian into polar co-ordinates

In Fig. 12.1, if lengths x and y are known, then the length of r can be obtained from Pythagoras' theorem (see Chapter 11) since OPQ is a right-angled triangle. Hence $r^2 = (x^2 + y^2)$

from which, $\boldsymbol{r = \sqrt{x^2 + y^2}}$

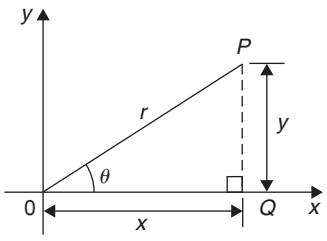


Figure 12.1

From trigonometric ratios (see Chapter 11),

$$\tan\theta = \frac{y}{x}$$

from which $\boldsymbol{\theta = \tan^{-1}\frac{y}{x}}$

$r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$ are the two formulae we need to change from Cartesian to polar co-ordinates. The angle θ, which may be expressed in degrees or radians, must **always** be measured from the positive x-axis, i.e., measured from the line OQ in Fig. 12.1. It is suggested that when changing from Cartesian to polar co-ordinates a diagram should always be sketched.

Problem 1. Change the Cartesian co-ordinates (3, 4) into polar co-ordinates.

A diagram representing the point (3, 4) is shown in Fig. 12.2.

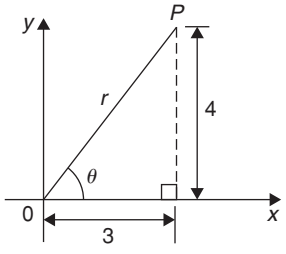


Figure 12.2

From Pythagoras' theorem, $r=\sqrt{3^2+4^2}=5$ (note that -5 has no meaning in this context). By trigonometric ratios, $\theta=\tan^{-1}\frac{4}{3}=53.13°$ or 0.927 rad.

[note that $53.13°=53.13\times(\pi/180)\,\text{rad}=0.927\,\text{rad}$]

Hence (3, 4) in Cartesian co-ordinates corresponds to (5, 53.13°) or (5, 0.927 rad) in polar co-ordinates.

Problem 2. Express in polar co-ordinates the position $(-4, 3)$.

A diagram representing the point using the Cartesian co-ordinates $(-4, 3)$ is shown in Fig. 12.3.

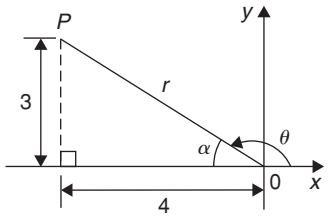


Figure 12.3

From Pythagoras' theorem, $r=\sqrt{4^2+3^2}=5$.

By trigonometric ratios, $\alpha=\tan^{-1}\frac{3}{4}=36.87°$ or 0.644 rad.

Hence $\theta=180°-36.87°=143.13°$ or $\theta=\pi-0.644=2.498$ rad.

Hence the position of point *P* in polar co-ordinate form is (5, 143.13°) or (5, 2.498 rad).

Problem 3. Express $(-5, -12)$ in polar co-ordinates.

A sketch showing the position $(-5, -12)$ is shown in Fig. 12.4.

$$r=\sqrt{5^2+12^2}=13$$

and $$\alpha=\tan^{-1}\frac{12}{5}$$

$$=67.38°\text{ or }1.176\,\text{rad}$$

Hence $$\theta=180°+67.38°=247.38°\text{ or}$$

$$\theta=\pi+1.176=4.318\,\text{rad}$$

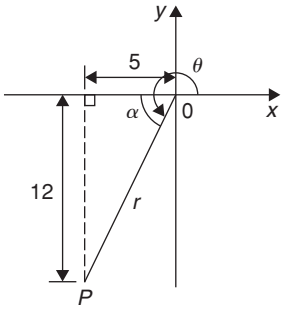


Figure 12.4

Thus (−5, −12) in Cartesian co-ordinates corresponds to (13, 247.38°) or (13, 4.318 rad) in polar co-ordinates.

Problem 4. Express $(2, -5)$ in polar co-ordinates.

A sketch showing the position $(2, -5)$ is shown in Fig. 12.5.

$$r=\sqrt{2^2+5^2}=\sqrt{29}=5.385\text{ correct to 3 decimal places}$$

$$\alpha=\tan^{-1}\frac{5}{2}=68.20°\text{ or }1.190\,\text{rad}$$

Hence $$\theta=360°-68.20°=291.80°\text{ or}$$

$$\theta=2\pi-1.190=5.093\,\text{rad}$$

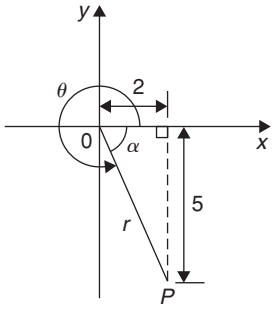


Figure 12.5

Thus (2, −5) in Cartesian co-ordinates corresponds to (5.385, 291.80°) or (5.385, 5.093 rad) in polar co-ordinates.

Now try the following exercise

Exercise 53 Further problems on changing from Cartesian into polar co-ordinates

In Problems 1 to 8, express the given Cartesian co-ordinates as polar co-ordinates, correct to 2 decimal places, in both degrees and in radians.

1. (3, 5) [(5.83, 59.04°) or (5.83, 1.03 rad)]
2. (6.18, 2.35) [(6.61, 20.82°) or (6.61, 0.36 rad)]
3. (−2, 4) [(4.47, 116.57°) or (4.47, 2.03 rad)]
4. (−5.4, 3.7) [(6.55, 145.58°) or (6.55, 2.54 rad)]
5. (−7, −3) [(7.62, 203.20°) or (7.62, 3.55 rad)]
6. (−2.4, −3.6) [(4.33, 236.31°) or (4.33, 4.12 rad)]
7. (5, −3) [(5.83, 329.04°) or (5.83, 5.74 rad)]
8. (9.6, −12.4) [(15.68, 307.75°) or (15.68, 5.37 rad)]

12.3 Changing from polar into Cartesian co-ordinates

From the right-angled triangle OPQ in Fig. 12.6.

$$\cos\theta = \frac{x}{r} \text{ and } \sin\theta = \frac{y}{r}, \text{ from}$$

trigonometric ratios

Hence $x = r\cos\theta$ and $y = r\sin\theta$

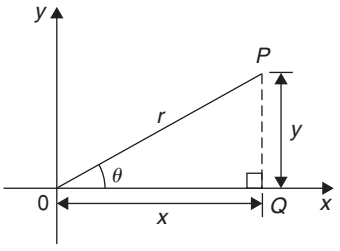


Figure 12.6

If lengths r and angle θ are known then $x = r\cos\theta$ and $y = r\sin\theta$ are the two formulae we need to change from polar to Cartesian co-ordinates.

Problem 5. Change (4, 32°) into Cartesian co-ordinates.

A sketch showing the position (4, 32°) is shown in Fig. 12.7.

Now $x = r\cos\theta = 4\cos 32° = 3.39$

and $y = r\sin\theta = 4\sin 32° = 2.12$

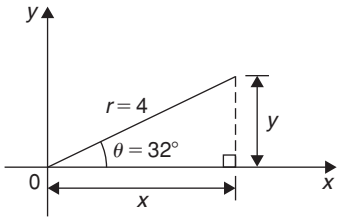


Figure 12.7

Hence (4, 32°) in polar co-ordinates corresponds to (3.39, 2.12) in Cartesian co-ordinates.

Problem 6. Express (6, 137°) in Cartesian co-ordinates.

A sketch showing the position (6, 137°) is shown in Fig. 12.8.

$$x = r\cos\theta = 6\cos 137° = -4.388$$

which corresponds to length OA in Fig. 12.8.

$$y = r\sin\theta = 6\sin 137° = 4.092$$

which corresponds to length AB in Fig. 12.8.

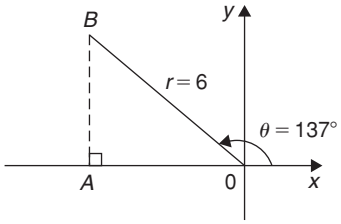


Figure 12.8

Thus (6, 137°) in polar co-ordinates corresponds to (−4.388, 4.092) in Cartesian co-ordinates.

(Note that when changing from polar to Cartesian co-ordinates it is not quite so essential to draw a sketch. Use of $x = r\cos\theta$ and $y = r\sin\theta$ automatically produces the correct signs.)

Problem 7. Express (4.5, 5.16 rad) in Cartesian co-ordinates.

A sketch showing the position (4.5, 5.16 rad) is shown in Fig. 12.9.

$$x = r\cos\theta = 4.5\cos 5.16 = 1.948$$

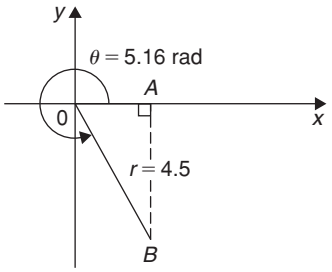


Figure 12.9

which corresponds to length OA in Fig. 12.9.

$$y = r\sin\theta = 4.5\sin 5.16 = -4.057$$

which corresponds to length AB in Fig. 12.9.

Thus (1.948, −4.057) in Cartesian co-ordinates corresponds to (4.5, 5.16 rad) in polar co-ordinates.

Now try the following exercise

Exercise 54 Further problems on changing polar into Cartesian co-ordinates

In Problems 1 to 8, express the given polar co-ordinates as Cartesian co-ordinates, correct to 3 decimal places.

1. (5, 75°) [(1.294, 4.830)]
2. (4.4, 1.12 rad) [(1.917, 3.960)]
3. (7, 140°) [(−5.362, 4.500)]
4. (3.6, 2.5 rad) [(−2.884, 2.154)]
5. (10.8, 210°) [(−9.353, −5.400)]
6. (4, 4 rad) [(−2.615, −3.207)]
7. (1.5, 300°) [(0.750, −1.299)]
8. (6, 5.5 rad) [(4.252, −4.233)]
9. Figure 12.10 shows 5 equally spaced holes on an 80 mm pitch circle diameter. Calculate their co-ordinates relative to axes $0x$ and $0y$ in (a) polar form, (b) Cartesian form.

   Calculate also the shortest distance between the centres of two adjacent holes.

   [(*a*) 40∠18°, 40∠90°, 40∠162°, 40∠234°, 40∠306°,

   (*b*) (38.04 + j12.36), (0 + j40), (−38.04 + j12.36), (−23.51 − j32.36), (23.51 − j32.36)

   47.02 mm]

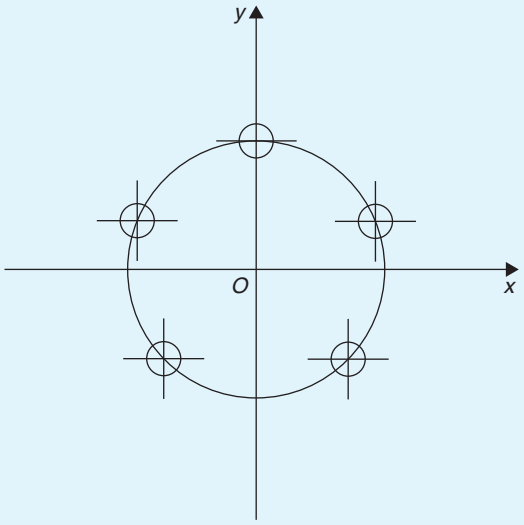


Figure 12.10

12.4 Use of Pol/Rec functions on calculators

Another name for Cartesian co-ordinates is **rectangular** co-ordinates. Many scientific notation calculators possess **Pol** and **Rec** functions. 'Rec' is an abbreviation of 'rectangular' (i.e., Cartesian) and 'Pol' is an abbreviation of 'polar'. Check the operation manual for your particular calculator to determine how to use these

two functions. They make changing from Cartesian to polar co-ordinates, and vice-versa, so much quicker and easier.

For example, with the Casio fx-83ES calculator, or similar, to change the Cartesian number (3, 4) into polar form, the following procedure is adopted:

1. Press 'shift' 2. Press 'Pol' 3. Enter 3
4. Enter 'comma' (obtained by 'shift' then))
5. Enter 4 6. Press)
7. Press = The answer is: $\boldsymbol{r = 5, \theta = 53.13°}$

Hence, (3, 4) in Cartesian form is the same as (5, 53.13°) in polar form.

If the angle is required in **radians**, then before repeating the above procedure press 'shift', 'mode' and then 4 to change your calculator to radian mode.

Similarly, to change the polar form number (7, 126°) into Cartesian or rectangular form, adopt the following procedure:

1. Press 'shift' 2. Press 'Rec' 3. Enter 7
4. Enter 'comma'
5. Enter 126 (assuming your calculator is in degrees mode)
6. Press) 7. Press =

The answer is: $\boldsymbol{X = -4.11}$, **and scrolling across,** $\boldsymbol{Y = 5.66}$, correct to 2 decimal places.

Hence, (7, 126°) in polar form is the same as (−4.11, 5.66) in rectangular or Cartesian form.

Now return to Exercises 53 and 54 in this chapter and use your calculator to determine the answers, and see how much more quickly they may be obtained.

Chapter 13

The circle and its properties

13.1 Introduction

A **circle** is a plain figure enclosed by a curved line, every point on which is equidistant from a point within, called the **centre**.

13.2 Properties of circles

(i) The distance from the centre to the curve is called the **radius**, r, of the circle (see OP in Fig. 13.1).

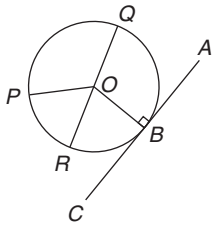


Figure 13.1

(ii) The boundary of a circle is called the **circumference**, c.

(iii) Any straight line passing through the centre and touching the circumference at each end is called the **diameter**, d (see QR in Fig. 13.1). Thus $\boldsymbol{d = 2r}$.

(iv) The ratio $\dfrac{\text{circumference}}{\text{diameter}} =$ a constant for any circle.
This constant is denoted by the Greek letter π (pronounced 'pie'), where $\pi = 3.14159$, correct to 5 decimal places.
Hence $c/d = \pi$ or $\boldsymbol{c = \pi d}$ or $\boldsymbol{c = 2\pi r}$.

(v) A **semicircle** is one half of the whole circle.

(vi) A **quadrant** is one quarter of a whole circle.

(vii) A **tangent** to a circle is a straight line which meets the circle in one point only and does not cut the circle when produced. AC in Fig. 13.1 is a tangent to the circle since it touches the curve at point B only. If radius OB is drawn, then angle ABO is a right angle.

(viii) A **sector** of a circle is the part of a circle between radii (for example, the portion OXY of Fig. 13.2 is a sector). If a sector is less than a semicircle it is called a **minor sector**, if greater than a semicircle it is called a **major sector**.

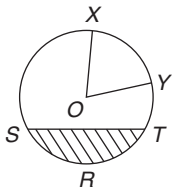


Figure 13.2

(ix) A **chord** of a circle is any straight line which divides the circle into two parts and is terminated at each end by the circumference. ST, in Fig. 13.2 is a chord.

(x) A **segment** is the name given to the parts into which a circle is divided by a chord. If the segment is less than a semicircle it is called a **minor segment** (see shaded area in Fig. 13.2). If the segment is greater than a semicircle it is called a **major segment** (see the unshaded area in Fig. 13.2).

(xi) An **arc** is a portion of the circumference of a circle. The distance SRT in Fig. 13.2 is called a **minor arc** and the distance $SXYT$ is called a **major arc**.

(xii) The angle at the centre of a circle, subtended by an arc, is double the angle at the circumference subtended by the same arc. With reference to Fig. 13.3, **Angle $AOC = 2 \times$ angle ABC**.

(xiii) The angle in a semicircle is a right angle (see angle BQP in Fig. 13.3).

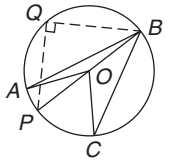


Figure 13.3

Problem 1. If the diameter of a circle is 75 mm, find its circumference.

Circumference, $c = \pi \times \text{diameter} = \pi d$
$= \pi(75) = \mathbf{235.6\,mm}$.

Problem 2. In Fig. 13.4, AB is a tangent to the circle at B. If the circle radius is 40 mm and $AB = 150$ mm, calculate the length AO.

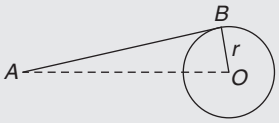


Figure 13.4

A tangent to a circle is at right angles to a radius drawn from the point of contact, i.e. $ABO = 90°$. Hence, using Pythagoras' theorem:

$$AO^2 = AB^2 + OB^2$$

$$AO = \sqrt{(AB^2 + OB^2)} = \sqrt{[(150)^2 + (40)^2]}$$

$$= \mathbf{155.2\,mm}$$

Now try the following exercise

Exercise 55 Further problems on properties of circles

1. If the radius of a circle is 41.3 mm, calculate the circumference of the circle. [259.5 mm]
2. Find the diameter of a circle whose perimeter is 149.8 cm. [47.68 cm]
3. A crank mechanism is shown in Fig. 13.5, where XY is a tangent to the circle at point X. If the circle radius OX is 10 cm and length OY is 40 cm, determine the length of the connecting rod XY. [38.73 cm]

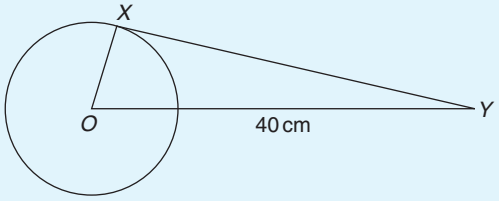


Figure 13.5

4. If the circumference of the earth is 40 000 km at the equator, calculate its diameter. [12 730 km]
5. Calculate the length of wire in the paper clip shown in Fig. 13.6. The dimensions are in millimetres. [97.13 mm]

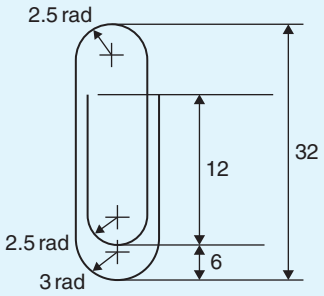


Figure 13.6

13.3 Radians and degrees

One **radian** is defined as the angle subtended at the centre of a circle by an arc equal in length to the radius.

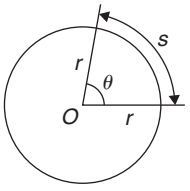


Figure 13.7

With reference to Fig. 13.7, for arc length s,

$$\theta \text{ radians} = \frac{s}{r}$$

When $s =$ whole circumference $(= 2\pi r)$ then $\theta = \frac{s}{r} = \frac{2\pi r}{r} = 2\pi$

i.e. $\mathbf{2\pi}$ **radians** $\mathbf{= 360°}$ or $\boldsymbol{\pi}$ **radians** $\mathbf{= 180°}$

Thus, $\mathbf{1\,rad = \frac{180°}{\pi} = 57.30°}$, correct to 2 decimal places.

Since π rad $= 180°$, then $\frac{\pi}{2} = 90°$, $\frac{\pi}{3} = 60°$, $\frac{\pi}{4} = 45°$, and so on.

Problem 3. Convert to radians: (a) 125° (b) 69°47′.

(a) Since $180° = \pi$ rad then $1° = \pi/180$ rad, therefore

$$125° = 125\left(\frac{\pi}{180}\right)^c = \mathbf{2.182\ rad}$$

(Note that ^{c} means 'circular measure' and indicates radian measure.)

(b) $69°47' = 69\frac{47°}{60} = 69.783°$

$$69.783° = 69.783\left(\frac{\pi}{180}\right)^c = \mathbf{1.218\ rad}$$

Problem 4. Convert to degrees and minutes: (a) 0.749 rad (b) $3\pi/4$ rad.

(a) Since π rad $= 180°$ then 1 rad $= 180°/\pi$, therefore

$$0.749 = 0.749\left(\frac{180}{\pi}\right)^\circ = 42.915°$$

$0.915° = (0.915 \times 60)' = 55'$, correct to the nearest minute, hence

$$\mathbf{0.749\ rad = 42°55'}$$

(b) Since $1\text{ rad} = \left(\frac{180}{\pi}\right)^\circ$ then

$$\frac{3\pi}{4}\text{ rad} = \frac{3\pi}{4}\left(\frac{180}{\pi}\right)^\circ = \frac{3}{4}(180)° = \mathbf{135°}$$

Problem 5. Express in radians, in terms of π, (a) 150° (b) 270° (c) 37.5°.

Since $180° = \pi$ rad then $1° = 180/\pi$, hence

(a) $150° = 150\left(\frac{\pi}{180}\right)\text{rad} = \mathbf{\frac{5\pi}{6}\ rad}$

(b) $270° = 270\left(\frac{\pi}{180}\right)\text{rad} = \mathbf{\frac{3\pi}{2}\ rad}$

(c) $37.5° = 37.5\left(\frac{\pi}{180}\right)\text{rad} = \frac{75\pi}{360}\text{rad} = \mathbf{\frac{5\pi}{24}\ rad}$

Now try the following exercise

Exercise 56 Further problems on radians and degrees

1. Convert to radians in terms of π: (a) 30° (b) 75° (c) 225°. $\left[\text{(a) } \frac{\pi}{6} \text{ (b) } \frac{5\pi}{12} \text{ (c) } \frac{5\pi}{4}\right]$
2. Convert to radians: (a) 48° (b) 84°51′ (c) 232°15′. [(a) 0.838 (b) 1.481 (c) 4.054]
3. Convert to degrees: (a) $\frac{5\pi}{6}$ rad (b) $\frac{4\pi}{9}$ rad (c) $\frac{7\pi}{12}$ rad. [(a) 150° (b) 80° (c) 105°]
4. Convert to degrees and minutes: (a) 0.0125 rad (b) 2.69 rad (c) 7.241 rad. [(a) 0°43′ (b) 154°8′ (c) 414°53′]

13.4 Arc length and area of circles and sectors

Arc length

From the definition of the radian in the previous section and Fig. 13.7,

$$\textbf{arc length, } \boldsymbol{s = r\theta} \quad \text{where } \theta \text{ is in radians}$$

Area of circle

For any circle, area $= \pi \times (\text{radius})^2$

i.e. $$\textbf{area} = \boldsymbol{\pi r^2}$$

Since $r = \frac{d}{2}$, then $\textbf{area} = \boldsymbol{\pi r^2}$ or $\boldsymbol{\frac{\pi d^2}{4}}$

Area of sector

Area of a sector $= \dfrac{\theta}{360}(\pi r^2)$ when θ is in degrees

$$= \frac{\theta}{2\pi}(\pi r^2) = \frac{1}{2}r^2\theta$$

when θ is in radians

Problem 6. A hockey pitch has a semicircle of radius 14.63 m around each goal net. Find the area enclosed by the semicircle, correct to the nearest square metre.

Area of a semicircle $= \dfrac{1}{2}\pi r^2$

When $r = 14.63$ m, area $= \dfrac{1}{2}\pi(14.63)^2$

i.e. **area of semicircle = 336 m^2**

Problem 7. Find the area of a circular metal plate, correct to the nearest square millimetre, having a diameter of 35.0 mm.

Area of a circle $= \pi r^2 = \dfrac{\pi d^2}{4}$

When $d = 35.0$ mm, area $= \dfrac{\pi(35.0)^2}{4}$

i.e. **area of circular plate = 962 mm^2**

Problem 8. Find the area of a circle having a circumference of 60.0 mm.

Circumference, $c = 2\pi r$

from which radius $r = \dfrac{c}{2\pi} = \dfrac{60.0}{2\pi} = \dfrac{30.0}{\pi}$

Area of a circle $= \pi r^2$

i.e. **area** $= \pi\left(\dfrac{30.0}{\pi}\right)^2 =$ **286.5 mm^2**

Problem 9. Find the length of arc of a circle of radius 5.5 cm when the angle subtended at the centre is 1.20 rad.

Length of arc, $s = r\theta$, where θ is in radians, hence

$s = (5.5)(1.20) =$ **6.60 cm**

Problem 10. Determine the diameter and circumference of a circle if an arc of length 4.75 cm subtends an angle of 0.91 rad.

Since $s = r\theta$ then $r = \dfrac{s}{\theta} = \dfrac{4.75}{0.91} = 5.22$ cm

Diameter $= 2 \times$ radius $= 2 \times 5.22 =$ **10.44 cm**

Circumference, $c = \pi d = \pi(10.44) =$ **32.80 cm**

Problem 11. If an angle of 125° is subtended by an arc of a circle of radius 8.4 cm, find the length of (a) the minor arc, and (b) the major arc, correct to 3 significant figures.

(a) Since $180° = \pi$ rad then $1° = \left(\dfrac{\pi}{180}\right)$ rad and $125° = 125\left(\dfrac{\pi}{180}\right)$ rad.

Length of minor arc,

$$s = r\theta = (8.4)(125)\left(\frac{\pi}{180}\right) =$$ **18.3 cm**,

correct to 3 significant figures.

(b) Length of major arc

$$= (\text{circumference} - \text{minor arc})$$

$$= 2\pi(8.4) - 18.3 = 34.5 \text{ cm},$$

correct to 3 significant figures.

(Alternatively, major arc $= r\theta$
$= 8.4(360 - 125)(\pi/180) =$ **34.5 cm**.)

Problem 12. Determine the angle, in degrees and minutes, subtended at the centre of a circle of diameter 42 mm by an arc of length 36 mm. Calculate also the area of the minor sector formed.

Since length of arc, $s = r\theta$ then $\theta = s/r$

$$\text{Radius, } r = \frac{\text{diameter}}{2} = \frac{42}{2} = 21 \text{ mm}$$

$$\text{hence } \theta = \frac{s}{r} = \frac{36}{21} = 1.7143 \text{ rad}$$

$1.7143 \text{ rad} = 1.7143 \times (180/\pi)° = 98.22° =$ **98°13′** = angle subtended at centre of circle.

Area of sector

$$= \tfrac{1}{2}r^2\theta = \tfrac{1}{2}(21)^2(1.7143) =$$ **378 mm^2**.

Problem 13. A football stadium floodlight can spread its illumination over an angle of 45° to a distance of 55 m. Determine the maximum area that is floodlit.

Floodlit area = area of sector

$$= \frac{1}{2}r^2\theta = \frac{1}{2}(55)^2\left(45 \times \frac{\pi}{180}\right)$$

$$= \mathbf{1188\,m^2}$$

Problem 14. An automatic garden spray produces a spray to a distance of 1.8 m and revolves through an angle α which may be varied. If the desired spray catchment area is to be 2.5 m^2, to what should angle α be set, correct to the nearest degree.

Area of sector $= \frac{1}{2}r^2\theta$, hence $2.5 = \frac{1}{2}(1.8)^2\alpha$

from which, $\alpha = \dfrac{2.5 \times 2}{1.8^2} = 1.5432\,\text{rad}$

$$1.5432\,\text{rad} = \left(1.5432 \times \frac{180^\circ}{\pi}\right) = 88.42^\circ$$

Hence **angle $\alpha = 88^\circ$**, correct to the nearest degree.

Problem 15. The angle of a tapered groove is checked using a 20 mm diameter roller as shown in Fig. 13.8. If the roller lies 2.12 mm below the top of the groove, determine the value of angle θ.

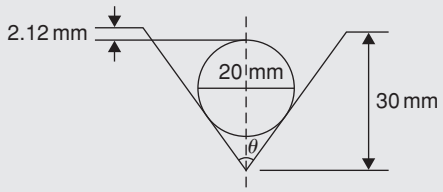


Figure 13.8

In Fig. 13.9, triangle ABC is right-angled at C (see Section 13.2 (vii)).

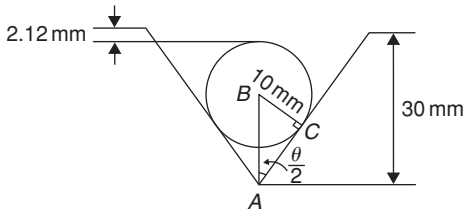


Figure 13.9

Length $BC = 10$ mm (i.e. the radius of the circle), and $AB = 30 - 10 - 2.12 = 17.88$ mm from Fig. 13.9.

Hence, $\sin\dfrac{\theta}{2} = \dfrac{10}{17.88}$ and $\dfrac{\theta}{2} = \sin^{-1}\left(\dfrac{10}{17.88}\right) = 34^\circ$

and **angle $\theta = 68^\circ$**

Now try the following exercise

Exercise 57 Further problems on arc length and area of circles and sectors

1. Calculate the area of a circle of radius 6.0 cm, correct to the nearest square centimetre. [113 cm^2]
2. The diameter of a circle is 55.0 mm. Determine its area, correct to the nearest square millimetre. [2376 mm^2]
3. The perimeter of a circle is 150 mm. Find its area, correct to the nearest square millimetre. [1790 mm^2]
4. Find the area of the sector, correct to the nearest square millimetre, of a circle having a radius of 35 mm, with angle subtended at centre of 75°. [802 mm^2]
5. An annulus has an outside diameter of 49.0 mm and an inside diameter of 15.0 mm. Find its area correct to 4 significant figures. [1709 mm^2]
6. Find the area, correct to the nearest square metre, of a 2 m wide path surrounding a circular plot of land 200 m in diameter. [1269 m^2]
7. A rectangular park measures 50 m by 40 m. A 3 m flower bed is made round the two longer sides and one short side. A circular fish pond of diameter 8.0 m is constructed in the centre of the park. It is planned to grass the remaining area. Find, correct to the nearest square metre, the area of grass. [1548 m^2]
8. Find the length of an arc of a circle of radius 8.32 cm when the angle subtended at the centre is 2.14 rad. Calculate also the area of the minor sector formed. [17.80 cm, 74.07 cm^2]
9. If the angle subtended at the centre of a circle of diameter 82 mm is 1.46 rad, find the lengths of the (a) minor arc (b) major arc. [(a) 59.86 mm (b) 197.8 mm]
10. A pendulum of length 1.5 m swings through an angle of 10° in a single swing. Find, in centimetres, the length of the arc traced by the pendulum bob. [26.2 cm]

11. Determine the length of the radius and circumference of a circle if an arc length of 32.6 cm subtends an angle of 3.76 rad.
[8.67 cm, 54.48 cm]

12. Determine the angle of lap, in degrees and minutes, if 180 mm of a belt drive are in contact with a pulley of diameter 250 mm.
[82°30′]

13. Determine the number of complete revolutions a motorcycle wheel will make in travelling 2 km, if the wheel's diameter is 85.1 cm.
[748]

14. The floodlights at a sports ground spread its illumination over an angle of 40° to a distance of 48 m. Determine (a) the angle in radians, and (b) the maximum area that is floodlit.
[(a) 0.698 rad (b) 804.1 m^2]

15. Determine (a) the shaded area in Fig. 13.10 (b) the percentage of the whole sector that the area of the shaded portion represents.
[(a) 396 mm^2 (b) 42.24%]

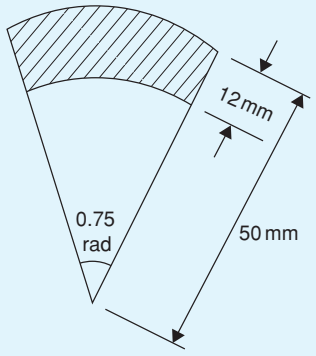


Figure 13.10

16. Determine the length of steel strip required to make the clip shown in Fig. 13.11.
[701.8 mm]

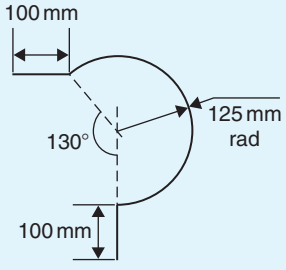


Figure 13.11

17. A 50° tapered hole is checked with a 40 mm diameter ball as shown in Fig. 13.12. Determine the length shown as x.
[7.74 mm]

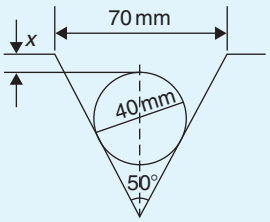


Figure 13.12

13.5 The equation of a circle

The simplest equation of a circle, centre at the origin, radius r, is given by:

$$x^2 + y^2 = r^2$$

For example, Fig. 13.13 shows a circle $x^2 + y^2 = 9$. More generally, the equation of a circle, centre (a, b), radius r, is given by:

$$(x-a)^2 + (y-b)^2 = r^2 \qquad (1)$$

Figure 13.14 shows a circle $(x-2)^2 + (y-3)^2 = 4$. The general equation of a circle is:

$$x^2 + y^2 + 2ex + 2fy + c = 0 \qquad (2)$$

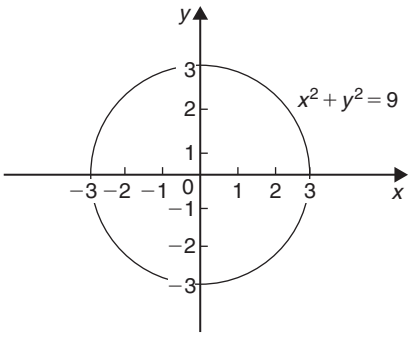


Figure 13.13

Multiplying out the bracketed terms in equation (1) gives:

$$x^2 - 2ax + a^2 + y^2 - 2by + b^2 = r^2$$

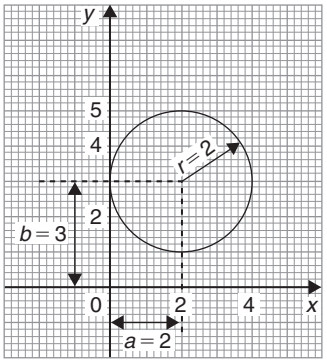


Figure 13.14

Comparing this with equation (2) gives:

$$2e = -2a, \text{ i.e. } \boldsymbol{a = -\frac{2e}{2}}$$

and $2f = -2b$, i.e. $\boldsymbol{b = -\frac{2f}{2}}$

and $c = a^2 + b^2 - r^2$,

i.e., $\boldsymbol{r = \sqrt{(a^2 + b^2 - c)}}$

Thus, for example, the equation

$$x^2 + y^2 - 4x - 6y + 9 = 0$$

represents a circle with centre $a = -\left(\frac{-4}{2}\right)$, $b = -\left(\frac{-6}{2}\right)$, i.e. at (2, 3) and radius $r = \sqrt{(2^2 + 3^2 - 9)} = 2$.

Hence $x^2 + y^2 - 4x - 6y + 9 = 0$ is the circle shown in Fig. 13.14 (which may be checked by multiplying out the brackets in the equation

$$(x - 2)^2 + (y - 3)^2 = 4$$

Problem 16. Determine (a) the radius, and (b) the co-ordinates of the centre of the circle given by the equation: $x^2 + y^2 + 8x - 2y + 8 = 0$.

$x^2 + y^2 + 8x - 2y + 8 = 0$ is of the form shown in equation (2),

where $a = -\left(\frac{8}{2}\right) = -4, b = -\left(\frac{-2}{2}\right) = 1$

and $r = \sqrt{[(-4)^2 + (1)^2 - 8]} = \sqrt{9} = 3$

Hence $x^2 + y^2 + 8x - 2y + 8 = 0$ represents a circle **centre (−4, 1)** and **radius 3**, as shown in Fig. 13.15.

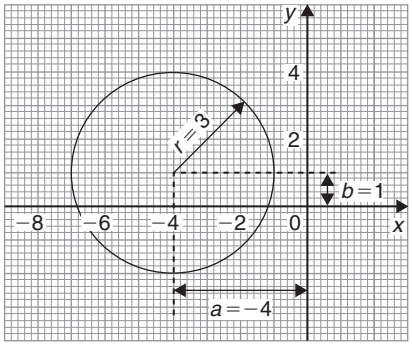


Figure 13.15

Alternatively, $x^2 + y^2 + 8x - 2y + 8 = 0$ may be rearranged as:

$$(x + 4)^2 + (y - 1)^2 - 9 = 0$$

i.e. $$(x + 4)^2 + (y - 1)^2 = 3^2$$

which represents a circle, **centre (−4, 1)** and **radius 3**, as stated above.

Problem 17. Sketch the circle given by the equation: $x^2 + y^2 - 4x + 6y - 3 = 0$.

The equation of a circle, centre (a, b), radius r is given by:

$$(x - a)^2 + (y - b)^2 = r^2$$

The general equation of a circle is

$$x^2 + y^2 + 2ex + 2fy + c = 0.$$

From above $a = -\frac{2e}{2}$, $b = -\frac{2f}{2}$ and

$$r = \sqrt{(a^2 + b^2 - c)}.$$

Hence if $x^2 + y^2 - 4x + 6y - 3 = 0$

then $a = -\left(\frac{-4}{2}\right) = 2, \quad b = -\left(\frac{6}{2}\right) = -3$

and $r = \sqrt{[(2)^2 + (-3)^2 - (-3)]}$

$= \sqrt{16} = 4$

Thus **the circle has centre (2, −3) and radius 4**, as shown in Fig. 13.16.

Alternatively, $x^2 + y^2 - 4x + 6y - 3 = 0$ may be rearranged as:

$$(x - 2)^2 + (y + 3)^2 - 3 - 13 = 0$$

i.e. $$(x - 2)^2 + (y + 3)^2 = 4^2$$

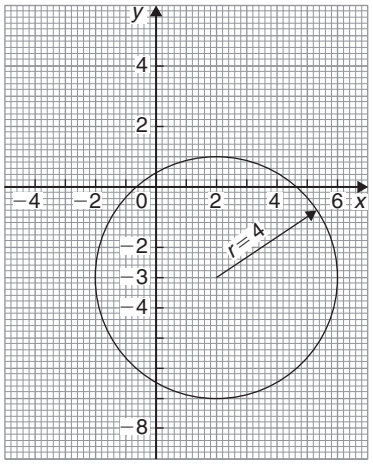


Figure 13.16

which represents a circle, **centre (2, −3)** and **radius 4**, as stated above.

Now try the following exercise

Exercise 58 Further problems on the equation of a circle

1. Determine the radius and the co-ordinates of the centre of the circle given by the equation $x^2+y^2+6x-2y-26=0$.
   [6, (−3, 1)]
2. Sketch the circle given by the equation $x^2+y^2-6x+4y-3=0$.
   [Centre at (3, −2), radius 4]
3. Sketch the curve $x^2+(y-1)^2-25=0$.
   [Circle, centre (0, 1), radius 5]
4. Sketch the curve $x=6\sqrt{[1-(y/6)^2]}$.
   [Circle, centre (0, 0), radius 6]

13.6 Linear and angular velocity

Linear velocity

Linear velocity v is defined as the rate of change of linear displacement s with respect to time t. For motion in a straight line:

$$\text{linear velocity} = \frac{\text{change of displacement}}{\text{change of time}}$$

i.e. $$v = \frac{s}{t} \qquad (1)$$

The unit of linear velocity is metres per second (m/s).

Angular velocity

The speed of revolution of a wheel or a shaft is usually measured in revolutions per minute or revolutions per second but these units do not form part of a coherent system of units. The basis in SI units is the angle turned through in one second.

Angular velocity is defined as the rate of change of angular displacement θ, with respect to time t. For an object rotating about a fixed axis at a constant speed:

$$\text{angular velocity} = \frac{\text{angle turned through}}{\text{time taken}}$$

i.e. $$\omega = \frac{\theta}{t} \qquad (2)$$

The unit of angular velocity is radians per second (rad/s). An object rotating at a constant speed of n revolutions per second subtends an angle of $2\pi n$ radians in one second, i.e., its angular velocity ω is given by:

$$\omega = 2\pi n \text{ rad/s} \qquad (3)$$

From page 124, $s=r\theta$ and from equation (2) above, $\theta=\omega t$

hence $$s = r(\omega t)$$

from which $$\frac{s}{t} = \omega r$$

However, from equation (1) $v = \frac{s}{t}$

hence $$v = \omega r \qquad (4)$$

Equation (4) gives the relationship between linear velocity v and angular velocity ω.

Problem 18. A wheel of diameter 540 mm is rotating at $\frac{1500}{\pi}$ rev/min. Calculate the angular velocity of the wheel and the linear velocity of a point on the rim of the wheel.

From equation (3), angular velocity $\omega = 2\pi n$ where n is the speed of revolution in rev/s. Since in this case

$n = \dfrac{1500}{\pi}$ rev/min $= \dfrac{1500}{60\pi} =$ rev/s, then

$$\textbf{angular velocity } \omega = 2\pi\left(\frac{1500}{60\pi}\right) = \mathbf{50\,rad/s}$$

The linear velocity of a point on the rim, $v = \omega r$, where r is the radius of the wheel, i.e.
$\dfrac{540}{2}$ mm $= \dfrac{0.54}{2}$ m $= 0.27$ m.

Thus **linear velocity** $v = \omega r = (50)(0.27)$

$$= \mathbf{13.5\,m/s}$$

Problem 19. A car is travelling at 64.8 km/h and has wheels of diameter 600 mm.

(a) Find the angular velocity of the wheels in both rad/s and rev/min.

(b) If the speed remains constant for 1.44 km, determine the number of revolutions made by the wheel, assuming no slipping occurs.

(a) Linear velocity $v = 64.8$ km/h

$$= 64.8\frac{\text{km}}{\text{h}} \times 1000\frac{\text{m}}{\text{km}} \times \frac{1}{3600}\frac{\text{h}}{\text{s}} = 18\,\text{m/s}.$$

The radius of a wheel $= \dfrac{600}{2} = 300$ mm

$$= 0.3\,\text{m}.$$

From equation (5), $v = \omega r$, from which,

$$\textbf{angular velocity } \omega = \frac{v}{r} = \frac{18}{0.3} = \mathbf{60\,rad/s}$$

From equation (4), angular velocity, $\omega = 2\pi n$, where n is in rev/s.

Hence angular speed $n = \dfrac{\omega}{2\pi} = \dfrac{60}{2\pi}$ rev/s

$$= 60 \times \frac{60}{2\pi}\,\text{rev/min}$$

$$= \mathbf{573\,rev/min}$$

(b) From equation (1), since $v = s/t$ then the time taken to travel 1.44 km, i.e. 1440 m at a constant speed of 18 m/s is given by:

$$\text{time } t = \frac{s}{v} = \frac{1440\,\text{m}}{18\,\text{m/s}} = 80\,\text{s}$$

Since a wheel is rotating at 573 rev/min, then in 80/60 minutes it makes

$$573\,\text{rev/min} \times \frac{80}{60}\,\text{min} = \mathbf{764\,revolutions}$$

Now try the following exercise

Exercise 59 Further problems on linear and angular velocity

1. A pulley driving a belt has a diameter of 300 mm and is turning at $2700/\pi$ revolutions per minute. Find the angular velocity of the pulley and the linear velocity of the belt assuming that no slip occurs.
   [$\omega = 90$ rad/s, $v = 13.5$ m/s]
2. A bicycle is travelling at 36 km/h and the diameter of the wheels of the bicycle is 500 mm. Determine the linear velocity of a point on the rim of one of the wheels of the bicycle, and the angular velocity of the wheels.
   [$v = 10$ m/s, $\omega = 40$ rad/s]
3. A train is travelling at 108 km/h and has wheels of diameter 800 mm.
   (a) Determine the angular velocity of the wheels in both rad/s and rev/min.
   (b) If the speed remains constant for 2.70 km, determine the number of revolutions made by a wheel, assuming no slipping occurs.
   [(a) 75 rad/s, 716.2 rev/min (b) 1074 revs]

13.7 Centripetal force

When an object moves in a circular path at constant speed, its direction of motion is continually changing and hence its velocity (which depends on both magnitude and direction) is also continually changing. Since acceleration is the (change in velocity)/(time taken), the object has an acceleration. Let the object be moving with a constant angular velocity of ω and a tangential velocity of magnitude v and let the change of velocity for a small change of angle of θ $(=\omega t)$ be V in Fig. 13.17. Then $v_2 - v_1 = V$. The vector diagram is shown in Fig. 13.17(b) and since the magnitudes of v_1 and v_2 are the same, i.e. v, the vector diagram is an isosceles triangle.

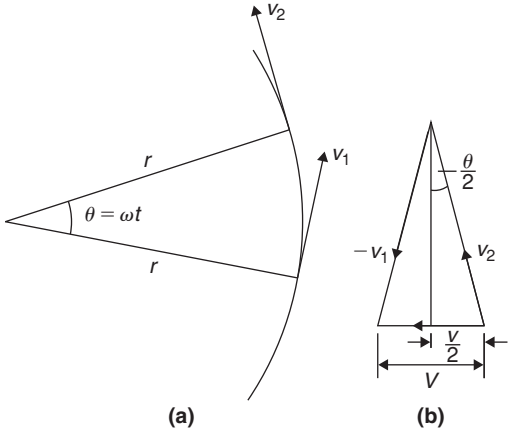


Figure 13.17

Bisecting the angle between v_2 and v_1 gives:

$$\sin\frac{\theta}{2} = \frac{V/2}{v_2} = \frac{V}{2v}$$

$$\text{i.e.} \quad V = 2v\sin\frac{\theta}{2} \tag{1}$$

Since $\theta = \omega t$ then

$$t = \frac{\theta}{\omega} \tag{2}$$

Dividing equation (1) by equation (2) gives:

$$\frac{V}{t} = \frac{2v\sin(\theta/2)}{(\theta/\omega)} = \frac{v\omega\sin(\theta/2)}{(\theta/2)}$$

For small angles $\dfrac{\sin(\theta/2)}{(\theta/2)} \approx 1$,

$$\text{hence} \quad \frac{V}{t} = \frac{\text{change of velocity}}{\text{change of time}}$$

$$= \text{acceleration } a = v\omega$$

However, $\omega = \dfrac{v}{r}$ (from Section 13.6)

$$\text{thus} \quad v\omega = v\cdot\frac{v}{r} = \frac{v^2}{r}$$

i.e. the acceleration a is $\dfrac{v^2}{r}$ and is towards the centre of the circle of motion (along V). It is called the **centripetal acceleration**. If the mass of the rotating object is m, then by Newton's second law, the **centripetal force is** $\boldsymbol{\dfrac{mv^2}{r}}$ and its direction is towards the centre of the circle of motion.

Problem 20. A vehicle of mass 750 kg travels around a bend of radius 150 m, at 50.4 km/h. Determine the centripetal force acting on the vehicle.

The centripetal force is given by $\dfrac{mv^2}{r}$ and its direction is towards the centre of the circle.

$$\text{Mass } m = 750\,\text{kg}, v = 50.4\,\text{km/h}$$
$$= \frac{50.4\times 1000}{60\times 60}\,\text{m/s}$$
$$= 14\,\text{m/s}$$

and radius $r = 150$ m,

thus **centripetal force** $= \dfrac{750(14)^2}{150} = \mathbf{980\,N}$.

Problem 21. An object is suspended by a thread 250 mm long and both object and thread move in a horizontal circle with a constant angular velocity of 2.0 rad/s. If the tension in the thread is 12.5 N, determine the mass of the object.

Centripetal force (i.e. tension in thread),

$$F = \frac{mv^2}{r} = 12.5\,\text{N}$$

Angular velocity $\omega = 2.0$ rad/s and radius $r = 250\,\text{mm} = 0.25$ m.

Since linear velocity $v = \omega r$, $v = (2.0)(0.25)$
$= 0.5$ m/s.

Since $F = \dfrac{mv^2}{r}$, then mass $m = \dfrac{Fr}{v^2}$,

i.e. **mass of object,** $\boldsymbol{m} = \dfrac{(12.5)(0.25)}{0.5^2} = \mathbf{12.5\ kg}$

Problem 22. An aircraft is turning at constant altitude, the turn following the arc of a circle of radius 1.5 km. If the maximum allowable acceleration of the aircraft is 2.5 g, determine the maximum speed of the turn in km/h. Take g as 9.8 m/s^2.

The acceleration of an object turning in a circle is $\dfrac{v^2}{r}$. Thus, to determine the maximum speed of turn, $\dfrac{v^2}{r} = 2.5\,g$, from which,

$$\text{velocity, } v = \sqrt{(2.5\,\text{gr})} = \sqrt{(2.5)(9.8)(1500)}$$
$$= \sqrt{36750} = 191.7\,\text{m/s}$$
$$\text{and } 191.7\,\text{m/s} = 191.7 \times \frac{60 \times 60}{1000}\,\text{km/h} = \mathbf{690\,km/h}$$

Now try the following exercise

Exercise 60 Further problems on centripetal force

1. Calculate the tension in a string when it is used to whirl a stone of mass 200 g round in a horizontal circle of radius 90 cm with a constant speed of 3 m/s. [2 N]

2. Calculate the centripetal force acting on a vehicle of mass 1 tonne when travelling around a bend of radius 125 m at 40 km/h. If this force should not exceed 750 N, determine the reduction in speed of the vehicle to meet this requirement. [988 N, 5.14 km/h]

3. A speed-boat negotiates an S-bend consisting of two circular arcs of radii 100 m and 150 m. If the speed of the boat is constant at 34 km/h, determine the change in acceleration when leaving one arc and entering the other. [1.49 m/s^2]

Revision Test 4

This Revision Test covers the material contained in Chapters 11 to 13. *The marks for each question are shown in brackets at the end of each question.*

1. A 2.0 m long ladder is placed against a perpendicular pylon with its foot 52 cm from the pylon. (a) Find how far up the pylon (correct to the nearest mm) the ladder reaches. (b) If the foot of the ladder is moved 10 cm towards the pylon how far does the top of the ladder rise? (7)

2. Evaluate correct to 4 significant figures:
   (a) $\cos 124°13'$ (b) $\cot 72.68°$ (4)

3. From a point on horizontal ground a surveyor measures the angle of elevation of a church spire as 15°. He moves 30 m nearer to the church and measures the angle of elevation as 20°. Calculate the height of the spire. (9)

4. If secant $\theta = 2.4613$ determine the acute angle θ (4)

5. Evaluate, correct to 3 significant figures:
$$\frac{3.5\,\text{cosec}\,31°17' - \cot(-12°)}{3\sec 79°41'} \quad (5)$$

6. A man leaves a point walking at 6.5 km/h in a direction E 20° N (i.e. a bearing of 70°). A cyclist leaves the same point at the same time in a direction E 40° S (i.e. a bearing of 130°) travelling at a constant speed. Find the average speed of the cyclist if the walker and cyclist are 80 km apart after 5 hours. (8)

7. A crank mechanism shown in Fig. RT4.1 comprises arm OP, of length 0.90 m, which rotates anti-clockwise about the fixed point O, and connecting rod PQ of length 4.20 m. End Q moves horizontally in a straight line OR.
   (a) If $\angle POR$ is initially zero, how far does end Q travel in $\frac{1}{4}$ revolution.
   (b) If $\angle POR$ is initially 40° find the angle between the connecting rod and the horizontal and the length OQ.
   (c) Find the distance Q moves (correct to the nearest cm) when $\angle POR$ changes from 40° to 140°. (16)

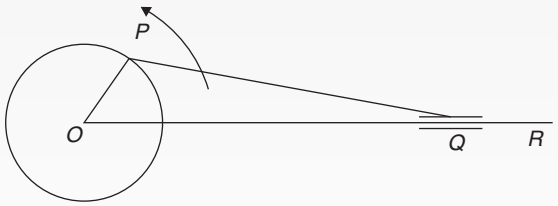


Figure RT4.1

8. Change the following Cartesian co-ordinates into polar co-ordinates, correct to 2 decimal places, in both degrees and in radians:
   (a) (−2.3, 5.4) (b) (7.6, −9.2) (10)

9. Change the following polar co-ordinates into Cartesian co-ordinates, correct to 3 decimal places: (a) (6.5, 132°) (b) (3, 3 rad) (6)

10. (a) Convert 2.154 radians into degrees and minutes.
    (b) Change $71°17'$ into radians. (4)

11. 140 mm of a belt drive is in contact with a pulley of diameter 180 mm which is turning at 300 revolutions per minute. Determine (a) the angle of lap, (b) the angular velocity of the pulley, and (c) the linear velocity of the belt assuming that no slipping occurs. (9)

12. Figure RT4.2 shows a cross-section through a circular water container where the shaded area represents the water in the container. Determine: (a) the depth, h, (b) the area of the shaded portion, and (c) the area of the unshaded area. (11)

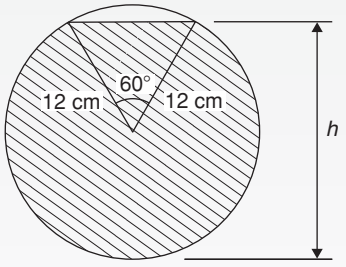


Figure RT4.2

13. Determine, (a) the co-ordinates of the centre of the circle, and (b) the radius, given the equation
$$x^2 + y^2 - 2x + 6y + 6 = 0 \quad (7)$$

Chapter 14

Trigonometric waveforms

14.1 Graphs of trigonometric functions

By drawing up tables of values from 0° to 360°, graphs of $y = \sin A$, $y = \cos A$ and $y = \tan A$ may be plotted. Values obtained with a calculator (correct to 3 decimal places—which is more than sufficient for plotting graphs), using 30° intervals, are shown below, with the respective graphs shown in Fig. 14.1.

(a) $y = \sin A$

A	0	30°	60°	90°	120°	150°	180°
$\sin A$	0	0.500	0.866	1.000	0.866	0.500	0

A	210°	240°	270°	300°	330°	360°
$\sin A$	−0.500	−0.866	−1.000	−0.866	−0.500	0

(b) $y = \cos A$

A	0	30°	60°	90°	120°	150°	180°
$\cos A$	1.000	0.866	0.500	0	−0.500	−0.866	−1.000

A	210°	240°	270°	300°	330°	360°
$\cos A$	−0.866	−0.500	0	0.500	0.866	1.000

(c) $y = \tan A$

A	0	30°	60°	90°	120°	150°	180°
$\tan A$	0	0.577	1.732	∞	−1.732	−0.577	0

A	210°	240°	270°	300°	330°	360°
$\tan A$	0.577	1.732	∞	−1.732	−0.577	0

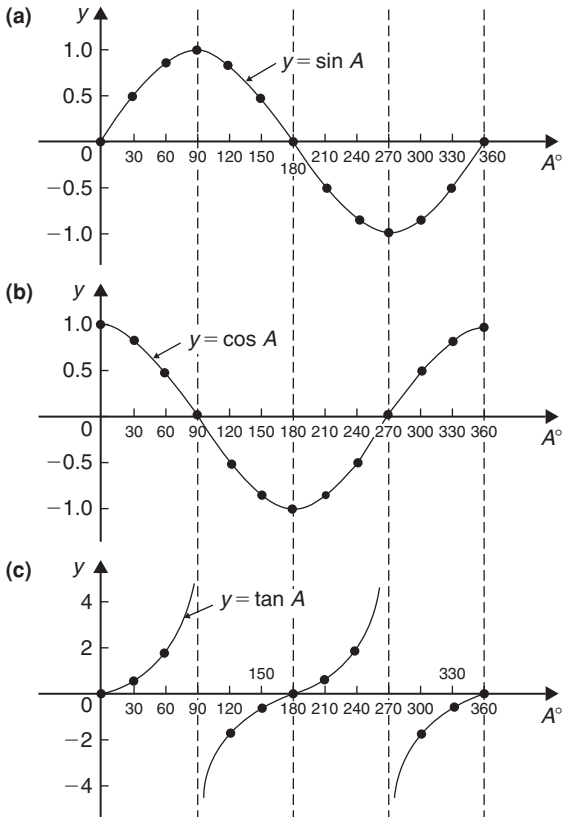


Figure 14.1

From Fig. 14.1 it is seen that:

(i) Sine and cosine graphs oscillate between peak values of ±1.

(ii) The cosine curve is the same shape as the sine curve but displaced by 90°.

(iii) The sine and cosine curves are continuous and they repeat at intervals of 360°; the tangent

curve appears to be discontinuous and repeats at intervals of 180°.

14.2 Angles of any magnitude

(i) Figure 14.2 shows rectangular axes *XX'* and *YY'* intersecting at origin 0. As with graphical work, measurements made to the right and above 0 are positive while those to the left and downwards are negative. Let *OA* be free to rotate about 0. By convention, when *OA* moves anticlockwise angular measurement is considered positive, and vice-versa.

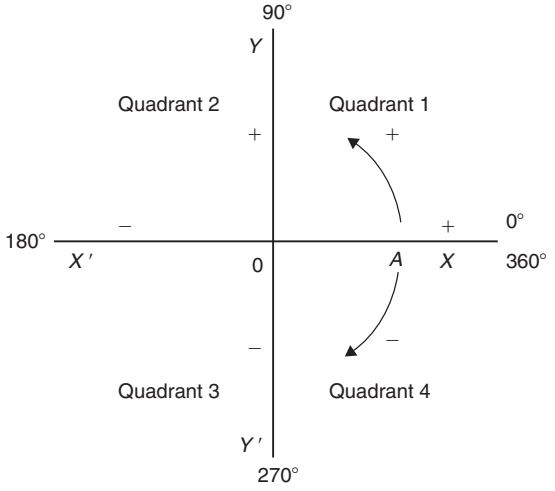


Figure 14.2

(ii) Let *OA* be rotated anticlockwise so that θ_1 is any angle in the first quadrant and let perpendicular *AB* be constructed to form the right-angled triangle *OAB* (see Fig. 14.3). Since all three sides of the triangle are positive, all six trigonometric ratios are positive in the first quadrant. (Note: *OA* is always positive since it is the radius of a circle.)

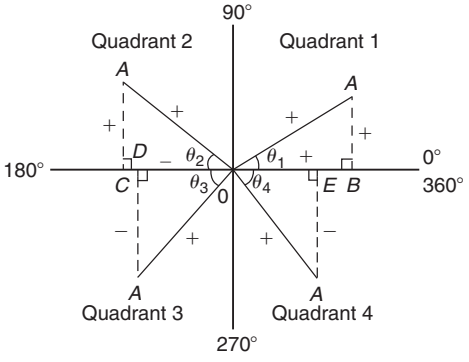


Figure 14.3

(iii) Let *OA* be further rotated so that θ_2 is any angle in the second quadrant and let *AC* be constructed to form the right-angled triangle *OAC*. Then:

$$\sin\theta_2 = \frac{+}{+} = + \qquad \cos\theta_2 = \frac{-}{+} = -$$

$$\tan\theta_2 = \frac{+}{-} = - \qquad \operatorname{cosec}\theta_2 = \frac{+}{+} = +$$

$$\sec\theta_2 = \frac{+}{-} = - \qquad \cot\theta_2 = \frac{-}{+} = -$$

(iv) Let *OA* be further rotated so that θ_3 is any angle in the third quadrant and let *AD* be constructed to form the right-angled triangle *OAD*. Then:

$$\sin\theta_3 = \frac{-}{+} = - \text{ (and hence } \operatorname{cosec}\theta_3 \text{ is } -)$$

$$\cos\theta_3 = \frac{-}{+} = - \text{ (and hence } \sec\theta_3 \text{ is } +)$$

$$\tan\theta_3 = \frac{-}{-} = + \text{ (and hence } \cot\theta_3 \text{ is } -)$$

(v) Let *OA* be further rotated so that θ_4 is any angle in the fourth quadrant and let *AE* be constructed to form the right-angled triangle *OAE*. Then:

$$\sin\theta_4 = \frac{-}{+} = - \text{ (and hence } \operatorname{cosec}\theta_4 \text{ is } -)$$

$$\cos\theta_4 = \frac{+}{+} = + \text{ (and hence } \sec\theta_4 \text{ is } +)$$

$$\tan\theta_4 = \frac{-}{+} = - \text{ (and hence } \cot\theta_4 \text{ is } -)$$

(vi) The results obtained in (ii) to (v) are summarized in Fig. 14.4. The letters underlined spell the word CAST when starting in the fourth quadrant and moving in an anticlockwise direction.

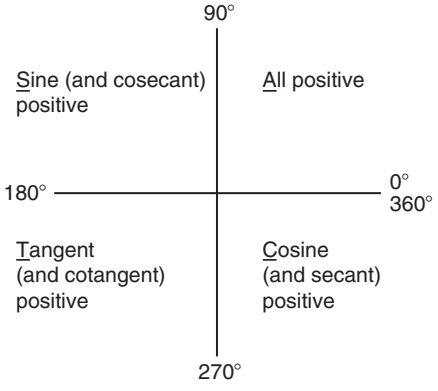


Figure 14.4

(vii) In the first quadrant of Fig. 14.1 all the curves have positive values; in the second only sine is positive; in the third only tangent is positive; in the fourth only cosine is positive (exactly as summarized in Fig. 14.4).

A knowledge of angles of any magnitude is needed when finding, for example, all the angles between 0° and 360° whose sine is, say, 0.3261. If 0.3261 is entered into a calculator and then the inverse sine key pressed (or $\sin^{-1}$ key) the answer 19.03° appears. However there is a second angle between 0° and 360° which the calculator does not give. Sine is also positive in the second quadrant (either from CAST or from Fig. 14.1(a)). The other angle is shown in Fig. 14.5 as angle θ where $\theta = 180° - 19.03° = 160.97°$. Thus 19.03° **and** 160.97° are the angles between 0° and 360° whose sine is 0.3261 (check that $\sin 160.97° = 0.3261$ on your calculator).

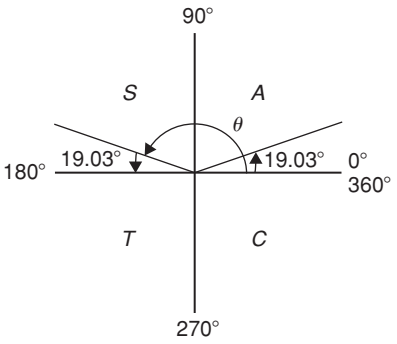


Figure 14.5

Be careful! Your calculator only gives you one of these answers. The second answer needs to be deduced from a knowledge of angles of any magnitude, as shown in the following problems.

Problem 1. Determine all the angles between 0° and 360° whose sine is −0.4638

The angles whose sine is −0.4638 occurs in the third and fourth quadrants since sine is negative in these quadrants (see Fig. 14.6(a)). From Fig. 14.6(b), $\theta = \sin^{-1} 0.4638 = 27°38'$.

Measured from 0°, the two angles between 0° and 360° whose sine is −0.4638 are 180° + 27°38′, i.e. **207°38′** and 360° − 27°38′, i.e. **332°22′**. (Note that a calculator generally only gives one answer, i.e. −27.632588°).

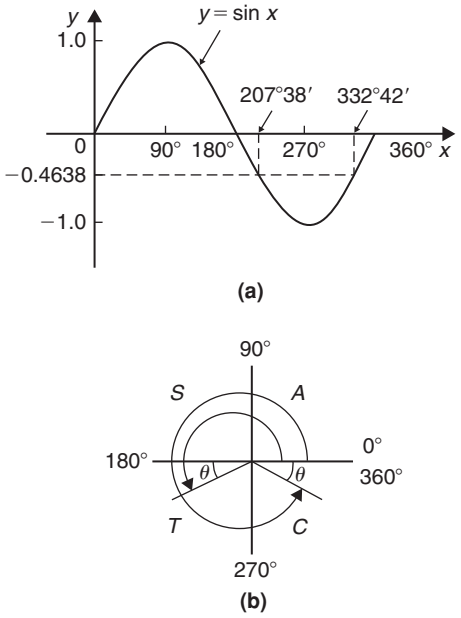


Figure 14.6

Problem 2. Determine all the angles between 0° and 360° whose tangent is 1.7629

A tangent is positive in the first and third quadrants (see Fig. 14.7(a)). From Fig. 14.7(b), $\theta = \tan^{-1} 1.7629 = 60°26'$. Measured from 0°, the two

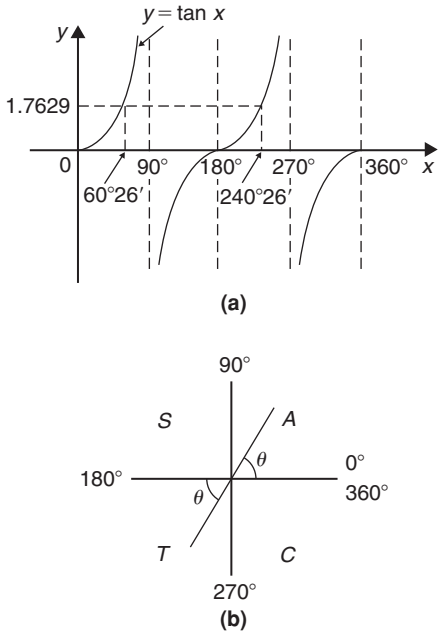


Figure 14.7

angles between 0° and 360° whose tangent is 1.7629 are **60°26′** and 180° + 60°26′, i.e. **240°26′**.

Problem 3. Solve $\sec^{-1}(-2.1499) = \alpha$ for angles of α between 0° and 360°.

Secant is negative in the second and third quadrants (i.e. the same as for cosine). From Fig. 14.8,

$$\theta = \sec^{-1} 2.1499 = \cos^{-1}\left(\frac{1}{2.1499}\right) = 62°17'.$$

Measured from 0°, the two angles between 0° and 360° whose secant is −2.1499 are

$$\alpha = 180° - 62°17' = \mathbf{117°43'} \quad \text{and}$$

$$\alpha = 180° + 62°17' = \mathbf{242°17'}$$

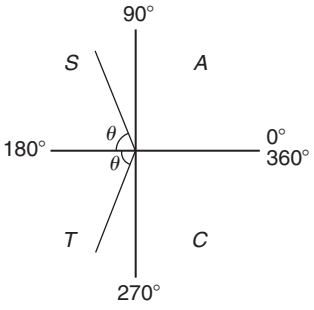


Figure 14.8

Problem 4. Solve $\cot^{-1} 1.3111 = \alpha$ for angles of α between 0° and 360°.

Cotangent is positive in the first and third quadrants (i.e. same as for tangent). From Fig. 14.9,

$$\theta = \cot^{-1} 1.3111 = \tan^{-1}\left(\frac{1}{1.3111}\right) = 37°20'.$$

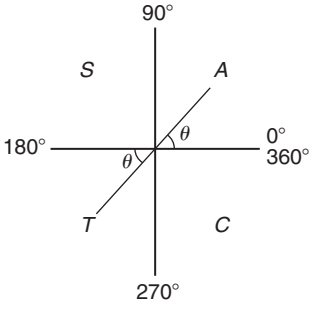


Figure 14.9

Hence $\alpha = \mathbf{37°20'}$

and $\alpha = 180° + 37°20' = \mathbf{217°20'}$

Now try the following exercise

Exercise 61 Further problems on evaluating trigonometric ratios of any magnitude

1. Find all the angles between 0° and 360° whose sine is −0.7321.
   [227°4′ and 312°56′]
2. Determine the angles between 0° and 360° whose cosecant is 2.5317.
   [23°16′ and 156°44′]
3. If cotangent $x = -0.6312$, determine the values of x in the range $0° \le x \le 360°$.
   [122°16′ and 302°16′]

In Problems 4 to 6 solve the given equations.

4. $\cos^{-1}(-0.5316) = t$
   [$t = 122°7'$ and 237°53′]
5. $\sec^{-1} 2.3162 = x$
   [$x = 64°25'$ and 295°35′]
6. $\tan^{-1} 0.8314 = \theta$
   [$\theta = 39°44'$ and 219°44′]

14.3 The production of a sine and cosine wave

In Fig. 14.10, let OR be a vector 1 unit long and free to rotate anticlockwise about O. In one revolution a circle is produced and is shown with 15° sectors. Each radius arm has a vertical and a horizontal component. For example, at 30°, the vertical component is TS and the horizontal component is OS.

From trigonometric ratios,

$$\sin 30° = \frac{TS}{TO} = \frac{TS}{1}, \text{ i.e. } TS = \sin 30°$$

$$\text{and } \cos 30° = \frac{OS}{TO} = \frac{OS}{1}, \text{ i.e. } OS = \cos 30°$$

The vertical component TS may be projected across to $T'S'$, which is the corresponding value of 30° on the graph of y against angle $x°$. If all such vertical components as TS are projected on to the

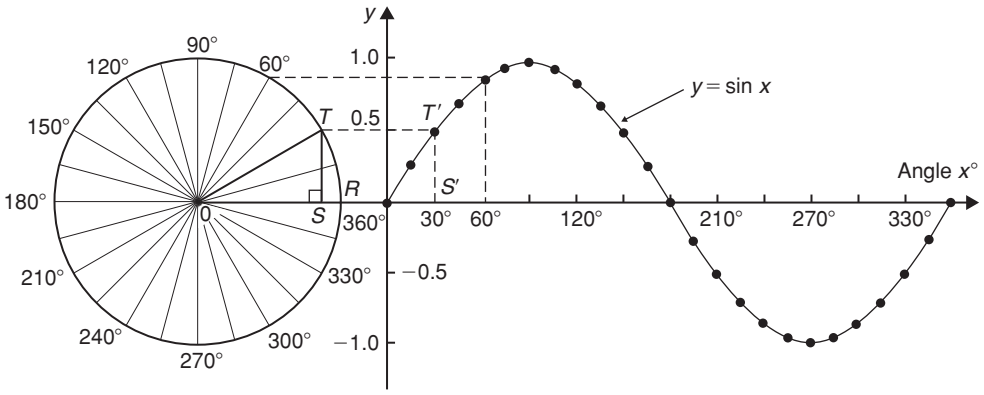


Figure 14.10

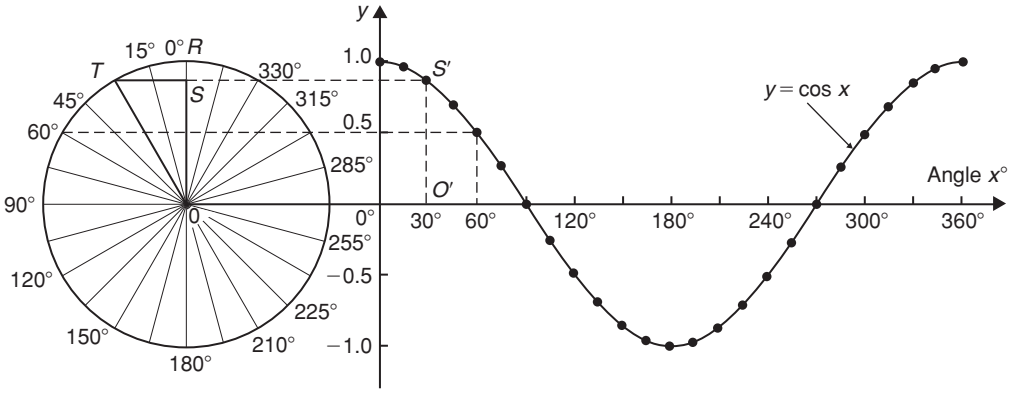


Figure 14.11

graph, then a **sine wave** is produced as shown in Fig. 14.10.

If all horizontal components such as OS are projected on to a graph of y against angle $x°$, then a **cosine wave** is produced. It is easier to visualize these projections by redrawing the circle with the radius arm OR initially in a vertical position as shown in Fig. 14.11.

From Figs. 14.10 and 14.11 it is seen that a cosine curve is of the same form as the sine curve but is displaced by 90° (or $\pi/2$ radians).

14.4 Sine and cosine curves

Graphs of sine and cosine waveforms

(i) A graph of $y = \sin A$ is shown by the broken line in Fig. 14.12 and is obtained by drawing up a table of values as in Section 14.1. A similar table may be produced for $y = \sin 2A$.

$A°$	$2A$	$\sin 2A$	$A°$	$2A$	$\sin 2A$
0	0	0	225	450	1.0
30	60	0.866	240	480	0.866
45	90	1.0	270	540	0
60	120	0.866	300	600	−0.866
90	180	0	315	630	−1.0
120	240	−0.866	330	660	−0.866
135	270	−1.0	360	720	0
150	300	−0.866			
180	360	0			
210	420	0.866			

A graph of $y = \sin 2A$ is shown in Fig. 14.12.

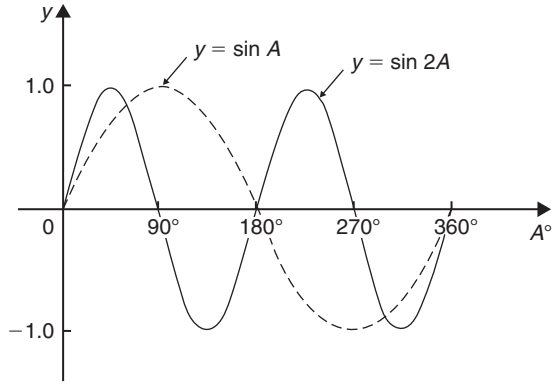


Figure 14.12

(ii) A graph of $y = \sin \frac{1}{2}A$ is shown in Fig. 14.13 using the following table of values.

$A°$	$\frac{1}{2}A$	$\sin \frac{1}{2}A$
0	0	0
30	15	0.259
60	30	0.500
90	45	0.707
120	60	0.866
150	75	0.966
180	90	1.00
210	105	0.966
240	120	0.866
270	135	0.707
300	150	0.500
330	165	0.259
360	180	0

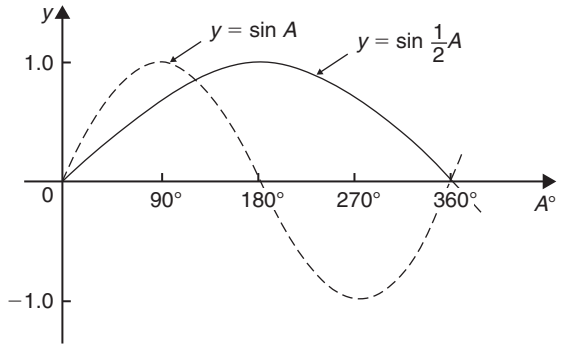


Figure 14.13

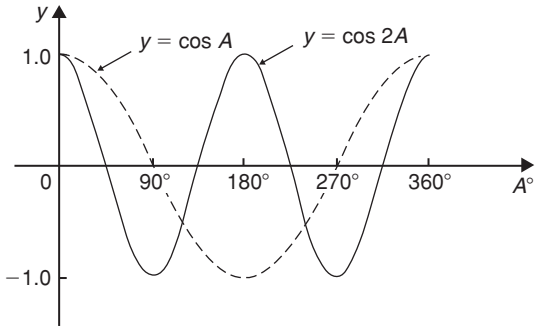


Figure 14.14

(iii) A graph of $y = \cos A$ is shown by the broken line in Fig. 14.14 and is obtained by drawing up a table of values. A similar table may be produced for $y = \cos 2A$ with the result as shown.

(iv) A graph of $y = \cos \frac{1}{2}A$ is shown in Fig. 14.15 which may be produced by drawing up a table of values, similar to above.

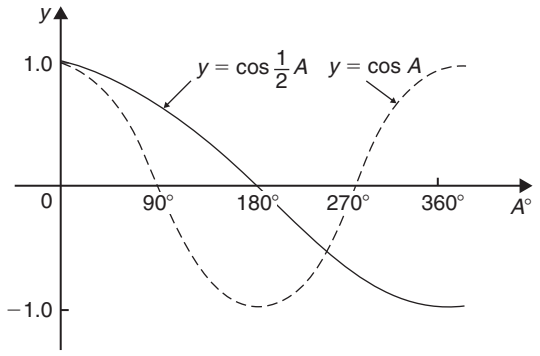


Figure 14.15

Periodic functions and period

(i) Each of the graphs shown in Figs. 14.12 to 14.15 will repeat themselves as angle A increases and are thus called **periodic functions**.

(ii) $y = \sin A$ and $y = \cos A$ repeat themselves every 360° (or 2π radians); thus 360° is called the **period** of these waveforms. $y = \sin 2A$ and $y = \cos 2A$ repeat themselves every 180° (or π radians); thus 180° is the period of these waveforms.

(iii) In general, if $y = \sin pA$ or $y = \cos pA$ (where p is a constant) then the period of the waveform is $360°/p$ (or $2\pi/p$ rad). Hence if $y = \sin 3A$ then the period is 360/3, i.e. 120°, and if $y = \cos 4A$ then the period is 360/4, i.e. 90°.

Amplitude

Amplitude is the name given to the maximum or peak value of a sine wave. Each of the graphs shown in Figs. 14.12 to 14.15 has an amplitude of +1 (i.e. they oscillate between +1 and −1). However, if $y = 4\sin A$, each of the values in the table is multiplied by 4 and the maximum value, and thus amplitude, is 4. Similarly, if $y = 5\cos 2A$, the amplitude is 5 and the period is 360°/2, i.e. 180°.

Problem 5. Sketch $y = \sin 3A$ between $A = 0°$ and $A = 360°$.

Amplitude = 1; period = 360°/3 = 120°.

A sketch of $y = \sin 3A$ is shown in Fig. 14.16.

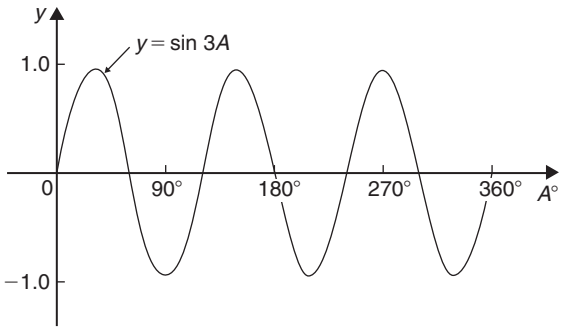


Figure 14.16

Problem 6. Sketch $y = 3\sin 2A$ from $A = 0$ to $A = 2\pi$ radians.

Amplitude = 3, period = $2\pi/2 = \pi$ rads (or 180°).

A sketch of $y = 3\sin 2A$ is shown in Fig. 14.17.

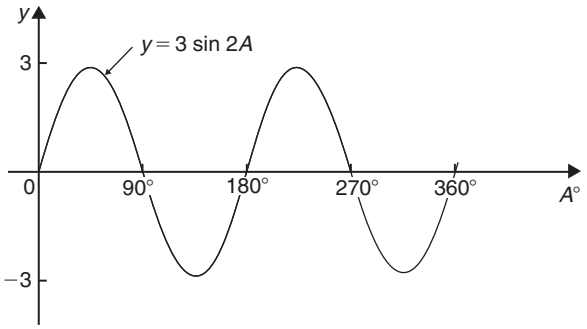


Figure 14.17

Problem 7. Sketch $y = 4\cos 2x$ from $x = 0°$ to $x = 360°$.

Amplitude = 4; period = 360°/2 = 180°.

A sketch of $y = 4\cos 2x$ is shown in Fig. 14.18.

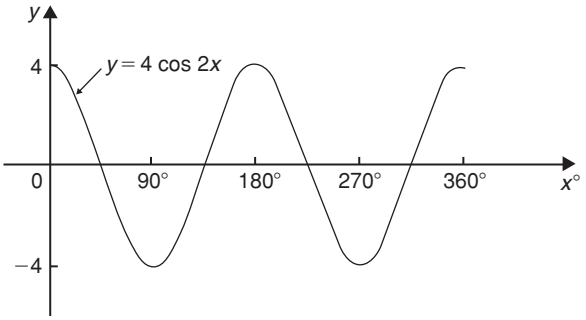


Figure 14.18

Problem 8. Sketch $y = 2\sin\frac{3}{5}A$ over one cycle.

Amplitude = 2; period = $\dfrac{360°}{\frac{3}{5}} = \dfrac{360° \times 5}{3} = 600°$.

A sketch of $y = 2\sin\frac{3}{5}A$ is shown in Fig. 14.19.

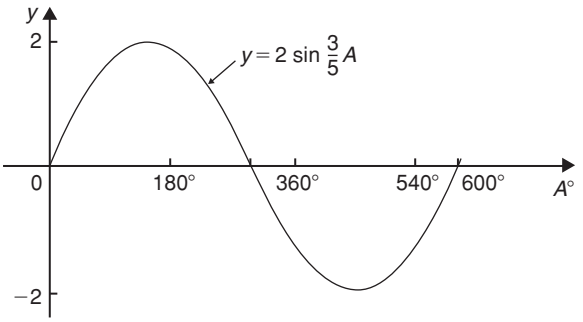


Figure 14.19

Lagging and leading angles

(i) A sine or cosine curve may not always start at 0°. To show this a periodic function is represented by $y = \sin(A \pm \alpha)$ or $y = \cos(A \pm \alpha)$ where α is a phase displacement compared with $y = \sin A$ or $y = \cos A$.

(ii) By drawing up a table of values, a graph of $y=\sin(A-60°)$ may be plotted as shown in Fig. 14.20. If $y=\sin A$ is assumed to start at 0° then $y=\sin(A-60°)$ starts 60° later (i.e. has a zero value 60° later). Thus $y=\sin(A-60°)$ is said to **lag** $y=\sin A$ by 60°.

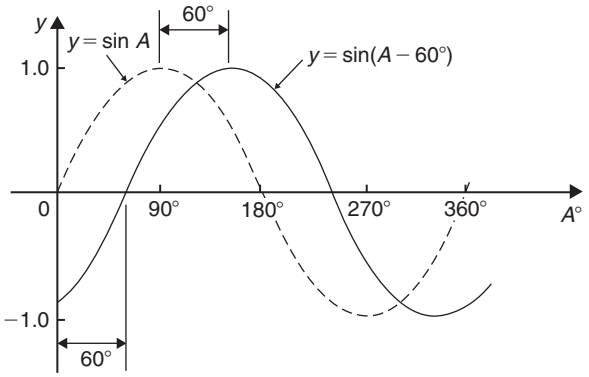


Figure 14.20

(iii) By drawing up a table of values, a graph of $y=\cos(A+45°)$ may be plotted as shown in Fig. 14.21. If $y=\cos A$ is assumed to start at 0° then $y=\cos(A+45°)$ starts 45° earlier (i.e. has a zero value 45° earlier). Thus $y=\cos(A+45°)$ is said to **lead** $y=\cos A$ by 45°.

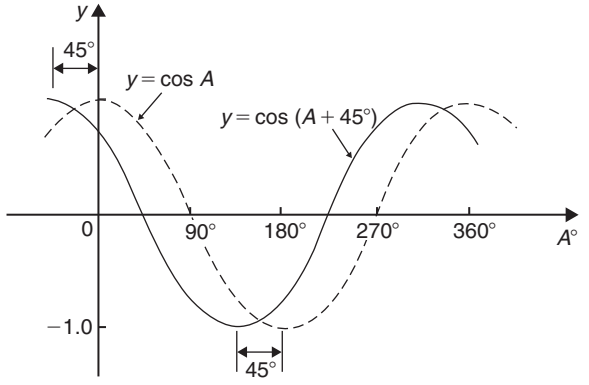


Figure 14.21

(iv) Generally, a graph of $y=\sin(A-\alpha)$ lags $y=\sin A$ by angle α, and a graph of $y=\sin(A+\alpha)$ leads $y=\sin A$ by angle α.

(v) A cosine curve is the same shape as a sine curve but starts 90° earlier, i.e. leads by 90°. Hence $\cos A=\sin(A+90°)$.

Problem 9. Sketch $y=5\sin(A+30°)$ from $A=0°$ to $A=360°$.

Amplitude $=5$; period $=360°/1=360°$.

$5\sin(A+30°)$ leads $5\sin A$ by 30° (i.e. starts 30° earlier).

A sketch of $y=5\sin(A+30°)$ is shown in Fig. 14.22.

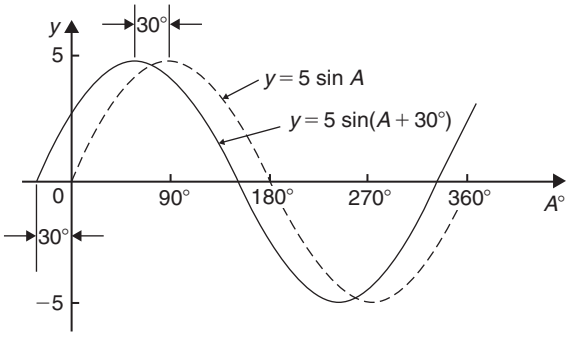


Figure 14.22

Problem 10. Sketch $y=7\sin(2A-\pi/3)$ in the range $0\leq A\leq 2\pi$.

Amplitude $=7$; period $=2\pi/2=\pi$ radians.

In general, $\boldsymbol{y=\sin(pt-\alpha)}$ **lags** $\boldsymbol{y=\sin pt}$ **by** $\boldsymbol{\alpha/p}$, hence $7\sin(2A-\pi/3)$ lags $7\sin 2A$ by $(\pi/3)/2$, i.e. $\pi/6$ rad or 30°.

A sketch of $y=7\sin(2A-\pi/3)$ is shown in Fig. 14.23.

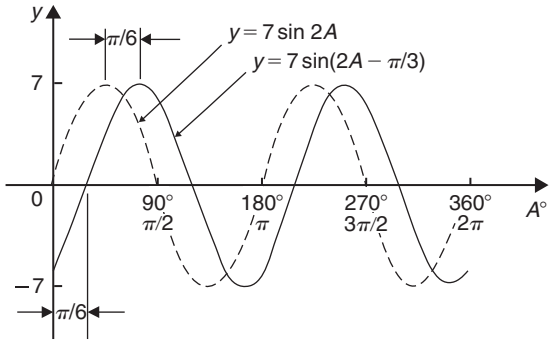


Figure 14.23

Problem 11. Sketch $y=2\cos(\omega t-3\pi/10)$ over one cycle.

Amplitude $=2$; period $=2\pi/\omega$ rad.

$2\cos(\omega t-3\pi/10)$ lags $2\cos\omega t$ by $3\pi/10\omega$ seconds.

A sketch of $y=2\cos(\omega t-3\pi/10)$ is shown in Fig. 14.24.

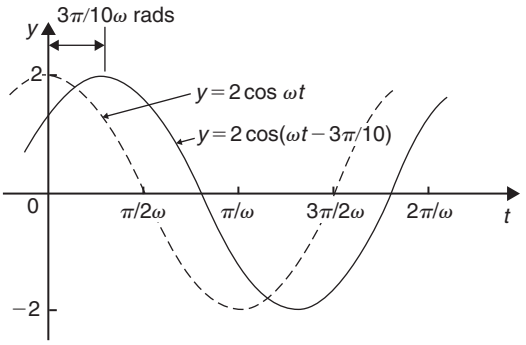


Figure 14.24

Graphs of $\sin^2 A$ and $\cos^2 A$

(i) A graph of $y = \sin^2 A$ is shown in Fig. 14.25 using the following table of values.

$A°$	$\sin A$	$(\sin A)^2 = \sin^2 A$
0	0	0
30	0.50	0.25
60	0.866	0.75
90	1.0	1.0
120	0.866	0.75
150	0.50	0.25
180	0	0
210	−0.50	0.25
240	−0.866	0.75
270	−1.0	1.0
300	−0.866	0.75
330	−0.50	0.25
360	0	0

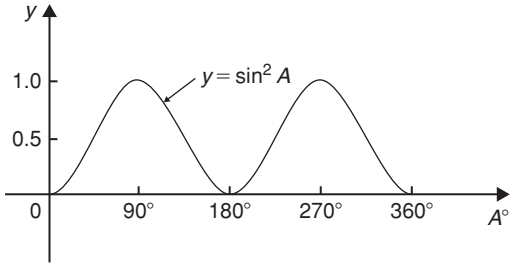


Figure 14.25

(ii) A graph of $y = \cos^2 A$ is shown in Fig. 14.26 obtained by drawing up a table of values, similar to above.

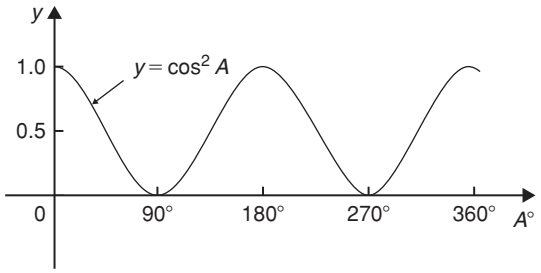


Figure 14.26

(iii) $y = \sin^2 A$ and $y = \cos^2 A$ are both periodic functions of period 180° (or π rad) and both contain only positive values. Thus a graph of $y = \sin^2 2A$ has a period 180°/2, i.e. 90°. Similarly, a graph of $y = 4\cos^2 3A$ has a maximum value of 4 and a period of 180°/3, i.e. 60°.

Problem 12. Sketch $y = 3\sin^2 \frac{1}{2}A$ in the range $0 < A < 360°$.

Maximum value = 3; period = 180°/(1/2) = 360°.
A sketch of $3\sin^2 \frac{1}{2}A$ is shown in Fig. 14.27.

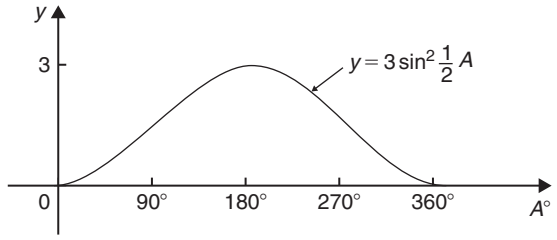


Figure 14.27

Problem 13. Sketch $y = 7\cos^2 2A$ between $A = 0°$ and $A = 360°$.

Maximum value = 7; period = 180°/2 = 90°.
A sketch of $y = 7\cos^2 2A$ is shown in Fig. 14.28.

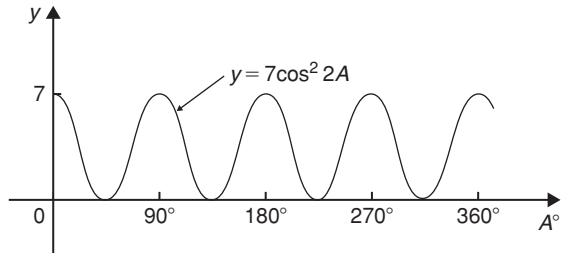


Figure 14.28

Now try the following exercise

Exercise 62 Further problems on sine and cosine curves

In Problems 1 to 9 state the amplitude and period of the waveform and sketch the curve between 0° and 360°.

1. $y = \cos 3A$ [1, 120°]
2. $y = 2\sin\frac{5x}{2}$ [2, 144°]
3. $y = 3\sin 4t$ [3, 90°]
4. $y = 3\cos\frac{\theta}{2}$ [3, 720°]
5. $y = \frac{7}{2}\sin\frac{3x}{8}$ $\left[\frac{7}{2}, 960°\right]$
6. $y = 6\sin(t - 45°)$ [6, 360°]
7. $y = 4\cos(2\theta + 30°)$ [4, 180°]
8. $y = 2\sin^2 2t$ [2, 90°]
9. $y = 5\cos^2\frac{3}{2}\theta$ [5, 120°]

14.5 Sinusoidal form $A\sin(\omega t \pm \alpha)$

In Fig. 14.29, let OR represent a vector that is free to rotate anticlockwise about O at a velocity of ω rad/s. A rotating vector is called a **phasor**. After a time t seconds OR will have turned through an angle ωt radians (shown as angle TOR in Fig. 14.29). If ST is constructed perpendicular to OR, then $\sin\omega t = ST/TO$, i.e. $ST = TO\sin\omega t$.

If all such vertical components are projected on to a graph of y against ωt, a sine wave results of amplitude OR (as shown in Section 14.3).

If phasor OR makes one revolution (i.e. 2π radians) in T seconds, then the angular velocity, $\omega = 2\pi/T$ rad/s, from which, $\boldsymbol{T = 2\pi/\omega}$ **seconds.** T is known as the **periodic time**.

The number of complete cycles occurring per second is called the **frequency, f**

$$\text{Frequency} = \frac{\text{number of cycles}}{\text{second}} = \frac{1}{T}$$

$$= \frac{\omega}{2\pi} \text{ i.e. } \boldsymbol{f = \frac{\omega}{2\pi}\ \text{Hz}}$$

Hence **angular velocity**, $\boldsymbol{\omega = 2\pi f}$ **rad/s**

Amplitude is the name given to the maximum or peak value of a sine wave, as explained in Section 14.3. The amplitude of the sine wave shown in Fig. 14.29 has an amplitude of 1.

A sine or cosine wave may not always start at 0°. To show this a periodic function is represented by $y = \sin(\omega t \pm \alpha)$ or $y = \cos(\omega t \pm \alpha)$, where α is a phase displacement compared with $y = \sin A$ or $y = \cos A$. A graph of $y = \sin(\omega t - \alpha)$ **lags** $y = \sin\omega t$ by angle

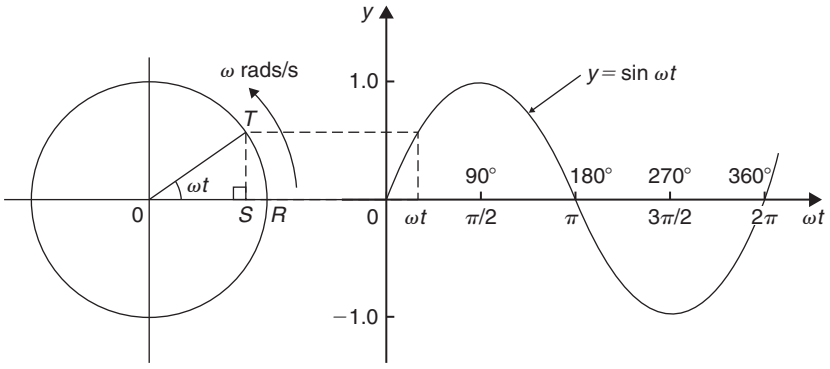


Figure 14.29

α, and a graph of $y = \sin(\omega t + \alpha)$ **leads** $y = \sin \omega t$ by angle α.

The angle ωt is measured in **radians** (i.e. $\left(\omega \frac{\text{rad}}{\text{s}}\right)(t\,\text{s}) = \omega t$ radians) hence angle α should also be in radians.

The relationship between degrees and radians is:

$$360^\circ = 2\pi \text{ radians or } \mathbf{180^\circ = \pi \text{ radians}}$$

Hence $1\text{ rad} = \frac{180}{\pi} = 57.30^\circ$ and, for example, $71^\circ = 71 \times \frac{\pi}{180} = 1.239\,\text{rad}$.

Given a general sinusoidal function $\boldsymbol{y = A\sin(\omega t \pm \alpha)}$, then

(i) A = amplitude

(ii) ω = angular velocity $= 2\pi f$ rad/s

(iii) $\frac{2\pi}{\omega}$ = periodic time T seconds

(iv) $\frac{\omega}{2\pi}$ = frequency, f hertz

(v) α = angle of lead or lag (compared with $y = A\sin\omega t$)

Problem 14. An alternating current is given by $i = 30\sin(100\pi t + 0.27)$ amperes. Find the amplitude, periodic time, frequency and phase angle (in degrees and minutes).

$i = 30\sin(100\pi t + 0.27)$ A, hence **amplitude = 30 A**

Angular velocity $\omega = 100\pi$, hence

$$\textbf{periodic time, } T = \frac{2\pi}{\omega} = \frac{2\pi}{100\pi} = \frac{1}{50}$$

$$= \mathbf{0.02\,s} \text{ or } \mathbf{20\,ms}$$

$$\textbf{Frequency, } f = \frac{1}{T} = \frac{1}{0.02} = \mathbf{50\,Hz}$$

$$\textbf{Phase angle, } \boldsymbol{\alpha} = 0.27\,\text{rad} = \left(0.27 \times \frac{180}{\pi}\right)^\circ$$

$$= \mathbf{15.47^\circ} \text{ or } \mathbf{15^\circ 28'} \textbf{ leading}$$

$$\boldsymbol{i = 30\sin(100\pi t)}$$

Problem 15. An oscillating mechanism has a maximum displacement of 2.5 m and a frequency of 60 Hz. At time $t = 0$ the displacement is 90 cm. Express the displacement in the general form $A\sin(\omega t \pm \alpha)$.

Amplitude = maximum displacement = 2.5 m.

Angular velocity, $\omega = 2\pi f = 2\pi(60) = 120\pi$ rad/s.

Hence displacement $= 2.5\sin(120\pi t + \alpha)$ m.

When $t = 0$, displacement = 90 cm = 0.90 m.

Hence $0.90 = 2.\sin(0 + \alpha)$

i.e. $\sin\alpha = \frac{0.90}{2.5} = 0.36$

Hence $\alpha = \arcsin 0.36 = 21.10^\circ = 21^\circ 6'$

$= 0.368\,\text{rad}$

Thus **displacement = 2.5 sin(120πt + 0.368) m**

Problem 16. The instantaneous value of voltage in an a.c. circuit at any time t seconds is given by $v = 340\sin(50\pi t - 0.541)$ volts. Determine:

(a) the amplitude, periodic time, frequency and phase angle (in degrees)

(b) the value of the voltage when $t = 0$

(c) the value of the voltage when $t = 10$ ms

(d) the time when the voltage first reaches 200 V, and

(e) the time when the voltage is a maximum.

Sketch one cycle of the waveform.

(a) **Amplitude = 340 V**

Angular velocity, $\omega = 50\pi$

$$\text{Hence } \textbf{periodic time, } T = \frac{2\pi}{\omega} = \frac{2\pi}{50\pi} = \frac{1}{25}$$

$$= \mathbf{0.04\,s} \text{ or } \mathbf{40\,ms}$$

$$\textbf{Frequency, } f = \frac{1}{T} = \frac{1}{0.04} = \mathbf{25}\,\text{Hz}$$

$$\textbf{Phase angle} = 0.541\text{rad} = \left(0.541 \times \frac{180}{\pi}\right)$$

$$= \mathbf{31^\circ\ lagging}\ v = 340\sin(50\pi t)$$

(b) **When $t = 0$,**

$$v = 340\sin(0 - 0.541) = 340\sin(-31^\circ)$$

$$= \mathbf{-175.1\,V}$$

(c) **When $t = 10$ ms**

$$\text{then } v = 340\sin\left(50\pi\frac{10}{10^3} - 0.541\right)$$

$$= 340\sin(1.0298) = 340\sin 59°$$

$$= \mathbf{291.4\,V}$$

(d) When $v = 200$ volts

$$\text{then } 200 = 340\sin(50\pi t - 0.541)$$

$$\frac{200}{340} = \sin(50\pi t - 0.541)$$

$$\text{Hence } (50\pi t - 0.541) = \arcsin\frac{200}{340}$$

$$= 36.03° \text{ or } 0.6288\,\text{rad}$$

$$50\pi t = 0.6288 + 0.541$$

$$= 1.1698$$

Hence when $v = 200$ V,

$$\textbf{time, } t = \frac{1.1698}{50\pi} = \mathbf{7.447\,ms}$$

(e) When the voltage is a maximum, $v = 340$ V.

$$\text{Hence } 340 = 340\sin(50\pi t - 0.541)$$

$$1 = \sin(50\pi t - 0.541)$$

$$50\pi t - 0.541 = \arcsin 1$$

$$= 90° \text{ or } 1.5708\,\text{rad}$$

$$50\pi t = 1.5708 + 0.541 = 2.1118$$

$$\text{Hence time, } t = \frac{2.1118}{50\pi} = \mathbf{13.44\,ms}$$

A sketch of $v = 340\sin(50\pi t - 0.541)$ volts is shown in Fig. 14.30.

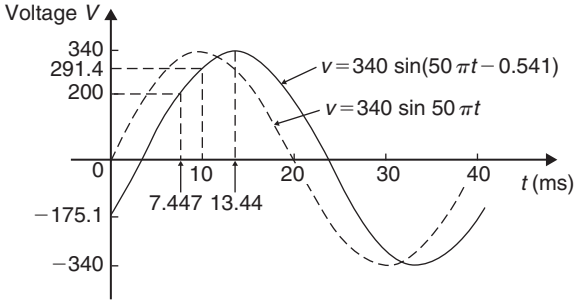


Figure 14.30

Now try the following exercise

Exercise 63 Further problems on the sinusoidal form $A\sin(\omega t \pm \alpha)$

In Problems 1 to 3 find the amplitude, periodic time, frequency and phase angle (stating whether it is leading or lagging $A\sin\omega t$) of the alternating quantities given.

1. $i = 40\sin(50\pi t + 0.29)$ mA

   [40, 0.04 s, 25 Hz, 0.29 rad (or 16°37′) leading $40\sin 50\pi t$]

2. $y = 75\sin(40t - 0.54)$ cm

   [75 cm, 0.157 s, 6.37 Hz, 0.54 rad (or 30°56′) lagging $75\sin 40t$]

3. $v = 300\sin(200\pi t - 0.412)$ V

   [300 V, 0.01 s, 100 Hz, 0.412 rad (or 23°36′) lagging $300\sin 200\pi t$]

4. A sinusoidal voltage has a maximum value of 120 V and a frequency of 50 Hz. At time $t = 0$, the voltage is (a) zero, and (b) 50 V. Express the instantaneous voltage v in the form $v = A\sin(\omega t \pm \alpha)$.

   [(a) $v = 120\sin 100\pi t$ volts
   (b) $v = 120\sin(100\pi t + 0.43)$ volts]

5. An alternating current has a periodic time of 25 ms and a maximum value of 20 A. When time $t = 0$, current $i = -10$ amperes. Express the current i in the form $i = A\sin(\omega t \pm \alpha)$.

   $\left[i = 20\sin\left(80\pi t - \frac{\pi}{6}\right) \text{ amperes}\right]$

6. An oscillating mechanism has a maximum displacement of 3.2 m and a frequency of 50 Hz. At time $t = 0$ the displacement is 150 cm. Express the displacement in the general form $A\sin(\omega t \pm \alpha)$.

   [$3.2\sin(100\pi t + 0.488)$ m]

7. The current in an a.c. circuit at any time t seconds is given by:

   $$i = 5\sin(100\pi t - 0.432) \text{ amperes}$$

   Determine (a) the amplitude, periodic time, frequency and phase angle (in degrees) (b) the value of current at $t = 0$ (c) the value of current at $t = 8$ ms (d) the time when the current is first a maximum (e) the time when the current first

reaches 3A. Sketch one cycle of the waveform showing relevant points.

[(a) 5 A, 20 ms, 50 Hz, 24°45′ lagging
(b) −2.093 A
(c) 4.363 A
(d) 6.375 ms
(e) 3.423 ms]

14.6 Harmonic synthesis with complex waveforms

A waveform that is not sinusoidal is called a **complex wave**. **Harmonic analysis** is the process of resolving a complex periodic waveform into a series of sinusoidal components of ascending order of frequency. Many of the waveforms met in practice can be represented by the following mathematical expression.

$$v = V_{1m}\sin(\omega t + \alpha_1) + V_{2m}\sin(2\omega t + \alpha_2) + \cdots + V_{nm}\sin(n\omega t + \alpha_n)$$

and the magnitude of their harmonic components together with their phase may be calculated using **Fourier series** (see Chapters 66 to 69). **Numerical methods** are used to analyse waveforms for which simple mathematical expressions cannot be obtained. A numerical method of harmonic analysis is explained in the Chapter 70 on page 637. In a laboratory, waveform analysis may be performed using a **waveform analyser** which produces a direct readout of the component waves present in a complex wave.

By adding the instantaneous values of the fundamental and progressive harmonics of a complex wave for given instants in time, the shape of a complex waveform can be gradually built up. This graphical procedure is known as **harmonic synthesis** (synthesis meaning 'the putting together of parts or elements so as to make up a complex whole').

Some examples of harmonic synthesis are considered in the following worked problems.

Problem 17. Use harmonic synthesis to construct the complex voltage given by:

$$v_1 = 100\sin\omega t + 30\sin 3\omega t \text{ volts.}$$

The waveform is made up of a fundamental wave of maximum value 100 V and frequency, $f = \omega/2\pi$ hertz and a third harmonic component of maximum value 30 V and frequency $= 3\omega/2\pi\,(=3f)$, the fundamental and third harmonics being initially in phase with each other.

In Fig. 14.31, the fundamental waveform is shown by the broken line plotted over one cycle, the periodic time T being $2\pi/\omega$ seconds. On the same axis is plotted $30\sin 3\omega t$, shown by the dotted line, having a maximum value of 30 V and for which three cycles are completed in time T seconds. At zero time, $30\sin 3\omega t$ is in phase with $100\sin\omega t$.

The fundamental and third harmonic are combined by adding ordinates at intervals to produce the waveform for v_1, as shown. For example, at time $T/12$ seconds, the fundamental has a value of 50 V and the third harmonic a value of 30 V. Adding gives a value of 80 V for waveform v_1 at time $T/12$ seconds. Similarly, at time $T/4$ seconds, the fundamental has a value of 100 V and the third harmonic a value of −30 V. After addition, the resultant waveform v_1 is 70 V at $T/4$. The procedure is continued between $t = 0$ and $t = T$ to produce the complex waveform for v_1. The negative half-cycle of waveform v_1 is seen to be identical in shape to the positive half-cycle.

If further odd harmonics of the appropriate amplitude and phase were added to v_1 a good approximation to a **square wave** would result.

Problem 18. Construct the complex voltage given by:

$$v_2 = 100\sin\omega t + 30\sin\left(3\omega t + \frac{\pi}{2}\right) \text{ volts.}$$

The peak value of the fundamental is 100 volts and the peak value of the third harmonic is 30 V. However the third harmonic has a phase displacement of $\frac{\pi}{2}$ radian leading (i.e. leading $30\sin 3\omega t$ by $\frac{\pi}{2}$ radian). Note that, since the periodic time of the fundamental is T seconds, the periodic time of the third harmonic is $T/3$ seconds, and a phase displacement of $\frac{\pi}{2}$ radian or $\frac{1}{4}$ cycle of the third harmonic represents a time interval of $(T/3) \div 4$, i.e. $T/12$ seconds.

Figure 14.32 shows graphs of $100\sin\omega t$ and $30\sin\left(3\omega t + \frac{\pi}{2}\right)$ over the time for one cycle of the fundamental. When ordinates of the two graphs are added at intervals, the resultant waveform v_2 is as shown. If the negative half-cycle in Fig. 14.32 is reversed it can be seen that the shape of the positive and negative half-cycles are identical.

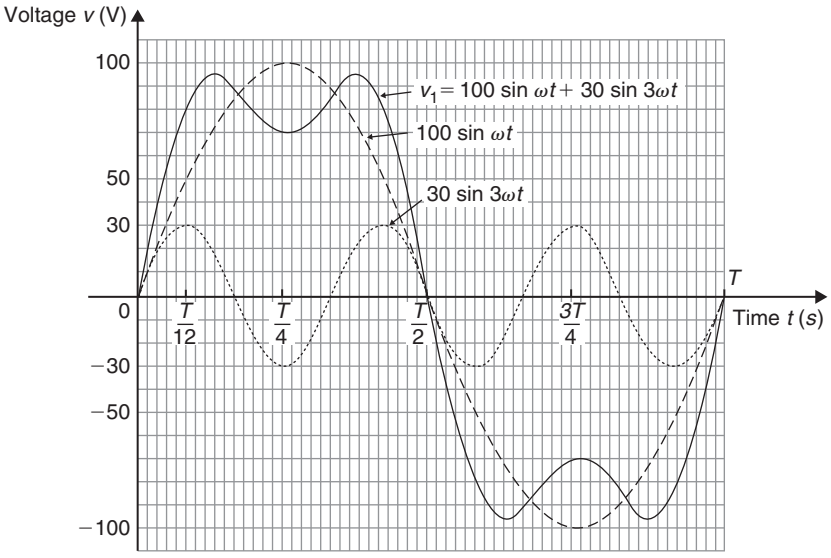


Figure 14.31

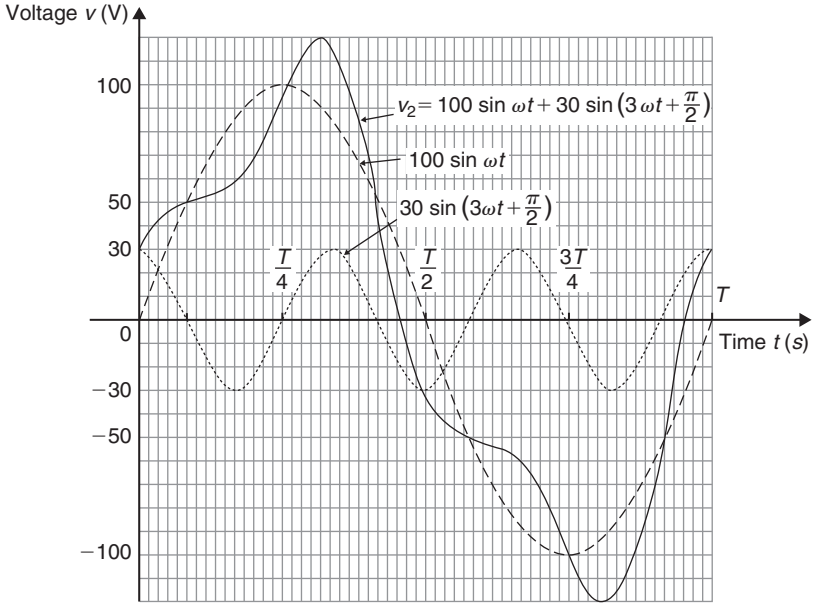


Figure 14.32

Problems 17 and 18 demonstrate that whenever odd harmonics are added to a fundamental waveform, whether initially in phase with each other or not, the positive and negative half-cycles of the resultant complex wave are identical in shape. This is a feature of waveforms containing the fundamental and odd harmonics.

Problem 19. Use harmonic synthesis to construct the complex current given by:

$$i_1 = 10\sin\omega t + 4\sin 2\omega t \text{ amperes.}$$

Current i_1 consists of a fundamental compon- ent, $10\sin\omega t$, and a second harmonic component, $4\sin 2\omega t$,

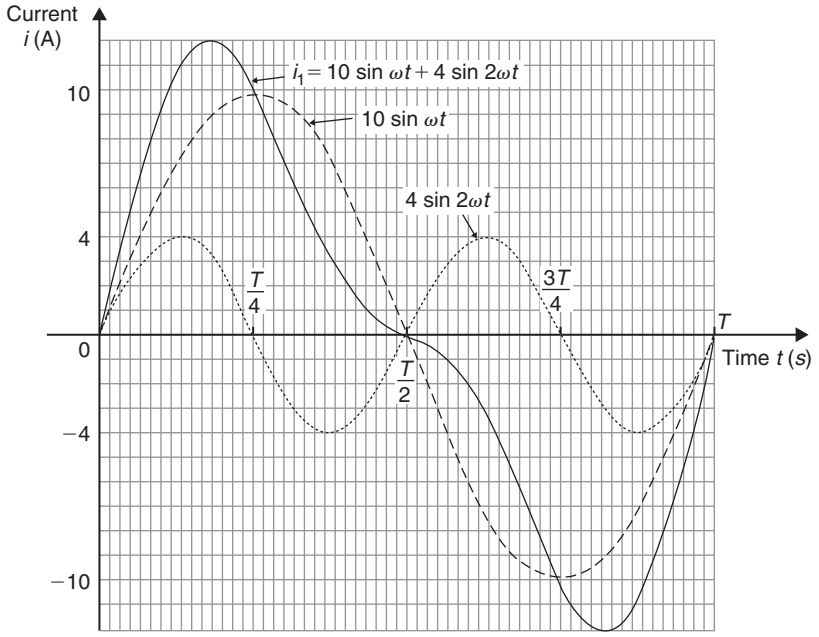


Figure 14.33

the components being initially in phase with each other. The fundamental and second harmonic are shown plotted separately in Fig. 14.33. By adding ordinates at intervals, the complex waveform representing i_1 is produced as shown. It is noted that if all the values in the negative half-cycle were reversed then this half-cycle would appear as a mirror image of the positive half-cycle about a vertical line drawn through time, $t = T/2$.

Problem 20. Construct the complex current given by:

$$i_2 = 10\sin\omega t + 4\sin\left(2\omega t + \frac{\pi}{2}\right) \text{ amperes.}$$

The fundamental component, $10\sin\omega t$, and the second harmonic component, having an amplitude of 4 A and a phase displacement of $\frac{\pi}{2}$ radian leading (i.e. leading $4\sin 2\omega t$ by $\frac{\pi}{2}$ radian or $T/8$ seconds), are shown plotted separately in Fig. 14.34. By adding ordinates at intervals, the complex waveform for i_2 is produced as shown. The positive and negative half-cycles of the resultant waveform are seen to be quite dissimilar.

From Problems 18 and 19 it is seen that whenever even harmonics are added to a fundamental component:

(a) if the harmonics are initially in phase, the negative half-cycle, when reversed, is a mirror image of the positive half-cycle about a vertical line drawn through time, $t = T/2$.

(b) if the harmonics are initially out of phase with each other, the positive and negative half-cycles are dissimilar.

These are features of waveforms containing the fundamental and even harmonics.

Problem 21. Use harmonic synthesis to construct the complex current expression given by:

$$i = 32 + 50\sin\omega t + 20\sin\left(2\omega t - \frac{\pi}{2}\right) \text{ mA.}$$

The current i comprises three components—a 32 mA d.c. component, a fundamental of amplitude 50 mA and a second harmonic of amplitude 20 mA, lagging by $\frac{\pi}{2}$ radian. The fundamental and second harmonic are shown separately in Fig. 14.35. Adding ordinates at intervals gives the complex waveform $50\sin\omega t + 20\sin\left(2\omega t - \frac{\pi}{2}\right)$.

This waveform is then added to the 32 mA d.c. component to produce the waveform i as shown. The effect of the d.c. component is to shift the whole wave 32 mA upward. The waveform approaches that expected from a **half-wave rectifier**.

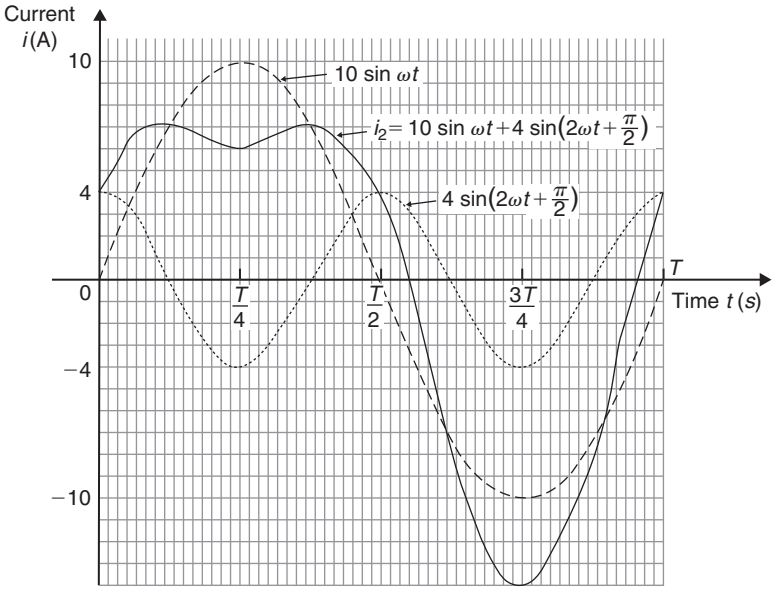


Figure 14.34

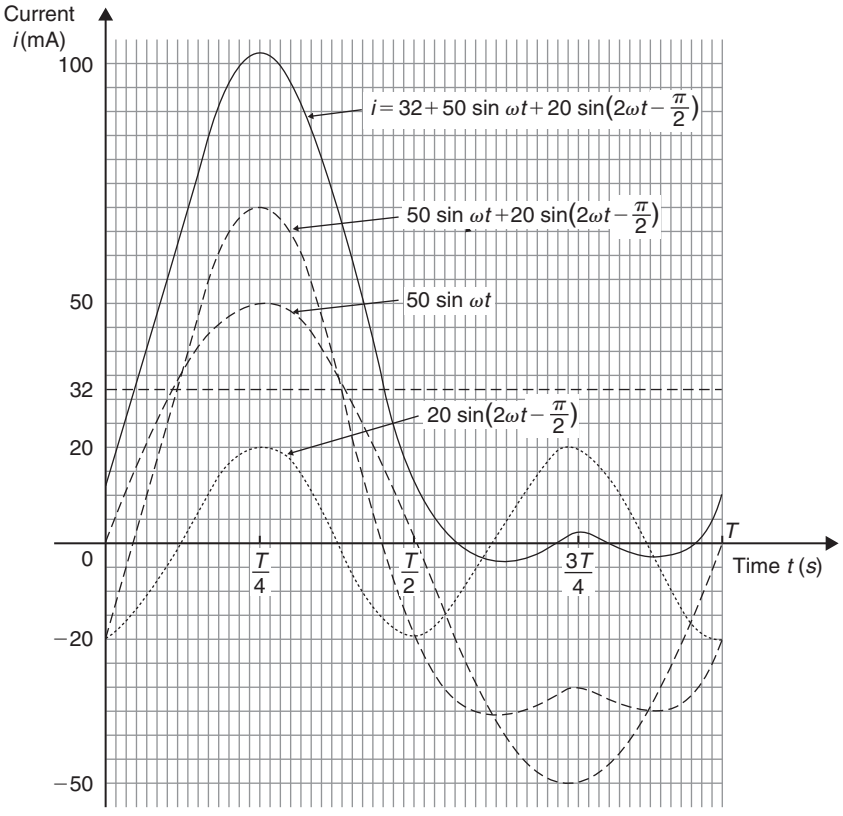


Figure 14.35

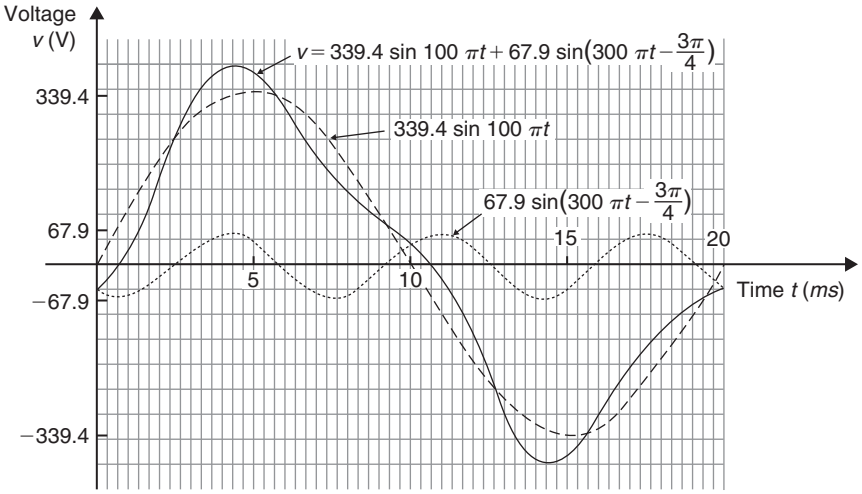


Figure 14.36

Problem 22. A complex waveform v comprises a fundamental voltage of 240 V rms and frequency 50 Hz, together with a 20% third harmonic which has a phase angle lagging by $3\pi/4$ rad at time $t = 0$. (a) Write down an expression to represent voltage v. (b) Use harmonic synthesis to sketch the complex waveform representing voltage v over one cycle of the fundamental component.

(a) A fundamental voltage having an rms value of 240 V has a maximum value, or amplitude of $\sqrt{2}\,(240)$ i.e. 339.4 V.

If the fundamental frequency is 50 Hz then angular velocity, $\omega = 2\pi f = 2\pi(50) = 100\pi$ rad/s. Hence the fundamental voltage is represented by $339.4\sin 100\pi t$ volts. Since the fundamental frequency is 50 Hz, the time for one cycle of the fundamental is given by $T = 1/f = 1/50$ s or 20 ms.

The third harmonic has an amplitude equal to 20% of 339.4 V, i.e. 67.9 V. The frequency of the third harmonic component is $3 \times 50 = 150$ Hz, thus the angular velocity is $2\pi(150)$, i.e. 300π rad/s. Hence the third harmonic voltage is represented by $67.9\sin(300\pi t - 3\pi/4)$ volts. Thus

$$\textbf{voltage, } v = \mathbf{339.4\sin 100\pi t} + \mathbf{67.9\sin(300\pi t - 3\pi/4)\ volts}$$

(b) One cycle of the fundamental, $339.4\sin 100\pi t$, is shown sketched in Fig. 14.36, together with three cycles of the third harmonic component, $67.9\sin(300\pi t - 3\pi/4)$ initially lagging by $3\pi/4$ rad. By adding ordinates at intervals, the complex waveform representing voltage is produced as shown.

Now try the following exercise

Exercise 64 Further problems on harmonic synthesis with complex waveforms

1. A complex current waveform i comprises a fundamental current of 50 A rms and frequency 100 Hz, together with a 24% third harmonic, both being in phase with each other at zero time. (a) Write down an expression to represent current i. (b) Sketch the complex waveform of current using harmonic synthesis over one cycle of the fundamental.

$$\left[\text{(a) } i = (70.71\sin 628.3t + 16.97\sin 1885t)\ A\right]$$

2. A complex voltage waveform v is comprised of a 212.1 V rms fundamental voltage at a frequency of 50 Hz, a 30% second harmonic component lagging by $\pi/2$ rad, and a 10% fourth harmonic component leading by $\pi/3$ rad. (a) Write down an expression to represent voltage v. (b) Sketch the complex voltage

waveform using harmonic synthesis over one cycle of the fundamental waveform.

$$\left[\begin{array}{l}\text{(a) } v = 300\sin 314.2t \\ \qquad + 90\sin(628.3t - \pi/2) \\ \qquad + 30\sin(1256.6t + \pi/3)\text{ V}\end{array}\right]$$

3. A voltage waveform is represented by:

$$v = 20 + 50\sin\omega t + 20\sin(2\omega t - \pi/2)\text{ volts.}$$

Draw the complex waveform over one cycle of the fundamental by using harmonic synthesis.

4. Write down an expression representing a current i having a fundamental component of amplitude 16 A and frequency 1 kHz, together with its third and fifth harmonics being respectively one-fifth and one-tenth the amplitude of the fundamental, all components being in phase at zero time. Sketch the complex current waveform for one cycle of the fundamental using harmonic synthesis.

$$\left[\begin{array}{c} i = 16\sin 2\pi 10^3 t + 3.2\sin 6\pi 10^3 t \\ + 1.6\sin \pi 10^4 t\text{ A}\end{array}\right]$$

5. A voltage waveform is described by

$$v = 200\sin 377t + 80\sin\left(1131t + \frac{\pi}{4}\right) + 20\sin\left(1885t - \frac{\pi}{3}\right)\text{ volts}$$

Determine (a) the fundamental and harmonic frequencies of the waveform (b) the percentage third harmonic and (c) the percentage fifth harmonic. Sketch the voltage waveform using harmonic synthesis over one cycle of the fundamental.

$$\left[\begin{array}{l}\text{(a) 60 Hz, 180 Hz, 300 Hz} \\ \text{(b) 40\%} \\ \text{(c) 10\%}\end{array}\right]$$

Chapter 15

Trigonometric identities and equations

15.1 Trigonometric identities

A trigonometric identity is a relationship that is true for all values of the unknown variable.

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \cot\theta = \frac{\cos\theta}{\sin\theta}, \sec\theta = \frac{1}{\cos\theta}$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} \text{ and } \cot\theta = \frac{1}{\tan\theta}$$

are examples of trigonometric identities from Chapter 11.

Applying Pythagoras' theorem to the right-angled triangle shown in Fig. 15.1 gives:

$$a^2 + b^2 = c^2 \qquad (1)$$

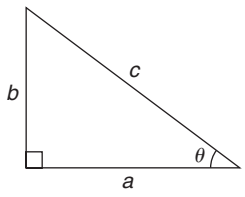


Figure 15.1

Dividing each term of equation (1) by c^2 gives:

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{c^2}{c^2}$$

i.e. $$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$$

$$(\cos\theta)^2 + (\sin\theta)^2 = 1$$

Hence $$\mathbf{\cos^2\theta + \sin^2\theta = 1} \qquad (2)$$

Dividing each term of equation (1) by a^2 gives:

$$\frac{a^2}{a^2} + \frac{b^2}{a^2} = \frac{c^2}{a^2}$$

i.e. $$1 + \left(\frac{b}{a}\right)^2 = \left(\frac{c}{a}\right)^2$$

Hence $$\mathbf{1 + \tan^2\theta = \sec^2\theta} \qquad (3)$$

Dividing each term of equation (1) by b^2 gives:

$$\frac{a^2}{b^2} + \frac{b^2}{b^2} = \frac{c^2}{b^2}$$

i.e. $$\left(\frac{a}{b}\right)^2 + 1 = \left(\frac{c}{b}\right)^2$$

Hence $$\mathbf{\cot^2\theta + 1 = \operatorname{cosec}^2\theta} \qquad (4)$$

Equations (2), (3) and (4) are three further examples of trigonometric identities. For the proof of further trigonometric identities, see Section 15.2.

15.2 Worked problems on trigonometric identities

Problem 1. Prove the identity $\sin^2\theta\cot\theta\sec\theta = \sin\theta$.

With trigonometric identities it is necessary to start with the left-hand side (LHS) and attempt to make it equal to

the right-hand side (RHS) or vice-versa. It is often useful to change all of the trigonometric ratios into sines and cosines where possible. Thus,

$$\begin{aligned}\text{LHS} &= \sin^2\theta\cot\theta\sec\theta \\ &= \sin^2\theta\left(\frac{\cos\theta}{\sin\theta}\right)\left(\frac{1}{\cos\theta}\right) \\ &= \sin\theta \text{ (by cancelling)} = \text{RHS}\end{aligned}$$

Problem 2. Prove that

$$\frac{\tan x+\sec x}{\sec x\left(1+\dfrac{\tan x}{\sec x}\right)} = 1.$$

$$\begin{aligned}\text{LHS} &= \frac{\tan x+\sec x}{\sec x\left(1+\dfrac{\tan x}{\sec x}\right)} \\ &= \frac{\dfrac{\sin x}{\cos x}+\dfrac{1}{\cos x}}{\left(\dfrac{1}{\cos x}\right)\left(1+\dfrac{\dfrac{\sin x}{\cos x}}{\dfrac{1}{\cos x}}\right)} \\ &= \frac{\dfrac{\sin x+1}{\cos x}}{\left(\dfrac{1}{\cos x}\right)\left[1+\left(\dfrac{\sin x}{\cos x}\right)\left(\dfrac{\cos x}{1}\right)\right]} \\ &= \frac{\dfrac{\sin x+1}{\cos x}}{\left(\dfrac{1}{\cos x}\right)[1+\sin x]} \\ &= \left(\frac{\sin x+1}{\cos x}\right)\left(\frac{\cos x}{1+\sin x}\right) \\ &= 1 \text{ (by cancelling)} = \text{RHS}\end{aligned}$$

Problem 3. Prove that $\dfrac{1+\cot\theta}{1+\tan\theta} = \cot\theta.$

$$\begin{aligned}\text{LHS} &= \frac{1+\cot\theta}{1+\tan\theta} \\ &= \frac{1+\dfrac{\cos\theta}{\sin\theta}}{1+\dfrac{\sin\theta}{\cos\theta}} = \frac{\dfrac{\sin\theta+\cos\theta}{\sin\theta}}{\dfrac{\cos\theta+\sin\theta}{\cos\theta}} \\ &= \left(\frac{\sin\theta+\cos\theta}{\sin\theta}\right)\left(\frac{\cos\theta}{\cos\theta+\sin\theta}\right) \\ &= \frac{\cos\theta}{\sin\theta} = \cot\theta = \text{RHS}\end{aligned}$$

Problem 4. Show that $\cos^2\theta-\sin^2\theta = 1-2\sin^2\theta.$

From equation (2), $\cos^2\theta+\sin^2\theta=1$, from which, $\cos^2\theta = 1-\sin^2\theta$.

Hence, LHS

$$\begin{aligned}&= \cos^2\theta-\sin^2\theta = (1-\sin^2\theta)-\sin^2\theta \\ &= 1-\sin^2\theta-\sin^2\theta = 1-2\sin^2\theta = \text{RHS}\end{aligned}$$

Problem 5. Prove that

$$\sqrt{\left(\frac{1-\sin x}{1+\sin x}\right)} = \sec x-\tan x.$$

$$\begin{aligned}\text{LHS} &= \sqrt{\left(\frac{1-\sin x}{1+\sin x}\right)} = \sqrt{\left\{\frac{(1-\sin x)(1-\sin x)}{(1+\sin x)(1-\sin x)}\right\}} \\ &= \sqrt{\left\{\frac{(1-\sin x)^2}{(1-\sin^2 x)}\right\}}\end{aligned}$$

Since $\cos^2 x+\sin^2 x=1$ then $1-\sin^2 x=\cos^2 x$

$$\begin{aligned}\text{LHS} &= \sqrt{\left\{\frac{(1-\sin x)^2}{(1-\sin^2 x)}\right\}} = \sqrt{\left\{\frac{(1-\sin x)^2}{\cos^2 x}\right\}} \\ &= \frac{1-\sin x}{\cos x} = \frac{1}{\cos x}-\frac{\sin x}{\cos x} \\ &= \sec x-\tan x = \text{RHS}\end{aligned}$$

Now try the following exercise

Exercise 65 Further problems on trigonometric identities

In Problems 1 to 6 prove the trigonometric identities.

1. $\sin x\cot x = \cos x$
2. $\dfrac{1}{\sqrt{(1-\cos^2\theta)}} = \operatorname{cosec}\theta$

3. $2\cos^2 A - 1 = \cos^2 A - \sin^2 A$
4. $\dfrac{\cos x - \cos^3 x}{\sin x} = \sin x \cos x$
5. $(1 + \cot\theta)^2 + (1 - \cot\theta)^2 = 2\,\text{cosec}^2\theta$
6. $\dfrac{\sin^2 x(\sec x + \text{cosec}\, x)}{\cos x \tan x} = 1 + \tan x$

15.3 Trigonometric equations

Equations which contain trigonometric ratios are called **trigonometric equations**. There are usually an infinite number of solutions to such equations; however, solutions are often restricted to those between 0° and 360°.

A knowledge of angles of any magnitude is essential in the solution of trigonometric equations and calculators cannot be relied upon to give all the solutions (as shown in Chapter 14). Fig. 15.2 shows a summary for angles of any magnitude.

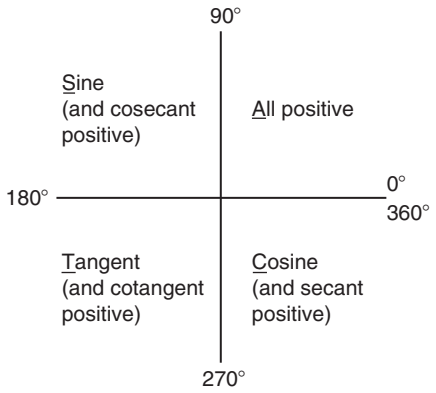


Figure 15.2

Equations of the type $a\sin^2 A + b\sin A + c = 0$

(i) **When $a = 0$**, $b\sin A + c = 0$, hence $\sin A = -\dfrac{c}{b}$ and $\boldsymbol{A = \sin^{-1}\left(-\dfrac{c}{b}\right)}$
There are two values of A between 0° and 360° which satisfy such an equation, provided $-1 \leq \dfrac{c}{b} \leq 1$ (see Problems 6 to 8).

(ii) **When $b = 0$**, $a\sin^2 A + c = 0$, hence $\sin^2 A = -\dfrac{c}{a}$, $\sin A = \sqrt{\left(-\dfrac{c}{a}\right)}$
and $\boldsymbol{A = \sin^{-1}\sqrt{\left(-\dfrac{c}{a}\right)}}$
If either a or c is a negative number, then the value within the square root sign is positive. Since when a square root is taken there is a positive and negative answer there are four values of A between 0° and 360° which satisfy such an equation, provided $-1 \leq \dfrac{c}{a} \leq 1$ (see Problems 9 and 10).

(iii) **When a, b and c are all non-zero**:
$a\sin^2 A + b\sin A + c = 0$ is a quadratic equation in which the unknown is $\sin A$. The solution of a quadratic equation is obtained either by factorizing (if possible) or by using the quadratic formula:

$$\boldsymbol{\sin A = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}}$$

(see Problems 11 and 12).

(iv) Often the trigonometric identities $\cos^2 A + \sin^2 A = 1$, $1 + \tan^2 A = \sec^2 A$ and $\cot^2 A + 1 = \text{cosec}^2 A$ need to be used to reduce equations to one of the above forms (see Problems 13 to 15).

15.4 Worked problems (i) on trigonometric equations

Problem 6. Solve the trigonometric equation $5\sin\theta + 3 = 0$ for values of θ from 0° to 360°.

$5\sin\theta + 3 = 0$, from which $\sin\theta = -\frac{3}{5} = -0.6000$

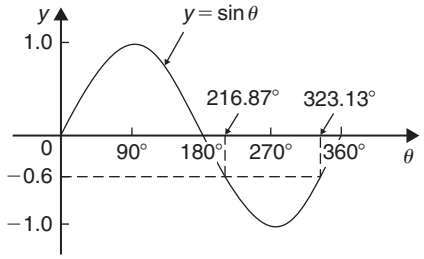
Hence $\theta = \sin^{-1}(-0.6000)$. Sine is negative in the third and fourth quadrants (see Fig. 15.3). The acute angle $\sin^{-1}(0.6000) = 36.87°$ (shown as α in Fig. 15.3(b)). Hence,

$\theta = 180° + 36.87°$, i.e. **216.87°** or

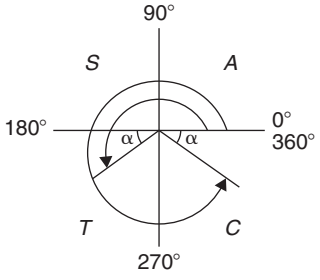
$\theta = 360° - 36.87°$, i.e. **323.13°**

Problem 7. Solve $1.5\tan x - 1.8 = 0$ for $0° \leq x \leq 360°$.

$1.5\tan x - 1.8 = 0$, from which
$\tan x = \dfrac{1.8}{1.5} = 1.2000$.
Hence $x = \tan^{-1} 1.2000$



(a)

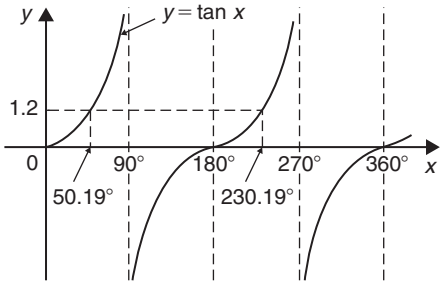


(b)

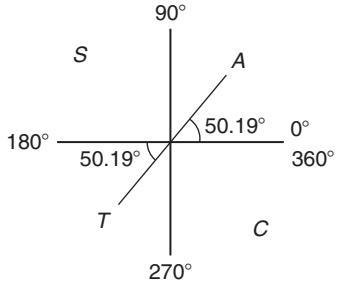
Figure 15.3

Tangent is positive in the first and third quadrants (see Fig. 15.4).
The acute angle $\tan^{-1} 1.2000 = 50.19°$. Hence,

$$x = \mathbf{50.19°} \text{ or } 180° + 50.19° = \mathbf{230.19°}$$



(a)



(b)

Figure 15.4

Problem 8. Solve for θ in the range $0° \leq \theta \leq 360°$ for $2\sin\theta = \cos\theta$

Dividing both sides by $\cos\theta$ gives: $\dfrac{2\sin\theta}{\cos\theta} = 1$

From Section 15.1, $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$,
hence $2\tan\theta = 1$
Dividing by 2 gives: $\tan\theta = \frac{1}{2}$
from which, $\theta = \tan^{-1}\frac{1}{2}$
Since tangent is positive in the first and third quadrants,
$\boldsymbol{\theta = 26.57°}$ and $\mathbf{206.57°}$

Problem 9. Solve $4\sec t = 5$ for values of t between 0° and 360°.

$4\sec t = 5$, from which $\sec t = \frac{5}{4} = 1.2500$
Hence $t = \sec^{-1} 1.2500$
Secant = (1/cosine) is positive in the first and fourth quadrants (see Fig. 15.5) The acute angle $\sec^{-1} 1.2500 = 36.87°$. Hence,

$$t = \mathbf{36.87°} \text{ or } 360° - 36.87° = \mathbf{323.13°}$$

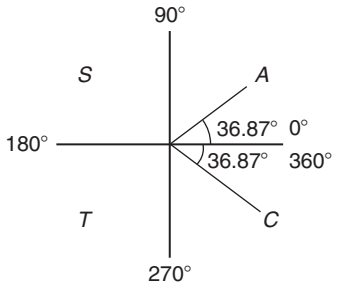


Figure 15.5

Now try the following exercise

Exercise 66 Further problems on trigonometric equations

In Problems 1 to 3 solve the equations for angles between 0° and 360°.

1. $4 - 7\sin\theta = 0$ [$\theta = 34.85°$ or $145.15°$]
2. $3\operatorname{cosec} A + 5.5 = 0$ [$A = 213.06°$ or $326.94°$]
3. $4(2.32 - 5.4\cot t) = 0$ [$t = 66.75°$ or $246.75°$]

In Problems 4 to 6, solve for θ in the range $0° \leq \theta \leq 360°$.

4. $\sec\theta = 2$ [60°, 300°]

5. $\cot\theta = 0.6$ [59°, 239°]

6. $\operatorname{cosec}\theta = 1.5$ [41.81°, 138.19°]

In Problems 7 to 9, solve for x in the range $-180° \leq x \leq 180°$.

7. $\sec x = -1.5$ [±131.81°]

8. $\cot x = 1.2$ [39.81°, −140.19°]

9. $\operatorname{cosec} x = -2$ [–30°, −150°]

In Problem 10 and 11, solve for θ in the range $0° \leq \theta \leq 360°$.

10. $3\sin\theta = 2\cos\theta$ [33.69°, 213.69°]

11. $5\cos\theta = -\sin\theta$ [101.31°, 281.31°]

15.5 Worked problems (ii) on trigonometric equations

Problem 10. Solve $2 - 4\cos^2 A = 0$ for values of A in the range $0° < A < 360°$.

$2 - 4\cos^2 A = 0$, from which $\cos^2 A = \frac{2}{4} = 0.5000$

Hence $\cos A = \sqrt{(0.5000)} = \pm 0.7071$ and $A = \cos^{-1}(\pm 0.7071)$.

Cosine is positive in quadrants one and four and negative in quadrants two and three. Thus in this case there are four solutions, one in each quadrant (see Fig. 15.6). The acute angle $\cos^{-1} 0.7071 = 45°$. Hence,

A = 45°, 135°, 225° or 315°

Problem 11. Solve $\frac{1}{2}\cot^2 y = 1.3$ for $0° < y < 360°$.

$\frac{1}{2}\cot^2 y = 1.3$, from which, $\cot^2 y = 2(1.3) = 2.6$

Hence $\cot y = \sqrt{2.6} = \pm 1.6125$, and $y = \cot^{-1}(\pm 1.6125)$. There are four solutions, one in each quadrant. The acute angle $\cot^{-1} 1.6125 = 31.81°$.

Hence **y = 31.81°, 148.19°, 211.81° or 328.19°**.

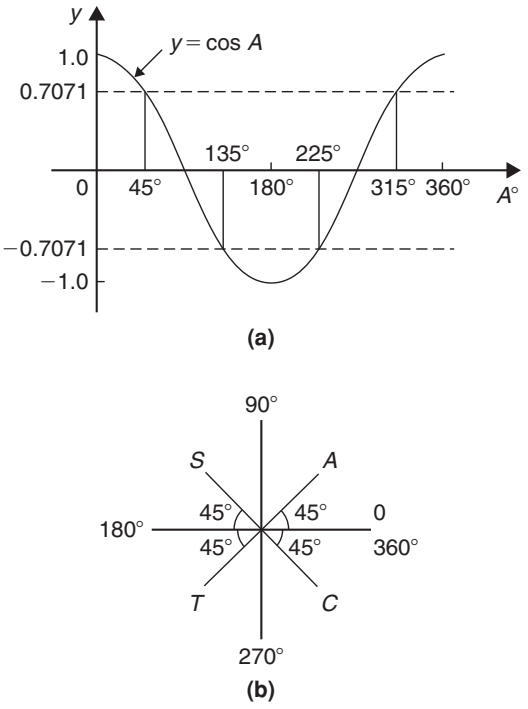


Figure 15.6

Now try the following exercise

Exercise 67 Further problems on trigonometric equations

In Problems 1 to 3 solve the equations for angles between 0° and 360°.

1. $5\sin^2 y = 3$

[y = 50.77°, 129.23°, 230.77° or 309.23°]

2. $\cos^2\theta = 0.25$

[θ = 60°, 120°, 240° or 300°]

3. $\tan^2 x = 3$

[θ = 60°, 120°, 240° or 300°]

4. $5 + 3\operatorname{cosec}^2 D = 8$

[D = 90° or 270°]

5. $2\cot^2\theta = 5$

[θ = 32.32°, 147.68°, 212.32° or 327.68°]

15.6 Worked problems (iii) on trigonometric equations

Problem 12. Solve the equation
$8\sin^2\theta + 2\sin\theta - 1 = 0$,
for all values of θ between 0° and 360°.

Factorizing $8\sin^2\theta + 2\sin\theta - 1 = 0$ gives
$(4\sin\theta - 1)(2\sin\theta + 1) = 0$.
Hence $4\sin\theta - 1 = 0$, from which, $\sin\theta = \frac{1}{4} = 0.2500$, or $2\sin\theta + 1 = 0$, from which, $\sin\theta = -\frac{1}{2} = -0.5000$. (Instead of factorizing, the quadratic formula can, of course, be used).

$\theta = \sin^{-1} 0.2500 = 14.48°$ or $165.52°$, since sine is positive in the first and second quadrants, or $\theta = \sin^{-1}(-0.5000) = 210°$ or $330°$, since sine is negative in the third and fourth quadrants. Hence

$$\boldsymbol{\theta = 14.48°, 165.52°, 210° \text{ or } 330°}$$

Problem 13. Solve $6\cos^2\theta + 5\cos\theta - 6 = 0$ for values of θ from 0° to 360°.

Factorizing $6\cos^2\theta + 5\cos\theta - 6 = 0$ gives
$(3\cos\theta - 2)(2\cos\theta + 3) = 0$.
Hence $3\cos\theta - 2 = 0$, from which, $\cos\theta = \frac{2}{3} = 0.6667$, or $2\cos\theta + 3 = 0$, from which, $\cos\theta = -\frac{3}{2} = -1.5000$.
The minimum value of a cosine is -1, hence the latter expression has no solution and is thus neglected. Hence,

$$\boldsymbol{\theta = \cos^{-1} 0.6667 = 48.18° \text{ or } 311.82°}$$

since cosine is positive in the first and fourth quadrants.

Now try the following exercise

Exercise 68 Further problems on trigonometric equations

In Problems 1 to 3 solve the equations for angles between 0° and 360°.

1. $15\sin^2 A + \sin A - 2 = 0$
   $[A = 19.47°, 160.53°, 203.58° \text{ or } 336.42°]$
2. $8\tan^2\theta + 2\tan\theta = 15$
   $[\theta = 51.34°, 123.69°, 231.34° \text{ or } 303.69°]$
3. $2\operatorname{cosec}^2 t - 5\operatorname{cosec} t = 12$
   $[t = 14.48°, 165.52°, 221.81° \text{ or } 318.19°]$
4. $2\cos^2\theta + 9\cos\theta - 5 = 0$
   $[\theta = 60° \text{ or } 300°]$

15.7 Worked problems (iv) on trigonometric equations

Problem 14. Solve $5\cos^2 t + 3\sin t - 3 = 0$ for values of t from 0° to 360°.

Since $\cos^2 t + \sin^2 t = 1$, $\cos^2 t = 1 - \sin^2 t$. Substituting for $\cos^2 t$ in $5\cos^2 t + 3\sin t - 3 = 0$ gives:

$$\begin{aligned} 5(1 - \sin^2 t) + 3\sin t - 3 &= 0 \\ 5 - 5\sin^2 t + 3\sin t - 3 &= 0 \\ -5\sin^2 t + 3\sin t + 2 &= 0 \\ 5\sin^2 t - 3\sin t - 2 &= 0 \end{aligned}$$

Factorizing gives $(5\sin t + 2)(\sin t - 1) = 0$. Hence $5\sin t + 2 = 0$, from which, $\sin t = -\frac{2}{5} = -0.4000$, or $\sin t - 1 = 0$, from which, $\sin t = 1$.
$t = \sin^{-1}(-0.4000) = 203.58°$ or $336.42°$, since sine is negative in the third and fourth quadrants, or $t = \sin^{-1} 1 = 90°$. Hence $\boldsymbol{t = 90°, 203.58° \text{ or } 336.42°}$ as shown in Fig. 15.7.

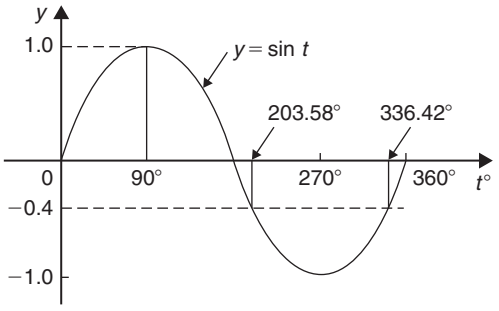


Figure 15.7

Problem 15. Solve $18\sec^2 A - 3\tan A = 21$ for values of A between 0° and 360°.

$1 + \tan^2 A = \sec^2 A$. Substituting for $\sec^2 A$ in $18\sec^2 A - 3\tan A = 21$ gives
$18(1 + \tan^2 A) - 3\tan A = 21$,

i.e. $18 + 18\tan^2 A - 3\tan A - 21 = 0$

$$18\tan^2 A - 3\tan A - 3 = 0$$

Factorizing gives $(6\tan A - 3)(3\tan A + 1) = 0$.
Hence $6\tan A - 3 = 0$, from which, $\tan A = \frac{3}{6} = 0.5000$ or $3\tan A + 1 = 0$, from which, $\tan A = -\frac{1}{3} = -0.3333$.
Thus $A = \tan^{-1}(0.5000) = 26.57°$ or $206.57°$, since tangent is positive in the first and third quadrants, or $A = \tan^{-1}(-0.3333) = 161.57°$ or $341.57°$, since tangent is negative in the second and fourth quadrants. Hence,

$A = 26.57°, 161.57°, 206.57°$ or $341.57°$

Problem 16. Solve $3\operatorname{cosec}^2\theta - 5 = 4\cot\theta$ in the range $0 < \theta < 360°$.

$\cot^2\theta + 1 = \operatorname{cosec}^2\theta$. Substituting for $\operatorname{cosec}^2\theta$ in $3\operatorname{cosec}^2\theta - 5 = 4\cot\theta$ gives:

$$3(\cot^2\theta + 1) - 5 = 4\cot\theta$$

$$3\cot^2\theta + 3 - 5 = 4\cot\theta$$

$$3\cot^2\theta - 4\cot\theta - 2 = 0$$

Since the left-hand side does not factorize the quadratic formula is used. Thus,

$$\cot\theta = \frac{-(-4) \pm \sqrt{[(-4)^2 - 4(3)(-2)]}}{2(3)}$$

$$= \frac{4 \pm \sqrt{(16 + 24)}}{6} = \frac{4 \pm \sqrt{40}}{6}$$

$$= \frac{10.3246}{6} \text{ or } -\frac{2.3246}{6}$$

Hence $\cot\theta = 1.7208$ or -0.3874, $\theta = \cot^{-1} 1.7208 = 30.17°$ or $210.17°$, since cotangent is positive in the first and third quadrants, or $\theta = \cot^{-1}(-0.3874) = 111.18°$ or $291.18°$, since cotangent is negative in the second and fourth quadrants.

Hence,

$\theta = 30.17°, 111.18°, 210.17°$ or $291.18°$

Now try the following exercise

Exercise 69 Further problems on trigonometric equations

In Problems 1 to 12 solve the equations for angles between 0° and 360°.

1. $2\cos^2\theta + \sin\theta = 1$ [$\theta = 90°, 210°, 330°$]
2. $4\cos^2 t + 5\sin t = 3$ [$t = 190.1°, 349.9°$]
3. $2\cos\theta - 4\sin^2\theta = 0$ [$\theta = 38.67°, 321.33°$]
4. $3\cos\theta + 2\sin^2\theta = 3$ [$\theta = 0°, 60°, 300°, 360°$]
5. $12\sin^2\theta - 6 = \cos\theta$ [$\theta = 48.19°, 138.59°,$ $221.41°$ or $311.81°$]
6. $16\sec x - 2 = 14\tan^2 x$ [$x = 52.53°$ or $307.07°$]
7. $4\cot^2 A - 6\operatorname{cosec} A + 6 = 0$ [$A = 90°$]
8. $5\sec t + 2\tan^2 t = 3$ [$t = 107.83°$ or $252.17°$]
9. $2.9\cos^2 a - 7\sin a + 1 = 0$ [$a = 27.83°$ or $152.17°$]
10. $3\operatorname{cosec}^2\beta = 8 - 7\cot\beta$ [$\beta = 60.17°, 161.02°,$ $240.17°$ or $341.02°$]
11. $\cot\theta = \sin\theta$ [$51.83°, 308.17°$]
12. $\tan\theta + 3\cot\theta = 5\sec\theta$ [$30°, 150°$]

Chapter 16

The relationship between trigonometric and hyperbolic functions

16.1 The relationship between trigonometric and hyperbolic functions

In Chapter 21, it is shown that

$$\cos\theta + j\sin\theta = e^{j\theta} \quad (1)$$

and $$\cos\theta - j\sin\theta = e^{-j\theta} \quad (2)$$

Adding equations (1) and (2) gives:

$$\cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta}) \quad (3)$$

Subtracting equation (2) from equation (1) gives:

$$\sin\theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta}) \quad (4)$$

Substituting $j\theta$ for θ in equations (3) and (4) gives:

$$\cos j\theta = \frac{1}{2}(e^{j(j\theta)} + e^{-j(j\theta)})$$

and $$\sin j\theta = \frac{1}{2j}(e^{j(j\theta)} - e^{-j(j\theta)})$$

Since $j^2 = -1$, $\cos j\theta = \frac{1}{2}(e^{-\theta} + e^{\theta}) = \frac{1}{2}(e^{\theta} + e^{-\theta})$

Hence from Chapter 5, $\cos j\theta = \cosh\theta$ (5)

Similarly, $$\sin j\theta = \frac{1}{2j}(e^{-\theta} - e^{\theta}) = -\frac{1}{2j}(e^{\theta} - e^{-\theta})$$

$$= \frac{-1}{j}\left[\frac{1}{2}(e^{\theta} - e^{-\theta})\right]$$

$$= -\frac{1}{j}\sinh\theta \quad \text{(see Chapter 5)}$$

But $$-\frac{1}{j} = -\frac{1}{j} \times \frac{j}{j} = -\frac{j}{j^2} = j,$$

hence $\sin j\theta = j\sinh\theta$ (6)

Equations (5) and (6) may be used to verify that in all standard trigonometric identities, $j\theta$ may be written for θ and the identity still remains true.

Problem 1. Verify that $\cos^2 j\theta + \sin^2 j\theta = 1$.

From equation (5), $\cos j\theta = \cosh\theta$, and from equation (6), $\sin j\theta = j\sinh\theta$.

Thus, $\cos^2 j\theta + \sin^2 j\theta = \cosh^2\theta + j^2\sinh^2\theta$, and since $j^2 = -1$,

$$\cos^2 j\theta + \sin^2 j\theta = \cosh^2\theta - \sinh^2\theta$$

But from Chapter 5, Problem 6,

$$\cosh^2\theta - \sinh^2\theta = 1,$$

hence $\mathbf{\cos^2 j\theta + \sin^2 j\theta = 1}$

Problem 2. Verify that $\sin j2A = 2\sin jA\cos jA$.

From equation (6), writing $2A$ for θ, $\sin j2A = j\sinh 2A$, and from Chapter 5, Table 5.1, page 45, $\sinh 2A = 2\sinh A\cosh A$.

Hence, $\sin j2A = j(2\sinh A\cosh A)$

But, $\sinh A = \frac{1}{2}(e^A - e^{-A})$ and $\cosh A = \frac{1}{2}(e^A + e^{-A})$

$$\text{Hence, } \sin j2A = j2\left(\frac{e^A - e^{-A}}{2}\right)\left(\frac{e^A + e^{-A}}{2}\right)$$

$$= -\frac{2}{j}\left(\frac{e^A - e^{-A}}{2}\right)\left(\frac{e^A + e^{-A}}{2}\right)$$

$$= -\frac{2}{j}\left(\frac{\sin j\theta}{j}\right)(\cos j\theta)$$

$$= 2\sin jA\cos jA \text{ since } j^2 = -1$$

i.e. $\mathbf{\sin j2A = 2\sin jA\cos jA}$

Now try the following exercise

Exercise 70 Further problems on the relationship between trigonometric and hyperbolic functions

Verify the following identities by expressing in exponential form.

1. $\sin j(A+B) = \sin jA\cos jB + \cos jA\sin jB$
2. $\cos j(A-B) = \cos jA\cos jB + \sin jA\sin jB$
3. $\cos j2A = 1 - 2\sin^2 jA$
4. $\sin jA\cos jB = \frac{1}{2}[\sin j(A+B) + \sin j(A-B)]$
5. $\sin jA - \sin jB = 2\cos j\left(\frac{A+B}{2}\right)\sin j\left(\frac{A-B}{2}\right)$

16.2 Hyperbolic identities

From Chapter 5, $\cosh\theta = \frac{1}{2}(e^\theta + e^{-\theta})$

Substituting $j\theta$ for θ gives:

$\cosh j\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta}) = \cos\theta$, from equation (3),

i.e. $\mathbf{\cosh j\theta = \cos\theta}$ (7)

Similarly, from Chapter 5,

$$\sinh\theta = \frac{1}{2}(e^\theta - e^{-\theta})$$

Substituting $j\theta$ for θ gives:

$\sinh j\theta = \frac{1}{2}(e^{j\theta} - e^{-j\theta}) = j\sin\theta$, from equation (4).

Hence $\mathbf{\sinh j\theta = j\sin\theta}$ (8)

$$\tan j\theta = \frac{\sin j\theta}{\cosh j\theta}$$

From equations (5) and (6),

$$\frac{\sin j\theta}{\cos j\theta} = \frac{j\sinh\theta}{\cosh\theta} = j\tanh\theta$$

Hence $\mathbf{\tan j\theta = j\tanh\theta}$ (9)

Similarly, $\tanh j\theta = \dfrac{\sinh j\theta}{\cosh j\theta}$

From equations (7) and (8),

$$\frac{\sinh j\theta}{\cosh j\theta} = \frac{j\sin\theta}{\cos\theta} = j\tan\theta$$

Hence $\mathbf{\tanh j\theta = j\tan\theta}$ (10)

Two methods are commonly used to verify hyperbolic identities. These are (a) by substituting $j\theta$ (and $j\phi$) in the corresponding trigonometric identity and using the relationships given in equations (5) to (10) (see Problems 3 to 5) and (b) by applying Osborne's rule given in Chapter 5, page 45.

Problem 3. By writing jA for θ in $\cot^2\theta + 1 = \operatorname{cosec}^2\theta$, determine the corresponding hyperbolic identity.

Substituting jA for θ gives:

$$\cot^2 jA + 1 = \operatorname{cosec}^2 jA,$$

i.e. $\dfrac{\cos^2 jA}{\sin^2 jA} + 1 = \dfrac{1}{\sin^2 jA}$

But from equation (5), $\cos jA = \cosh A$

and from equation (6), $\sin jA = j\sinh A$.

Hence $\dfrac{\cosh^2 A}{j^2\sinh^2 A} + 1 = \dfrac{1}{j^2\sinh^2 A}$

and since $j^2 = -1$, $-\dfrac{\cosh^2 A}{\sinh^2 A} + 1 = -\dfrac{1}{\sinh^2 A}$

Multiplying throughout by -1, gives:

$$\frac{\cosh^2 A}{\sinh^2 A} - 1 = \frac{1}{\sinh^2 A}$$

i.e. $\mathbf{\coth^2 A - 1 = \operatorname{cosech}^2 A}$

Problem 4. By substituting jA and jB for θ and ϕ respectively in the trigonometric identity for $\cos\theta - \cos\phi$, show that

$$\cosh A - \cosh B = 2\sinh\left(\frac{A+B}{2}\right)\sinh\left(\frac{A-B}{2}\right)$$

$$\cos\theta - \cos\phi = -2\sin\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$$

(see Chapter 17, page 172)

thus $\cos jA - \cos jB$

$$= -2\sin j\left(\frac{A+B}{2}\right)\sin j\left(\frac{A-B}{2}\right)$$

But from equation (5), $\cos jA = \cosh A$

and from equation (6), $\sin jA = j\sinh A$

Hence, $\cosh A - \cosh B$

$$= -2j\sinh\left(\frac{A+B}{2}\right) j\sinh\left(\frac{A-B}{2}\right)$$

$$= -2j^2\sinh\left(\frac{A+B}{2}\right)\sinh\left(\frac{A-B}{2}\right)$$

But $j^2 = -1$, hence

$$\mathbf{\cosh A - \cosh B = 2\sinh\left(\frac{A+B}{2}\right)\sinh\left(\frac{A-B}{2}\right)}$$

Problem 5. Develop the hyperbolic identity corresponding to $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ by writing jA for θ.

Substituting jA for θ gives:

$$\sin 3jA = 3\sin jA - 4\sin^3 jA$$

and since from equation (6),

$$\sin jA = j\sinh A,$$

$$j\sinh 3A = 3j\sinh A - 4j^3\sinh^3 A$$

Dividing throughout by j gives:

$$\sinh 3A = 3\sinh A - j^2 4\sinh^3 A$$

But $j^2 = -1$, hence

$$\mathbf{\sinh 3A = 3\sinh A + 4\sinh^3 A}$$

[An examination of Problems 3 to 5 shows that whenever the trigonometric identity contains a term which is the product of two sines, or the implied product of two sine (e.g. $\tan^2\theta = \sin^2\theta/\cos^2\theta$, thus $\tan^2\theta$ is the implied product of two sines), the sign of the corresponding term in the hyperbolic function changes. This relationship between trigonometric and hyperbolic functions is known as Osborne's rule, as discussed in Chapter 5, page 45].

Now try the following exercise

Exercise 71 Further problems on hyperbolic identities

In Problems 1 to 9, use the substitution $A = j\theta$ (and $B = j\phi$) to obtain the hyperbolic identities corresponding to the trigonometric identities given.

1. $1 + \tan^2 A = \sec^2 A$ $[1 - \tanh^2\theta = \operatorname{sech}^2\theta]$
2. $\cos(A+B) = \cos A\cos B - \sin A\sin B$ $[\cosh(\theta+\phi) = \cosh\theta\cosh\phi + \sinh\theta\sinh\phi]$
3. $\sin(A-B) = \sin A\cos B - \cos A\sin B$ $[\sinh(\theta+\phi) = \sinh\theta\cosh\phi - \cosh\theta\sinh\phi]$
4. $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$ $\left[\tanh 2\theta = \dfrac{2\tanh\theta}{1+\tanh^2\theta}\right]$

5. $\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$

$$\left[\cosh\theta\cosh\phi = \frac{1}{2}[\sinh(\theta+\phi) - \sinh(\theta-\phi)]\right]$$

6. $\sin^3 A = \frac{3}{4}\sin A - \frac{1}{4}\sin 3A$

$$\left[\sinh^3\theta = \frac{1}{4}\sinh 3\theta - \frac{3}{4}\sinh\theta\right]$$

7. $\cot^2 A(\sec^2 A - 1) = 1$

$$[\coth^2\theta(1 - \operatorname{sech}^2\theta) = 1]$$

Chapter 17

Compound angles

17.1 Compound angle formulae

An electric current i may be expressed as $i = 5\sin(\omega t - 0.33)$ amperes. Similarly, the displacement x of a body from a fixed point can be expressed as $x = 10\sin(2t + 0.67)$ metres. The angles $(\omega t - 0.33)$ and $(2t + 0.67)$ are called **compound angles** because they are the sum or difference of two angles. The **compound angle formulae** for sines and cosines of the sum and difference of two angles A and B are:

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$
$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

(Note, $\sin(A+B)$ is **not** equal to $(\sin A + \sin B)$, and so on.)

The formulae stated above may be used to derive two further compound angle formulae:

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$
$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

The compound-angle formulae are true for all values of A and B, and by substituting values of A and B into the formulae they may be shown to be true.

Problem 1. Expand and simplify the following expressions:
(a) $\sin(\pi + \alpha)$ (b) $-\cos(90° + \beta)$
(c) $\sin(A-B) - \sin(A+B)$

(a) $\sin(\pi + \alpha) = \sin\pi\cos\alpha + \cos\pi\sin\alpha$ (from the formula for $\sin(A+B)$)
$= (0)(\cos\alpha) + (-1)\sin\alpha = \mathbf{-\sin\alpha}$

(b) $-\cos(90° + \beta)$
$= -[\cos 90°\cos\beta - \sin 90°\sin\beta]$
$= -[(0)(\cos\beta) - (1)\sin\beta] = \mathbf{\sin\beta}$

(c) $\sin(A-B) - \sin(A+B)$
$= [\sin A\cos B - \cos A\sin B]$
$\quad - [\sin A\cos B + \cos A\sin B]$
$= \mathbf{-2\cos A\sin B}$

Problem 2. Prove that
$$\cos(y-\pi) + \sin\left(y + \frac{\pi}{2}\right) = 0.$$

$\cos(y-\pi) = \cos y\cos\pi + \sin y\sin\pi$
$= (\cos y)(-1) + (\sin y)(0)$
$= -\cos y$

$\sin\left(y + \frac{\pi}{2}\right) = \sin y\cos\frac{\pi}{2} + \cos y\sin\frac{\pi}{2}$
$= (\sin y)(0) + (\cos y)(1) = \cos y$

Hence $\cos(y-\pi) + \sin\left(y + \frac{\pi}{2}\right)$
$= (-\cos y) + (\cos y) = \mathbf{0}$

Problem 3. Show that
$$\tan\left(x + \frac{\pi}{4}\right)\tan\left(x - \frac{\pi}{4}\right) = -1.$$

$$\tan\left(x+\frac{\pi}{4}\right)=\frac{\tan x+\tan\frac{\pi}{4}}{1-\tan x\tan\frac{\pi}{4}}$$

from the formula for $\tan(A+B)$

$$=\frac{\tan x+1}{1-(\tan x)(1)}=\left(\frac{1+\tan x}{1-\tan x}\right)$$

since $\tan\frac{\pi}{4}=1$

$$\tan\left(x-\frac{\pi}{4}\right)=\frac{\tan x-\tan\frac{\pi}{4}}{1+\tan x\tan\frac{\pi}{4}}=\left(\frac{\tan x-1}{1+\tan x}\right)$$

Hence $\tan\left(x+\frac{\pi}{4}\right)\tan\left(x-\frac{\pi}{4}\right)$

$$=\left(\frac{1+\tan x}{1-\tan x}\right)\left(\frac{\tan x-1}{1+\tan x}\right)$$

$$=\frac{\tan x-1}{1-\tan x}=\frac{-(1-\tan x)}{1-\tan x}=\mathbf{-1}$$

Problem 4. If $\sin P=0.8142$ and $\cos Q=0.4432$ evaluate, correct to 3 decimal places:
(a) $\sin(P-Q)$, (b) $\cos(P+Q)$ and
(c) $\tan(P+Q)$, using the compound-angle formulae.

Since $\sin P=0.8142$ then
$P=\sin^{-1}0.8142=54.51°$.
Thus $\cos P=\cos 54.51°=0.5806$ and
$\tan P=\tan 54.51°=1.4025$

Since $\cos Q=0.4432$, $Q=\cos^{-1}0.4432=63.69°$.
Thus $\sin Q=\sin 63.69°=0.8964$ and
$\tan Q=\tan 63.69°=2.0225$

(a) $\sin(P-Q)$

$$=\sin P\cos Q-\cos P\sin Q$$
$$=(0.8142)(0.4432)-(0.5806)(0.8964)$$
$$=0.3609-0.5204=\mathbf{-0.160}$$

(b) $\cos(P+Q)$

$$=\cos P\cos Q-\sin P\sin Q$$
$$=(0.5806)(0.4432)-(0.8142)(0.8964)$$
$$=0.2573-0.7298=\mathbf{-0.473}$$

(c) $\tan(P+Q)$

$$=\frac{\tan P+\tan Q}{1-\tan P\tan Q}=\frac{(1.4025)+(2.0225)}{1-(1.4025)(2.0225)}$$
$$=\frac{3.4250}{-1.8366}=\mathbf{-1.865}$$

Problem 5. Solve the equation

$$4\sin(x-20°)=5\cos x$$

for values of x between 0° and 90°.

$$4\sin(x-20°)=4[\sin x\cos 20°-\cos x\sin 20°],$$

from the formula for $\sin(A-B)$

$$=4[\sin x(0.9397)-\cos x(0.3420)]$$
$$=3.7588\sin x-1.3680\cos x$$

Since $4\sin(x-20°)=5\cos x$ then
$3.7588\sin x-1.3680\cos x=5\cos x$
Rearranging gives:

$$3.7588\sin x=5\cos x+1.3680\cos x$$
$$=6.3680\cos x$$

and $\frac{\sin x}{\cos x}=\frac{6.3680}{3.7588}=1.6942$

i.e. $\tan x=1.6942$, and $x=\tan^{-1}1.6942=59.449°$ or **59°27′**

[Check: LHS $=4\sin(59.449°-20°)$

$$=4\sin 39.449°=2.542$$

RHS $=5\cos x=5\cos 59.449°=2.542$]

Now try the following exercise

Exercise 72 Further problems on compound angle formulae

1. Reduce the following to the sine of one angle:
   (a) $\sin 37°\cos 21°+\cos 37°\sin 21°$
   (b) $\sin 7t\cos 3t-\cos 7t\sin 3t$
   [(a) $\sin 58°$ (b) $\sin 4t$]

2. Reduce the following to the cosine of one angle:
   (a) $\cos 71°\cos 33°-\sin 71°\sin 33°$
   (b) $\cos\frac{\pi}{3}\cos\frac{\pi}{4}+\sin\frac{\pi}{3}\sin\frac{\pi}{4}$
   [(a) $\cos 104°\equiv-\cos 76°$ (b) $\cos\frac{\pi}{12}$]

3. Show that:
(a) $\sin\left(x+\frac{\pi}{3}\right)+\sin\left(x+\frac{2\pi}{3}\right)=\sqrt{3}\cos x$
and
(b) $-\sin\left(\frac{3\pi}{2}-\phi\right)=\cos\phi$

4. Prove that:
(a) $\sin\left(\theta+\frac{\pi}{4}\right)-\sin\left(\theta-\frac{3\pi}{4}\right)$
$=\sqrt{2}(\sin\theta+\cos\theta)$
(b) $\frac{\cos(270°+\theta)}{\cos(360°-\theta)}=\tan\theta$

5. Given $\cos A=0.42$ and $\sin B=0.73$ evaluate (a) $\sin(A-B)$, (b) $\cos(A-B)$, (c) $\tan(A+B)$, correct to 4 decimal places.
[(a) 0.3136 (b) 0.9495 (c) −2.4687]

In Problems 6 and 7, solve the equations for values of θ between 0° and 360°.

6. $3\sin(\theta+30°)=7\cos\theta$
[64.72° or 244.72°]

7. $4\sin(\theta-40°)=2\sin\theta$
[67.52° or 247.52°]

17.2 Conversion of $a\sin\omega t+b\cos\omega t$ into $R\sin(\omega t+\alpha)$

(i) $R\sin(\omega t+\alpha)$ represents a sine wave of maximum value R, periodic time $2\pi/\omega$, frequency $\omega/2\pi$ and leading $R\sin\omega t$ by angle α. (See Chapter 14).

(ii) $R\sin(\omega t+\alpha)$ may be expanded using the compound-angle formula for $\sin(A+B)$, where $A=\omega t$ and $B=\alpha$. Hence,

$$\begin{aligned}R\sin(\omega t+\alpha)&=R[\sin\omega t\cos\alpha+\cos\omega t\sin\alpha]\\&=R\sin\omega t\cos\alpha+R\cos\omega t\sin\alpha\\&=(R\cos\alpha)\sin\omega t+(R\sin\alpha)\cos\omega t\end{aligned}$$

(iii) If $a=R\cos\alpha$ and $b=R\sin\alpha$, where a and b are constants, then $R\sin(\omega t+\alpha)=a\sin\omega t+b\cos\omega t$, i.e. a sine and cosine function of the same frequency when added produce a sine wave of the same frequency (which is further demonstrated in Chapter 25).

(iv) Since $a=R\cos\alpha$, then $\cos\alpha=a/R$, and since $b=R\sin\alpha$, then $\sin\alpha=b/R$.

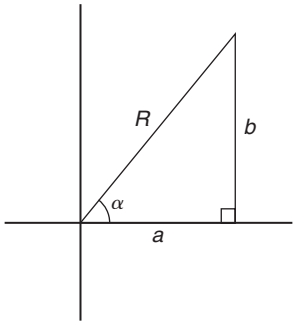


Figure 17.1

If the values of a and b are known then the values of R and α may be calculated. The relationship between constants a, b, R and α are shown in Fig. 17.1.

From Fig. 17.1, by Pythagoras' theorem:

$$R=\sqrt{a^2+b^2}$$

and from trigonometric ratios:

$$\alpha=\tan^{-1}b/a$$

Problem 6. Find an expression for $3\sin\omega t+4\cos\omega t$ in the form $R\sin(\omega t+\alpha)$ and sketch graphs of $3\sin\omega t$, $4\cos\omega t$ and $R\sin(\omega t+\alpha)$ on the same axes.

Let $3\sin\omega t+4\cos\omega t=R\sin(\omega t+\alpha)$

then $3\sin\omega t+4\cos\omega t$

$$\begin{aligned}&=R[\sin\omega t\cos\alpha+\cos\omega t\sin\alpha]\\&=(R\cos\alpha)\sin\omega t+(R\sin\alpha)\cos\omega t\end{aligned}$$

Equating coefficients of $\sin\omega t$ gives:

$$3=R\cos\alpha,\ \text{from which, }\cos\alpha=\frac{3}{R}$$

Equating coefficients of $\cos\omega t$ gives:

$$4=R\sin\alpha,\ \text{from which, }\sin\alpha=\frac{4}{R}$$

There is only one quadrant where both $\sin\alpha$ **and** $\cos\alpha$ are positive, and this is the first, as shown in Fig. 17.2. From Fig. 17.2, by Pythagoras' theorem:

$$R = \sqrt{(3^2 + 4^2)} = 5$$

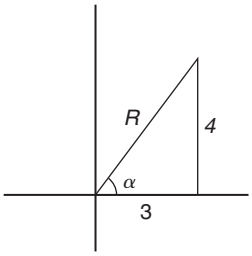


Figure 17.2

From trigonometric ratios: $\alpha = \tan^{-1}\frac{4}{3} = 53.13°$ or 0.927 radians.

Hence $3\sin\omega t + 4\cos\omega t = 5\sin(\omega t + 0.927)$.

A sketch of $3\sin\omega t$, $4\cos\omega t$ and $5\sin(\omega t + 0.927)$ is shown in Fig. 17.3.

Two periodic functions of the same frequency may be combined by,

(a) plotting the functions graphically and combining ordinates at intervals, or

(b) by resolution of phasors by drawing or calculation.

Problem 6, together with Problems 7 and 8 following, demonstrate a third method of combining waveforms.

Problem 7. Express $4.6\sin\omega t - 7.3\cos\omega t$ in the form $R\sin(\omega t + \alpha)$.

Let $4.6\sin\omega t - 7.3\cos\omega t = R\sin(\omega t + \alpha)$.

then $4.6\sin\omega t - 7.3\cos\omega t$

$$= R[\sin\omega t\cos\alpha + \cos\omega t\sin\alpha]$$

$$= (R\cos\alpha)\sin\omega t + (R\sin\alpha)\cos\omega t$$

Equating coefficients of $\sin\omega t$ gives:

$$4.6 = R\cos\alpha, \text{ from which, } \cos\alpha = \frac{4.6}{R}$$

Equating coefficients of $\cos\omega t$ gives:

$$-7.3 = R\sin\alpha, \text{ from which, } \sin\alpha = \frac{-7.3}{R}$$

There is only one quadrant where cosine is positive **and** sine is negative, i.e. the fourth quadrant, as shown in Fig. 17.4. By Pythagoras' theorem:

$$R = \sqrt{[(4.6)^2 + (-7.3)^2]} = 8.628$$

By trigonometric ratios:

$$\alpha = \tan^{-1}\left(\frac{-7.3}{4.6}\right)$$

$$= -57.78° \text{ or } -1.008 \text{ radians.}$$

Hence

$4.6\sin\omega t - 7.3\cos\omega t = 8.628\sin(\omega t - 1.008)$.

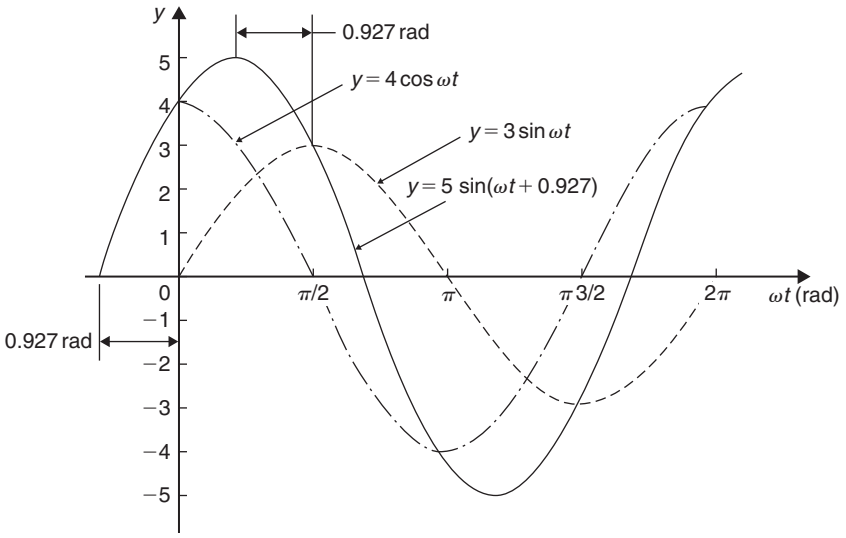


Figure 17.3

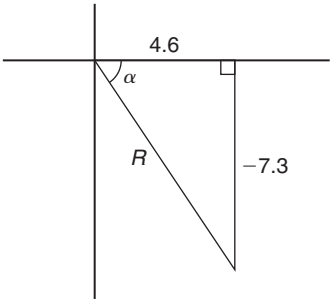


Figure 17.4

Problem 8. Express $-2.7\sin\omega t - 4.1\cos\omega t$ in the form $R\sin(\omega t + \alpha)$.

Let $-2.7\sin\omega t - 4.1\cos\omega t = R\sin(\omega t + \alpha)$

$= R[\sin\omega t\cos\alpha + \cos\omega t\sin\alpha]$

$= (R\cos\alpha)\sin\omega t + (R\sin\alpha)\cos\omega t$

Equating coefficients gives:

$$-2.7 = R\cos\alpha, \text{ from which, } \cos\alpha = \frac{-2.7}{R}$$

and $$-4.1 = R\sin\alpha, \text{ from which, } \sin\alpha = \frac{-4.1}{R}$$

There is only one quadrant in which both cosine **and** sine are negative, i.e. the third quadrant, as shown in Fig. 17.5. From Fig. 17.5,

$$R = \sqrt{[(-2.7)^2 + (-4.1)^2]} = 4.909$$

and $$\theta = \tan^{-1}\frac{4.1}{2.7} = 56.63°$$

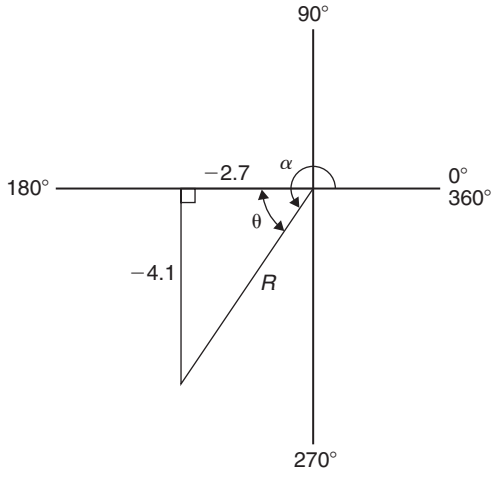


Figure 17.5

Hence $\alpha = 180° + 56.63° = 236.63°$ or 4.130 radians. **Thus**,

$$\mathbf{-2.7\sin\omega t - 4.1\cos\omega t = 4.909\sin(\omega t + 4.130)}.$$

An angle of 236.63° is the same as −123.37° or −2.153 radians.
Hence $-2.7\sin\omega t - 4.1\cos\omega t$ may be expressed also as $\mathbf{4.909\sin(\omega t - 2.153)}$, which is preferred since it is the **principal value** (i.e. $-\pi \le \alpha \le \pi$).

Problem 9. Express $3\sin\theta + 5\cos\theta$ in the form $R\sin(\theta + \alpha)$, and hence solve the equation $3\sin\theta + 5\cos\theta = 4$, for values of θ between 0° and 360°.

Let $3\sin\theta + 5\cos\theta = R\sin(\theta + \alpha)$

$= R[\sin\theta\cos\alpha + \cos\theta\sin\alpha]$

$= (R\cos\alpha)\sin\theta + (R\sin\alpha)\cos\theta$

Equating coefficients gives:

$$3 = R\cos\alpha, \text{ from which, } \cos\alpha = \frac{3}{R}$$

and $$5 = R\sin\alpha, \text{ from which, } \sin\alpha = \frac{5}{R}$$

Since both $\sin\alpha$ and $\cos\alpha$ are positive, R lies in the first quadrant, as shown in Fig. 17.6.

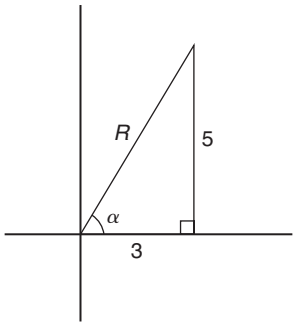


Figure 17.6

From Fig. 17.6, $R = \sqrt{(3^2 + 5^2)} = 5.831$ and $\alpha = \tan^{-1}\frac{5}{3} = 59.03°$.
Hence $3\sin\theta + 5\cos\theta = 5.831\sin(\theta + 59.03°)$

However $3\sin\theta + 5\cos\theta = 4$

Thus $5.831\sin(\theta + 59.03°) = 4$, from which

$$(\theta + 59.03°) = \sin^{-1}\left(\frac{4}{5.831}\right)$$

i.e. $\theta + 59.03° = 43.32°$ or $136.68°$

Hence $\theta = 43.32° - 59.03° = -15.71°$

or $\theta = 136.68° - 59.03° = 77.65°$

Since $-15.71°$ is the same as $-15.71° + 360°$, i.e. $344.29°$, then the solutions are $\boldsymbol{\theta = 77.65°}$ **or** $\boldsymbol{344.29°}$, which may be checked by substituting into the original equation.

Problem 10. Solve the equation $3.5\cos A - 5.8\sin A = 6.5$ for $0° \leq A \leq 360°$.

Let $3.5\cos A - 5.8\sin A = R\sin(A + \alpha)$

$= R[\sin A\cos\alpha + \cos A\sin\alpha]$

$= (R\cos\alpha)\sin A + (R\sin\alpha)\cos A$

Equating coefficients gives:

$3.5 = R\sin\alpha$, from which, $\sin\alpha = \dfrac{3.5}{R}$

and $-5.8 = R\cos\alpha$, from which, $\cos\alpha = \dfrac{-5.8}{R}$

There is only one quadrant in which both sine is positive **and** cosine is negative, i.e. the second, as shown in Fig. 17.7.

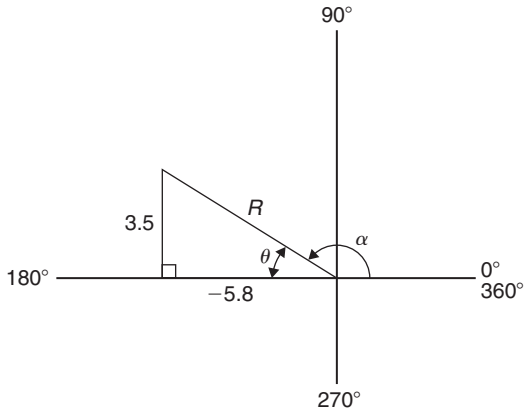


Figure 17.7

From Fig. 17.7, $R = \sqrt{[(3.5)^2 + (-5.8)^2]} = 6.774$ and $\theta = \tan^{-1}\dfrac{3.5}{5.8} = 31.12°$.

Hence $\alpha = 180° - 31.12° = 148.88°$.

Thus

$3.5\cos A - 5.8\sin A = 6.774\sin(A + 144.88°) = 6.5$

Hence $\sin(A + 148.88°) = \dfrac{6.5}{6.774}$, from which,

$(A + 148.88°) = \sin^{-1}\dfrac{6.5}{6.774}$

$= 73.65°$ or $106.35°$

Thus $A = 73.65° - 148.88° = -75.23°$

$\equiv (-75.23° + 360°) = 284.77°$

or $A = 106.35° - 148.88° = -42.53°$

$\equiv (-42.53° + 360°) = 317.47°$

The solutions are thus $\boldsymbol{A = 284.77°}$ **or** $\boldsymbol{317.47°}$, which may be checked in the original equation.

Now try the following exercise

Exercise 73 Further problems on the conversion of $a\sin\omega t + b\cos\omega t$ into $R\sin(\omega t + \alpha)$

In Problems 1 to 4, change the functions into the form $R\sin(\omega t \pm \alpha)$.

1. $5\sin\omega t + 8\cos\omega t$ $[9.434\sin(\omega t + 1.012)]$
2. $4\sin\omega t - 3\cos\omega t$ $[5\sin(\omega t - 0.644)]$
3. $-7\sin\omega t + 4\cos\omega t$ $[8.062\sin(\omega t + 2.622)]$
4. $-3\sin\omega t - 6\cos\omega t$ $[6.708\sin(\omega t - 2.034)]$
5. Solve the following equations for values of θ between $0°$ and $360°$: (a) $2\sin\theta + 4\cos\theta = 3$ (b) $12\sin\theta - 9\cos\theta = 7$. [(a) $74.44°$ or $338.70°$ (b) $64.69°$ or $189.05°$]
6. Solve the following equations for $0° < A < 360°$: (a) $3\cos A + 2\sin A = 2.8$ (b) $12\cos A - 4\sin A = 11$. [(a) $72.73°$ or $354.63°$ (b) $11.15°$ or $311.98°$]
7. Solve the following equations for values of θ between $0°$ and $360°$: (a) $3\sin\theta + 4\cos\theta = 3$ (b) $2\cos\theta + \sin\theta = 2$. [(a) $90°$ or $343.74°$ (b) $0°$, $53.14°$]

8. Solve the following equations for values of θ between $0°$ and $360°$: (a) $6\cos\theta + \sin\theta = \sqrt{3}$ (b) $2\sin 3\theta + 8\cos 3\theta = 1$.

$$\left[\begin{array}{l}\text{(a) } 82.9°, 296° \\ \text{(b) } 32.36°, 97°, 152.36°, 217°, \\ \quad 272.36° \text{ and } 337°\end{array}\right]$$

9. The third harmonic of a wave motion is given by $4.3\cos 3\theta - 6.9\sin 3\theta$. Express this in the form $R\sin(3\theta \pm \alpha)$. $[8.13\sin(3\theta + 2.584)]$

10. The displacement x metres of a mass from a fixed point about which it is oscillating is given by $x = 2.4\sin\omega t + 3.2\cos\omega t$, where t is the time in seconds. Express x in the form $R\sin(\omega t + \alpha)$. $[x = 4.0\sin(\omega t + 0.927)\text{m}]$

11. Two voltages, $v_1 = 5\cos\omega t$ and $v_2 = -8\sin\omega t$ are inputs to an analogue circuit. Determine an expression for the output voltage if this is given by $(v_1 + v_2)$.

$[9.434\sin(\omega t + 2.583)]$

17.3 Double angles

(i) If, in the compound-angle formula for $\sin(A+B)$, we let $B = A$ then

$$\mathbf{\sin 2A = 2\sin A\cos A}$$

Also, for example,

$$\sin 4A = 2\sin 2A\cos 2A$$

and $\sin 8A = 2\sin 4A\cos 4A$, and so on.

(ii) If, in the compound-angle formula for $\cos(A+B)$, we let $B = A$ then

$$\mathbf{\cos 2A = \cos^2 A - \sin^2 A}$$

Since $\cos^2 A + \sin^2 A = 1$, then $\cos^2 A = 1 - \sin^2 A$, and $\sin^2 A = 1 - \cos^2 A$, and two further formula for $\cos 2A$ can be produced.

Thus $\cos 2A = \cos^2 A - \sin^2 A$

$= (1 - \sin^2 A) - \sin^2 A$

i.e. $\mathbf{\cos 2A = 1 - 2\sin^2 A}$

and $\cos 2A = \cos^2 A - \sin^2 A$

$= \cos^2 A - (1 - \cos^2 A)$

i.e. $\mathbf{\cos 2A = 2\cos^2 A - 1}$

Also, for example,

$$\cos 4A = \cos^2 2A - \sin^2 2A \text{ or}$$
$$1 - 2\sin^2 2A \text{ or}$$
$$2\cos^2 2A - 1$$

and $\cos 6A = \cos^2 3A - \sin^2 3A$ or

$$1 - 2\sin^2 3A \text{ or}$$
$$2\cos^2 3A - 1,$$

and so on.

(iii) If, in the compound-angle formula for $\tan(A+B)$, we let $B = A$ then

$$\mathbf{\tan 2A = \frac{2\tan A}{1 - \tan^2 A}}$$

Also, for example,

$$\tan 4A = \frac{2\tan 2A}{1 - \tan^2 2A}$$

and $\tan 5A = \dfrac{2\tan\frac{5}{2}A}{1 - \tan^2\frac{5}{2}A}$ and so on.

Problem 11. $I_3\sin 3\theta$ is the third harmonic of a waveform. Express the third harmonic in terms of the first harmonic $\sin\theta$, when $I_3 = 1$.

When $I_3 = 1$,

$I_3\sin 3\theta = \sin 3\theta = \sin(2\theta + \theta)$

$= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta,$

from the $\sin(A+B)$ formula

$= (2\sin\theta\cos\theta)\cos\theta + (1 - 2\sin^2\theta)\sin\theta,$

from the double angle expansions

$= 2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta$

$= 2\sin\theta(1 - \sin^2\theta) + \sin\theta - 2\sin^3\theta,$

(since $\cos^2\theta = 1 - \sin^2\theta$)

$= 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta$

i.e. $\mathbf{\sin 3\theta = 3\sin\theta - 4\sin^3\theta}$

Problem 12. Prove that $\dfrac{1 - \cos 2\theta}{\sin 2\theta} = \tan\theta$.

$$\text{LHS} = \frac{1-\cos 2\theta}{\sin 2\theta} = \frac{1-(1-2\sin^2\theta)}{2\sin\theta\cos\theta}$$
$$= \frac{2\sin^2\theta}{2\sin\theta\cos\theta} = \frac{\sin\theta}{\cos\theta}$$
$$= \tan\theta = \text{RHS}$$

Problem 13. Prove that
$\cot 2x + \operatorname{cosec} 2x = \cot x$.

$$\text{LHS} = \cot 2x + \operatorname{cosec} 2x = \frac{\cos 2x}{\sin 2x} + \frac{1}{\sin 2x}$$
$$= \frac{\cos 2x + 1}{\sin 2x}$$
$$= \frac{(2\cos^2 x - 1) + 1}{\sin 2x}$$
$$= \frac{2\cos^2 x}{\sin 2x} = \frac{2\cos^2 x}{2\sin x\cos x}$$
$$= \frac{\cos x}{\sin x} = \cot x = \text{RHS}$$

Problem 14. Solve the equation $\cos 2\theta + 3\sin\theta = 2$ for θ in the range $0° \le \theta \le 360°$.

Replacing the double angle term with the relationship $\cos 2\theta = 1 - 2\sin^2\theta$ gives:

$$1 - 2\sin^2\theta + 3\sin\theta = 2$$

Rearranging gives: $-2\sin^2\theta + 3\sin\theta - 1 = 0$
or $2\sin^2\theta - 3\sin\theta + 1 = 0$
which is a quadratic in $\sin\theta$
Using the quadratic formula or by factorising gives:

$$(2\sin\theta - 1)(\sin\theta - 1) = 0$$

from which, $2\sin\theta - 1 = 0$ or $\sin\theta - 1 = 0$
and $\sin\theta = \frac{1}{2}$ or $\sin\theta = 1$
from which, $\boldsymbol{\theta = 30°}$ **or** $\boldsymbol{150°}$ **or** $\boldsymbol{90°}$

Now try the following exercise

Exercise 74 Further problems on double angles

1. The power p in an electrical circuit is given by $p = \frac{v^2}{R}$. Determine the power in terms of V, R and $\cos 2t$ when $v = V\cos t$.
$\left[\frac{V^2}{2R}(1+\cos 2t)\right]$

2. Prove the following identities:
   (a) $1 - \frac{\cos 2\phi}{\cos^2\phi} = \tan^2\phi$
   (b) $\frac{1+\cos 2t}{\sin^2 t} = 2\cot^2 t$
   (c) $\frac{(\tan 2x)(1+\tan x)}{\tan x} = \frac{2}{1-\tan x}$
   (d) $2\operatorname{cosec} 2\theta\cos 2\theta = \cot\theta - \tan\theta$

3. If the third harmonic of a waveform is given by $V_3\cos 3\theta$, express the third harmonic in terms of the first harmonic $\cos\theta$, when $V_3 = 1$.
$[\cos 3\theta = 4\cos^3\theta - 3\cos\theta]$

In Problems 4 to 8, solve for θ in the range $-180° \le \theta \le 180°$

4. $\cos 2\theta = \sin\theta$ $[-90°, 30°, 150°]$

5. $3\sin 2\theta + 2\cos\theta = 0$
$[-160.47°, -90°, -19.47°, 90°]$

6. $\sin 2\theta + \cos\theta = 0$
$[-150°, -90°, -30°, 90°]$

7. $\cos 2\theta + 2\sin\theta = -3$ $[-90°]$

8. $\tan\theta + \cot\theta = 2$ $[45°, -135°]$

17.4 Changing products of sines and cosines into sums or differences

(i) $\sin(A+B) + \sin(A-B) = 2\sin A\cos B$ (from the formulae in Section 17.1)

i.e. $\boldsymbol{\sin A\cos B}$
$$\boldsymbol{= \tfrac{1}{2}[\sin(A+B) + \sin(A-B)]} \qquad (1)$$

(ii) $\sin(A+B) - \sin(A-B) = 2\cos A\sin B$

i.e. $\boldsymbol{\cos A\sin B}$
$$\boldsymbol{= \tfrac{1}{2}[\sin(A+B) - \sin(A-B)]} \qquad (2)$$

(iii) $\cos(A+B) + \cos(A-B) = 2\cos A\cos B$

i.e. $\boldsymbol{\cos A\cos B}$
$$\boldsymbol{= \tfrac{1}{2}[\cos(A+B) + \cos(A-B)]} \qquad (3)$$

(iv) $\cos(A+B)-\cos(A-B)=-2\sin A\sin B$

i.e. $\sin A\sin B$

$$=-\tfrac{1}{2}[\cos(A+B)-\cos(A-B)] \qquad (4)$$

Problem 15. Express $\sin 4x\cos 3x$ as a sum or difference of sines and cosines.

From equation (1),

$$\sin 4x\cos 3x=\tfrac{1}{2}[\sin(4x+3x)+\sin(4x-3x)]$$
$$=\tfrac{1}{2}(\sin 7x+\sin x)$$

Problem 16. Express $2\cos 5\theta\sin 2\theta$ as a sum or difference of sines or cosines.

From equation (2),

$$2\cos 5\theta\sin 2\theta=2\left\{\frac{1}{2}[\sin(5\theta+2\theta)-\sin(5\theta-2\theta)]\right\}$$
$$=\sin 7\theta-\sin 3\theta$$

Problem 17. Express $3\cos 4t\cos t$ as a sum or difference of sines or cosines.

From equation (3),

$$3\cos 4t\cos t=3\left\{\frac{1}{2}[\cos(4t+t)+\cos(4t-t)]\right\}$$
$$=\frac{3}{2}(\cos 5t+\cos 3t)$$

Thus, if the integral $\int 3\cos 4t\cos t\,\mathrm{d}t$ was required (for integration see Chapter 37), then

$$\int 3\cos 4t\cos t\,\mathrm{d}t=\int\frac{3}{2}(\cos 5t+\cos 3t)\,\mathrm{d}t$$
$$=\frac{3}{2}\left[\frac{\sin 5t}{5}+\frac{\sin 3t}{3}\right]+c$$

Problem 18. In an alternating current circuit, voltage $v=5\sin\omega t$ and current $i=10\sin(\omega t-\pi/6)$. Find an expression for the instantaneous power p at time t given that $p=vi$, expressing the answer as a sum or difference of sines and cosines.

$$p=vi=(5\sin\omega t)[10\sin(\omega t-\pi/6)]$$
$$=50\sin\omega t\sin(\omega t-\pi/6)$$

From equation (4),

$$50\sin\omega t\sin(\omega t-\pi/6)$$
$$=(50)\left[-\tfrac{1}{2}\left\{\cos(\omega t+\omega t-\pi/6)-\cos[\omega t-(\omega t-\pi/6)]\right\}\right]$$
$$=-25\{\cos(2\omega t-\pi/6)-\cos\pi/6\}$$

i.e. instantaneous power,

$$p=25[\cos\pi/6-\cos(2\omega t-\pi/6)]$$

Now try the following exercise

Exercise 75 Further problems on changing products of sines and cosines into sums or differences

In Problems 1 to 5, express as sums or differences:

1. $\sin 7t\cos 2t$ $\left[\tfrac{1}{2}(\sin 9t+\sin 5t)\right]$
2. $\cos 8x\sin 2x$ $\left[\tfrac{1}{2}(\sin 10x-\sin 6x)\right]$
3. $2\sin 7t\sin 3t$ $[\cos 4t-\cos 10t]$
4. $4\cos 3\theta\cos\theta$ $[2(\cos 4\theta+\cos 2\theta)]$
5. $3\sin\frac{\pi}{3}\cos\frac{\pi}{6}$ $\left[\frac{3}{2}\left(\sin\frac{\pi}{2}+\sin\frac{\pi}{6}\right)\right]$
6. Determine $\int 2\sin 3t\cos t\,\mathrm{d}t$. $\left[-\frac{\cos 4t}{4}-\frac{\cos 2t}{2}+c\right]$
7. Evaluate $\int_0^{\frac{\pi}{2}} 4\cos 5x\cos 2x\,\mathrm{d}x$. $\left[-\frac{20}{21}\right]$
8. Solve the equation: $2\sin 2\phi\sin\phi=\cos\phi$ in the range $\phi=0$ to $\phi=180°$. [30°, 90° or 150°]

17.5 Changing sums or differences of sines and cosines into products

In the compound-angle formula let,

$$(A+B)=X$$

and

$$(A-B)=Y$$

Solving the simultaneous equations gives:

$$A = \frac{X+Y}{2} \text{ and } B = \frac{X-Y}{2}$$

Thus $\sin(A+B)+\sin(A-B) = 2\sin A\cos B$ becomes,

$$\sin X + \sin Y = 2\sin\left(\frac{X+Y}{2}\right)\cos\left(\frac{X-Y}{2}\right) \quad (5)$$

Similarly,

$$\sin X - \sin Y = 2\cos\left(\frac{X+Y}{2}\right)\sin\left(\frac{X-Y}{2}\right) \quad (6)$$

$$\cos X + \cos Y = 2\cos\left(\frac{X+Y}{2}\right)\cos\left(\frac{X-Y}{2}\right) \quad (7)$$

$$\cos X - \cos Y = -2\sin\left(\frac{X+Y}{2}\right)\sin\left(\frac{X-Y}{2}\right) \quad (8)$$

Problem 19. Express $\sin 5\theta + \sin 3\theta$ as a product.

From equation (5),

$$\sin 5\theta + \sin 3\theta = 2\sin\left(\frac{5\theta+3\theta}{2}\right)\cos\left(\frac{5\theta-3\theta}{2}\right)$$
$$= \mathbf{2\sin 4\theta\cos\theta}$$

Problem 20. Express $\sin 7x - \sin x$ as a product.

From equation (6),

$$\sin 7x - \sin x = 2\cos\left(\frac{7x+x}{2}\right)\sin\left(\frac{7x-x}{2}\right)$$
$$= \mathbf{2\cos 4x\sin 3x}$$

Problem 21. Express $\cos 2t - \cos 5t$ as a product.

From equation (8),

$$\cos 2t - \cos 5t = -2\sin\left(\frac{2t+5t}{2}\right)\sin\left(\frac{2t-5t}{2}\right)$$
$$= -2\sin\frac{7}{2}t\sin\left(-\frac{3}{2}t\right) = \mathbf{2\sin\frac{7}{2}t\sin\frac{3}{2}t}$$

$$\left(\text{since } \sin\left(-\frac{3}{2}t\right) = -\sin\frac{3}{2}t\right)$$

Problem 22. Show that $\dfrac{\cos 6x + \cos 2x}{\sin 6x + \sin 2x} = \cot 4x$.

From equation (7),

$$\cos 6x + \cos 2x = 2\cos 4x\cos 2x$$

From equation (5),

$$\sin 6x + \sin 2x = 2\sin 4x\cos 2x$$

Hence

$$\frac{\cos 6x + \cos 2x}{\sin 6x + \sin 2x} = \frac{2\cos 4x\cos 2x}{2\sin 4x\cos 2x}$$
$$= \frac{\cos 4x}{\sin 4x} = \mathbf{\cot 4x}$$

Problem 23. Solve the equation $\cos 4\theta + \cos 2\theta = 0$ for θ in the range $0° \le \theta \le 360°$.

From equation (7),

$$\cos 4\theta + \cos 2\theta = 2\cos\left(\frac{4\theta+2\theta}{2}\right)\cos\left(\frac{4\theta-2\theta}{2}\right)$$

Hence, $2\cos 3\theta\cos\theta = 0$

Dividing by 2 gives: $\cos 3\theta\cos\theta = 0$

Hence, either $\cos 3\theta = 0$ or $\cos\theta = 0$

Thus, $3\theta = \cos^{-1}0$ or $\theta = \cos^{-1}0$

from which, $3\theta = 90°$ or $270°$ or $450°$ or $630°$ or $810°$ or $990°$

and $\boldsymbol{\theta = 30°, 90°, 150°, 210°, 270° \text{ or } 330°}$

Now try the following exercise

Exercise 76 Further problems on changing sums or differences of sines and cosines into products

In Problems 1 to 5, express as products:

1. $\sin 3x + \sin x$ [$2\sin 2x\cos x$]
2. $\frac{1}{2}(\sin 9\theta - \sin 7\theta)$ [$\cos 8\theta\sin\theta$]
3. $\cos 5t + \cos 3t$ [$2\cos 4t\cos t$]
4. $\frac{1}{8}(\cos 5t - \cos t)$ [$-\frac{1}{4}\sin 3t\sin 2t$]
5. $\frac{1}{2}\left(\cos\frac{\pi}{3} + \cos\frac{\pi}{4}\right)$ $\left[\cos\frac{7\pi}{24}\cos\frac{\pi}{24}\right]$
6. Show that:
   (a) $\dfrac{\sin 4x - \sin 2x}{\cos 4x + \cos 2x} = \tan x$
   (b) $\frac{1}{2}\{\sin(5x-\alpha) - \sin(x+\alpha)\} = \cos 3x\sin(2x-\alpha)$

In Problems 7 and 8, solve for θ in the range $0° \leq \theta \leq 180°$.

7. $\cos 6\theta + \cos 2\theta = 0$
   [22.5°, 45°, 67.5°, 112.5°, 135°, 157.5°]

8. $\sin 3\theta - \sin \theta = 0$
   [0°, 45°, 135°, 180°]

In Problems 9 and 10, solve in the range 0° to 360°.

9. $\cos 2x = 2 \sin x$ [21.47° or 158.53°]

10. $\sin 4t + \sin 2t = 0$
    [0°, 60°, 90°, 120°, 180°, 240°, 270°, 300°, 360°]

17.6 Power waveforms in a.c. circuits

(a) Purely resistive a.c. circuits

Let a voltage $v = V_m \sin \omega t$ be applied to a circuit comprising resistance only. The resulting current is $i = I_m \sin \omega t$, and the corresponding instantaneous power, p, is given by:

$$p = vi = (V_m \sin \omega t)(I_m \sin \omega t)$$

i.e. $p = V_m I_m \sin^2 \omega t$

From double angle formulae of Section 17.3,

$$\cos 2A = 1 - 2\sin^2 A, \text{ from which,}$$
$$\sin^2 A = \tfrac{1}{2}(1 - \cos 2A) \text{ thus}$$
$$\sin^2 \omega t = \tfrac{1}{2}(1 - \cos 2\omega t)$$

Then power $p = V_m I_m \left[\tfrac{1}{2}(l - \cos 2\omega t)\right]$

i.e. $\boldsymbol{p = \tfrac{1}{2} V_m I_m (1 - \cos 2\omega t)}$

The waveforms of v, i and p are shown in Fig. 17.8. The waveform of power repeats itself after π/ω seconds and hence the power has a frequency twice that of voltage and current. The power is always positive, having a maximum value of $V_m I_m$. The average or mean value of the power is $\frac{1}{2} V_m I_m$.

The rms value of voltage $V = 0.707 V_m$, i.e. $V = \dfrac{V_m}{\sqrt{2}}$, from which, $V_m = \sqrt{2}\, V$.

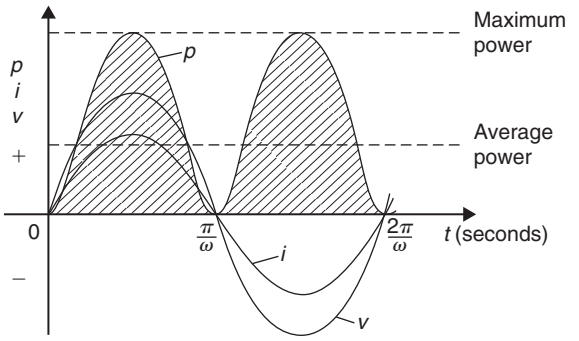


Figure 17.8

Similarly, the rms value of current, $I = \dfrac{I_m}{\sqrt{2}}$, from which, $I_m = \sqrt{2}\, I$. Hence the average power, P, developed in a purely resistive a.c. circuit is given by $P = \frac{1}{2} V_m I_m = \frac{1}{2}(\sqrt{2}V)(\sqrt{2}I) = VI$ watts.

Also, power $P = I^2 R$ or V^2/R as for a d.c. circuit, since $V = IR$.

Summarizing, the average power P in a purely resistive a.c. circuit given by

$$\boldsymbol{P = VI = I^2 R = \frac{V^2}{R}}$$

where V and I are rms values.

(b) Purely inductive a.c. circuits

Let a voltage $v = V_m \sin \omega t$ be applied to a circuit containing pure inductance (theoretical case). The resulting current is $i = I_m \sin\left(\omega t - \dfrac{\pi}{2}\right)$ since current lags voltage by $\dfrac{\pi}{2}$ radians or 90° in a purely inductive circuit, and the corresponding instantaneous power, p, is given by:

$$p = vi = (V_m \sin \omega t) I_m \sin\left(\omega t - \frac{\pi}{2}\right)$$

i.e. $p = V_m I_m \sin \omega t \sin\left(\omega t - \dfrac{\pi}{2}\right)$

However,

$$\sin\left(\omega t - \frac{\pi}{2}\right) = -\cos \omega t \text{ thus}$$
$$p = -V_m I_m \sin \omega t \cos \omega t.$$

Rearranging gives:

$$p = -\tfrac{1}{2} V_m I_m (2 \sin \omega t \cos \omega t).$$

However, from double-angle formulae,

$$2 \sin \omega t \cos \omega t = \sin 2\omega t.$$

Thus **power**, $\boldsymbol{p = -\tfrac{1}{2} V_m I_m \sin 2\omega t}$.

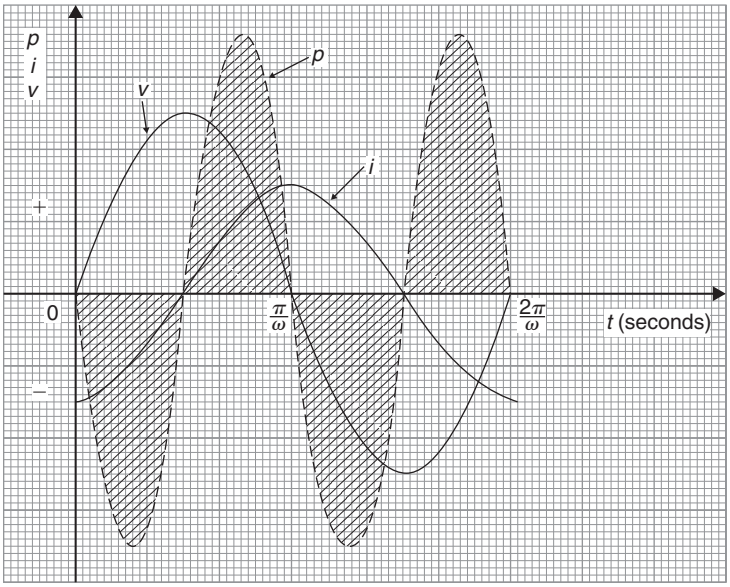


Figure 17.9

The waveforms of v, i and p are shown in Fig. 17.9. The frequency of power is twice that of voltage and current. For the power curve shown in Fig. 17.9, the area above the horizontal axis is equal to the area below, thus over a complete cycle the average power P is zero. It is noted that when v and i are both positive, power p is positive and energy is delivered from the source to the inductance; when v and i have opposite signs, power p is negative and energy is returned from the inductance to the source.

In general, when the current through an inductance is increasing, energy is transferred from the circuit to the magnetic field, but this energy is returned when the current is decreasing.

Summarizing, **the average power P in a purely inductive a.c. circuit is zero**.

(c) Purely capacitive a.c. circuits

Let a voltage $v = V_m \sin\omega t$ be applied to a circuit containing pure capacitance. The resulting current is $i = I_m \sin\left(\omega t + \frac{\pi}{2}\right)$, since current leads voltage by 90° in a purely capacitive circuit, and the corresponding instantaneous power, p, is given by:

$$p = vi = (V_m \sin\omega t) I_m \sin\left(\omega t + \frac{\pi}{2}\right)$$

i.e. $$p = V_m I_m \sin\omega t \sin\left(\omega t + \frac{\pi}{2}\right)$$

However, $$\sin\left(\omega t + \frac{\pi}{2}\right) = \cos\omega t$$

thus $$p = V_m I_m \sin\omega t \cos\omega t$$

Rearranging gives $p = \frac{1}{2} V_m I_m (2 \sin\omega t \cos\omega t)$.

Thus **power**, $\boldsymbol{p = \frac{1}{2} V_m I_m \sin 2\omega t}$.

The waveforms of v, i and p are shown in Fig. 17.10. Over a complete cycle the average power P is zero. When the voltage across a capacitor is increasing, energy is transferred from the circuit to the electric field, but this energy is returned when the voltage is decreasing.

Summarizing, **the average power P in a purely capacitive a.c. circuit is zero**.

(d) *R–L* or *R–C* a.c. circuits

Let a voltage $v = V_m \sin\omega t$ be applied to a circuit containing resistance and inductance or resistance and capacitance. Let the resulting current be $i = I_m \sin(\omega t + \phi)$, where phase angle ϕ will be positive for an R–C circuit and negative for an R–L circuit. The corresponding instantaneous power, p, is given by:

$$p = vi = (V_m \sin\omega t) I_m \sin(\omega t + \phi)$$

i.e. $$p = V_m I_m \sin\omega t \sin(\omega t + \phi)$$

Products of sine functions may be changed into differences of cosine functions as shown in Section 17.4,
i.e. $\sin A \sin B = -\frac{1}{2}[\cos(A + B) - \cos(A - B)]$.

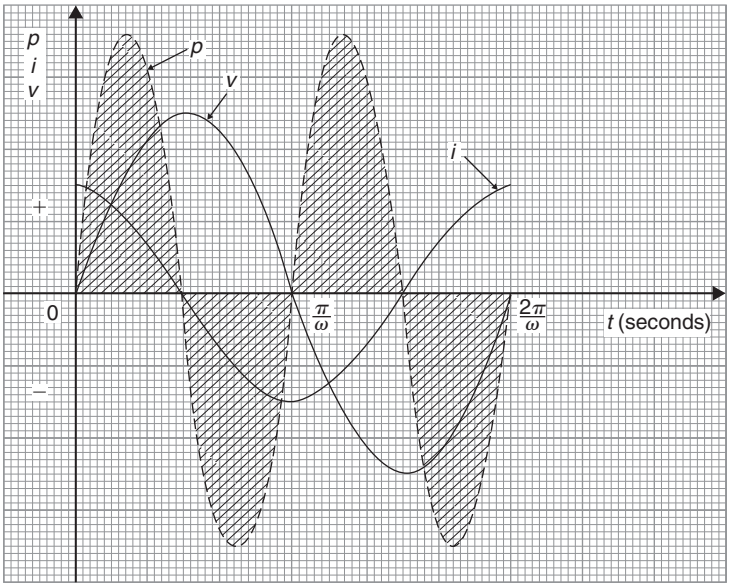


Figure 17.10

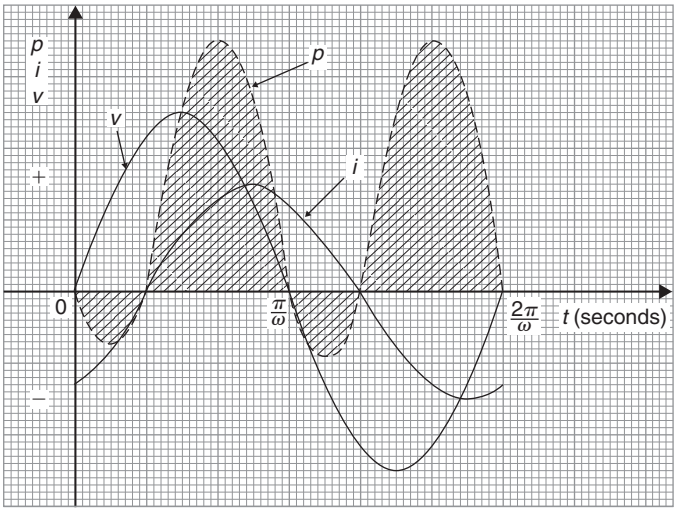


Figure 17.11

Substituting $\omega t = A$ and $(\omega t + \phi) = B$ gives:

power, $p = V_m I_m\{-\frac{1}{2}[\cos(\omega t + \omega t + \phi) - \cos(\omega t - (\omega t + \phi))]\}$

i.e. $p = \frac{1}{2} V_m I_m[\cos(-\phi) - \cos(2\omega t + \phi)]$

However, $\cos(-\phi) = \cos\phi$

Thus $\boldsymbol{p = \frac{1}{2} V_m I_m[\cos\phi - \cos(2\omega t + \phi)]}$

The instantaneous power p thus consists of

(i) a sinusoidal term, $-\frac{1}{2} V_m I_m \cos(2\omega t + \phi)$ which has a mean value over a cycle of zero, and

(ii) a constant term, $\frac{1}{2} V_m I_m \cos\phi$ (since ϕ is constant for a particular circuit).

Thus the average value of power, $P = \frac{1}{2} V_m I_m \cos\phi$.

Since $V_m = \sqrt{2}\,V$ and $I_m = \sqrt{2}\,I$, average power,

$$P = \tfrac{1}{2}(\sqrt{2}\,V)(\sqrt{2}\,I)\cos\phi$$

i.e. $\boldsymbol{P = V I \cos\phi}$

The waveforms of v, i and p, are shown in Fig. 17.11 for an R–L circuit. The waveform of power is seen to

pulsate at twice the supply frequency. The areas of the power curve (shown shaded) above the horizontal time axis represent power supplied to the load; the small areas below the axis represent power being returned to the supply from the inductance as the magnetic field collapses.

A similar shape of power curve is obtained for an R–C circuit, the small areas below the horizontal axis representing power being returned to the supply from the charged capacitor. The difference between the areas above and below the horizontal axis represents the heat loss due to the circuit resistance. Since power is dissipated only in a pure resistance, the alternative equations for power, $P = I_R^2 R$, may be used, where I_R is the rms current flowing through the resistance.

Summarizing, **the average power P in a circuit containing resistance and inductance and/or capacitance, whether in series or in parallel, is given by $P = VI\cos\phi$** or $P = I_R^2 R$ (V, I and I_R being rms values).

Revision Test 5

This Revision Test covers the material contained in Chapters 14 to 17. *The marks for each question are shown in brackets at the end of each question.*

1. Solve the following equations in the range 0° to 360°.

   (a) $\sin^{-1}(-0.4161) = x$

   (b) $\cot^{-1}(2.4198) = \theta$ (8)

2. Sketch the following curves labelling relevant points:

   (a) $y = 4\cos(\theta + 45°)$

   (b) $y = 5\sin(2t - 60°)$ (8)

3. The current in an alternating current circuit at any time t seconds is given by:

   $$i = 120\sin(100\pi t + 0.274) \text{ amperes.}$$

   Determine

   (a) the amplitude, periodic time, frequency and phase angle (with reference to $120\sin 100\pi t$)

   (b) the value of current when $t = 0$

   (c) the value of current when $t = 6$ ms

   (d) the time when the current first reaches 80 A

   Sketch one cycle of the oscillation. (19)

4. A complex voltage waveform v is comprised of a 141.4 V rms fundamental voltage at a frequency of 100 Hz, a 35% third harmonic component leading the fundamental voltage at zero time by $\pi/3$ radians, and a 20% fifth harmonic component lagging the fundamental at zero time by $\pi/4$ radians.

   (a) Write down an expression to represent voltage v.

   (b) Draw the complex voltage waveform using harmonic synthesis over one cycle of the fundamental waveform using scales of 12 cm for the time for one cycle horizontally and 1 cm = 20 V vertically. (15)

5. Prove the following identities:

   (a) $\sqrt{\left[\dfrac{1-\cos^2\theta}{\cos^2\theta}\right]} = \tan\theta$

   (b) $\cos\left(\dfrac{3\pi}{2} + \phi\right) = \sin\phi$

   (c) $\dfrac{\sin^2 x}{1+\cos 2x} = \frac{1}{2}\tan^2 x$ (9)

6. Solve the following trigonometric equations in the range $0° \le x \le 360°$:

   (a) $4\cos x + 1 = 0$

   (b) $3.25\operatorname{cosec} x = 5.25$

   (c) $5\sin^2 x + 3\sin x = 4$

   (d) $2\sec^2\theta + 5\tan\theta = 3$ (18)

7. Solve the equation $5\sin(\theta - \pi/6) = 8\cos\theta$ for values $0 \le \theta \le 2\pi$. (8)

8. Express $5.3\cos t - 7.2\sin t$ in the form $R\sin(t + \alpha)$. Hence solve the equation $5.3\cos t - 7.2\sin t = 4.5$ in the range $0 \le t \le 2\pi$. (12)

9. Determine $\int 2\cos 3t \sin t \, dt$. (3)

Chapter 18

Functions and their curves

18.1 Standard curves

When a mathematical equation is known, co-ordinates may be calculated for a limited range of values, and the equation may be represented pictorially as a graph, within this range of calculated values. Sometimes it is useful to show all the characteristic features of an equation, and in this case a sketch depicting the equation can be drawn, in which all the important features are shown, but the accurate plotting of points is less important. This technique is called 'curve sketching' and can involve the use of differential calculus, with, for example, calculations involving turning points.

If, say, y depends on, say, x, then y is said to be a function of x and the relationship is expressed as $y = f(x)$; x is called the independent variable and y is the dependent variable.

In engineering and science, corresponding values are obtained as a result of tests or experiments.

Here is a brief resumé of standard curves, some of which have been met earlier in this text.

(i) Straight Line

The general equation of a straight line is $y = mx + c$, where m is the gradient $\left(\text{i.e.}\ \dfrac{dy}{dx}\right)$ and c is the y-axis intercept.

Two examples are shown in Fig. 18.1

(ii) Quadratic Graphs

The general equation of a quadratic graph is $y = ax^2 + bx + c$, and its shape is that of a parabola.

The simplest example of a quadratic graph, $y = x^2$, is shown in Fig. 18.2.

(iii) Cubic Equations

The general equation of a cubic graph is $y = ax^3 + bx^2 + cx + d$.

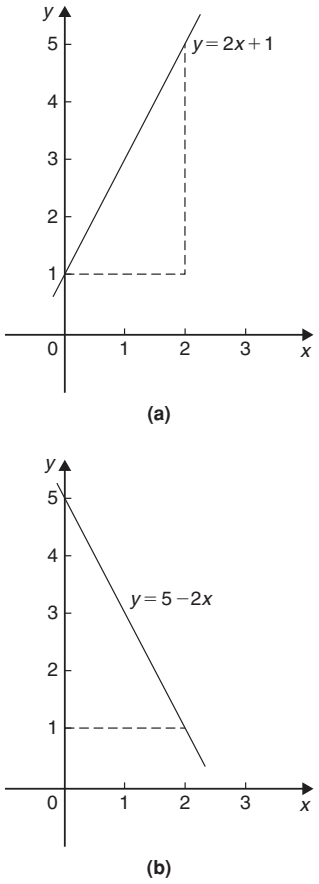


Figure 18.1

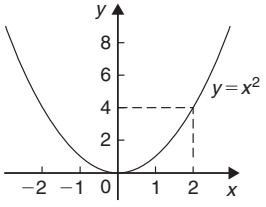


Figure 18.2

The simplest example of a cubic graph, $y = x^3$, is shown in Fig. 18.3.

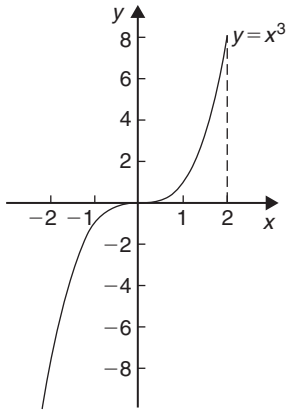


Figure 18.3

(iv) Trigonometric Functions (see Chapter 14, page 134)

Graphs of $y = \sin\theta$, $y = \cos\theta$ and $y = \tan\theta$ are shown in Fig. 18.4.

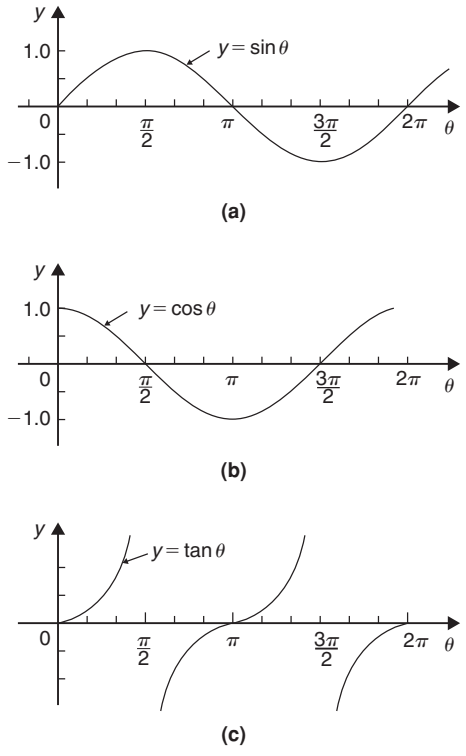


Figure 18.4

(v) Circle (see Chapter 13, page 122)

The simplest equation of a circle is $x^2 + y^2 = r^2$, with centre at the origin and radius r, as shown in Fig. 18.5.

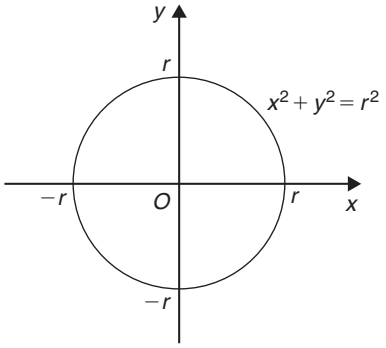


Figure 18.5

More generally, the equation of a circle, centre (a, b), radius r, is given by:

$$(x - a)^2 + (y - b)^2 = r^2$$

Figure 18.6 shows a circle

$$(x - 2)^2 + (y - 3)^2 = 4$$

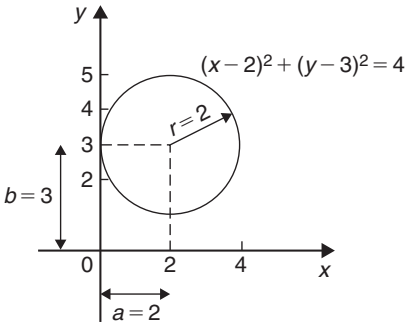


Figure 18.6

(vi) Ellipse

The equation of an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and the general shape is as shown in Fig. 18.7.

The length AB is called the **major axis** and CD the **minor axis**.

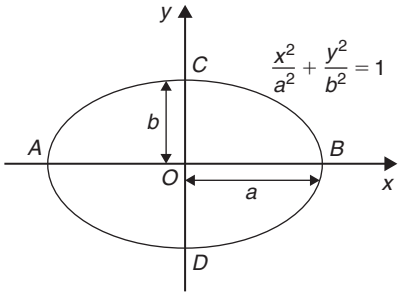


Figure 18.7

In the above equation, 'a' is the semi-major axis and 'b' is the semi-minor axis.

(Note that if $b=a$, the equation becomes $\frac{x^2}{a^2}+\frac{y^2}{a^2}=1$, i.e. $x^2+y^2=a^2$, which is a circle of radius a).

(vii) Hyperbola

The equation of a hyperbola is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

and the general shape is shown in Fig. 18.8. The curve is seen to be symmetrical about both the x- and y-axes. The distance AB in Fig. 18.8 is given by $2a$.

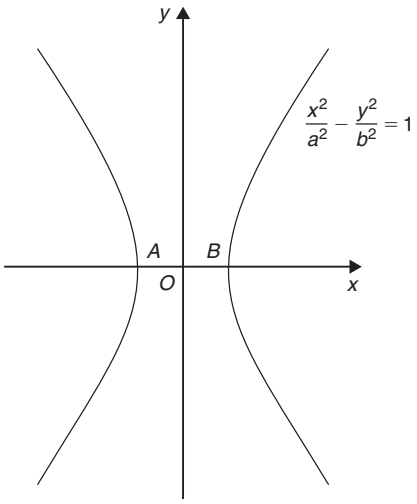


Figure 18.8

(viii) Rectangular Hyperbola

The equation of a rectangular hyperbola is $xy=c$ or $y=\frac{c}{x}$ and the general shape is shown in Fig. 18.9.

(ix) Logarithmic Function (see Chapter 3, page 26)

$y=\ln x$ and $y=\lg x$ are both of the general shape shown in Fig. 18.10.

(x) Exponential Functions (see Chapter 4, page 30)

$y=e^x$ is of the general shape shown in Fig. 18.11.

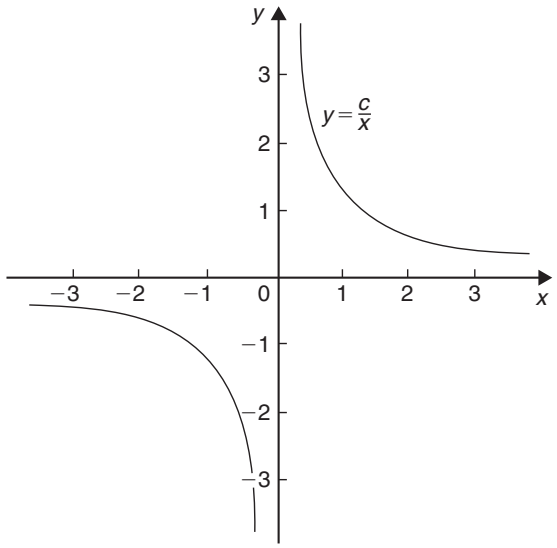


Figure 18.9

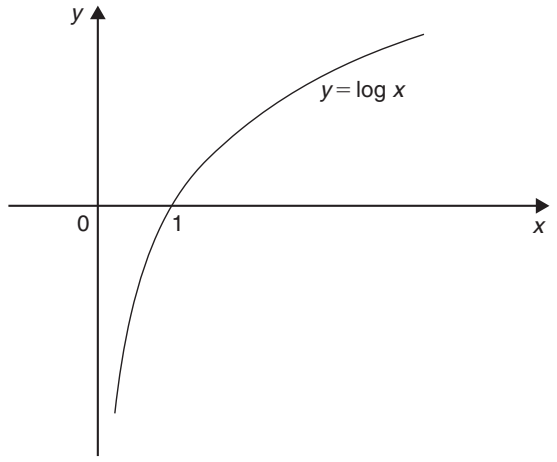


Figure 18.10

(xi) Polar Curves

The equation of a polar curve is of the form $r=f(\theta)$. An example of a polar curve, $r=a\sin\theta$, is shown in Fig. 18.12.

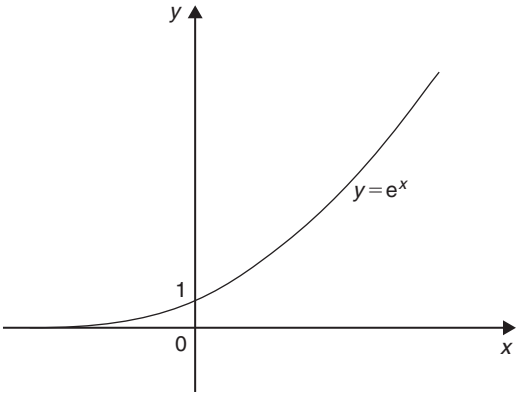


Figure 18.11

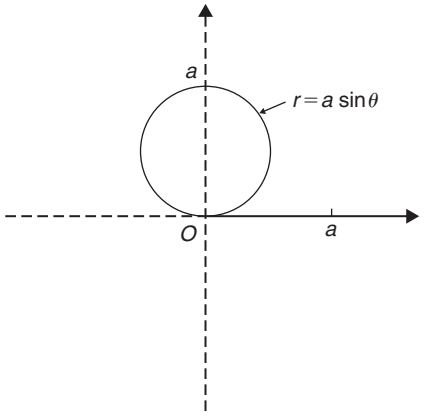
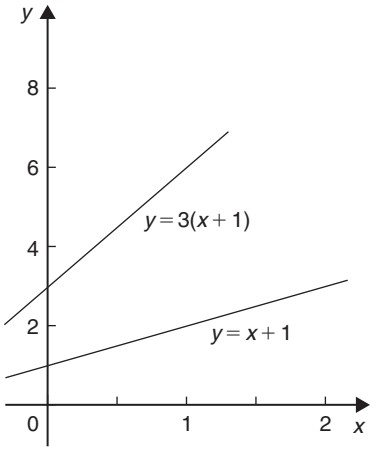
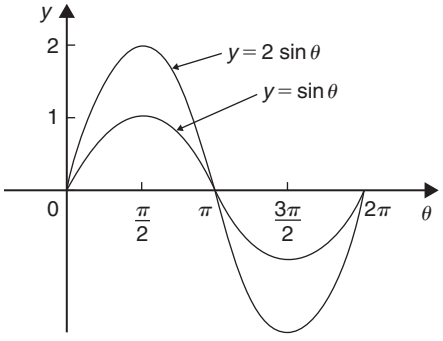


Figure 18.12



(a)



(b)

Figure 18.13

18.2 Simple transformations

From the graph of $y = f(x)$ it is possible to deduce the graphs of other functions which are transformations of $y = f(x)$. For example, knowing the graph of $y = f(x)$, can help us draw the graphs of $y = af(x)$, $y = f(x) + a$, $y = f(x + a)$, $y = f(ax)$, $y = -f(x)$ and $y = f(-x)$.

(i) $y = af(x)$

For each point (x_1, y_1) on the graph of $y = f(x)$ there exists a point (x_1, ay_1) on the graph of $y = af(x)$. Thus the graph of $y = af(x)$ can be obtained by stretching $y = f(x)$ parallel to the y-axis by a scale factor 'a'.

Graphs of $y = x + 1$ and $y = 3(x + 1)$ are shown in Fig. 18.13(a) and graphs of $y = \sin\theta$ and $y = 2\sin\theta$ are shown in Fig. 18.13(b).

(ii) $y = f(x) + a$

The graph of $y = f(x)$ is translated by 'a' units parallel to the y-axis to obtain $y = f(x) + a$. For example, if $f(x) = x$, $y = f(x) + 3$ becomes $y = x + 3$, as shown in Fig. 18.14(a). Similarly, if $f(\theta) = \cos\theta$, then $y = f(\theta) + 2$ becomes $y = \cos\theta + 2$, as shown in Fig. 18.14(b). Also, if $f(x) = x^2$, then $y = f(x) + 3$ becomes $y = x^2 + 3$, as shown in Fig. 18.14(c).

(iii) $y = f(x + a)$

The graph of $y = f(x)$ is translated by 'a' units parallel to the x-axis to obtain $y = f(x + a)$. If 'a' > 0 it moves $y = f(x)$ in the negative direction on the x-axis (i.e. to the left), and if 'a' < 0 it moves $y = f(x)$ in the positive direction on the x-axis (i.e. to the right). For example, if $f(x) = \sin x$, $y = f\left(x - \frac{\pi}{3}\right)$ becomes $y = \sin\left(x - \frac{\pi}{3}\right)$ as shown in Fig. 18.15(a) and $y = \sin\left(x + \frac{\pi}{4}\right)$ is shown in Fig. 18.15(b).

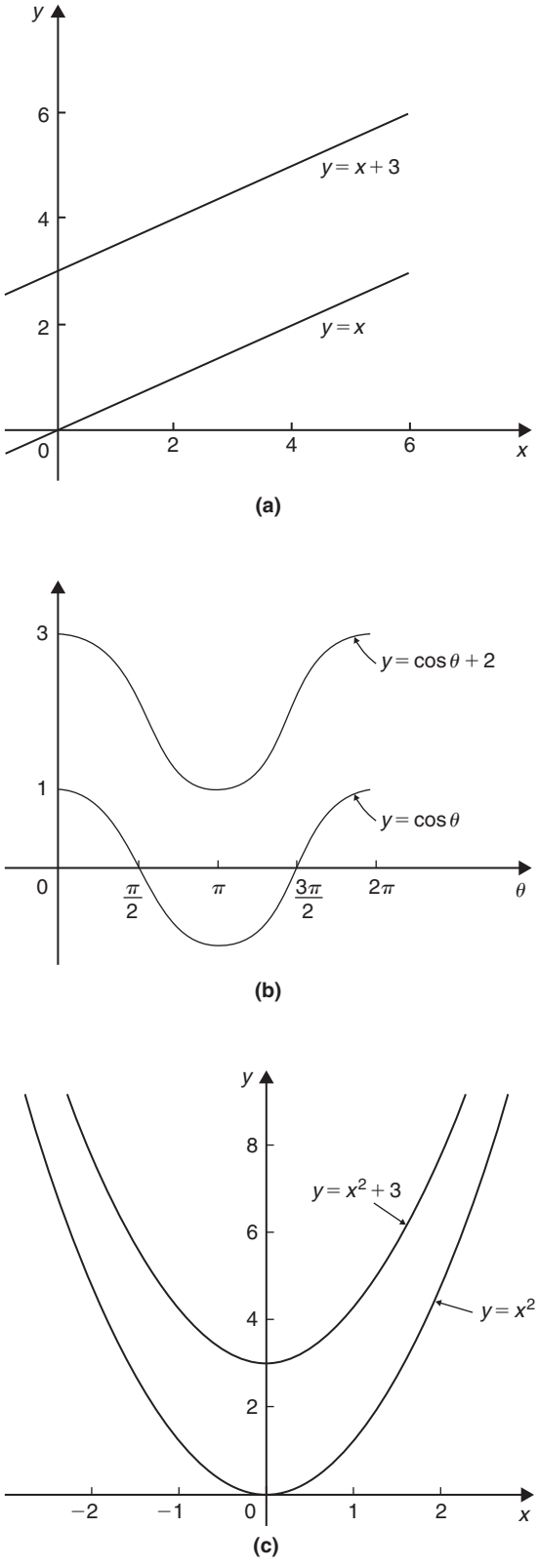


Figure 18.14

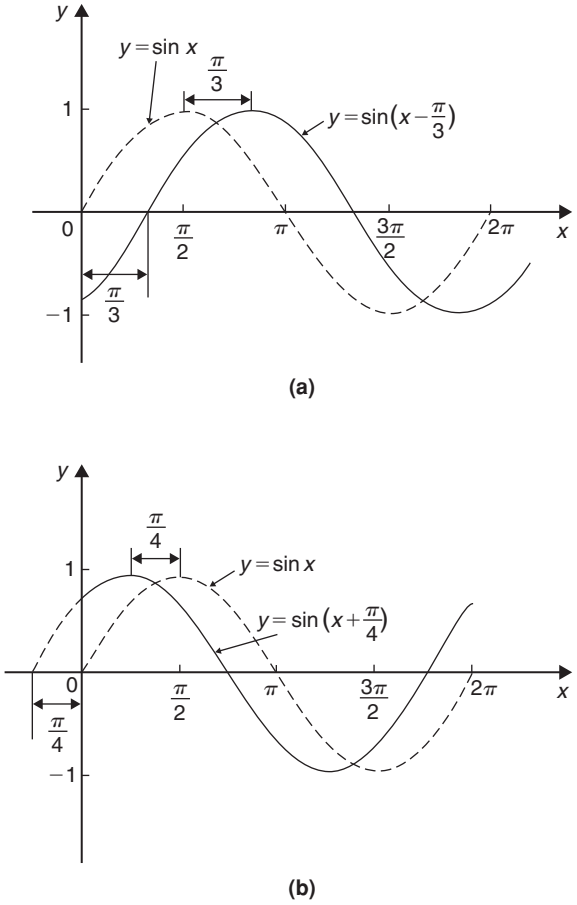


Figure 18.15

Similarly graphs of $y = x^2$, $y = (x-1)^2$ and $y = (x+2)^2$ are shown in Fig. 18.16.

(iv) $y = f(ax)$

For each point (x_1, y_1) on the graph of $y = f(x)$, there exists a point $\left(\frac{x_1}{a}, y_1\right)$ on the graph of $y = f(ax)$. Thus the graph of $y = f(ax)$ can be obtained by stretching $y = f(x)$ parallel to the x-axis by a scale factor $\frac{1}{a}$

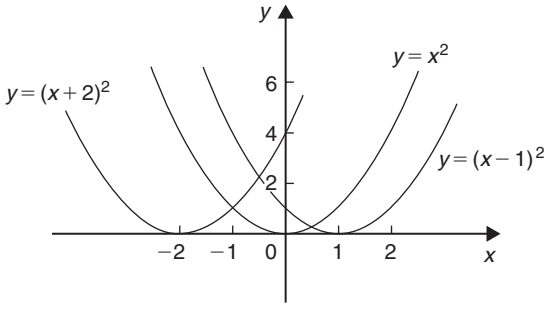


Figure 18.16

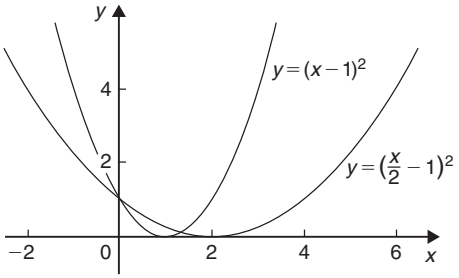
For example, if $f(x)=(x-1)^2$, and $a=\frac{1}{2}$, then $f(ax)=\left(\frac{x}{2}-1\right)^2$.

Both of these curves are shown in Fig. 18.17(a).

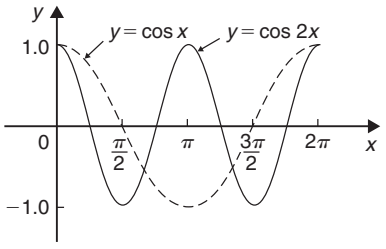
Similarly, $y=\cos x$ and $y=\cos 2x$ are shown in Fig. 18.17(b).

(v) $y=-f(x)$

The graph of $y=-f(x)$ is obtained by reflecting $y=f(x)$ in the x-axis. For example, graphs of $y=e^x$ and $y=-e^x$ are shown in Fig. 18.18(a) and graphs of $y=x^2+2$ and $y=-(x^2+2)$ are shown in Fig. 18.18(b).

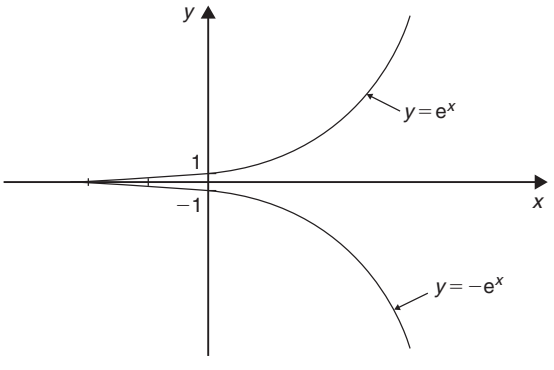


(a)



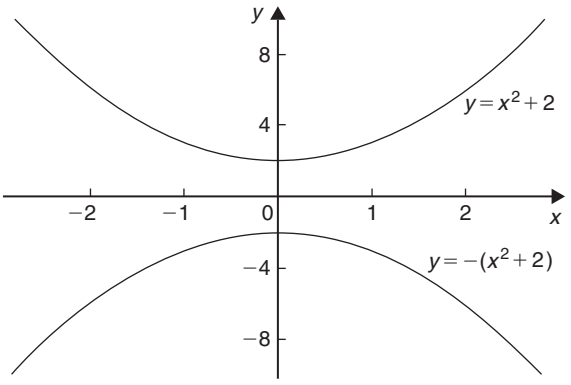
(b)

Figure 18.17



(a)

Figure 18.18

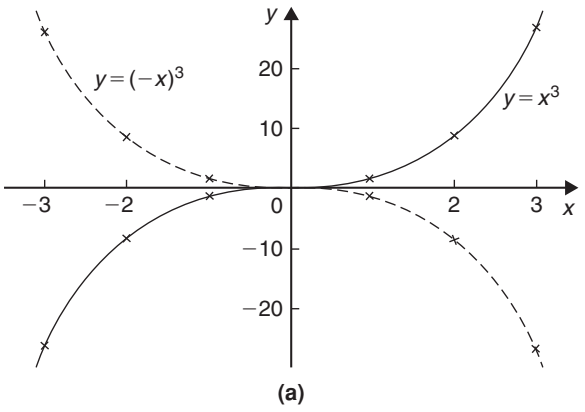


(b)

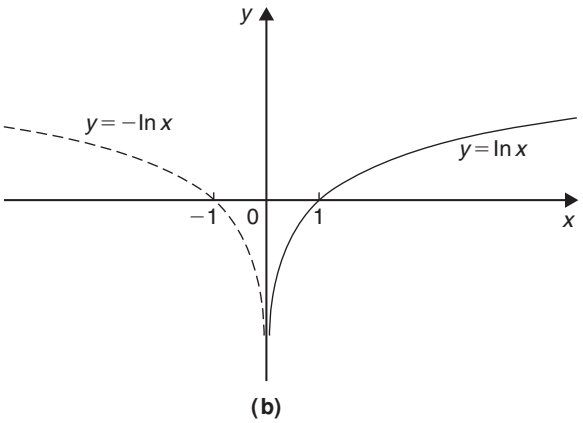
Figure 18.18 (*Continued*)

(vi) $y=f(-x)$

The graph of $y=f(-x)$ is obtained by reflecting $y=f(x)$ in the y-axis. For example, graphs of $y=x^3$ and $y=(-x)^3=-x^3$ are shown in Fig. 18.19(a)



(a)



(b)

Figure 18.19

and graphs of $y = \ln x$ and $y = -\ln x$ are shown in Fig. 18.19(b).

Problem 1. Sketch the following graphs, showing relevant points:

(a) $y = (x-4)^2$ (b) $y = x^3 - 8$

(a) In Fig. 18.20 a graph of $y = x^2$ is shown by the broken line. The graph of $y = (x-4)^2$ is of the form $y = f(x+a)$. Since $a = -4$, then $y = (x-4)^2$ is translated 4 units to the right of $y = x^2$, parallel to the x-axis.

(See Section (iii) above).

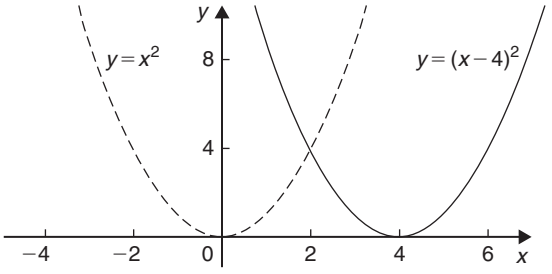


Figure 18.20

(b) In Fig. 18.21 a graph of $y = x^3$ is shown by the broken line. The graph of $y = x^3 - 8$ is of the form $y = f(x) + a$. Since $a = -8$, then $y = x^3 - 8$ is translated 8 units down from $y = x^3$, parallel to the y-axis.

(See Section (ii) above).

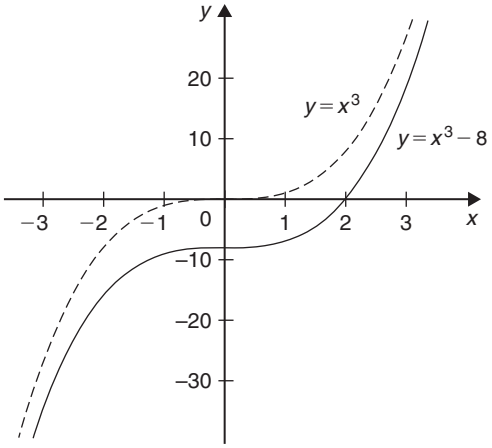


Figure 18.21

Problem 2. Sketch the following graphs, showing relevant points:

(a) $y = 5 - (x+2)^3$ (b) $y = 1 + 3\sin 2x$

(a) Figure 18.22(a) shows a graph of $y = x^3$. Figure 18.22(b) shows a graph of $y = (x+2)^3$ (see $f(x+a)$, Section (iii) above).

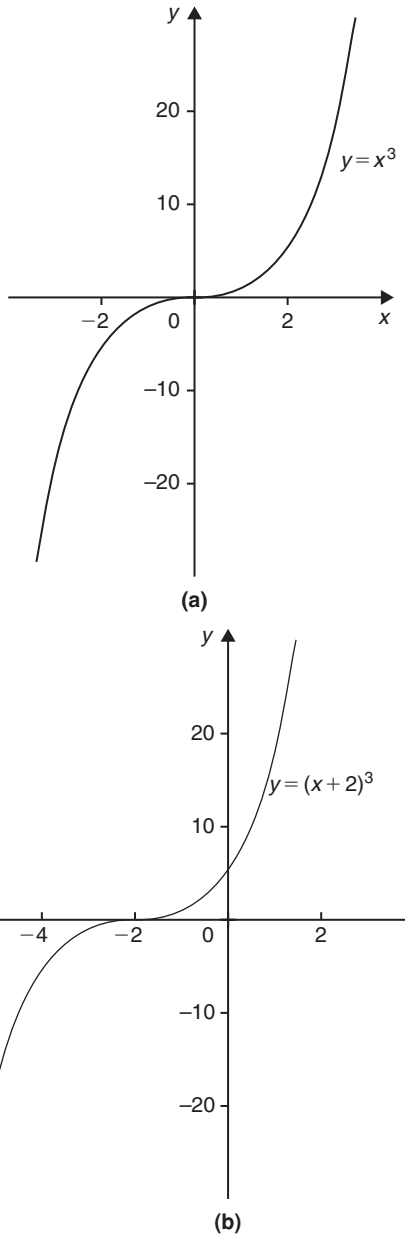


Figure 18.22

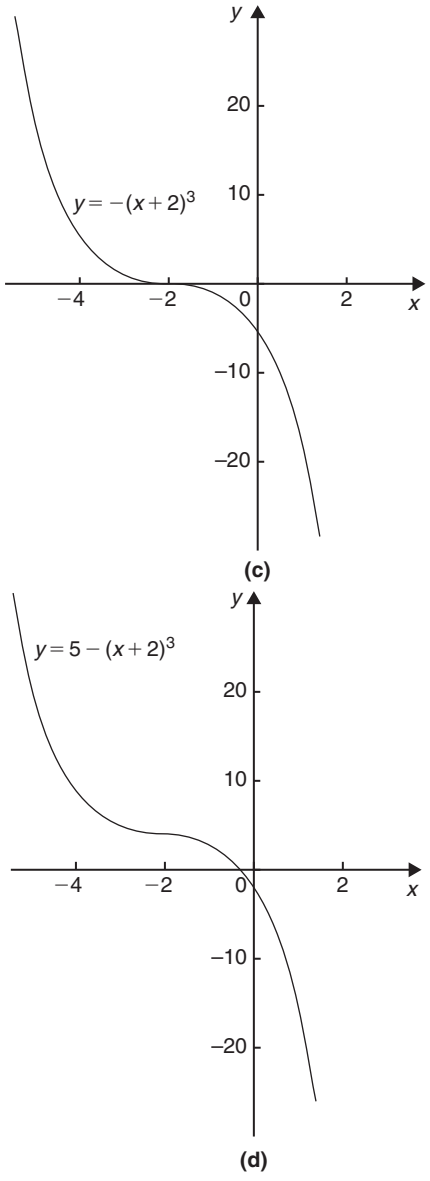


Figure 18.22 (*Continued*)

Figure 18.22(c) shows a graph of $y = -(x+2)^3$ (see $-f(x)$, Section (v) above). Figure 18.22(d) shows the graph of $y = 5-(x+2)^3$ (see $f(x)+a$, Section (ii) above).

(b) Figure 18.23(a) shows a graph of $y = \sin x$. Figure 18.23(b) shows a graph of $y = \sin 2x$ (see $f(ax)$, Section (iv) above).

Figure 18.23(c) shows a graph of $y = 3\sin 2x$ (see a $f(x)$, Section (i) above). Figure 18.23(d) shows a graph of $y = 1+3\sin 2x$ (see $f(x)+a$, Section (ii) above).

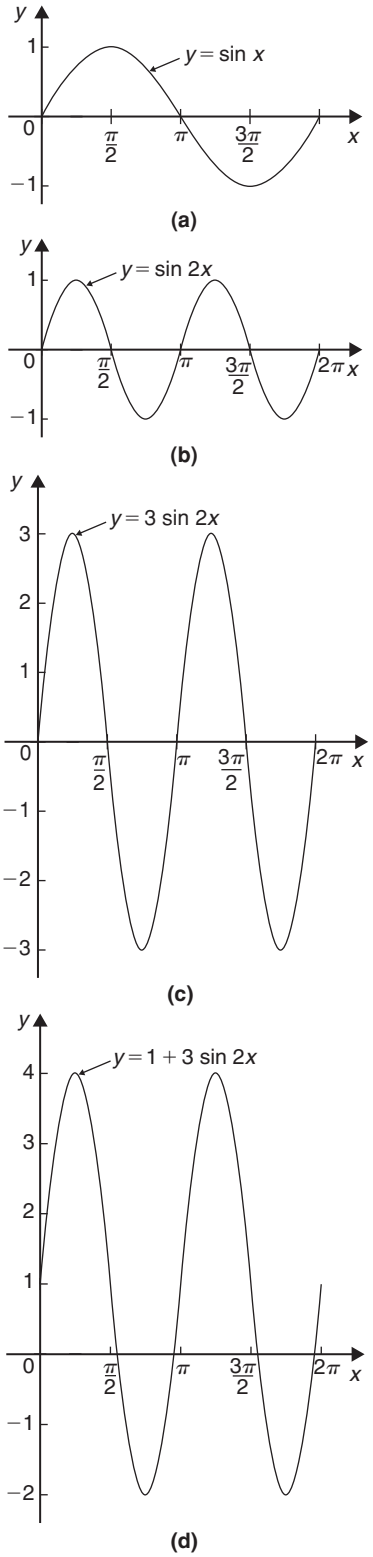


Figure 18.23

Now try the following exercise

Exercise 77 Further problems on simple transformations with curve sketching

Sketch the following graphs, showing relevant points:
(Answers on page 200, Fig. 18.39)

1. $y=3x-5$
2. $y=-3x+4$
3. $y=x^2+3$
4. $y=(x-3)^2$
5. $y=(x-4)^2+2$
6. $y=x-x^2$
7. $y=x^3+2$
8. $y=1+2\cos 3x$
9. $y=3-2\sin\left(x+\frac{\pi}{4}\right)$
10. $y=2\ln x$

18.3 Periodic functions

A function $f(x)$ is said to be **periodic** if $f(x+T)=f(x)$ for all values of x, where T is some positive number. T is the interval between two successive repetitions and is called the period of the function $f(x)$. For example, $y=\sin x$ is periodic in x with period 2π since $\sin x=\sin(x+2\pi)=\sin(x+4\pi)$, and so on. Similarly, $y=\cos x$ is a periodic function with period 2π since $\cos x=\cos(x+2\pi)=\cos(x+4\pi)$, and so on. In general, if $y=\sin\omega t$ or $y=\cos\omega t$ then the period of the waveform is $2\pi/\omega$. The function shown in Fig. 18.24 is also periodic of period 2π and is defined by:

$$f(x)=\begin{cases}-1, & \text{when } -\pi\le x\le 0\\ 1, & \text{when } 0\le x\le\pi\end{cases}$$

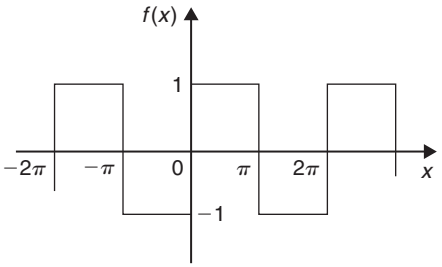


Figure 18.24

18.4 Continuous and discontinuous functions

If a graph of a function has no sudden jumps or breaks it is called a **continuous function**, examples being the graphs of sine and cosine functions. However, other graphs make finite jumps at a point or points in the interval. The square wave shown in Fig. 18.24 has **finite discontinuities** as $x=\pi$, 2π, 3π, and so on, and is therefore a discontinuous function. $y=\tan x$ is another example of a discontinuous function.

18.5 Even and odd functions

Even functions

A function $y=f(x)$ is said to be even if $f(-x)=f(x)$ for all values of x. Graphs of even functions are always symmetrical about the y-axis (i.e. is a mirror image). Two examples of even functions are $y=x^2$ and $y=\cos x$ as shown in Fig. 18.25.

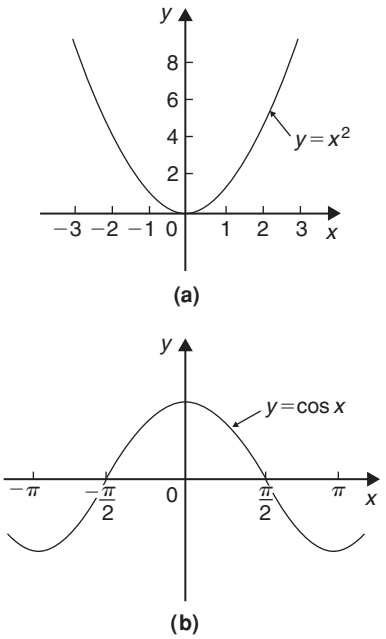


Figure 18.25

Odd functions

A function $y = f(x)$ is said to be odd if $f(-x) = -f(x)$ for all values of x. Graphs of odd functions are always symmetrical about the origin. Two examples of odd functions are $y = x^3$ and $y = \sin x$ as shown in Fig. 18.26.

Many functions are neither even nor odd, two such examples being shown in Fig. 18.27.

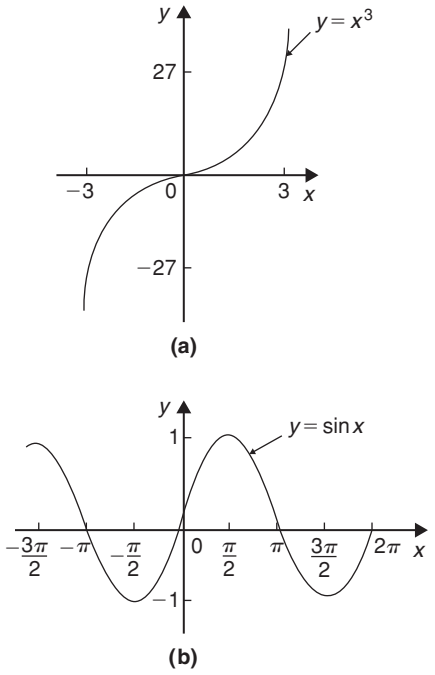


Figure 18.26

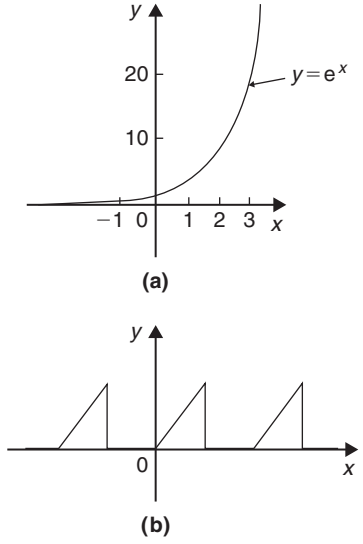


Figure 18.27

Problem 3. Sketch the following functions and state whether they are even or odd functions:

(a) $y = \tan x$

(b) $f(x) = \begin{cases} 2, & \text{when } 0 \le x \le \dfrac{\pi}{2} \\ -2, & \text{when } \dfrac{\pi}{2} \le x \le \dfrac{3\pi}{2}, \\ 2, & \text{when } \dfrac{3\pi}{2} \le x \le 2\pi \end{cases}$ and is periodic of period 2π.

(a) A graph of $y = \tan x$ is shown in Fig. 18.28(a) and is symmetrical about the origin and is thus an **odd function** (i.e. $\tan(-x) = -\tan x$).

(b) A graph of $f(x)$ is shown in Fig. 18.28(b) and is symmetrical about the $f(x)$ axis hence the function is an **even** one, ($f(-x) = f(x)$).

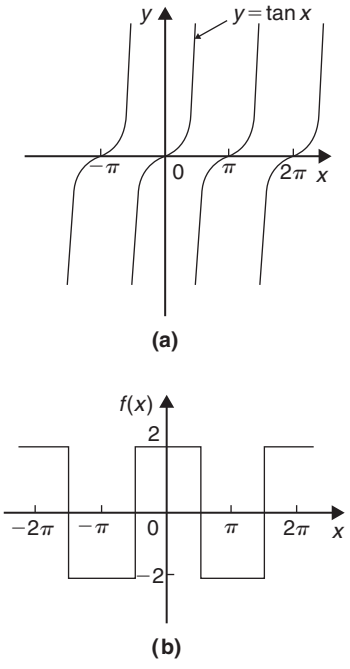


Figure 18.28

Problem 4. Sketch the following graphs and state whether the functions are even, odd or neither even nor odd:

(a) $y = \ln x$

(b) $f(x) = x$ in the range $-\pi$ to π and is periodic of period 2π.

(a) A graph of $y = \ln x$ is shown in Fig. 18.29(a) and the curve is neither symmetrical about the y-axis nor symmetrical about the origin and is thus **neither even nor odd**.

(b) A graph of $y = x$ in the range $-\pi$ to π is shown in Fig. 18.29(b) and is symmetrical about the origin and is thus an **odd function**.

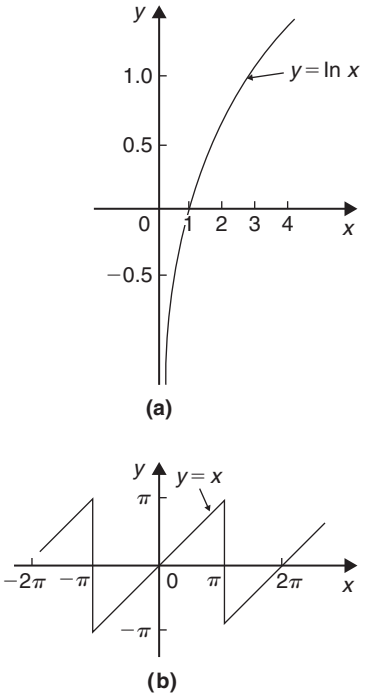


Figure 18.29

Now try the following exercise

Exercise 78 Further problems on even and odd functions

In Problems 1 and 2 determine whether the given functions are even, odd or neither even nor odd.

1. (a) x^4 (b) $\tan 3x$ (c) $2e^{3t}$ (d) $\sin^2 x$

[(a) even (b) odd (c) neither (d) even]

2. (a) $5t^3$ (b) $e^x + e^{-x}$ (c) $\dfrac{\cos\theta}{\theta}$ (d) e^x

[(a) odd (b) even (c) odd (d) neither]

3. State whether the following functions, which are periodic of period 2π, are even or odd:

(a) $f(\theta) = \begin{cases} \theta, & \text{when } -\pi \le \theta \le 0 \\ -\theta, & \text{when } 0 \le \theta \le \pi \end{cases}$

(b) $f(x) = \begin{cases} x, & \text{when } -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2} \\ 0, & \text{when } \dfrac{\pi}{2} \le x \le \dfrac{3\pi}{2} \end{cases}$

[(a) even (b) odd]

18.6 Inverse functions

If y is a function of x, the graph of y against x can be used to find x when any value of y is given. Thus the graph also expresses that x is a function of y. Two such functions are called **inverse functions**.

In general, given a function $y = f(x)$, its inverse may be obtained by interchanging the roles of x and y and then transposing for y. The inverse function is denoted by $y = f^{-1}(x)$.

For example, if $y = 2x + 1$, the inverse is obtained by

(i) transposing for x, i.e. $x = \dfrac{y-1}{2} = \dfrac{y}{2} - \dfrac{1}{2}$ and

(ii) interchanging x and y, giving the inverse as $y = \dfrac{x}{2} - \dfrac{1}{2}$

Thus if $f(x) = 2x + 1$, then $f^{-1}(x) = \dfrac{x}{2} - \dfrac{1}{2}$

A graph of $f(x) = 2x + 1$ and its inverse $f^{-1}(x) = \dfrac{x}{2} - \dfrac{1}{2}$ is shown in Fig. 18.30 and $f^{-1}(x)$ is seen to be a reflection of $f(x)$ in the line $y = x$.

Similarly, if $y = x^2$, the inverse is obtained by

(i) transposing for x, i.e. $x = \pm\sqrt{y}$ and

(ii) interchanging x and y, giving the inverse $y = \pm\sqrt{x}$.

Hence the inverse has two values for every value of x. Thus $f(x) = x^2$ does not have a single inverse. In such a case the domain of the original function may be restricted to $y = x^2$ for $x > 0$. Thus the inverse is then $y = +\sqrt{x}$. A graph of $f(x) = x^2$ and its inverse

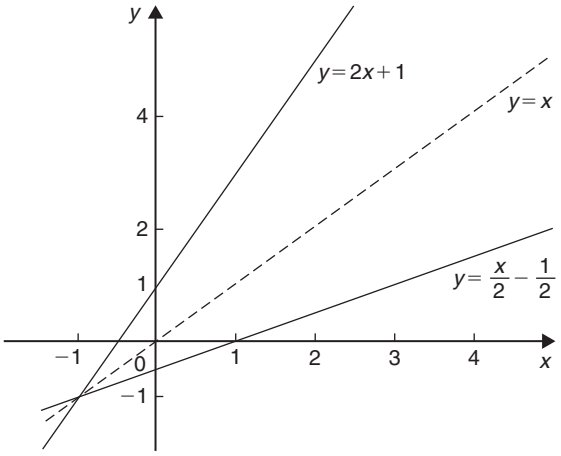


Figure 18.30

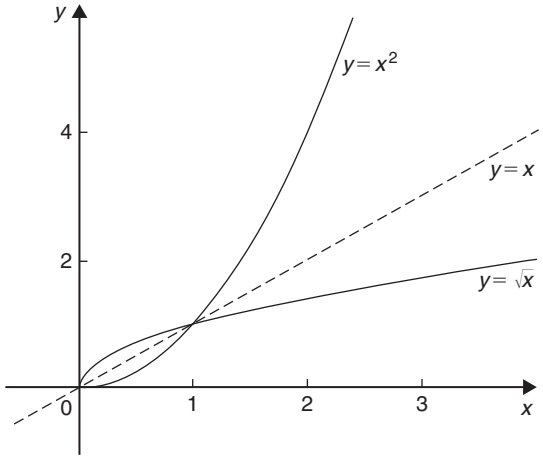


Figure 18.31

$f^{-1}(x)=\sqrt{x}$ for $x>0$ is shown in Fig. 18.31 and, again, $f^{-1}(x)$ is seen to be a reflection of $f(x)$ in the line $y=x$.

It is noted from the latter example, that not all functions have an inverse. An inverse, however, can be determined if the range is restricted.

Problem 5. Determine the inverse for each of the following functions:
(a) $f(x)=x-1$ (b) $f(x)=x^2-4 \quad (x>0)$
(c) $f(x)=x^2+1$

(a) If $y=f(x)$, then $y=x-1$
Transposing for x gives $x=y+1$
Interchanging x and y gives $y=x+1$
Hence if $f(x)=x-1$, then $\boldsymbol{f^{-1}(x)=x+1}$

(b) If $y=f(x)$, then $y=x^2-4 \quad (x>0)$
Transposing for x gives $x=\sqrt{y+4}$
Interchanging x and y gives $y=\sqrt{x+4}$
Hence if $f(x)=x^2-4 \quad (x>0)$ then
$\boldsymbol{f^{-1}(x)=\sqrt{x+4}}$ **if** $\boldsymbol{x>-4}$

(c) If $y=f(x)$, then $y=x^2+1$
Transposing for x gives $x=\sqrt{y-1}$
Interchanging x and y gives $y=\sqrt{x-1}$, which has two values.
Hence there is no inverse of $\boldsymbol{f(x)=x^2+1}$, since the domain of $f(x)$ is not restricted.

Inverse trigonometric functions

If $y=\sin x$, then x is the angle whose sine is y. Inverse trigonometrical functions are denoted by prefixing the function with 'arc' or, more commonly, $^{-1}$. Hence transposing $y=\sin x$ for x gives $x=\sin^{-1} y$. Interchanging x and y gives the inverse $y=\sin^{-1} x$.

Similarly, $y=\cos^{-1} x$, $y=\tan^{-1} x$, $y=\sec^{-1} x$, $y=\text{cosec}^{-1} x$ and $y=\cot^{-1} x$ are all inverse trigonometric functions. The angle is always expressed in radians.

Inverse trigonometric functions are periodic so it is necessary to specify the smallest or principal value of the angle. For $\sin^{-1} x$, $\tan^{-1} x$, $\text{cosec}^{-1} x$ and $\cot^{-1} x$, the principal value is in the range $-\frac{\pi}{2}<y<\frac{\pi}{2}$. For $\cos^{-1} x$ and $\sec^{-1} x$ the principal value is in the range $0<y<\pi$.

Graphs of the six inverse trigonometric functions are shown in Fig. 33.1, page 335.

Problem 6. Determine the principal values of
(a) arcsin 0.5 (b) arctan(−1)
(c) $\arccos\left(-\frac{\sqrt{3}}{2}\right)$ (d) arccosec$(\sqrt{2})$

Using a calculator,

(a) $\arcsin 0.5 \equiv \sin^{-1} 0.5 = 30°$

$$= \frac{\pi}{6}\ \textbf{rad} \text{ or } \textbf{0.5236 rad}$$

(b) $\arctan(-1) \equiv \tan^{-1}(-1) = -45°$

$$= -\frac{\pi}{4}\ \textbf{rad} \text{ or } \textbf{−0.7854 rad}$$

(c) $\arccos\left(-\frac{\sqrt{3}}{2}\right) \equiv \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = 150°$

$$= \frac{5\pi}{6}\,\textbf{rad or } \textbf{2.6180 rad}$$

(d) $\operatorname{arccosec}(\sqrt{2}) = \arcsin\left(\frac{1}{\sqrt{2}}\right)$

$$\equiv \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45°$$

$$= \frac{\pi}{4}\,\textbf{rad or } \textbf{0.7854 rad}$$

Problem 7. Evaluate (in radians), correct to 3 decimal places: $\sin^{-1}0.30 + \cos^{-1}0.65$.

$\sin^{-1}0.30 = 17.4576° = 0.3047\,\text{rad}$

$\cos^{-1}0.65 = 49.4584° = 0.8632\,\text{rad}$

Hence $\sin^{-1}0.30 + \cos^{-1}0.65$
$= 0.3047 + 0.8632 = \textbf{1.168}$, correct to 3 decimal places.

Now try the following exercise

Exercise 79 Further problems on inverse functions

Determine the inverse of the functions given in Problems 1 to 4.

1. $f(x) = x + 1$ $[f^{-1}(x) = x - 1]$
2. $f(x) = 5x - 1$ $\left[f^{-1}(x) = \frac{1}{5}(x+1)\right]$
3. $f(x) = x^3 + 1$ $[f^{-1}(x) = \sqrt[3]{x-1}]$
4. $f(x) = \frac{1}{x} + 2$ $\left[f^{-1}(x) = \frac{1}{x-2}\right]$

Determine the principal value of the inverse functions in Problems 5 to 11.

5. $\sin^{-1}(-1)$ $\left[-\frac{\pi}{2}\text{ or } -1.5708\,\text{rad}\right]$
6. $\cos^{-1}0.5$ $\left[\frac{\pi}{3}\text{ or } 1.0472\,\text{rad}\right]$
7. $\tan^{-1}1$ $\left[\frac{\pi}{4}\text{ or } 0.7854\,\text{rad}\right]$
8. $\cot^{-1}2$ [0.4636 rad]
9. $\operatorname{cosec}^{-1}2.5$ [0.4115 rad]
10. $\sec^{-1}1.5$ [0.8411 rad]
11. $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ $\left[\frac{\pi}{4}\text{ or } 0.7854\,\text{rad}\right]$
12. Evaluate x, correct to 3 decimal places:
$$x = \sin^{-1}\frac{1}{3} + \cos^{-1}\frac{4}{5} - \tan^{-1}\frac{8}{9}$$
[0.257]
13. Evaluate y, correct to 4 significant figures:
$$y = 3\sec^{-1}\sqrt{2} - 4\operatorname{cosec}^{-1}\sqrt{2} + 5\cot^{-1}2$$
[1.533]

18.7 Asymptotes

If a table of values for the function $y = \frac{x+2}{x+1}$ is drawn up for various values of x and then y plotted against x, the graph would be as shown in Fig. 18.32. The straight lines AB, i.e. $x = -1$, and CD, i.e. $y = 1$, are known as **asymptotes**.

An asymptote to a curve is defined as a straight line to which the curve approaches as the distance from the origin increases. Alternatively, an asymptote can be considered as a tangent to the curve at infinity.

Asymptotes parallel to the *x*- and *y*-axes

There is a simple rule which enables asymptotes parallel to the x- and y-axis to be determined. For a curve $y = f(x)$:

(i) the asymptotes parallel to the x-axis are found by equating the coefficient of the highest power of x to zero.

(ii) the asymptotes parallel to the y-axis are found by equating the coefficient of the highest power of y to zero.

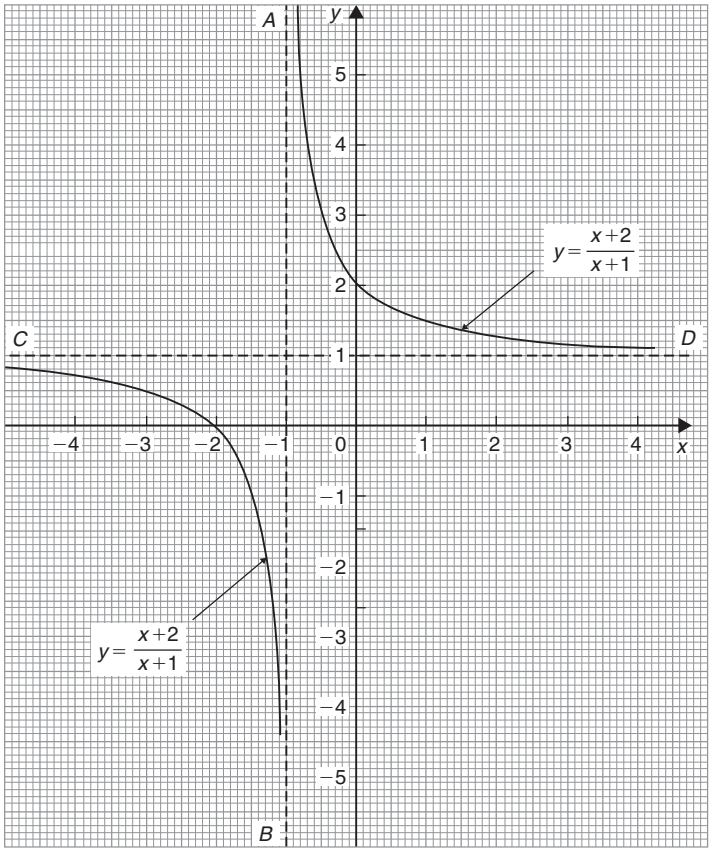


Figure 18.32

With the above example $y=\dfrac{x+2}{x+1}$, rearranging gives:

$$y(x+1)=x+2$$

i.e. $yx+y-x-2=0$ (1)

and $x(y-1)+y-2=0$

The coefficient of the highest power of x (in this case x^1) is $(y-1)$. Equating to zero gives: $y-1=0$

From which, $\boldsymbol{y=1}$, which is an asymptote of $y=\dfrac{x+2}{x+1}$ as shown in Fig. 18.32.

Returning to equation (1): $yx+y-x-2=0$

from which, $y(x+1)-x-2=0.$

The coefficient of the highest power of y (in this case y^1) is $(x+1)$. Equating to zero gives: $x+1=0$ from which, $\boldsymbol{x=-1}$, which is another asymptote of $y=\dfrac{x+2}{x+1}$ as shown in Fig. 18.32.

Problem 8. Determine the asymptotes for the function $y=\dfrac{x-3}{2x+1}$ and hence sketch the curve.

Rearranging $y=\dfrac{x-3}{2x+1}$ gives: $y(2x+1)=x-3$

i.e. $2xy+y=x-3$

or $2xy+y-x+3=0$

and $x(2y-1)+y+3=0$

Equating the coefficient of the highest power of x to zero gives: $2y-1=0$ from which, $\boldsymbol{y=\frac{1}{2}}$ which is an asymptote.

Since $y(2x+1)=x-3$ then equating the coefficient of the highest power of y to zero gives: $2x+1=0$ from which, $\boldsymbol{x=-\frac{1}{2}}$ which is also an asymptote.

When $x=0$, $y=\dfrac{x-3}{2x+1}=\dfrac{-3}{1}=-3$ and when $y=0$, $0=\dfrac{x-3}{2x+1}$ from which, $x-3=0$ and $x=3$.

A sketch of $y=\dfrac{x-3}{2x+1}$ is shown in Fig. 18.33.

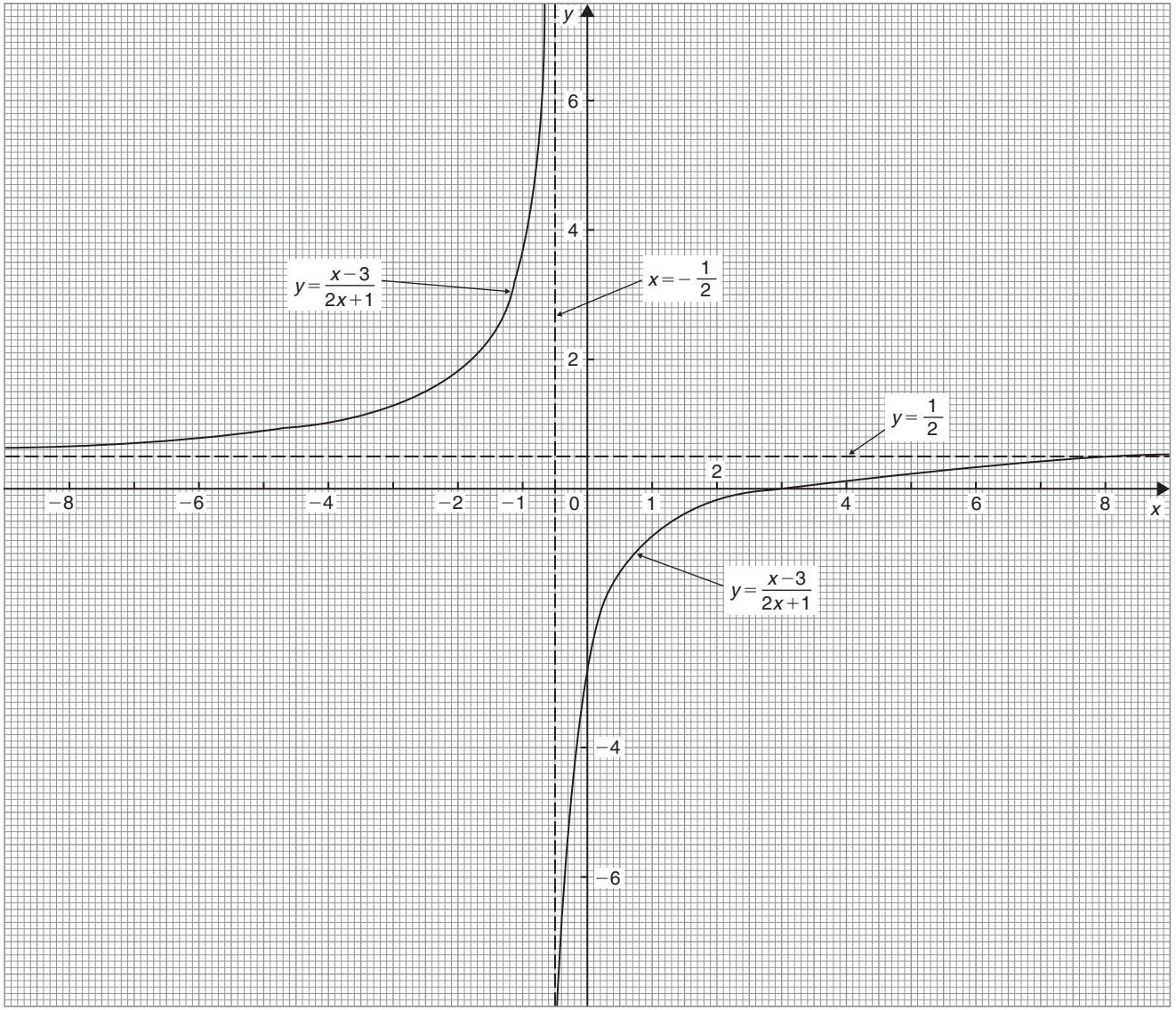

y
x
6
4
2
−4
−6
−8
−6
−4
−2
−1
0
1
2
4
6
8
$y = \frac{x-3}{2x+1}$
$x = -\frac{1}{2}$
$y = \frac{1}{2}$
$y = \frac{x-3}{2x+1}$

Figure 18.33

Problem 9. Determine the asymptotes parallel to the x- and y-axes for the function $x^2y^2=9(x^2+y^2)$.

Asymptotes parallel to the x-axis:

Rearranging $x^2y^2=9(x^2+y^2)$ gives

$$x^2y^2-9x^2-9y^2=0$$

hence $x^2(y^2-9)-9y^2=0$

Equating the coefficient of the highest power of x to zero gives $y^2-9=0$ from which, $y^2=9$ and $\boldsymbol{y=\pm 3}$.

Asymptotes parallel to the y-axis:

Since $x^2y^2-9x^2-9y^2=0$

then $y^2(x^2-9)-9x^2=0$

Equating the coefficient of the highest power of y to zero gives $x^2-9=0$ from which, $x^2=9$ and $\boldsymbol{x=\pm 3}$.

Hence asymptotes occur at $\boldsymbol{y=\pm 3}$ and $\boldsymbol{x=\pm 3}$.

Other asymptotes

To determine asymptotes other than those parallel to x- and y-axes a simple procedure is:

(i) substitute $y=mx+c$ in the given equation

(ii) simplify the expression

(iii) equate the coefficients of the two highest powers of x to zero and determine the values of m and c. $y=mx+c$ gives the asymptote.

Problem 10. Determine the asymptotes for the function: $y(x+1)=(x-3)(x+2)$ and sketch the curve.

Following the above procedure:

(i) Substituting $y=mx+c$ into $y(x+1)=(x-3)\ (x+2)$ gives:

$$(mx+c)(x+1)=(x-3)(x+2)$$

(ii) Simplifying gives

$$mx^2+mx+cx+c=x^2-x-6$$

and $(m-1)x^2+(m+c+1)x+c+6=0$

(iii) Equating the coefficient of the highest power of x to zero gives $m-1=0$ from which, $\boldsymbol{m=1}$.

Equating the coefficient of the next highest power of x to zero gives $m+c+1=0$.

and since $m=1$, $1+c+1=0$ from which, $\boldsymbol{c=-2}$.

Hence $y=mx+c=1x-2$.

i.e. $\boldsymbol{y=x-2}$ **is an asymptote**.

To determine any asymptotes parallel to the x-axis:

Rearranging $y(x+1)=(x-3)(x+2)$

gives $yx+y=x^2-x-6$

The coefficient of the highest power of x (i.e. x^2) is 1. Equating this to zero gives $1=0$ which is not an equation of a line. Hence there is no asymptote parallel to the x-axis.

To determine any asymptotes parallel to the y-axis:

Since $y(x+1)=(x-3)(x+2)$ the coefficient of the highest power of y is $x+1$. Equating this to zero gives $x+1=0$, from which, $x=-1$. Hence $\boldsymbol{x=-1}$ is an asymptote.

When $\boldsymbol{x=0}$, $y(1)=(-3)(2)$, i.e. $\boldsymbol{y=-6}$.

When $\boldsymbol{y=0}$, $0=(x-3)(x+2)$, i.e. $\boldsymbol{x=3}$ and $\boldsymbol{x=-2}$.

A sketch of the function $y(x+1)=(x-3)(x+2)$ is shown in Fig. 18.34.

Problem 11. Determine the asymptotes for the function $x^3-xy^2+2x-9=0$.

Following the procedure:

(i) Substituting $y=mx+c$ gives $x^3-x(mx+c)^2+2x-9=0$.

(ii) Simplifying gives

$$x^3-x[m^2x^2+2mcx+c^2]+2x-9=0$$

i.e. $x^3-m^2x^3-2mcx^2-c^2x+2x-9=0$

and $x^3(1-m^2)-2mcx^2-c^2x+2x-9=0$

(iii) Equating the coefficient of the highest power of x (i.e. x^3 in this case) to zero gives $1-m^2=0$, from which, $m=\pm 1$.

Equating the coefficient of the next highest power of x (i.e. x^2 in this case) to zero gives $-2mc=0$, from which, $c=0$.

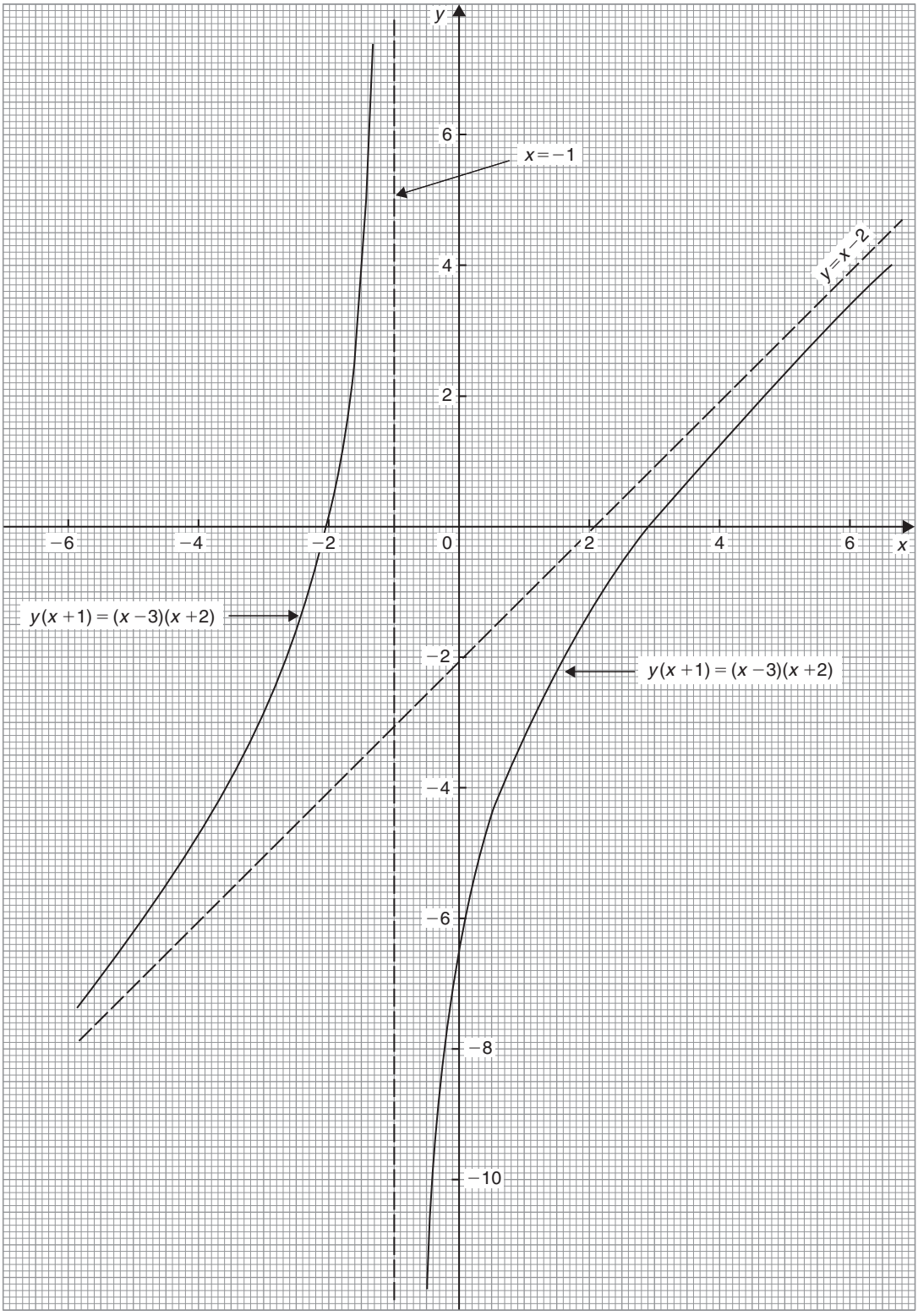


Figure 18.34

Hence $y = mx + c = \pm 1x + 0$, i.e. **$y = x$ and $y = -x$ are asymptotes**.

To determine any asymptotes parallel to the x- and y-axes for the function $x^3 - xy^2 + 2x - 9 = 0$:

Equating the coefficient of the highest power of x term to zero gives $1 = 0$ which is not an equation of a line. Hence there is no asymptote parallel with the x-axis.

Equating the coefficient of the highest power of y term to zero gives $-x = 0$ from which, $x = 0$.

Hence $x = 0$, $y = x$ and $y = -x$ are asymptotes for the function $x^3 - xy^2 + 2x - 9 = 0$.

Problem 12. Find the asymptotes for the function $y = \dfrac{x^2 + 1}{x}$ and sketch a graph of the function.

Rearranging $y = \dfrac{x^2 + 1}{x}$ gives $yx = x^2 + 1$.

Equating the coefficient of the highest power x term to zero gives $1 = 0$, hence there is no asymptote parallel to the x-axis.

Equating the coefficient of the highest power y term to zero gives $x = 0$.

Hence there is an asymptote at $x = 0$ (i.e. the y-axis).

To determine any other asymptotes we substitute $y = mx + c$ into $yx = x^2 + 1$ which gives

$$(mx + c)x = x^2 + 1$$

i.e. $$mx^2 + cx = x^2 + 1$$

and $$(m - 1)x^2 + cx - 1 = 0$$

Equating the coefficient of the highest power x term to zero gives $m - 1 = 0$, from which $m = 1$.

Equating the coefficient of the next highest power x term to zero gives $c = 0$. Hence $y = mx + c = 1x + 0$, i.e. **$y = x$ is an asymptote**.

A sketch of $y = \dfrac{x^2 + 1}{x}$ is shown in Fig. 18.35.

It is possible to determine maximum/minimum points on the graph (see Chapter 28).

Since $$y = \frac{x^2 + 1}{x} = \frac{x^2}{x} + \frac{1}{x} = x + x^{-1}$$

then $$\frac{dy}{dx} = 1 - x^{-2} = 1 - \frac{1}{x^2} = 0$$

for a turning point.

Hence $1 = \dfrac{1}{x^2}$ and $x^2 = 1$, from which, $x = \pm 1$.

When $x = 1$,

$$y = \frac{x^2 + 1}{x} = \frac{1 + 1}{1} = 2$$

and when $x = -1$,

$$y = \frac{(-1)^2 + 1}{-1} = -2$$

i.e. $(1, 2)$ and $(-1, -2)$ are the co-ordinates of the turning points. $\dfrac{d^2y}{dx^2} = 2x^{-3} = \dfrac{2}{x^3}$; when $x = 1$, $\dfrac{d^2y}{dx^2}$ is positive, which indicates a minimum point and when $x = -1$, $\dfrac{d^2y}{dx^2}$ is negative, which indicates a maximum point, as shown in Fig. 18.35.

Now try the following exercise

Exercise 80 Further problems on asymptotes

In Problems 1 to 3, determine the asymptotes parallel to the x- and y-axes.

1. $y = \dfrac{x - 2}{x + 1}$ $[y = 1, x = -1]$

2. $y^2 = \dfrac{x}{x - 3}$ $[x = 3, y = 1 \text{ and } y = -1]$

3. $y = \dfrac{x(x + 3)}{(x + 2)(x + 1)}$ $[x = -1, x = -2 \text{ and } y = 1]$

In Problems 4 and 5, determine all the asymptotes.

4. $8x - 10 + x^3 - xy^2 = 0$ $[x = 0, y = x \text{ and } y = -x]$

5. $x^2(y^2 - 16) = y$ $[y = 4, y = -4 \text{ and } x = 0]$

In Problems 6 and 7, determine the asymptotes and sketch the curves.

6. $y = \dfrac{x^2 - x - 4}{x + 1}$ $[x = -1, y = x - 2,$ see Fig 18.40, page 202$]$

7. $xy^2 - x^2y + 2x - y = 5$ $[x = 0, y = 0, y = x,$ see Fig. 18.41, page 202$]$

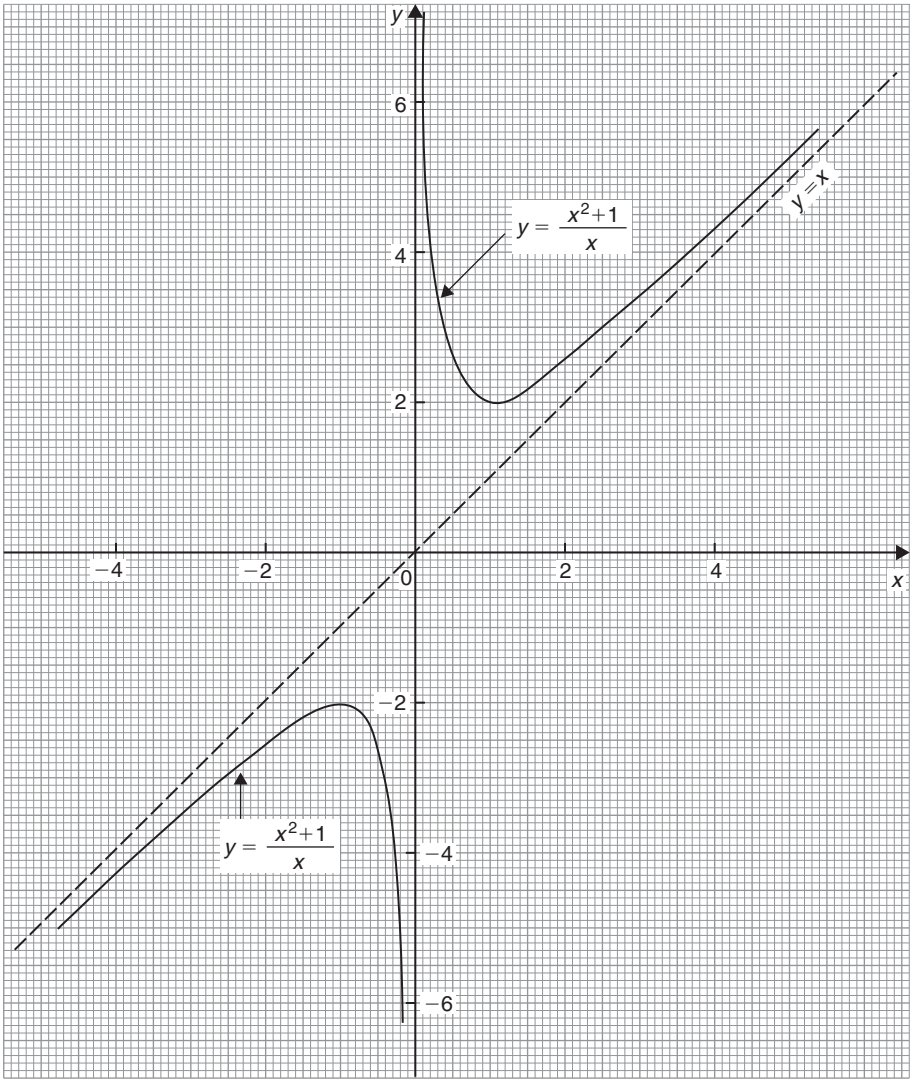


Figure 18.35

18.8 Brief guide to curve sketching

The following steps will give information from which the graphs of many types of functions $y = f(x)$ can be sketched.

(i) Use calculus to determine the location and nature of maximum and minimum points (see Chapter 28)

(ii) Determine where the curve cuts the x- and y-axes

(iii) Inspect the equation for symmetry.

(a) If the equation is unchanged when $-x$ is substituted for x, the graph will be symmetrical about the y-axis (i.e. it is an **even function**).

(b) If the equation is unchanged when $-y$ is substituted for y, the graph will be symmetrical about the x-axis.

(c) If $f(-x) = -f(x)$, the graph is symmetrical about the origin (i.e. it is an **odd function**).

(iv) Check for any asymptotes.

18.9 Worked problems on curve sketching

Problem 13. Sketch the graphs of

(a) $y=2x^2+12x+20$

(b) $y=-3x^2+12x-15$

(a) $y=2x^2+12x+20$ is a parabola since the equation is a quadratic. To determine the turning point:

Gradient $=\dfrac{dy}{dx}=4x+12=0$ for a turning point.

Hence $4x=-12$ and $x=-3$.

When $x=-3$, $y=2(-3)^2+12(-3)+20=2$.

Hence $(-3, 2)$ are the co-ordinates of the turning point

$\dfrac{d^2y}{dx^2}=4$, which is positive, hence $(-3, 2)$ is a minimum point.

When $x=0$, $y=20$, hence the curve cuts the y-axis at $y=20$.

Thus knowing the curve passes through $(-3, 2)$ and $(0, 20)$ and appreciating the general shape of a parabola results in the sketch given in Fig. 18.36.

(b) $y=-3x^2+12x-15$ is also a parabola (but 'upside down' due to the minus sign in front of the x^2 term).

Gradient $=\dfrac{dy}{dx}=-6x+12=0$ for a turning point.

Hence $6x=12$ and $x=2$.

When $x=2$, $y=-3(2)^2+12(2)-15=-3$.

Hence $(2, -3)$ are the co-ordinates of the turning point

$\dfrac{d^2y}{dx^2}=-6$, which is negative, hence $(2, -3)$ is a maximum point.

When $x=0$, $y=-15$, hence the curve cuts the axis at $y=-15$.

The curve is shown sketched in Fig. 18.36.

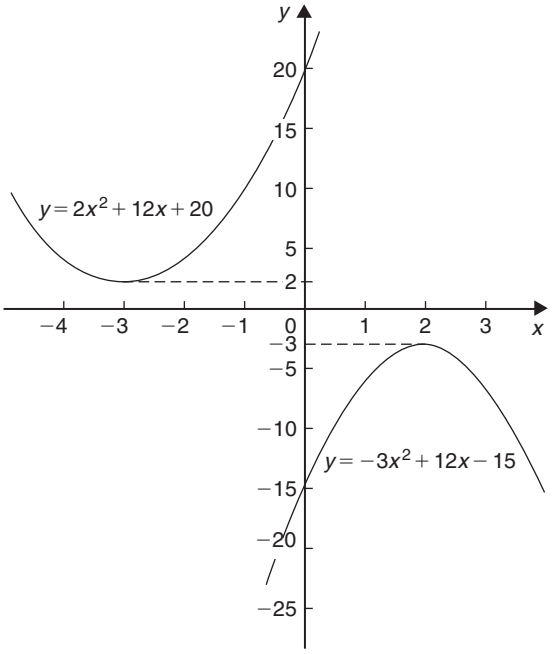


Figure 18.36

Problem 14. Sketch the curves depicting the following equations:

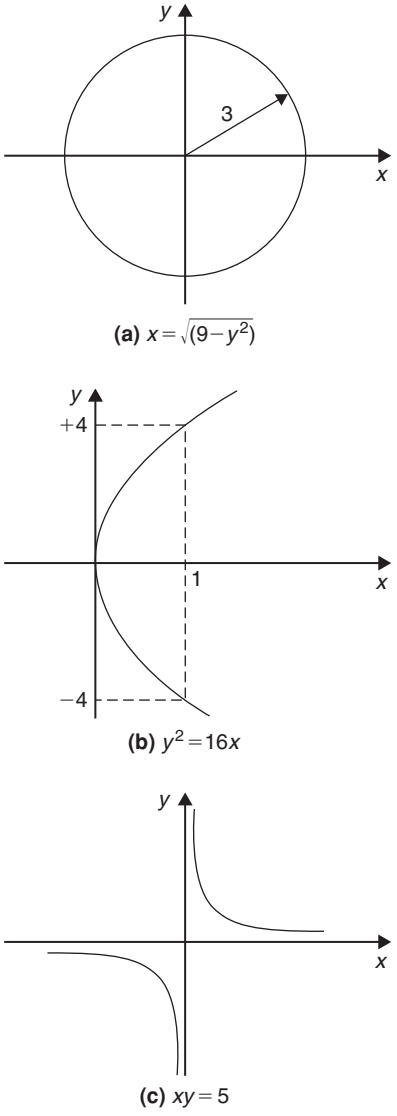
(a) $x=\sqrt{9-y^2}$ (b) $y^2=16x$

(c) $xy=5$

(a) Squaring both sides of the equation and transposing gives $x^2+y^2=9$. Comparing this with the standard equation of a circle, centre origin and radius a, i.e. $x^2+y^2=a^2$, shows that $x^2+y^2=9$ represents a circle, centre origin and radius 3. A sketch of this circle is shown in Fig. 18.37(a).

(b) The equation $y^2=16x$ is symmetrical about the x-axis and having its vertex at the origin (0, 0). Also, when $x=1$, $y=\pm 4$. A sketch of this parabola is shown in Fig. 18.37(b).

(c) The equation $y=\dfrac{a}{x}$ represents a rectangular hyperbola lying entirely within the first and third quadrants. Transposing $xy=5$ gives $y=\dfrac{5}{x}$, and therefore represents the rectangular hyperbola shown in Fig. 18.37(c).



(a) $x=\sqrt{(9-y^2)}$

(b) $y^2=16x$

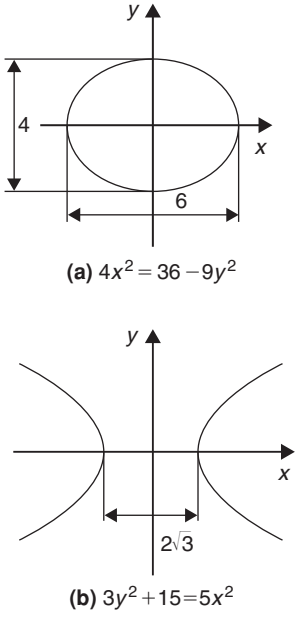
(c) $xy=5$

Figure 18.37

Problem 15. Sketch the curves depicting the following equations:

(a) $4x^2=36-9y^2$ (b) $3y^2+15=5x^2$

(a) By dividing throughout by 36 and transposing, the equation $4x^2=36-9y^2$ can be written as $\frac{x^2}{9}+\frac{y^2}{4}=1$. The equation of an ellipse is of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $2a$ and $2b$ represent the length of the axes of the ellipse. Thus $\frac{x^2}{3^2}+\frac{y^2}{2^2}=1$ represents an ellipse, having its axes coinciding with the x- and y-axes of a rectangular co-ordinate system, the major axis being 2(3), i.e. 6 units long and the minor axis 2(2), i.e. 4 units long, as shown in Fig. 18.38(a).



(a) $4x^2=36-9y^2$

(b) $3y^2+15=5x^2$

Figure 18.38

(b) Dividing $3y^2+15=5x^2$ throughout by 15 and transposing gives $\frac{x^2}{3}-\frac{y^2}{5}=1$. The equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ represents a hyperbola which is symmetrical about both the x- and y-axes, the distance between the vertices being given by $2a$.

Thus a sketch of $\frac{x^2}{3}-\frac{y^2}{5}=1$ is as shown in Fig. 18.38(b), having a distance of $2\sqrt{3}$ between its vertices.

Problem 16. Describe the shape of the curves represented by the following equations:

(a) $x=2\sqrt{\left[1-\left(\frac{y}{2}\right)^2\right]}$ (b) $\frac{y^2}{8}=2x$

(c) $y=6\left(1-\frac{x^2}{16}\right)^{1/2}$

(a) Squaring the equation gives $x^2=4\left[1-\left(\frac{y}{2}\right)^2\right]$ and transposing gives $x^2=4-y^2$, i.e.

$x^2+y^2=4$. Comparing this equation with $x^2+y^2=a^2$ shows that $x^2+y^2=4$ is the equation of a **circle** having centre at the origin (0, 0) and of radius 2 units.

(b) Transposing $\frac{y^2}{8}=2x$ gives $y=4\sqrt{x}$. Thus $\frac{y^2}{8}=2x$ is the equation of a **parabola** having its axis of symmetry coinciding with the x-axis and its vertex at the origin of a rectangular co-ordinate system.

(c) $y=6\left(1-\frac{x^2}{16}\right)^{1/2}$ can be transposed to $\frac{y}{6}=\left(1-\frac{x^2}{16}\right)^{1/2}$ and squaring both sides gives $\frac{y^2}{36}=1-\frac{x^2}{16}$, i.e. $\frac{x^2}{16}+\frac{y^2}{36}=1$.

This is the equation of an **ellipse**, centre at the origin of a rectangular co-ordinate system, the major axis coinciding with the y-axis and being $2\sqrt{36}$, i.e. 12 units long. The minor axis coincides with the x-axis and is $2\sqrt{16}$, i.e. 8 units long.

Problem 17. Describe the shape of the curves represented by the following equations:

(a) $\frac{x}{5}=\sqrt{\left[1+\left(\frac{y}{2}\right)^2\right]}$ (b) $\frac{y}{4}=\frac{15}{2x}$

(a) Since $\frac{x}{5}=\sqrt{\left[1+\left(\frac{y}{2}\right)^2\right]}$

$\frac{x^2}{25}=1+\left(\frac{y}{2}\right)^2$

i.e. $\frac{x^2}{25}-\frac{y^2}{4}=1$

This is a **hyperbola** which is symmetrical about both the x- and y-axes, the vertices being $2\sqrt{25}$, i.e. 10 units apart.

(With reference to Section 18.1 (vii), a is equal to ± 5)

(b) The equation $\frac{y}{4}=\frac{15}{2x}$ is of the form $y=\frac{a}{x}$, $a=\frac{60}{2}=30$.

This represents a **rectangular hyperbola**, symmetrical about both the x- and y-axis, and lying entirely in the first and third quadrants, similar in shape to the curves shown in Fig. 18.9.

Now try the following exercise

Exercise 81 Further problems on curve sketching

1. Sketch the graphs of (a) $y=3x^2+9x+\frac{7}{4}$ (b) $y=-5x^2+20x+50$.

[(a) Parabola with minimum value at $\left(-\frac{3}{2}, -5\right)$ and passing through $\left(0, 1\frac{3}{4}\right)$. (b) Parabola with maximum value at (2, 70) and passing through (0, 50).]

In Problems 2 to 8, sketch the curves depicting the equations given.

2. $x=4\sqrt{\left[1-\left(\frac{y}{4}\right)^2\right]}$

[circle, centre (0, 0), radius 4 units]

3. $\sqrt{x}=\frac{y}{9}$

[parabola, symmetrical about x-axis, vertex at (0, 0)]

4. $y^2=\frac{x^2-16}{4}$

[hyperbola, symmetrical about x- and y-axes, distance between vertices 8 units along x-axis]

5. $\frac{y^2}{5}=5-\frac{x^2}{2}$

[ellipse, centre (0, 0), major axis 10 units along y-axis, minor axis $2\sqrt{10}$ units along x-axis]

6. $x=3\sqrt{1+y^2}$

[hyperbola, symmetrical about x- and y-axes, distance between vertices 6 units along x-axis]

7. $x^2y^2=9$

[rectangular hyperbola, lying in first and third quadrants only]

8. $x = \frac{1}{3}\sqrt{(36 - 18y^2)}$

[ellipse, centre (0, 0), major axis 4 units along x-axis, minor axis $2\sqrt{2}$ units along y-axis]

9. Sketch the circle given by the equation $x^2 + y^2 - 4x + 10y + 25 = 0$.

[Centre at (2, −5), radius 2]

In Problems 10 to 15 describe the shape of the curves represented by the equations given.

10. $y = \sqrt{[3(1 - x^2)]}$

[ellipse, centre (0, 0), major axis $2\sqrt{3}$ units along y-axis, minor axis 2 units along x-axis]

11. $y = \sqrt{[3(x^2 - 1)]}$

[hyperbola, symmetrical about x- and y-axes, vertices 2 units apart along x-axis]

12. $y = \sqrt{9 - x^2}$

[circle, centre (0, 0), radius 3 units]

13. $y = 7x^{-1}$

[rectangular hyperbola, lying in first and third quadrants, symmetrical about x- and y-axes]

14. $y = (3x)^{1/2}$

[parabola, vertex at (0, 0), symmetrical about the x-axis]

15. $y^2 - 8 = -2x^2$

[ellipse, centre (0, 0), major axis $2\sqrt{8}$ units along the y-axis, minor axis 4 units along the x-axis]

Graphical solutions to Exercise 77, page 186

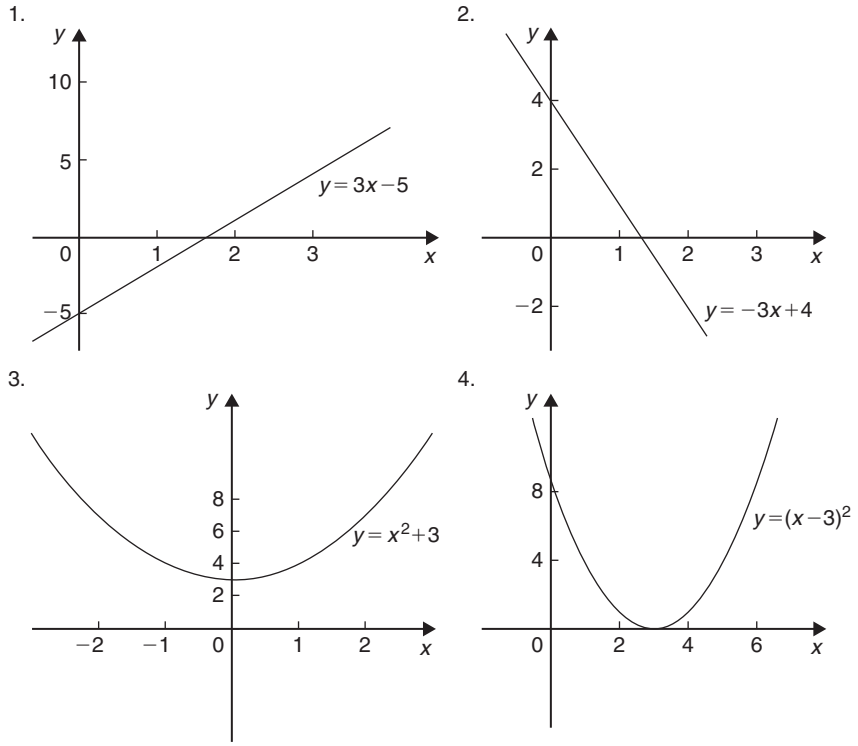


Figure 18.39

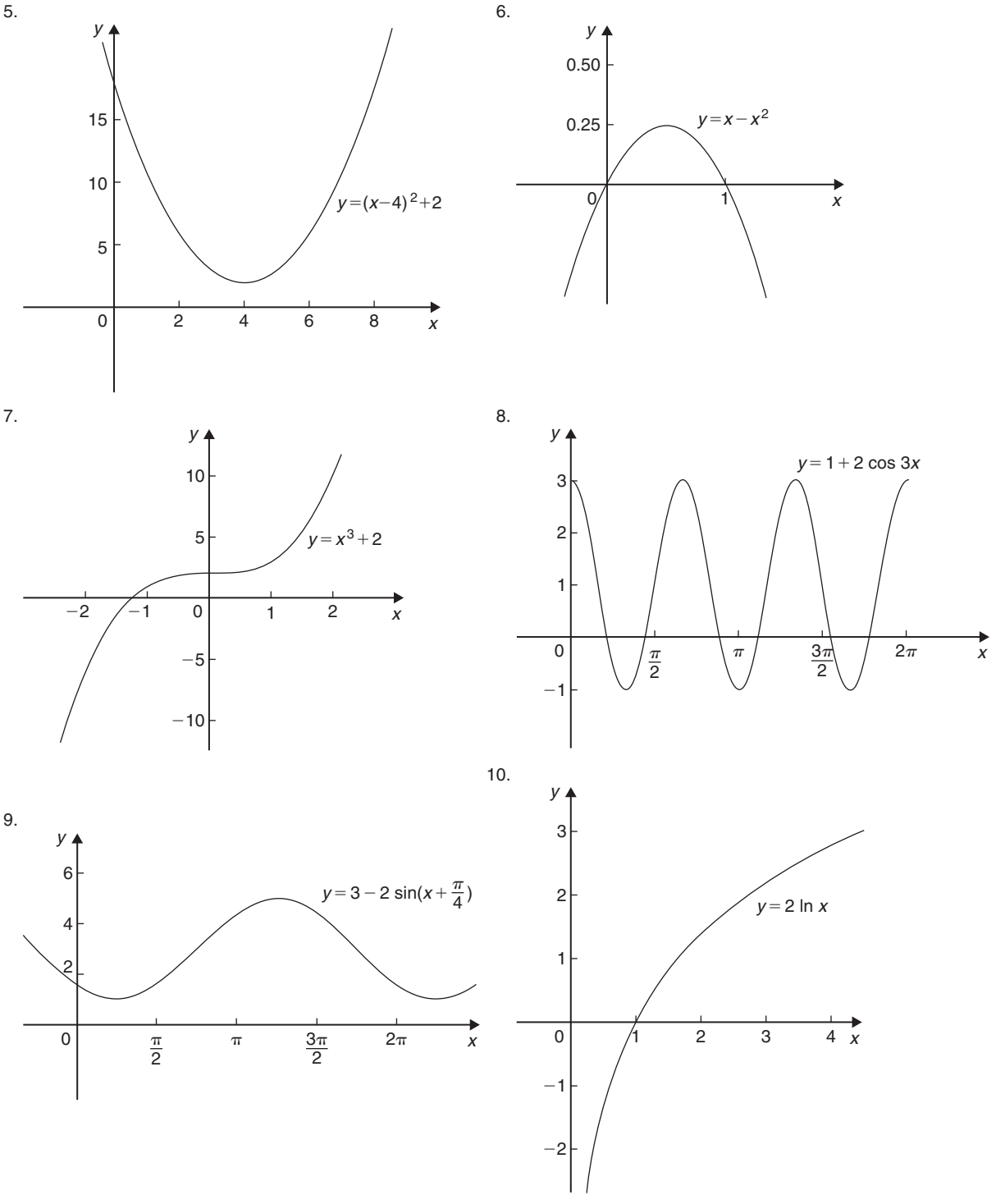


Figure 18.39 (*Continued*)

Graphical solutions to Problems 6 and 7, Exercise 80, page 195

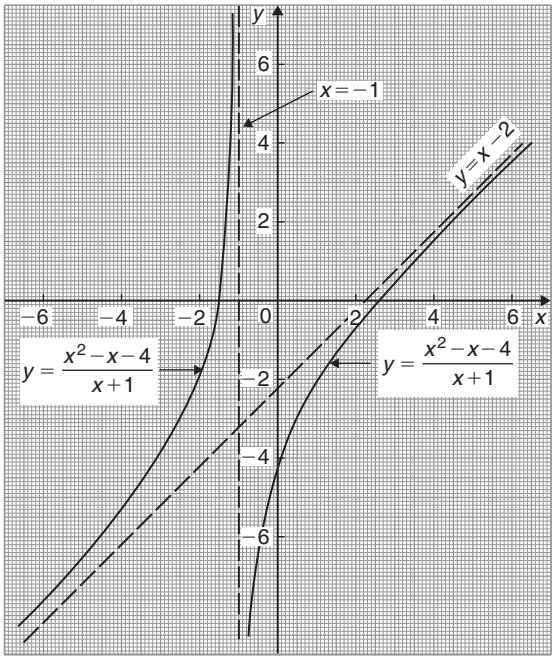


Figure 18.40

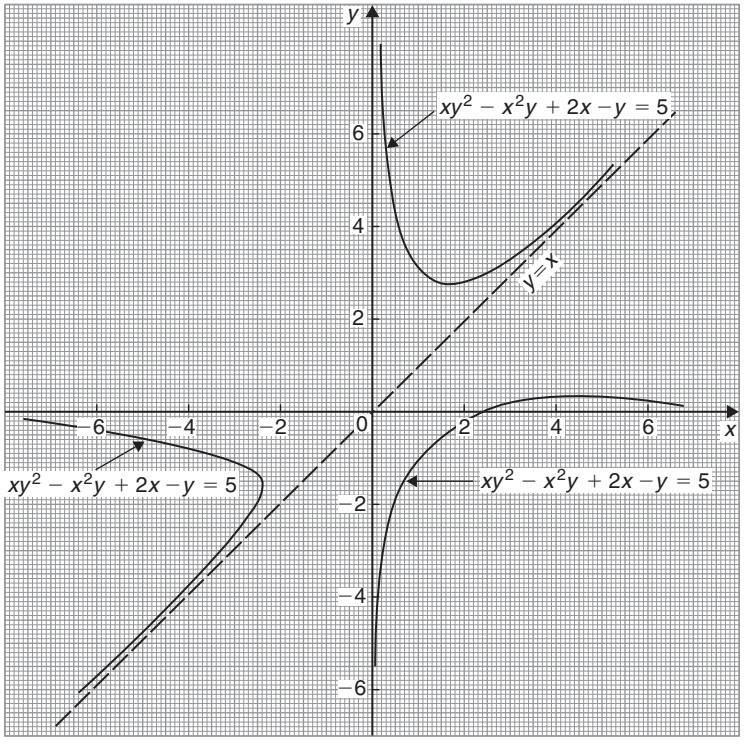


Figure 18.41

Chapter 19

Irregular areas, volumes and mean values of waveforms

19.1 Areas of irregular figures

Areas of irregular plane surfaces may be approximately determined by using (a) a planimeter, (b) the trapezoidal rule, (c) the mid-ordinate rule, and (d) Simpson's rule. Such methods may be used, for example, by engineers estimating areas of indicator diagrams of steam engines, surveyors estimating areas of plots of land or naval architects estimating areas of water planes or transverse sections of ships.

(a) **A planimeter** is an instrument for directly measuring small areas bounded by an irregular curve.

(b) **Trapezoidal rule**

To determine the areas *PQRS* in Fig. 19.1:

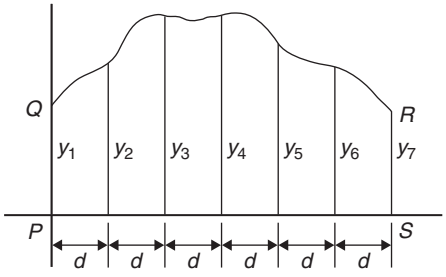


Figure 19.1

(i) Divide base *PS* into any number of equal intervals, each of width *d* (the greater the number of intervals, the greater the accuracy).

(ii) Accurately measure ordinates y_1, y_2, y_3, etc.

(iii) Areas *PQRS*

$$= d\left[\frac{y_1 + y_7}{2} + y_2 + y_3 + y_4 + y_5 + y_6\right]$$

In general, the trapezoidal rule states:

$$\text{Area} = \left(\begin{matrix}\text{width of}\\ \text{interval}\end{matrix}\right)\left[\frac{1}{2}\left(\begin{matrix}\text{first} + \\ \text{last}\\ \text{ordinate}\end{matrix}\right) + \begin{matrix}\text{sum of}\\ \text{remaining}\\ \text{ordinates}\end{matrix}\right]$$

(c) **Mid-ordinate rule**

To determine the area *ABCD* of Fig. 19.2:

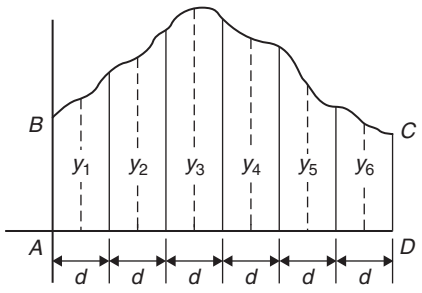


Figure 19.2

(i) Divide base *AD* into any number of equal intervals, each of width *d* (the greater the number of intervals, the greater the accuracy).

(ii) Erect ordinates in the middle of each interval (shown by broken lines in Fig. 19.2).

(iii) Accurately measure ordinates y_1, y_2, y_3, etc.

(iv) Area $ABCD = d(y_1 + y_2 + y_3 + y_4 + y_5 + y_6)$

In general, the mid-ordinate rule states:

$$\text{Area} = \begin{pmatrix}\text{width of}\\ \text{interval}\end{pmatrix}\begin{pmatrix}\text{sum of}\\ \text{mid-ordinates}\end{pmatrix}$$

(d) **Simpson's rule**

To determine the area $PQRS$ of Fig. 19.1:

(i) Divide base PS into an **even** number of intervals, each of width d (the greater the number of intervals, the greater the accuracy).

(ii) Accurately measure ordinates y_1, y_2, y_3, etc.

(iii) Area $PQRS = \frac{d}{3}[(y_1 + y_7) + 4(y_2 + y_4 + y_6) + 2(y_3 + y_5)]$

In general, Simpson's rule states:

$$\text{Area} = \frac{1}{3}\begin{pmatrix}\text{width of}\\ \text{interval}\end{pmatrix}\left[\begin{pmatrix}\text{first + last}\\ \text{ordinate}\end{pmatrix} + 4\begin{pmatrix}\text{sum of even}\\ \text{ordinates}\end{pmatrix} + 2\begin{pmatrix}\text{sum of remaining}\\ \text{odd ordinates}\end{pmatrix}\right]$$

Problem 1. A car starts from rest and its speed is measured every second for 6 s:

Time t (s)	0	1	2	3	4	5	6
Speed v (m/s)	0	2.5	5.5	8.75	12.5	17.5	24.0

Determine the distance travelled in 6 seconds (i.e. the area under the v/t graph), by (a) the trapezoidal rule, (b) the mid-ordinate rule, and (c) Simpson's rule.

A graph of speed/time is shown in Fig. 19.3.

(a) **Trapezoidal rule** (see para. (b) above)

The time base is divided into 6 strips each of width 1 s, and the length of the ordinates measured.

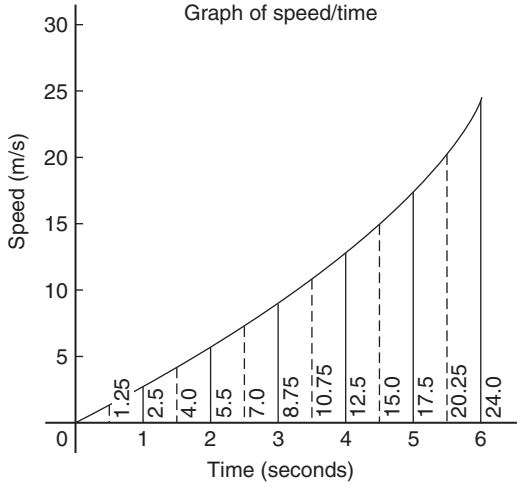


Figure 19.3

Thus

$$\textbf{area} = (1)\left[\left(\frac{0 + 24.0}{2}\right) + 2.5 + 5.5 + 8.75 + 12.5 + 17.5\right]$$

$$= \mathbf{58.75\,m}$$

(b) **Mid-ordinate rule** (see para. (c) above)

The time base is divided into 6 strips each of width 1 second.

Mid-ordinates are erected as shown in Fig. 19.3 by the broken lines. The length of each mid-ordinate is measured. Thus

$$\textbf{area} = (1)[1.25 + 4.0 + 7.0 + 10.75 + 15.0 + 20.25]$$

$$= \mathbf{58.25\,m}$$

(c) **Simpson's rule** (see para. (d) above)

The time base is divided into 6 strips each of width 1 s, and the length of the ordinates measured. Thus

$$\textbf{area} = \tfrac{1}{3}(1)[(0 + 24.0) + 4(2.5 + 8.75 + 17.5) + 2(5.5 + 12.5)]$$

$$= \mathbf{58.33\,m}$$

Problem 2. A river is 15 m wide. Soundings of the depth are made at equal intervals of 3 m across the river and are as shown below.

Depth (m)	0	2.2	3.3	4.5	4.2	2.4	0

Calculate the cross-sectional area of the flow of water at this point using Simpson's rule.

From para. (d) above,

$$\text{Area} = \tfrac{1}{3}(3)[(0+0)+4(2.2+4.5+2.4) + 2(3.3+4.2)]$$
$$= (1)[0+36.4+15] = \mathbf{51.4\,m^2}$$

Now try the following exercise

Exercise 82 Further problems on areas of irregular figures

1. Plot a graph of $y = 3x - x^2$ by completing a table of values of y from $x = 0$ to $x = 3$. Determine the area enclosed by the curve, the x-axis and ordinate $x = 0$ and $x = 3$ by (a) the trapezoidal rule, (b) the mid-ordinate rule and (c) by Simpson's rule. [4.5 square units]
2. Plot the graph of $y = 2x^2 + 3$ between $x = 0$ and $x = 4$. Estimate the area enclosed by the curve, the ordinates $x = 0$ and $x = 4$, and the x-axis by an approximate method. [54.7 square units]
3. The velocity of a car at one second intervals is given in the following table:

time t (s)	0	1	2	3	4	5	6
velocity v (m/s)	0	2.0	4.5	8.0	14.0	21.0	29.0

Determine the distance travelled in 6 seconds (i.e. the area under the v/t graph) using Simpson's rule. [63.33 m]

4. The shape of a piece of land is shown in Fig. 19.4. To estimate the area of the land, a surveyor takes measurements at intervals of 50 m, perpendicular to the straight portion with the results shown (the dimensions being in metres). Estimate the area of the land in hectares (1 ha = 10^4 m^2). [4.70 ha]

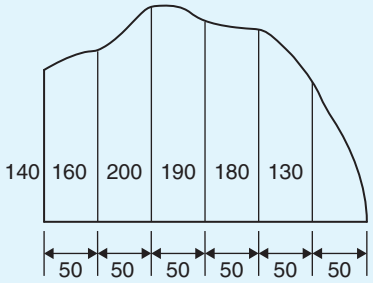


Figure 19.4

5. The deck of a ship is 35 m long. At equal intervals of 5 m the width is given by the following table:

Width (m)	0	2.8	5.2	6.5	5.8	4.1	3.0	2.3

Estimate the area of the deck. [143 m^2]

19.2 Volumes of irregular solids

If the cross-sectional areas $A_1, A_2, A_3, \ldots$ of an irregular solid bounded by two parallel planes are known at equal intervals of width d (as shown in Fig. 19.5), then by Simpson's rule:

$$\textbf{volume}, V = \frac{d}{3}[(A_1 + A_7) + 4(A_2 + A_4 + A_6) + 2(A_3 + A_5)]$$

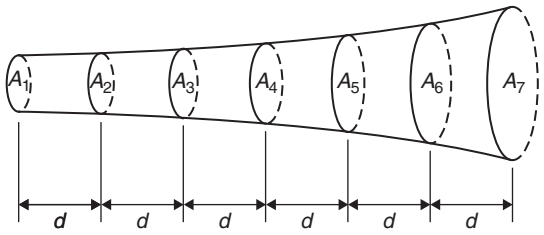


Figure 19.5

Problem 3. A tree trunk is 12 m in length and has a varying cross-section. The cross-sectional areas at intervals of 2 m measured from one end are:

0.52, 0.55, 0.59, 0.63, 0.72, 0.84, 0.97 m^2

Estimate the volume of the tree trunk.

A sketch of the tree trunk is similar to that shown in Fig. 19.5 above, where $d = 2\,\text{m}$, $A_1 = 0.52\,\text{m}^2$, $A_2 = 0.55\,\text{m}^2$, and so on.

Using Simpson's rule for volumes gives:

$$\text{Volume} = \tfrac{2}{3}[(0.52 + 0.97) + 4(0.55 + 0.63 + 0.84) + 2(0.59 + 0.72)]$$

$$= \tfrac{2}{3}[1.49 + 8.08 + 2.62] = \mathbf{8.13\,m^3}$$

Problem 4. The areas of seven horizontal cross-sections of a water reservoir at intervals of 10 m are:

210, 250, 320, 350, 290, 230, 170 m^2

Calculate the capacity of the reservoir in litres.

Using Simpson's rule for volumes gives:

$$\text{Volume} = \frac{10}{3}[(210 + 170) + 4(250 + 350 + 230) + 2(320 + 290)]$$

$$= \frac{10}{3}[380 + 3320 + 1220]$$

$$= \mathbf{16400\,m^3}$$

$16400\,\text{m}^3 = 16400 \times 10^6\,\text{cm}^3$ and since $1\,\text{litre} = 1000\,\text{cm}^3$,

$$\text{capacity of reservoir} = \frac{16400 \times 10^6}{1000}\ \text{litres}$$

$$= 16400000$$

$$= \mathbf{1.64 \times 10^7\ litres}$$

Now try the following exercise

Exercise 83 Further problems on volumes of irregular solids

1. The areas of equidistantly spaced sections of the underwater form of a small boat are as follows:

   1.76, 2.78, 3.10, 3.12, 2.61, 1.24, 0.85 m^2

   Determine the underwater volume if the sections are 3 m apart. [42.59 m^3]

2. To estimate the amount of earth to be removed when constructing a cutting the cross-sectional area at intervals of 8 m were estimated as follows:

   0, 2.8, 3.7, 4.5, 4.1, 2.6, 0 m^3

   Estimate the volume of earth to be excavated. [147 m^3]

3. The circumference of a 12 m long log of timber of varying circular cross-section is measured at intervals of 2 m along its length and the results are:

Distance from one end (m)	Circumference (m)
0	2.80
2	3.25
4	3.94
6	4.32
8	5.16
10	5.82
12	6.36

   Estimate the volume of the timber in cubic metres. [20.42 m^3]

19.3 The mean or average value of a waveform

The mean or average value, y, of the waveform shown in Fig. 19.6 is given by:

$$y = \frac{\textbf{area under curve}}{\textbf{length of base, } b}$$

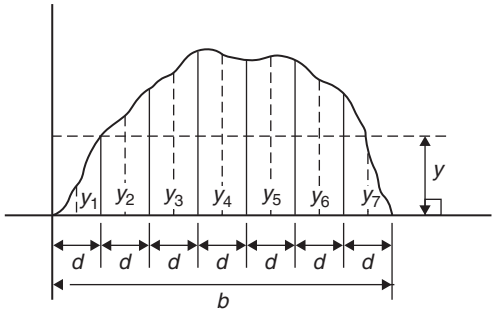


Figure 19.6

If the mid-ordinate rule is used to find the area under the curve, then:

$$y = \frac{\text{sum of mid-ordinates}}{\text{number of mid-ordinates}}$$

$$\left(= \frac{y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7}{7} \quad \text{for Fig. 19.6}\right)$$

For a **sine wave**, the mean or average value:

(i) over one complete cycle is zero (see Fig. 19.7(a)),

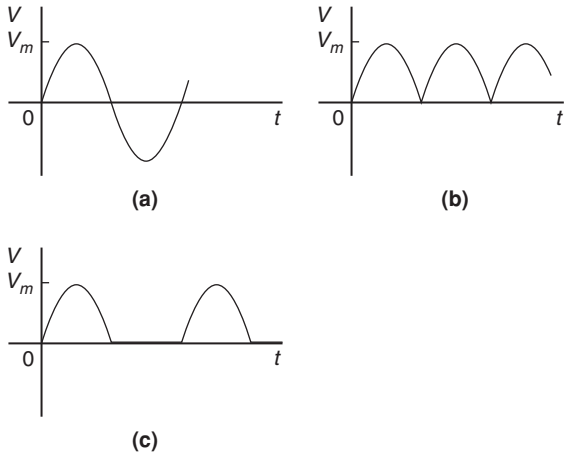


Figure 19.7

(ii) over half a cycle is **0.637 × maximum value**, or **(2/π) × maximum value**,

(iii) of a full-wave rectified waveform (see Fig. 19.7(b)) is **0.637 × maximum value**,

(iv) of a half-wave rectified waveform (see Fig. 19.7(c)) is **0.318 × maximum value**, or **(1/π) maximum value**.

Problem 5. Determine the average values over half a cycle of the periodic waveforms shown in Fig. 19.8.

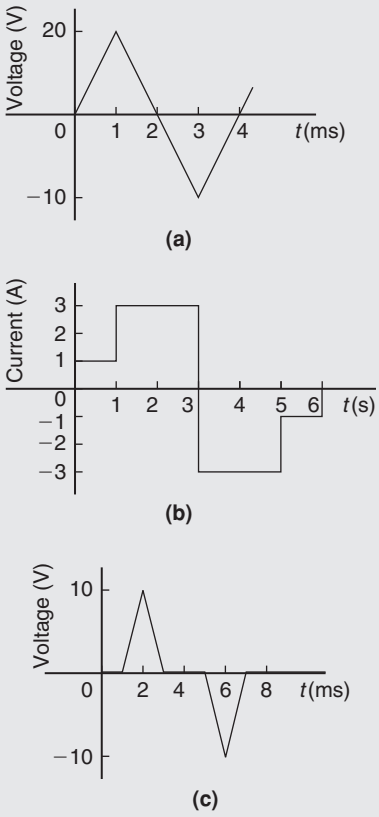


Figure 19.8

(a) Area under triangular waveform (a) for a half cycle is given by:

$$\text{Area} = \tfrac{1}{2}\,(\text{base})\,(\text{perpendicular height})$$

$$= \tfrac{1}{2}(2 \times 10^{-3})(20)$$

$$= 20 \times 10^{-3}\,\text{Vs}$$

Average value of waveform

$$= \frac{\text{area under curve}}{\text{length of base}}$$

$$= \frac{20 \times 10^{-3}\,\text{Vs}}{2 \times 10^{-3}\,\text{s}}$$

$$= \mathbf{10\,V}$$

(b) Area under waveform (b) for a half cycle $= (1 \times 1) + (3 \times 2) = 7\,\text{As}$.

$$\text{Average value of waveform} = \frac{\text{area under curve}}{\text{length of base}} = \frac{7\,\text{As}}{3\,\text{s}} = \mathbf{2.33\,A}$$

(c) A half cycle of the voltage waveform (c) is completed in 4 ms.

$$\text{Area under curve} = \tfrac{1}{2}\{(3-1)10^{-3}\}(10) = 10 \times 10^{-3}\,\text{Vs}$$

$$\text{Average value of waveform} = \frac{\text{area under curve}}{\text{length of base}} = \frac{10 \times 10^{-3}\,\text{Vs}}{4 \times 10^{-3}\,\text{s}} = \mathbf{2.5\,V}$$

Problem 6. Determine the mean value of current over one complete cycle of the periodic waveforms shown in Fig. 19.9.

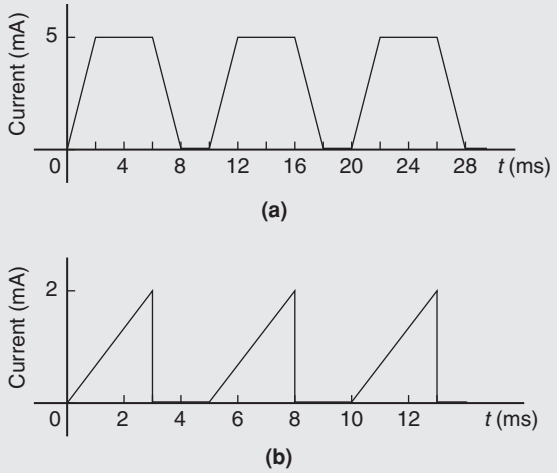


Figure 19.9

(a) One cycle of the trapezoidal waveform (a) is completed in 10 ms (i.e. the periodic time is 10 ms).

Area under curve = area of trapezium

$= \frac{1}{2}$ (sum of parallel sides) (perpendicular distance between parallel sides)

$= \frac{1}{2}\{(4+8) \times 10^{-3}\}(5 \times 10^{-3})$

$= 30 \times 10^{-6}\,\text{As}$

Mean value over one cycle

$$= \frac{\text{area under curve}}{\text{length of base}} = \frac{30 \times 10^{-6}\,\text{As}}{10 \times 10^{-3}\,\text{s}} = \mathbf{3\,mA}$$

(b) One cycle of the sawtooth waveform (b) is completed in 5 ms.

$$\text{Area under curve} = \tfrac{1}{2}(3 \times 10^{-3})(2) = 3 \times 10^{-3}\,\text{As}$$

Mean value over one cycle

$$= \frac{\text{area under curve}}{\text{length of base}} = \frac{3 \times 10^{-3}\,\text{As}}{5 \times 10^{-3}\,\text{s}} = \mathbf{0.6\,A}$$

Problem 7. The power used in a manufacturing process during a 6 hour period is recorded at intervals of 1 hour as shown below.

Time (h)	0	1	2	3	4	5	6
Power (kW)	0	14	29	51	45	23	0

Plot a graph of power against time and, by using the mid-ordinate rule, determine (a) the area under the curve and (b) the average value of the power.

The graph of power/time is shown in Fig. 19.10.

(a) The time base is divided into 6 equal intervals, each of width 1 hour. Mid-ordinates are erected (shown by broken lines in Fig. 19.10) and measured. The values are shown in Fig. 19.10.

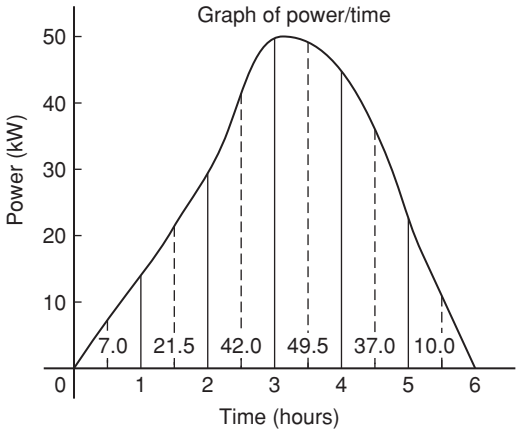


Figure 19.10

Area under curve = (width of interval) × (sum of mid-ordinates)
= (1)[7.0 + 21.5 + 42.0 + 49.5 + 37.0 + 10.0]
= **167 kWh** (i.e. a measure of electrical energy)

(b) Average value of waveform

$$= \frac{\text{area under curve}}{\text{length of base}}$$

$$= \frac{167\,\text{kWh}}{6\,\text{h}} = \mathbf{27.83\,kW}$$

Alternatively, average value

$$= \frac{\text{sum of mid-ordinates}}{\text{number of mid-ordinates}}$$

Problem 8. Fig. 19.11 shows a sinusoidal output voltage of a full-wave rectifier. Determine, using the mid-ordinate rule with 6 intervals, the mean output voltage.

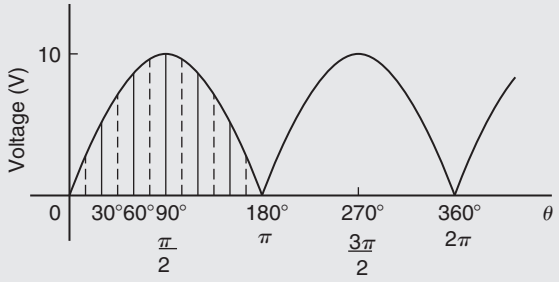


Figure 19.11

One cycle of the output voltage is completed in π radians or 180°. The base is divided into 6 intervals, each of width 30°. The mid-ordinate of each interval will lie at 15°, 45°, 75°, etc.
At 15° the height of the mid-ordinate is $10\sin 15° = 2.588$ V.
At 45° the height of the mid-ordinate is $10\sin 45° = 7.071$ V, and so on.
The results are tabulated below:

Mid-ordinate	Height of mid-ordinate
15°	$10\sin 15° = 2.588$ V
45°	$10\sin 45° = 7.071$ V
75°	$10\sin 75° = 9.659$ V
105°	$10\sin 105° = 9.659$ V
135°	$10\sin 135° = 7.071$ V
165°	$10\sin 165° = 2.588$ V
sum of mid-ordinates = 38.636 V	

Mean or average value of output voltage

$$= \frac{\text{sum of mid-ordinates}}{\text{number of mid-ordinates}}$$

$$= \frac{38.636}{6}$$

$$= \mathbf{6.439\,V}$$

(With a larger number of intervals a more accurate answer may be obtained.) For a sine wave the actual mean value is 0.637 × maximum value, which in this problem gives 6.37 V.

Problem 9. An indicator diagram for a steam engine is shown in Fig. 19.12. The base line has been divided into 6 equally spaced intervals and the lengths of the 7 ordinates measured with the results shown in centimetres. Determine (a) the area of the indicator diagram using Simpson's rule, and (b) the mean pressure in the cylinder given that 1 cm represents 100 kPa.

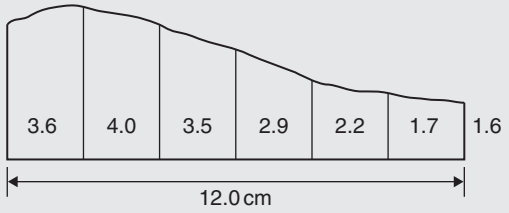


Figure 19.12

(a) The width of each interval is $\frac{12.0}{6}$ cm. Using Simpson's rule,

$$\text{area} = \tfrac{1}{3}(2.0)[(3.6+1.6)+4(4.0+2.9+1.7)+2(3.5+2.2)]$$

$$= \tfrac{2}{3}[5.2+34.4+11.4]$$

$$= \mathbf{34\,cm^2}$$

(b) Mean height of ordinates

$$= \frac{\text{area of diagram}}{\text{length of base}} = \frac{34}{12}$$

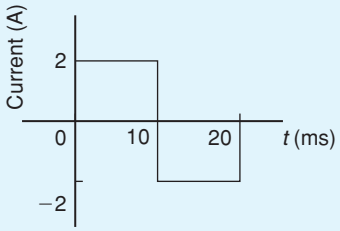
$$= 2.83\,\text{cm}$$

Since 1 cm represents 100 kPa, the mean pressure in the cylinder
$= 2.83\,\text{cm} \times 100\,\text{kPa/cm} = \mathbf{283\,kPa}$.

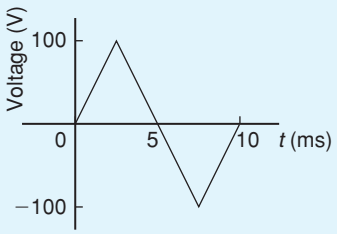
Now try the following exercise

Exercise 84 Further problems on mean or average values of waveforms

1. Determine the mean value of the periodic waveforms shown in Fig. 19.13 over a half cycle. [(a) 2 A (b) 50 V (c) 2.5 A]

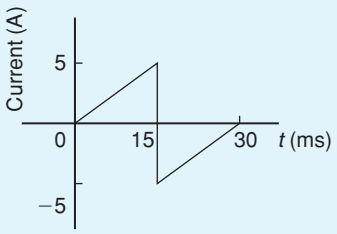


(a)



(b)

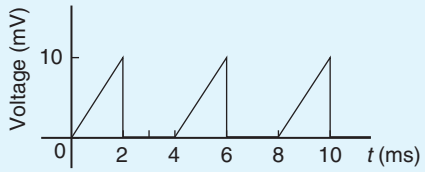
Figure 19.13



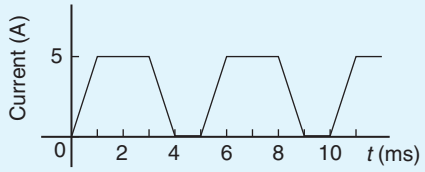
(c)

Figure 19.13 (*Continued*)

2. Find the average value of the periodic waveforms shown in Fig. 19.14 over one complete cycle. [(a) 2.5 V (b) 3 A]



(a)



(b)

Figure 19.14

3. An alternating current has the following values at equal intervals of 5 ms

Time (ms)	0	5	10	15	20	25	30
Current (A)	0	0.9	2.6	4.9	5.8	3.5	0

Plot a graph of current against time and estimate the area under the curve over the 30 ms period using the mid-ordinate rule and determine its mean value.

[0.093 As, 3.1 A]

4. Determine, using an approximate method, the average value of a sine wave of maximum value 50 V for (a) a half cycle and (b) a complete cycle. [(a) 31.83 V (b) 0]

5. An indicator diagram of a steam engine is 12 cm long. Seven evenly spaced ordinates, including the end ordinates, are measured as follows:

5.90, 5.52, 4.22, 3.63, 3.32, 3.24, 3.16 cm

Determine the area of the diagram and the mean pressure in the cylinder if 1 cm represents 90 kPa. [49.13 cm^2, 368.5 kPa]

Revision Test 6

This Revision Test covers the material contained in Chapters 18 and 19. *The marks for each question are shown in brackets at the end of each question.*

1. Sketch the following graphs, showing the relevant points:

   (a) $y=(x-2)^2$ (c) $x^2+y^2-2x+4y-4=0$

   (b) $y=3-\cos 2x$ (d) $9x^2-4y^2=36$

   (e) $f(x)=\begin{cases} -1 & -\pi \le x \le -\dfrac{\pi}{2} \\ x & -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2} \\ 1 & \dfrac{\pi}{2} \le x \le \pi \end{cases}$ (15)

2. Determine the inverse of $f(x)=3x+1$ (3)

3. Evaluate, correct to 3 decimal places:
   $2\tan^{-1}1.64+\sec^{-1}2.43-3\,\text{cosec}^{-1}3.85$ (3)

4. Determine the asymptotes for the following function and hence sketch the curve:
   $$y=\frac{(x-1)(x+4)}{(x-2)(x-5)}$$ (8)

5. Plot a graph of $y=3x^2+5$ from $x=1$ to $x=4$. Estimate, correct to 2 decimal places, using 6 intervals, the area enclosed by the curve, the ordinates $x=1$ and $x=4$, and the x-axis by (a) the trapezoidal rule, (b) the mid-ordinate rule, and (c) Simpson's rule. (11)

6. A circular cooling tower is 20 m high. The inside diameter of the tower at different heights is given in the following table:

Height (m)	0	5.0	10.0	15.0	20.0
Diameter (m)	16.0	13.3	10.7	8.6	8.0

   Determine the area corresponding to each diameter and hence estimate the capacity of the tower in cubic metres. (5)

7. A vehicle starts from rest and its velocity is measured every second for 6 seconds, with the following results:

Time t (s)	0	1	2	3	4	5	6
Velocity v (m/s)	0	1.2	2.4	3.7	5.2	6.0	9.2

   Using Simpson's rule, calculate (a) the distance travelled in 6 s (i.e. the area under the v/t graph) and (b) the average speed over this period. (5)

Chapter 20

Complex numbers

20.1 Cartesian complex numbers

There are several applications of complex numbers in science and engineering, in particular in electrical alternating current theory and in mechanical vector analysis.

There are two main forms of complex number – **Cartesian form and polar form** – and both are explained in this chapter.

If we can add, subtract, multiply and divide complex numbers in both forms and represent the numbers on an Argand diagram then a.c. theory and vector analysis become considerably easier.

(i) If the quadratic equation $x^2+2x+5=0$ is solved using the quadratic formula then,

$$x=\frac{-2\pm\sqrt{[(2)^2-(4)(1)(5)]}}{2(1)}$$

$$=\frac{-2\pm\sqrt{[-16]}}{2}=\frac{-2\pm\sqrt{[(16)(-1)]}}{2}$$

$$=\frac{-2\pm\sqrt{16}\sqrt{-1}}{2}=\frac{-2\pm4\sqrt{-1}}{2}$$

$$=-1\pm2\sqrt{-1}$$

It is not possible to evaluate $\sqrt{-1}$ in real terms. However, if an operator j is defined as $j=\sqrt{-1}$ then the solution may be expressed as $x=-1\pm j2$.

(ii) $-1+j2$ and $-1-j2$ are known as **complex numbers**. Both solutions are of the form $a+jb$, 'a' being termed the **real part** and jb the **imaginary part**. A complex number of the form $a+jb$ is called **Cartesian complex number**.

(iii) In pure mathematics the symbol i is used to indicate $\sqrt{-1}$ (i being the first letter of the word imaginary). However i is the symbol of electric current in engineering, and to avoid possible confusion the next letter in the alphabet, j, is used to represent $\sqrt{-1}$.

Problem 1. Solve the quadratic equation $x^2+4=0$.

Since $x^2+4=0$ then $x^2=-4$ and $x=\sqrt{-4}$.

i.e., $x=\sqrt{[(-1)(4)]}=\sqrt{(-1)}\sqrt{4}=j(\pm2)$

$=\mathbf{\pm j2}$, (since $j=\sqrt{-1}$)

(Note that $\pm j2$ may also be written $\mathbf{\pm 2j}$).

Problem 2. Solve the quadratic equation $2x^2+3x+5=0$.

Using the quadratic formula,

$$x=\frac{-3\pm\sqrt{[(3)^2-4(2)(5)]}}{2(2)}$$

$$=\frac{-3\pm\sqrt{-31}}{4}=\frac{-3\pm\sqrt{(-1)}\sqrt{31}}{4}$$

$$=\frac{-3\pm j\sqrt{31}}{4}$$

Hence $\boldsymbol{x=-\frac{3}{4}\pm j\frac{\sqrt{31}}{4}}$ or $\mathbf{-0.750\pm j1.392}$, correct to 3 decimal places.

(Note, a graph of $y=2x^2+3x+5$ does not cross the x-axis and hence $2x^2+3x+5=0$ has no real roots.)

Problem 3. Evaluate

(a) j^3 (b) j^4 (c) j^{23} (d) $\dfrac{-4}{j^9}$

(a) $j^3 = j^2 \times j = (-1) \times j = \mathbf{-j}$, since $j^2 = -1$

(b) $j^4 = j^2 \times j^2 = (-1) \times (-1) = \mathbf{1}$

(c) $j^{23} = j \times j^{22} = j \times (j^2)^{11} = j \times (-1)^{11}$

$= j \times (-1) = \mathbf{-j}$

(d) $j^9 = j \times j^8 = j \times (j^2)^4 = j \times (-1)^4$

$= j \times 1 = \mathbf{j}$

Hence $\dfrac{-4}{j^9} = \dfrac{-4}{j} = \dfrac{-4}{j} \times \dfrac{-j}{-j} = \dfrac{4j}{-j^2}$

$= \dfrac{4j}{-(-1)} = \mathbf{4j}$ or $\boldsymbol{j4}$

Now try the following exercise

Exercise 85 Further problems on the introduction to cartesian complex numbers

In Problems 1 to 9, solve the quadratic equations.

1. $x^2 + 25 = 0$ $[\pm j5]$
2. $x^2 - 2x + 2 = 0$ $[x = 1 \pm j]$
3. $x^2 - 4x + 5 = 0$ $[x = 2 \pm j]$
4. $x^2 - 6x + 10 = 0$ $[x = 3 \pm j]$
5. $2x^2 - 2x + 1 = 0$ $[x = 0.5 \pm j0.5]$
6. $x^2 - 4x + 8 = 0$ $[x = 2 \pm j2]$
7. $25x^2 - 10x + 2 = 0$ $[x = 0.2 \pm j0.2]$
8. $2x^2 + 3x + 4 = 0$

$\left[-\dfrac{3}{4} \pm j\dfrac{\sqrt{23}}{4} \text{ or } -0.750 \pm j1.199\right]$

9. $4t^2 - 5t + 7 = 0$

$\left[\dfrac{5}{8} \pm j\dfrac{\sqrt{87}}{8} \text{ or } 0.625 \pm j1.166\right]$

10. Evaluate (a) j^8 (b) $-\dfrac{1}{j^7}$ (c) $\dfrac{4}{2j^{13}}$

[(a) 1 (b) $-j$ (c) $-j2$]

20.2 The Argand diagram

A complex number may be represented pictorially on rectangular or cartesian axes. The horizontal (or x) axis is used to represent the real axis and the vertical (or y) axis is used to represent the imaginary axis. Such a diagram is called an **Argand diagram**. In Fig. 20.1, the point A represents the complex number $(3 + j2)$ and is obtained by plotting the co-ordinates $(3, j2)$ as in graphical work. Figure 20.1 also shows the Argand points B, C and D representing the complex numbers $(-2 + j4)$, $(-3 - j5)$ and $(1 - j3)$ respectively.

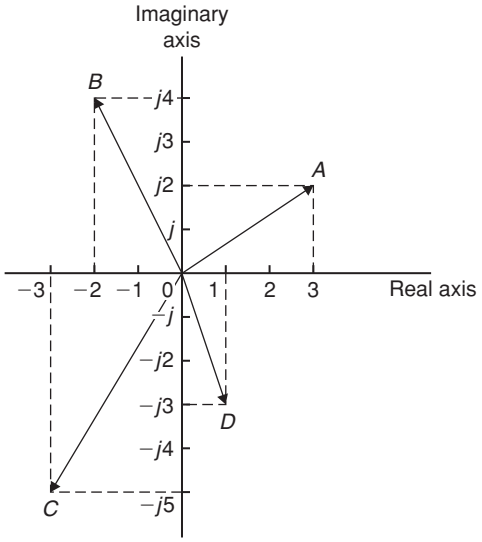


Figure 20.1

20.3 Addition and subtraction of complex numbers

Two complex numbers are added/subtracted by adding/subtracting separately the two real parts and the two imaginary parts.

For example, if $Z_1 = a + jb$ and $Z_2 = c + jd$,

then $Z_1 + Z_2 = (a + jb) + (c + jd)$

$= (a + c) + j(b + d)$

and $Z_1 - Z_2 = (a + jb) - (c + jd)$

$= (a - c) + j(b - d)$

Thus, for example,

$$(2+j3)+(3-j4)=2+j3+3-j4$$
$$=\mathbf{5-j1}$$

and $(2+j3)-(3-j4)=2+j3-3+j4$
$$=\mathbf{-1+j7}$$

The addition and subtraction of complex numbers may be achieved graphically as shown in the Argand diagram of Fig. 20.2. $(2+j3)$ is represented by vector $\boldsymbol{OP}$ and $(3-j4)$ by vector $\boldsymbol{OQ}$. In Fig. 20.2(a) by vector addition (i.e. the diagonal of the parallelogram) $\boldsymbol{OP+OQ=OR}$. R is the point $(5, -j1)$.

Hence $(2+j3)+(3-j4)=\mathbf{5-j1}$.

In Fig. 20.2(b), vector $\boldsymbol{OQ}$ is reversed (shown as OQ') since it is being subtracted. (Note $\boldsymbol{OQ}=3-j4$ and $\boldsymbol{OQ'}=-(3-j4)=-3+j4$).
$\boldsymbol{OP-OQ=OP+OQ'=OS}$ is found to be the Argand point $(-1, j7)$.

Hence $(2+j3)-(3-j4)=\mathbf{-1+j7}$

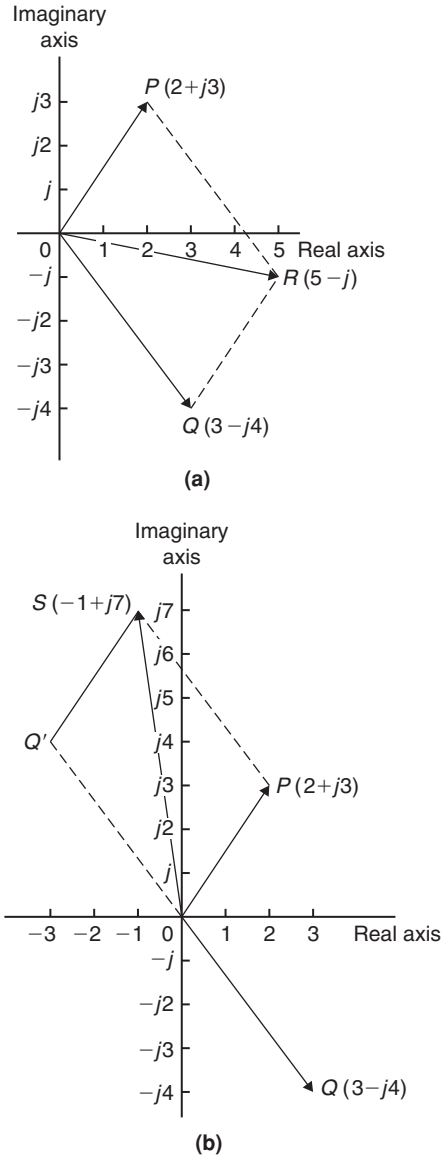


Figure 20.2

Problem 4. Given $Z_1=2+j4$ and $Z_2=3-j$ determine (a) Z_1+Z_2, (b) Z_1-Z_2, (c) Z_2-Z_1 and show the results on an Argand diagram.

(a) $Z_1+Z_2=(2+j4)+(3-j)$
$$=(2+3)+j(4-1)=\mathbf{5+j3}$$

(b) $Z_1-Z_2=(2+j4)-(3-j)$
$$=(2-3)+j(4-(-1))=\mathbf{-1+j5}$$

(c) $Z_2-Z_1=(3-j)-(2+j4)$
$$=(3-2)+j(-1-4)=\mathbf{1-j5}$$

Each result is shown in the Argand diagram of Fig. 20.3.

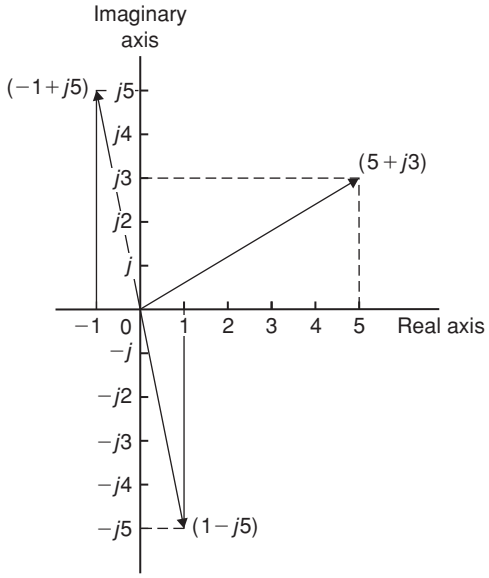


Figure 20.3

20.4 Multiplication and division of complex numbers

(i) **Multiplication of complex numbers** is achieved by assuming all quantities involved are real and then using $j^2=-1$ to simplify.

Hence $(a+jb)(c+jd)$

$$=ac+a(jd)+(jb)c+(jb)(jd)$$

$$=ac+jad+jbc+j^2bd$$

$$=(ac-bd)+j(ad+bc),$$

since $j^2=-1$

Thus $(3+j2)(4-j5)$

$$=12-j15+j8-j^210$$

$$=(12-(-10))+j(-15+8)$$

$$=\mathbf{22-j7}$$

(ii) The **complex conjugate** of a complex number is obtained by changing the sign of the imaginary part. Hence the complex conjugate of $a+jb$ is $a-jb$. The product of a complex number and its complex conjugate is always a real number.

For example,

$$(3+j4)(3-j4)=9-j12+j12-j^216$$
$$=9+16=25$$

$[(a+jb)(a-jb)$ may be evaluated 'on sight' as $a^2+b^2]$.

(iii) **Division of complex numbers** is achieved by multiplying both numerator and denominator by the complex conjugate of the denominator.

For example,

$$\frac{2-j5}{3+j4}=\frac{2-j5}{3+j4}\times\frac{(3-j4)}{(3-j4)}$$

$$=\frac{6-j8-j15+j^220}{3^2+4^2}$$

$$=\frac{-14-j23}{25}=\frac{\mathbf{-14}}{\mathbf{25}}-j\frac{\mathbf{23}}{\mathbf{25}}$$

or $\mathbf{-0.56-j0.92}$

Problem 5. If $Z_1=1-j3$, $Z_2=-2+j5$ and $Z_3=-3-j4$, determine in $a+jb$ form:

(a) Z_1Z_2 (b) $\dfrac{Z_1}{Z_3}$

(c) $\dfrac{Z_1Z_2}{Z_1+Z_2}$ (d) $Z_1Z_2Z_3$

(a) $Z_1Z_2=(1-j3)(-2+j5)$

$$=-2+j5+j6-j^215$$

$$=(-2+15)+j(5+6), \text{ since } j^2=-1,$$

$$=\mathbf{13+j11}$$

(b) $\dfrac{Z_1}{Z_3}=\dfrac{1-j3}{-3-j4}=\dfrac{1-j3}{-3-j4}\times\dfrac{-3+j4}{-3+j4}$

$$=\frac{-3+j4+j9-j^212}{3^2+4^2}$$

$$=\frac{9+j13}{25}=\frac{\mathbf{9}}{\mathbf{25}}+j\frac{\mathbf{13}}{\mathbf{25}}$$

or $\mathbf{0.36+j0.52}$

(c) $\dfrac{Z_1Z_2}{Z_1+Z_2}=\dfrac{(1-j3)(-2+j5)}{(1-j3)+(-2+j5)}$

$$=\frac{13+j11}{-1+j2}, \text{ from part (a),}$$

$$=\frac{13+j11}{-1+j2}\times\frac{-1-j2}{-1-j2}$$

$$=\frac{-13-j26-j11-j^222}{1^2+2^2}$$

$$=\frac{9-j37}{5}=\frac{\mathbf{9}}{\mathbf{5}}-j\frac{\mathbf{37}}{\mathbf{5}} \text{ or } \mathbf{1.8-j7.4}$$

(d) $Z_1Z_2Z_3=(13+j11)(-3-j4)$, since

$Z_1Z_2=13+j11$, from part (a)

$$=-39-j52-j33-j^244$$

$$=(-39+44)-j(52+33)$$

$$=\mathbf{5-j85}$$

Problem 6. Evaluate:

(a) $\dfrac{2}{(1+j)^4}$ (b) $j\left(\dfrac{1+j3}{1-j2}\right)^2$

(a) $(1+j)^2=(1+j)(1+j)=1+j+j+j^2$

$=1+j+j-1=j2$

$(1+j)^4=[(1+j)^2]^2=(j2)^2=j^24=-4$

Hence $\dfrac{2}{(1+j)^4}=\dfrac{2}{-4}=-\dfrac{\mathbf{1}}{\mathbf{2}}$

(b) $\dfrac{1+j3}{1-j2}=\dfrac{1+j3}{1-j2}\times\dfrac{1+j2}{1+j2}$

$=\dfrac{1+j2+j3+j^26}{1^2+2^2}=\dfrac{-5+j5}{5}$

$=-1+j1=-1+j$

$\left(\dfrac{1+j3}{1-j2}\right)^2=(-1+j)^2=(-1+j)(-1+j)$

$=1-j-j+j^2=-j2$

Hence $j\left(\dfrac{1+j3}{1-j2}\right)^2=j(-j2)=-j^22=\mathbf{2}$,

since $j^2=-1$

Now try the following exercise

Exercise 86 Further problems on operations involving Cartesian complex numbers

1. Evaluate (a) $(3+j2)+(5-j)$ and (b) $(-2+j6)-(3-j2)$ and show the results on an Argand diagram.

   [(a) $8+j$ (b) $-5+j8$]

2. Write down the complex conjugates of (a) $3+j4$, (b) $2-j$.

   [(a) $3-j4$ (b) $2+j$]

3. If $z=2+j$ and $w=3-j$ evaluate (a) $z+w$ (b) $w-z$ (c) $3z-2w$ (d) $5z+2w$ (e) $j(2w-3z)$ (f) $2jw-jz$

   [(a) 5 (b) $1-j2$ (c) $j5$ (d) $16+j3$ (e) 5 (f) $3+j4$]

In Problems 4 to 8 evaluate in $a+jb$ form given $Z_1=1+j2$, $Z_2=4-j3$, $Z_3=-2+j3$ and $Z_4=-5-j$.

4. (a) $Z_1+Z_2-Z_3$ (b) $Z_2-Z_1+Z_4$

   [(a) $7-j4$ (b) $-2-j6$]

5. (a) Z_1Z_2 (b) Z_3Z_4

   [(a) $10+j5$ (b) $13-j13$]

6. (a) $Z_1Z_3+Z_4$ (b) $Z_1Z_2Z_3$

   [(a) $-13-j2$ (b) $-35+j20$]

7. (a) $\dfrac{Z_1}{Z_2}$ (b) $\dfrac{Z_1+Z_3}{Z_2-Z_4}$

   $\left[\text{(a) } \dfrac{-2}{25}+j\dfrac{11}{25} \quad \text{(b) } \dfrac{-19}{85}+j\dfrac{43}{85}\right]$

8. (a) $\dfrac{Z_1Z_3}{Z_1+Z_3}$ (b) $Z_2+\dfrac{Z_1}{Z_4}+Z_3$

   $\left[\text{(a) } \dfrac{3}{26}+j\dfrac{41}{26} \quad \text{(b) } \dfrac{45}{26}-j\dfrac{9}{26}\right]$

9. Evaluate (a) $\dfrac{1-j}{1+j}$ (b) $\dfrac{1}{1+j}$

   $\left[\text{(a) } -j \quad \text{(b) } \dfrac{1}{2}-j\dfrac{1}{2}\right]$

10. Show that $\dfrac{-25}{2}\left(\dfrac{1+j2}{3+j4}-\dfrac{2-j5}{-j}\right)=57+j24$

20.5 Complex equations

If two complex numbers are equal, then their real parts are equal and their imaginary parts are equal. Hence if $a+jb=c+jd$, then $a=c$ and $b=d$.

Problem 7. Solve the complex equations:
(a) $2(x+jy)=6-j3$
(b) $(1+j2)(-2-j3)=a+jb$

(a) $2(x+jy)=6-j3$ hence $2x+j2y=6-j3$

Equating the real parts gives:

$2x=6$, i.e. $\boldsymbol{x=3}$

Equating the imaginary parts gives:

$2y=-3$, i.e. $\boldsymbol{y=-\frac{3}{2}}$

(b) $(1+j2)(-2-j3)=a+jb$

$-2-j3-j4-j^2 6=a+jb$

Hence $4-j7=a+jb$

Equating real and imaginary terms gives:

$\boldsymbol{a=4}$ and $\boldsymbol{b=-7}$

Problem 8. Solve the equations:

(a) $(2-j3)=\sqrt{(a+jb)}$

(b) $(x-j2y)+(y-j3x)=2+j3$

(a) $(2-j3)=\sqrt{(a+jb)}$

Hence $(2-j3)^2=a+jb$,

i.e. $(2-j3)(2-j3)=a+jb$

Hence $4-j6-j6+j^2 9=a+jb$

and $-5-j12=a+jb$

Thus $\boldsymbol{a=-5}$ and $\boldsymbol{b=-12}$

(b) $(x-j2y)+(y-j3x)=2+j3$

Hence $(x+y)+j(-2y-3x)=2+j3$

Equating real and imaginary parts gives:

$$x+y=2 \qquad (1)$$

and $$-3x-2y=3 \qquad (2)$$

i.e. two simultaneous equations to solve.

Multiplying equation (1) by 2 gives:

$$2x+2y=4 \qquad (3)$$

Adding equations (2) and (3) gives:

$-x=7$, i.e., $\boldsymbol{x=-7}$

From equation (1), $\boldsymbol{y=9}$, which may be checked in equation (2).

Now try the following exercise

Exercise 87 Further problems on complex equations

In Problems 1 to 4 solve the complex equations.

1. $(2+j)(3-j2)=a+jb$ $[a=8, b=-1]$

2. $\dfrac{2+j}{1-j}=j(x+jy)$ $\left[x=\dfrac{3}{2}, y=-\dfrac{1}{2}\right]$

3. $(2-j3)=\sqrt{(a+jb)}$ $[a=-5, b=-12]$

4. $(x-j2y)-(y-jx)=2+j$ $[x=3, y=1]$

5. If $Z=R+j\omega L+1/j\omega C$, express Z in $(a+jb)$ form when $R=10$, $L=5$, $C=0.04$ and $\omega=4$. $[Z=10+j13.75]$

20.6 The polar form of a complex number

(i) Let a complex number z be $x+jy$ as shown in the Argand diagram of Fig. 20.4. Let distance OZ be r and the angle OZ makes with the positive real axis be θ.

From trigonometry, $x=r\cos\theta$ and

$y=r\sin\theta$

Hence $Z=x+jy=r\cos\theta+jr\sin\theta$

$=r(\cos\theta+j\sin\theta)$

$Z=r(\cos\theta+j\sin\theta)$ is usually abbreviated to $Z=r\angle\theta$ which is known as the **polar form** of a complex number.

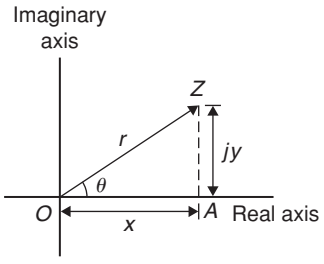


Figure 20.4

(ii) r is called the **modulus** (or magnitude) of Z and is written as mod Z or $|Z|$.

r is determined using Pythagoras' theorem on triangle OAZ in Fig. 20.4,

i.e. $r=\sqrt{(x^2+y^2)}$

(iii) θ is called the **argument** (or amplitude) of Z and is written as arg Z.

By trigonometry on triangle OAZ,

$$\arg Z = \theta = \tan^{-1}\frac{y}{x}$$

(iv) Whenever changing from cartesian form to polar form, or vice-versa, a sketch is invaluable for determining the quadrant in which the complex number occurs.

Problem 9. Determine the modulus and argument of the complex number $Z = 2 + j3$, and express Z in polar form.

$Z = 2 + j3$ lies in the first quadrant as shown in Fig. 20.5.

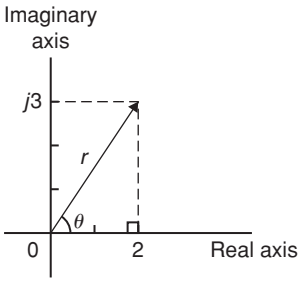


Figure 20.5

Modulus, $|Z| = r = \sqrt{(2^2 + 3^2)} = \sqrt{13}$ or **3.606**, correct to 3 decimal places.

Argument, $\arg Z = \theta = \tan^{-1}\frac{3}{2}$

$$= \mathbf{56.31°} \text{ or } \mathbf{56°19'}$$

In polar form, $2 + j3$ is written as $\mathbf{3.606\angle 56.31°}$.

Problem 10. Express the following complex numbers in polar form:

(a) $3 + j4$ (b) $-3 + j4$

(c) $-3 - j4$ (d) $3 - j4$

(a) $3 + j4$ is shown in Fig. 20.6 and lies in the first quadrant.

Modulus, $r = \sqrt{(3^2 + 4^2)} = 5$ and argument $\theta = \tan^{-1}\frac{4}{3} = 53.13°$.

Hence $\mathbf{3 + j4 = 5\angle 53.13°}$

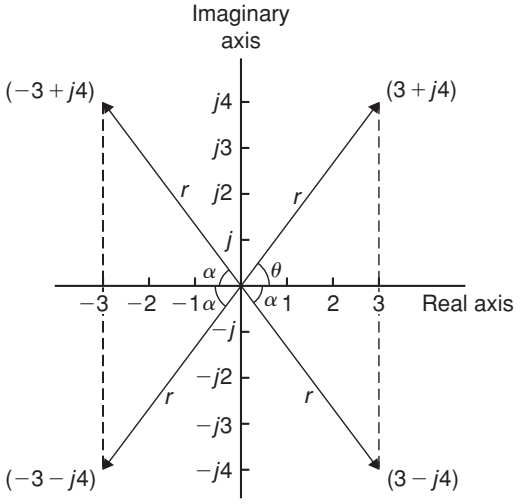


Figure 20.6

(b) $-3 + j4$ is shown in Fig. 20.6 and lies in the second quadrant.

Modulus, $r = 5$ and angle $\alpha = 53.13°$, from part (a).

Argument $= 180° - 53.13° = 126.87°$ (i.e. the argument must be measured from the positive real axis).

Hence $\mathbf{-3 + j4 = 5\angle 126.87°}$

(c) $-3 - j4$ is shown in Fig. 20.6 and lies in the third quadrant.

Modulus, $r = 5$ and $\alpha = 53.13°$, as above.

Hence the argument $= 180° + 53.13° = 233.13°$, which is the same as $-126.87°$.

Hence $(-3 - j4) = 5\angle 233.13°$ or $5\angle -126.87°$

(By convention the **principal value** is normally used, i.e. the numerically least value, such that $-\pi < \theta < \pi$).

(d) $3 - j4$ is shown in Fig. 20.6 and lies in the fourth quadrant.

Modulus, $r = 5$ and angle $\alpha = 53.13°$, as above.

Hence $(3 - j4) = 5\angle -53.13°$

Problem 11. Convert (a) $4\angle 30°$ (b) $7\angle -145°$ into $a + jb$ form, correct to 4 significant figures.

(a) $4\angle 30°$ is shown in Fig. 20.7(a) and lies in the first quadrant.

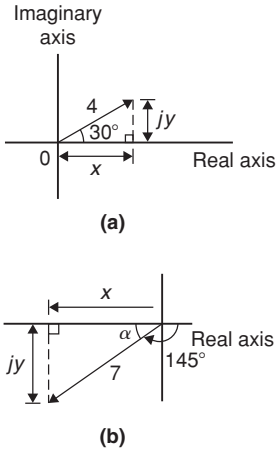


Figure 20.7

Using trigonometric ratios, $x = 4\cos 30° = 3.464$ and $y = 4\sin 30° = 2.000$.

Hence $4\angle 30° = 3.464 + j2.000$

(b) $7\angle 145°$ is shown in Fig. 20.7(b) and lies in the third quadrant.

Angle $\alpha = 180° - 145° = 35°$

Hence $x = 7\cos 35° = 5.734$

and $y = 7\sin 35° = 4.015$

Hence $7\angle -145° = -5.734 - j4.015$

Alternatively

$7\angle -145° = 7\cos(-145°) + j7\sin(-145°)$

$= \mathbf{-5.734 - j4.015}$

Calculator

Using the '**Pol**' and '**Rec**' functions on a calculator enables changing from Cartesian to polar and vice-versa to be achieved more quickly.

Since complex numbers are used with vectors and with electrical engineering a.c. theory, it is essential that the calculator can be used quickly and accurately.

20.7 Multiplication and division in polar form

If $Z_1 = r_1\angle\theta_1$ and $Z_2 = r_2\angle\theta_2$ then:

(i) $\mathbf{Z_1 Z_2 = r_1 r_2 \angle(\theta_1 + \theta_2)}$ and

(ii) $\dfrac{\mathbf{Z_1}}{\mathbf{Z_2}} = \dfrac{\mathbf{r_1}}{\mathbf{r_2}}\angle(\boldsymbol{\theta_1 - \theta_2})$

Problem 12. Determine, in polar form:

(a) $8\angle 25° \times 4\angle 60°$

(b) $3\angle 16° \times 5\angle -44° \times 2\angle 80°$

(a) $8\angle 25° \times 4\angle 60° = (8\times 4)\angle(25° + 60°) = \mathbf{32\angle 85°}$

(b) $3\angle 16° \times 5\angle -44° \times 2\angle 80°$

$= (3\times 5\times 2)\angle[16° + (-44°) + 80°] = \mathbf{30\angle 52°}$

Problem 13. Evaluate in polar form

(a) $\dfrac{16\angle 75°}{2\angle 15°}$ (b) $\dfrac{10\angle\frac{\pi}{4} \times 12\angle\frac{\pi}{2}}{6\angle -\frac{\pi}{3}}$

(a) $\dfrac{16\angle 75°}{2\angle 15°} = \dfrac{16}{2}\angle(75° - 15°) = \mathbf{8\angle 60°}$

(b) $\dfrac{10\angle\frac{\pi}{4} \times 12\angle\frac{\pi}{2}}{6\angle -\frac{\pi}{3}} = \dfrac{10\times 12}{6}\angle\left(\dfrac{\pi}{4} + \dfrac{\pi}{2} - \left(-\dfrac{\pi}{3}\right)\right)$

$= \mathbf{20\angle\dfrac{13\pi}{12}}$ or $\mathbf{20\angle -\dfrac{11\pi}{12}}$ or

$\mathbf{20\angle 195°}$ or $\mathbf{20\angle -165°}$

Problem 14. Evaluate, in polar form $2\angle 30° + 5\angle -45° - 4\angle 120°$.

Addition and subtraction in polar form is not possible directly. Each complex number has to be converted into cartesian form first.

$2\angle 30° = 2(\cos 30° + j\sin 30°)$

$= 2\cos 30° + j2\sin 30° = 1.732 + j1.000$

$5\angle -45° = 5(\cos(-45°) + j\sin(-45°))$

$= 5\cos(-45°) + j5\sin(-45°)$

$= 3.536 - j3.536$

$4\angle 120° = 4(\cos 120° + j\sin 120°)$

$= 4\cos 120° + j4\sin 120°$

$= -2.000 + j3.464$

Hence $2\angle 30° + 5\angle -45° - 4\angle 120°$

$= (1.732 + j1.000) + (3.536 - j3.536)$
$\quad - (-2.000 + j3.464)$
$= 7.268 - j6.000$, which lies in the fourth quadrant
$= \sqrt{[(7.268)^2 + (6.000)^2]} \angle \tan^{-1}\left(\frac{-6.000}{7.268}\right)$
$= \mathbf{9.425\angle -39.54°}$

Now try the following exercise

Exercise 88 Further problems on polar form

1. Determine the modulus and argument of (a) $2 + j4$ (b) $-5 - j2$ (c) $j(2 - j)$.

[(a) 4.472, 63.43° (b) 5.385, −158.20° (c) 2.236, 63.43°]

In Problems 2 and 3 express the given Cartesian complex numbers in polar form, leaving answers in surd form.

2. (a) $2 + j3$ (b) -4 (c) $-6 + j$

[(a) $\sqrt{13}\angle 56.31°$ (b) $4\angle 180°$ (c) $\sqrt{37}\angle 170.54°$]

3. (a) $-j3$ (b) $(-2 + j)^3$ (c) $j^3(1 - j)$

[(a) $3\angle -90°$ (b) $\sqrt{125}\angle 100.30°$ (c) $\sqrt{2}\angle -135°$]

In Problems 4 and 5 convert the given polar complex numbers into $(a + jb)$ form giving answers correct to 4 significant figures.

4. (a) $5\angle 30°$ (b) $3\angle 60°$ (c) $7\angle 45°$

[(a) $4.330 + j2.500$ (b) $1.500 + j2.598$ (c) $4.950 + j4.950$]

5. (a) $6\angle 125°$ (b) $4\angle \pi$ (c) $3.5\angle -120°$

[(a) $-3.441 + j4.915$ (b) $-4.000 + j0$ (c) $-1.750 - j3.031$]

In Problems 6 to 8, evaluate in polar form.

6. (a) $3\angle 20° \times 15\angle 45°$
(b) $2.4\angle 65° \times 4.4\angle -21°$

[(a) $45\angle 65°$ (b) $10.56\angle 44°$]

7. (a) $6.4\angle 27° \div 2\angle -15°$
(b) $5\angle 30° \times 4\angle 80° \div 10\angle -40°$

[(a) $3.2\angle 42°$ (b) $2\angle 150°$]

8. (a) $4\angle\frac{\pi}{6} + 3\angle\frac{\pi}{8}$
(b) $2\angle 120° + 5.2\angle 58° - 1.6\angle -40°$

[(a) $6.986\angle 26.79°$ (b) $7.190\angle 85.77°$]

20.8 Applications of complex numbers

There are several applications of complex numbers in science and engineering, in particular in electrical alternating current theory and in mechanical vector analysis.

The effect of multiplying a phasor by j is to rotate it in a positive direction (i.e. anticlockwise) on an Argand diagram through 90° without altering its length. Similarly, multiplying a phasor by $-j$ rotates the phasor through −90°. These facts are used in a.c. theory since certain quantities in the phasor diagrams lie at 90° to each other. For example, in the $R-L$ series circuit shown in Fig. 20.8(a), V_L leads I by 90° (i.e. I lags V_L by 90°) and may be written as jV_L, the vertical axis being regarded as the imaginary axis of an Argand diagram. Thus $V_R + jV_L = V$ and since $V_R = IR$, $V = IX_L$ (where X_L is the inductive reactance, $2\pi fL$ ohms) and $V = IZ$ (where Z is the impedance) then $R + jX_L = Z$.

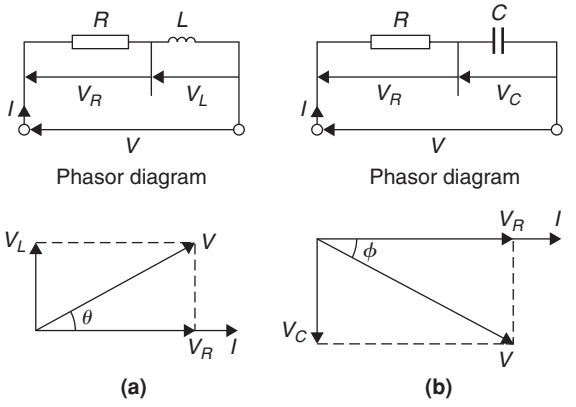


Figure 20.8

Similarly, for the $R-C$ circuit shown in Fig. 20.8(b), V_C lags I by 90° (i.e. I leads V_C by 90°) and $V_R - jV_C = V$, from which $R - jX_C = Z$ (where X_C is the capacitive reactance $\frac{1}{2\pi fC}$ ohms).

Problem 15. Determine the resistance and series inductance (or capacitance) for each of the following impedances, assuming a frequency of 50 Hz:

(a) $(4.0 + j7.0)\ \Omega$ (b) $-j20\ \Omega$

(c) $15\angle -60°\ \Omega$

(a) Impedance, $Z = (4.0 + j7.0)\ \Omega$ hence, **resistance = 4.0 Ω** and reactance = 7.00 Ω.

Since the imaginary part is positive, the reactance is inductive,

i.e. $X_L = 7.0\ \Omega$

Since $X_L = 2\pi fL$ then **inductance**,

$$L = \frac{X_L}{2\pi f} = \frac{7.0}{2\pi(50)} = \mathbf{0.0223\,H} \text{ or } \mathbf{22.3\,mH}$$

(b) Impedance, $Z = j20$, i.e. $Z = (0 - j20)\ \Omega$ hence **resistance = 0** and reactance = 20 Ω. Since the imaginary part is negative, the reactance is capacitive, i.e., $X_C = 20\ \Omega$ and since $X_C = \frac{1}{2\pi fC}$ then:

$$\textbf{capacitance, } C = \frac{1}{2\pi fX_C} = \frac{1}{2\pi(50)(20)}\text{F}$$

$$= \frac{10^6}{2\pi(50)(20)}\mu\text{F} = \mathbf{159.2\,\mu F}$$

(c) Impedance, Z

$$= 15\angle -60° = 15[\cos(-60°) + j\sin(-60°)]$$

$$= 7.50 - j12.99\ \Omega$$

Hence **resistance = 7.50 Ω** and capacitive reactance, $X_C = 12.99\ \Omega$

Since $X_C = \frac{1}{2\pi fC}$ then **capacitance**,

$$C = \frac{1}{2\pi fX_C} = \frac{10^6}{2\pi(50)(12.99)}\mu\text{F}$$

$$= \mathbf{245\,\mu F}$$

Problem 16. An alternating voltage of 240 V, 50 Hz is connected across an impedance of $(60 - j100)\ \Omega$. Determine (a) the resistance (b) the capacitance (c) the magnitude of the impedance and its phase angle and (d) the current flowing.

(a) Impedance $Z = (60 - j100)\ \Omega$.

Hence **resistance = 60 Ω**

(b) Capacitive reactance $X_C = 100\ \Omega$ and since $X_C = \frac{1}{2\pi fC}$ then

$$\text{capacitance, } C = \frac{1}{2\pi fX_C} = \frac{1}{2\pi(50)(100)}$$

$$= \frac{10^6}{2\pi(50)(100)}\mu\text{F}$$

$$= \mathbf{31.83\,\mu F}$$

(c) Magnitude of impedance,

$$|Z| = \sqrt{[(60)^2 + (-100)^2]} = \mathbf{116.6\,\Omega}$$

$$\text{Phase angle, } \arg Z = \tan^{-1}\left(\frac{-100}{60}\right) = \mathbf{-59.04°}$$

(d) Current flowing, $I = \frac{V}{Z} = \frac{240\angle 0°}{116.6\angle -59.04°}$

$$= \mathbf{2.058\angle 59.04°\ A}$$

The circuit and phasor diagrams are as shown in Fig. 20.8(b).

Problem 17. For the parallel circuit shown in Fig. 20.9, determine the value of current I and its phase relative to the 240 V supply, using complex numbers.

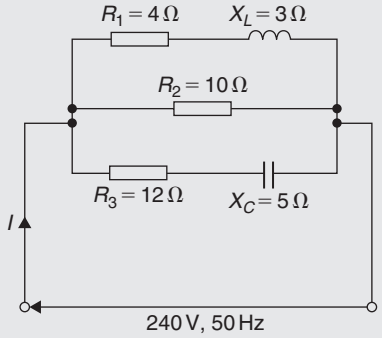


Figure 20.9

Current $I = \dfrac{V}{Z}$. Impedance Z for the three-branch parallel circuit is given by:

$$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3},$$

where $Z_1 = 4 + j3$, $Z_2 = 10$ and $Z_3 = 12 - j5$

Admittance, $Y_1 = \dfrac{1}{Z_1} = \dfrac{1}{4 + j3}$

$$= \frac{1}{4 + j3} \times \frac{4 - j3}{4 - j3} = \frac{4 - j3}{4^2 + 3^2}$$

$$= 0.160 - j0.120 \text{ siemens}$$

Admittance, $Y_2 = \dfrac{1}{Z_2} = \dfrac{1}{10} = 0.10$ siemens

Admittance, $Y_3 = \dfrac{1}{Z_3} = \dfrac{1}{12 - j5}$

$$= \frac{1}{12 - j5} \times \frac{12 + j5}{12 + j5} = \frac{12 + j5}{12^2 + 5^2}$$

$$= 0.0710 + j0.0296 \text{ siemens}$$

Total admittance, $Y = Y_1 + Y_2 + Y_3$

$$= (0.160 - j0.120) + (0.10) + (0.0710 + j0.0296)$$

$$= 0.331 - j0.0904$$

$$= 0.343\angle -15.28° \text{ siemens}$$

Current $I = \dfrac{V}{Z} = VY$

$$= (240\angle 0°)(0.343\angle -15.28°)$$

$$= \mathbf{82.32\angle -15.28° A}$$

Problem 18. Determine the magnitude and direction of the resultant of the three coplanar forces given below, when they act at a point.

Force A, 10 N acting at 45° from the positive horizontal axis.

Force B, 87 N acting at 120° from the positive horizontal axis.

Force C, 15 N acting at 210° from the positive horizontal axis.

The space diagram is shown in Fig. 20.10. The forces may be written as complex numbers.

Thus force A, $f_A = 10\angle 45°$, force B, $f_B = 8\angle 120°$ and force C, $f_C = 15\angle 210°$.

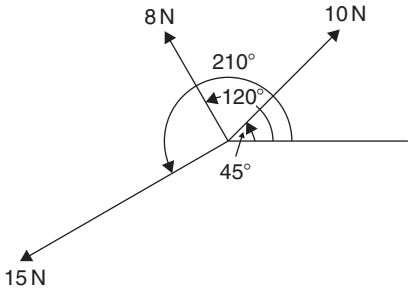


Figure 20.10

The resultant force

$$= f_A + f_B + f_C$$

$$= 10\angle 45° + 8\angle 120° + 15\angle 210°$$

$$= 10(\cos 45° + j \sin 45°) + 8(\cos 120° + j \sin 120°) + 15(\cos 210° + j \sin 210°)$$

$$= (7.071 + j7.071) + (-4.00 + j6.928) + (-12.99 - j7.50)$$

$$= -9.919 + j6.499$$

Magnitude of resultant force

$$= \sqrt{[(-9.919)^2 + (6.499)^2]} = \mathbf{11.86 N}$$

Direction of resultant force

$$= \tan^{-1}\left(\frac{6.499}{-9.919}\right) = \mathbf{146.77°}$$

(since $-9.919 + j6.499$ lies in the second quadrant).

Now try the following exercise

Exercise 89 Further problems on applications of complex numbers

1. Determine the resistance R and series inductance L (or capacitance C) for each of the following impedances assuming the frequency to be 50 Hz.

   (a) $(3 + j8)\,\Omega$ (b) $(2 - j3)\,\Omega$
   (c) $j14\,\Omega$ (d) $8\angle -60°\,\Omega$

   $$\begin{bmatrix} \text{(a) } R = 3\,\Omega, L = 25.5\,\text{mH} \\ \text{(b) } R = 2\,\Omega, C = 1061\,\mu\text{F} \\ \text{(c) } R = 0, L = 44.56\,\text{mH} \\ \text{(d) } R = 4\,\Omega, C = 459.4\,\mu\text{F} \end{bmatrix}$$

2. Two impedances, $Z_1 = (3 + j6)\,\Omega$ and $Z_2 = (4 - j3)\,\Omega$ are connected in series to a supply voltage of 120 V. Determine the magnitude of the current and its phase angle relative to the voltage.

[15.76 A, 23.20° lagging]

3. If the two impedances in Problem 2 are connected in parallel determine the current flowing and its phase relative to the 120 V supply voltage. [27.25 A, 3.37° lagging]

4. A series circuit consists of a 12 Ω resistor, a coil of inductance 0.10 H and a capacitance of 160 μF. Calculate the current flowing and its phase relative to the supply voltage of 240 V, 50 Hz. Determine also the power factor of the circuit. [14.42 A, 43.85° lagging, 0.721]

5. For the circuit shown in Fig. 20.11, determine the current I flowing and its phase relative to the applied voltage. [14.6 A, 2.51° leading]

6. Determine, using complex numbers, the magnitude and direction of the resultant of the coplanar forces given below, which are acting at a point. Force A, 5 N acting horizontally, Force B, 9 N acting at an angle of 135° to force A, Force C, 12 N acting at an angle of 240° to force A. [8.394 N, 208.68° from force A]

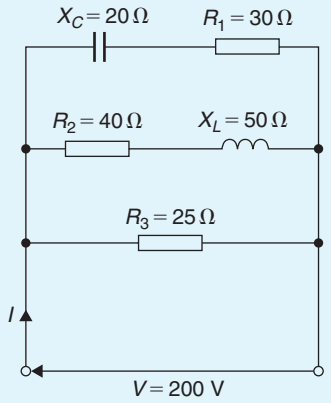


Figure 20.11

7. A delta-connected impedance Z_A is given by:

$$Z_A = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_2}$$

Determine Z_A in both Cartesian and polar form given $Z_1 = (10 + j0)\,\Omega$, $Z_2 = (0 - j10)\,\Omega$ and $Z_3 = (10 + j10)\,\Omega$.

[$(10 + j20)\,\Omega$, $22.36\angle 63.43°\,\Omega$]

8. In the hydrogen atom, the angular momentum, p, of the de Broglie wave is given by: $p\psi = -\left(\frac{jh}{2\pi}\right)(\pm jm\psi)$. Determine an expression for p. $\left[\pm\frac{mh}{2\pi}\right]$

9. An aircraft P flying at a constant height has a velocity of $(400 + j300)$ km/h. Another aircraft Q at the same height has a velocity of $(200 - j600)$ km/h. Determine (a) the velocity of P relative to Q, and (b) the velocity of Q relative to P. Express the answers in polar form, correct to the nearest km/h.

$$\begin{bmatrix}\text{(a) } 922\text{ km/h at } 77.47° \\ \text{(b) } 922\text{ km/h at } -102.53°\end{bmatrix}$$

10. Three vectors are represented by P, $2\angle 30°$, Q, $3\angle 90°$ and R, $4\angle -60°$. Determine in polar form the vectors represented by (a) $P + Q + R$, (b) $P - Q - R$.

$$\begin{bmatrix}\text{(a) } 3.770\angle 8.17° \\ \text{(b) } 1.488\angle 100.37°\end{bmatrix}$$

11. In a Schering bridge circuit,
$Z_X = (R_X - jX_{C_X})$, $Z_2 = -jX_{C_2}$,
$Z_3 = \dfrac{(R_3)(-jX_{C_3})}{(R_3 - jX_{C_3})}$ and $Z_4 = R_4$
where $X_C = \dfrac{1}{2\pi fC}$
At balance: $(Z_X)(Z_3) = (Z_2)(Z_4)$.
Show that at balance $R_X = \dfrac{C_3 R_4}{C_2}$ and $C_X = \dfrac{C_2 R_3}{R_4}$

Chapter 21

De Moivre's theorem

21.1 Introduction

From multiplication of complex numbers in polar form,

$$(r\angle\theta) \times (r\angle\theta) = r^2\angle 2\theta$$

Similarly, $(r\angle\theta) \times (r\angle\theta) \times (r\angle\theta) = r^3\angle 3\theta$, and so on. In general, **De Moivre's theorem** states:

$$[r\angle\theta]^n = r^n\angle n\theta$$

The theorem is true for all positive, negative and fractional values of n. The theorem is used to determine powers and roots of complex numbers.

21.2 Powers of complex numbers

For example $[3\angle 20°]^4 = 3^4\angle(4 \times 20°) = 81\angle 80°$ by De Moivre's theorem.

Problem 1. Determine, in polar form
(a) $[2\angle 35°]^5$ (b) $(-2 + j3)^6$.

(a) $[2\angle 35°]^5 = 2^5\angle(5 \times 35°)$, from De Moivre's theorem

$= \mathbf{32\angle 175°}$

(b) $(-2 + j3) = \sqrt{[(-2)^2 + (3)^2]}\angle \tan^{-1}\dfrac{3}{-2}$

$= \sqrt{13}\angle 123.69°$, since $-2 + j3$ lies in the second quadrant

$(-2 + j3)^6 = [\sqrt{13}\angle 123.69°]^6$

$= (\sqrt{13})^6\angle(6 \times 123.69°)$, by De Moivre's theorem

$= 2197\angle 742.14°$

$= 2197\angle 382.14°$ (since $742.14° \equiv 742.14° - 360° = 382.14°$)

$= \mathbf{2197\angle 22.14°}$ (since $382.14° \equiv 382.14° - 360° = 22.14°$)

or $\mathbf{2197\angle 22°8'}$

Problem 2. Determine the value of $(-7 + j5)^4$, expressing the result in polar and rectangular forms.

$(-7 + j5) = \sqrt{[(-7)^2 + 5^2]}\angle \tan^{-1}\dfrac{5}{-7}$

$= \sqrt{74}\angle 144.46°$

(Note, by considering the Argand diagram, $-7 + j5$ must represent an angle in the second quadrant and **not** in the fourth quadrant.)

Applying De Moivre's theorem:

$(-7 + j5)^4 = [\sqrt{74}\angle 144.46°]^4$

$= \sqrt{74^4}\angle 4 \times 144.46°$

$= 5476\angle 577.84°$

$= \mathbf{5476\angle 217.84°}$

or $\mathbf{5476\angle 217°50'}$ in polar form

Since $r\angle\theta = r\cos\theta + jr\sin\theta$,

$5476\angle 217.84° = 5476\cos 217.84° + j5476\sin 217.84°$

$= -4325 - j3359$

i.e. $\mathbf{(-7 + j5)^4 = -4325 - j3359}$ in rectangular form

Now try the following exercise

Exercise 90 Further problems on powers of complex numbers

1. Determine in polar form (a) $[1.5\angle 15°]^5$ (b) $(1+j2)^6$.
   [(a) $7.594\angle 75°$ (b) $125\angle 20.61°$]
2. Determine in polar and cartesian forms (a) $[3\angle 41°]^4$ (b) $(-2-j)^5$.
   [(a) $81\angle 164°, -77.86+j22.33$ (b) $55.90\angle -47.18°, 38-j41$]
3. Convert $(3-j)$ into polar form and hence evaluate $(3-j)^7$, giving the answer in polar form. [$\sqrt{10}\angle -18.43°, 3162\angle -129°$]

In problems 4 to 7, express in both polar and rectangular forms.

4. $(6+j5)^3$ [$476.4\angle 119.42°, -234+j415$]
5. $(3-j8)^5$ [$45530\angle 12.78°, 44400+j10070$]
6. $(-2+j7)^4$ [$2809\angle 63.78°, 1241+j2520$]
7. $(-16-j9)^6$ [$(38.27\times 10^6)\angle 176.15°, 10^6(-38.18+j2.570)$]

21.3 Roots of complex numbers

The **square root** of a complex number is determined by letting $n=1/2$ in De Moivre's theorem,

i.e. $$\sqrt{[r\angle\theta]} = [r\angle\theta]^{\frac{1}{2}} = r^{\frac{1}{2}}\angle\frac{1}{2}\theta = \sqrt{r}\angle\frac{\theta}{2}$$

There are two square roots of a real number, equal in size but opposite in sign.

Problem 3. Determine the two square roots of the complex number $(5+j12)$ in polar and cartesian forms and show the roots on an Argand diagram.

$$(5+j12) = \sqrt{[5^2+12^2]}\angle\tan^{-1}\left(\frac{12}{5}\right)$$
$$= 13\angle 67.38°$$

When determining square roots two solutions result. To obtain the second solution one way is to express $13\angle 67.38°$ also as $13\angle(67.38°+360°)$, i.e. $13\angle 427.38°$. When the angle is divided by 2 an angle less than 360° is obtained.

Hence

$$\sqrt{(5+j12)} = \sqrt{[13\angle 67.38°]} \text{ and } \sqrt{[13\angle 427.38°]}$$
$$= [13\angle 67.38°]^{\frac{1}{2}} \text{ and } [13\angle 427.38°]^{\frac{1}{2}}$$
$$= 13^{\frac{1}{2}}\angle\left(\frac{1}{2}\times 67.38°\right) \text{ and}$$
$$13^{\frac{1}{2}}\angle\left(\frac{1}{2}\times 427.38°\right)$$
$$= \sqrt{13}\angle 33.69° \text{ and } \sqrt{13}\angle 213.69°$$
$$= 3.61\angle 33.69° \text{ and } 3.61\angle 213.69°$$

Thus, in polar form, the two roots are $3.61\angle 33.69°$ and $3.61\angle -146.31°$.

$$\sqrt{13}\angle 33.69° = \sqrt{13}(\cos 33.69° + j\sin 33.69°)$$
$$= 3.0 + j2.0$$
$$\sqrt{13}\angle 213.69° = \sqrt{13}(\cos 213.69° + j\sin 213.69°)$$
$$= -3.0 - j2.0$$

Thus, in cartesian form the two roots are $\pm(3.0+j2.0)$.

From the Argand diagram shown in Fig. 21.1 the two roots are seen to be 180° apart, which is always true when finding square roots of complex numbers.

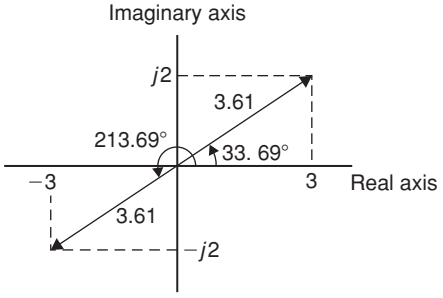


Figure 21.1

In general, **when finding the n^{th} root of a complex number, there are n solutions**. For example, there are three solutions to a cube root, five solutions to a fifth root, and so on. In the solutions to the roots of a complex number, the modulus, r, is always the same, but the

arguments, θ, are different. It is shown in Problem 3 that arguments are symmetrically spaced on an Argand diagram and are $(360/n)°$ apart, where n is the number of the roots required. Thus if one of the solutions to the cube root of a complex number is, say, $5\angle 20°$, the other two roots are symmetrically spaced $(360/3)°$, i.e. 120° from this root and the three roots are $5\angle 20°$, $5\angle 140°$ and $5\angle 260°$.

Problem 4. Find the roots of $[(5+j3)]^{\frac{1}{2}}$ in rectangular form, correct to 4 significant figures.

$$(5+j3) = \sqrt{34}\angle 30.96°$$

Applying De Moivre's theorem:

$$(5+j3)^{\frac{1}{2}} = \sqrt{34^{\frac{1}{2}}}\angle\tfrac{1}{2} \times 30.96°$$

$$= \mathbf{2.415\angle 15.48°} \text{ or } \mathbf{2.415\angle 15°29'}$$

The second root may be obtained as shown above, i.e. having the same modulus but displaced $(360/2)°$ from the first root.

Thus, $(5+j3)^{\frac{1}{2}} = 2.415\angle(15.48° + 180°)$

$$= \mathbf{2.415\angle 195.48°}$$

In rectangular form:

$$2.415\angle 15.48° = 2.415\cos 15.48° + j2.415\sin 15.48°$$

$$= \mathbf{2.327 + j0.6446}$$

and $$2.415\angle 195.48° = 2.415\cos 195.48° + j2.415\sin 195.48°$$

$$= \mathbf{-2.327 - j0.6446}$$

Hence $[(5+j3)]^{\frac{1}{2}} = \mathbf{2.415\angle 15.48°}$ and $\mathbf{2.415\angle 195.48°}$ or $\mathbf{\pm(2.327 + j0.6446)}$.

Problem 5. Express the roots of $(-14+j3)^{\frac{-2}{5}}$ in polar form.

$$(-14+j3) = \sqrt{205}\angle 167.905°$$

$$(-14+j3)^{\frac{-2}{5}} = \sqrt{205^{\frac{-2}{5}}}\angle\left[\left(-\frac{2}{5}\right) \times 167.905°\right]$$

$$= 0.3449\angle -67.164°$$

$$\text{or } 0.3449\angle -67°10'$$

There are five roots to this complex number,

$$\left(x^{\frac{-2}{5}} = \frac{1}{x^{\frac{2}{5}}} = \frac{1}{\sqrt[5]{x^2}}\right)$$

The roots are symmetrically displaced from one another $(360/5)°$, i.e. 72° apart round an Argand diagram.

Thus the required roots are $\mathbf{0.3449\angle -67°10'}$, $\mathbf{0.3449\angle 4°50'}$, $\mathbf{0.3449\angle 76°50'}$, $\mathbf{0.3449\angle 148°50'}$ and $\mathbf{0.3449\angle 220°50'}$.

Now try the following exercise

Exercise 91 Further problems on the roots of complex numbers

In Problems 1 to 3 determine the two square roots of the given complex numbers in Cartesian form and show the results on an Argand diagram.

1. (a) $1+j$ (b) j

$$\begin{bmatrix}\text{(a) } \pm(1.099 + j0.455)\\ \text{(b) } \pm(0.707 + j0.707)\end{bmatrix}$$

2. (a) $3-j4$ (b) $-1-j2$

$$\begin{bmatrix}\text{(a) } \pm(2-j)\\ \text{(b) } \pm(0.786 - j1.272)\end{bmatrix}$$

3. (a) $7\angle 60°$ (b) $12\angle\dfrac{3\pi}{2}$

$$\begin{bmatrix}\text{(a) } \pm(2.291 + j1.323)\\ \text{(b) } \pm(-2.449 + j2.449)\end{bmatrix}$$

In Problems 4 to 7, determine the moduli and arguments of the complex roots.

4. $(3+j4)^{\frac{1}{3}}$

$$\begin{bmatrix}\text{Moduli 1.710, arguments 17.71°,}\\ \text{137.71° and 257.71°}\end{bmatrix}$$

5. $(-2+j)^{\frac{1}{4}}$

[Modulus 1.223, arguments 38.36°, 128.36°, 218.36° and 308.36°]

6. $(-6-j5)^{\frac{1}{2}}$

[Modulus 2.795, arguments 109.90°, 289.90°]

7. $(4-j3)^{\frac{-2}{3}}$

[Modulus 0.3420, arguments 24.58°, 144.58° and 264.58°]

8. For a transmission line, the characteristic impedance Z_0 and the propagation coefficient γ are given by:

$$Z_0 = \sqrt{\left(\frac{R+j\omega L}{G+j\omega C}\right)} \quad \text{and}$$

$$\gamma = \sqrt{[(R+j\omega L)(G+j\omega C)]}$$

Given $R=25\,\Omega$, $L=5\times10^{-3}\,\text{H}$, $G=80\times10^{-6}$ siemens, $C=0.04\times10^{-6}\,\text{F}$ and $\omega=2000\pi$ rad/s, determine, in polar form, Z_0 and γ. [$Z_0=390.2\angle-10.43°\,\Omega$, $\gamma=0.1029\angle61.92°$]

21.4 The exponential form of a complex number

Certain mathematical functions may be expressed as power series (for example, by Maclaurin's series—see Chapter 8), three examples being:

$$\text{(i)}\quad e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\cdots \qquad (1)$$

$$\text{(ii)}\quad \sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots \qquad (2)$$

$$\text{(iii)}\quad \cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots \qquad (3)$$

Replacing x in equation (1) by the imaginary number $j\theta$ gives:

$$e^{j\theta} = 1+j\theta+\frac{(j\theta)^2}{2!}+\frac{(j\theta)^3}{3!}+\frac{(j\theta)^4}{4!}+\frac{(j\theta)^5}{5!}+\cdots$$

$$= 1+j\theta+\frac{j^2\theta^2}{2!}+\frac{j^3\theta^3}{3!}+\frac{j^4\theta^4}{4!}+\frac{j^5\theta^5}{5!}+\cdots$$

By definition, $j=\sqrt{(-1)}$, hence $j^2=-1$, $j^3=-j$, $j^4=1$, $j^5=j$, and so on.

Thus $e^{j\theta} = 1+j\theta-\frac{\theta^2}{2!}-j\frac{\theta^3}{3!}+\frac{\theta^4}{4!}+j\frac{\theta^5}{5!}-\cdots$

Grouping real and imaginary terms gives:

$$e^{j\theta} = \left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\cdots\right) + j\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\cdots\right)$$

However, from equations (2) and (3):

$$\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\cdots\right) = \cos\theta$$

and $$\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\cdots\right) = \sin\theta$$

Thus $$\mathbf{e^{j\theta} = \cos\theta + j\sin\theta} \qquad (4)$$

Writing $-\theta$ for θ in equation (4), gives:

$$e^{j(-\theta)} = \cos(-\theta) + j\sin(-\theta)$$

However, $\cos(-\theta)=\cos\theta$ and $\sin(-\theta)=-\sin\theta$

Thus $$\mathbf{e^{-j\theta} = \cos\theta - j\sin\theta} \qquad (5)$$

The polar form of a complex number z is: $z=r(\cos\theta+j\sin\theta)$. But, from equation (4), $\cos\theta+j\sin\theta=e^{j\theta}$.

Therefore $$\mathbf{z = re^{j\theta}}$$

When a complex number is written in this way, it is said to be expressed in **exponential form**.

There are therefore three ways of expressing a complex number:

1. $z=(a+jb)$, called **Cartesian** or **rectangular form**,
2. $z=r(\cos\theta+j\sin\theta)$ or $r\angle\theta$, called **polar form**, and
3. $z=re^{j\theta}$ called **exponential form**.

The exponential form is obtained from the polar form. For example, $4\angle30°$ becomes $4e^{j\frac{\pi}{6}}$ in exponential form. (Note that in $re^{j\theta}$, θ must be in radians.)

Problem 6. Change $(3-j4)$ into (a) polar form, (b) exponential form.

(a) $(3-j4) = \mathbf{5\angle -53.13^\circ}$ or $\mathbf{5\angle -0.927}$ in polar form

(b) $(3-j4) = 5\angle -0.927 = \mathbf{5e^{-j0.927}}$ in exponential form

Problem 7. Convert $7.2e^{j1.5}$ into rectangular form.

$7.2e^{j1.5} = 7.2\angle 1.5\,\text{rad} (= 7.2\angle 85.94^\circ)$ in polar form

$= 7.2\cos 1.5 + j7.2\sin 1.5$

$= \mathbf{(0.509 + j7.182)}$ in rectangular form

Problem 8. Express $z = 2e^{1+j\frac{\pi}{3}}$ in Cartesian form.

$z = (2e^1)\left(e^{j\frac{\pi}{3}}\right)$ by the laws of indices

$= (2e^1)\angle\frac{\pi}{3}$ (or $2e\angle 60^\circ$)in polar form

$= 2e\left(\cos\frac{\pi}{3} + j\sin\frac{\pi}{3}\right)$

$= \mathbf{(2.718 + j4.708)}$ in Cartesian form

Problem 9. Change $6e^{2-j3}$ into $(a+jb)$ form.

$6e^{2-j3} = (6e^2)(e^{-j3})$ by the laws of indices

$= 6e^2\angle -3$ rad (or $6e^2\angle -171.89^0$) in polar form

$= 6e^2[\cos(-3) + j\sin(-3)]$

$= \mathbf{(-43.89 - j6.26)}$ in $(a+jb)$ form

Problem 10. If $z = 4e^{j1.3}$, determine $\ln z$ (a) in Cartesian form, and (b) in polar form.

If $z = re^{j\theta}$ then $\ln z = \ln(re^{j\theta})$

$= \ln r + \ln e^{j\theta}$

i.e. $\ln z = \ln r + j\theta$,

by the laws of logarithms

(a) Thus if $z = 4e^{j1.3}$ then $\ln z = \ln(4e^{j1.3})$

$= \mathbf{\ln 4 + j1.3}$

(or $\mathbf{1.386 + j1.300}$) in Cartesian form.

(b) $(1.386 + j1.300) = \mathbf{1.90\angle 43.17^\circ}$ or $\mathbf{1.90\angle 0.753}$ in polar form.

Problem 11. Given $z = 3e^{1-j}$, find $\ln z$ in polar form.

If $z = 3e^{1-j}$, then

$\ln z = \ln(3e^{1-j})$

$= \ln 3 + \ln e^{1-j}$

$= \ln 3 + 1 - j$

$= (1 + \ln 3) - j$

$= 2.0986 - j1.0000$

$= \mathbf{2.325\angle -25.48^\circ}$ or $\mathbf{2.325\angle -0.445}$

Problem 12. Determine, in polar form, $\ln(3+j4)$.

$\ln(3+j4) = \ln[5\angle 0.927] = \ln[5e^{j0.927}]$

$= \ln 5 + \ln(e^{j0.927})$

$= \ln 5 + j0.927$

$= 1.609 + j0.927$

$= \mathbf{1.857\angle 29.95^\circ}$ or $\mathbf{1.857\angle 0.523}$

Now try the following exercise

Exercise 92 Further problems on the exponential form of complex numbers

1. Change $(5+j3)$ into exponential form. $[5.83e^{j0.54}]$
2. Convert $(-2.5+j4.2)$ into exponential form. $[4.89e^{j2.11}]$
3. Change $3.6e^{j2}$ into cartesian form. $[-1.50 + j3.27]$
4. Express $2e^{3+j\frac{\pi}{6}}$ in $(a+jb)$ form. $[34.79 + j20.09]$
5. Convert $1.7e^{1.2-j2.5}$ into rectangular form. $[-4.52 - j3.38]$

6. If $z = 7e^{j2.1}$, determine $\ln z$ (a) in Cartesian form, and (b) in polar form.

$$\begin{bmatrix} \text{(a) } \ln 7 + j2.1 \\ \text{(b) } 2.86\angle 47.18^\circ \text{or} \\ 2.86\angle 0.82 \end{bmatrix}$$

7. Given $z = 4e^{1.5-j2}$, determine $\ln z$ in polar form. [$3.51\angle -34.72^\circ$ or $3.51\angle -0.61$]

8. Determine in polar form (a) $\ln(2 + j5)$ (b) $\ln(-4 - j3)$

$$\begin{bmatrix} \text{(a) } 2.06\angle 35.26^\circ \text{or} \\ 2.06\angle 0.615 \\ \text{(b) } 4.11\angle 66.96^\circ \text{or} \\ 4.11\angle 1.17 \end{bmatrix}$$

9. When displaced electrons oscillate about an equilibrium position the displacement x is given by the equation:

$$x = Ae^{\left\{-\frac{ht}{2m} + j\frac{\sqrt{(4mf-h^2)}}{2m-a}t\right\}}$$

Determine the real part of x in terms of t, assuming $(4mf - h^2)$ is positive.

$$\left[Ae^{-\frac{ht}{2m}}\cos\left(\frac{\sqrt{(4mf-h^2)}}{2m-a}\right)t\right]$$

Chapter 22

The theory of matrices and determinants

22.1 Matrix notation

Matrices and determinants are mainly used for the solution of linear simultaneous equations. The theory of matrices and determinants is dealt with in this chapter and this theory is then used in Chapter 23 to solve simultaneous equations.

The coefficients of the variables for linear simultaneous equations may be shown in matrix form. The coefficients of x and y in the simultaneous equations

$$x + 2y = 3$$
$$4x - 5y = 6$$

become $\begin{pmatrix} 1 & 2 \\ 4 & -5 \end{pmatrix}$ in matrix notation.

Similarly, the coefficients of p, q and r in the equations

$$1.3p - 2.0q + r = 7$$
$$3.7p + 4.8q - 7r = 3$$
$$4.1p + 3.8q + 12r = -6$$

become $\begin{pmatrix} 1.3 & -2.0 & 1 \\ 3.7 & 4.8 & -7 \\ 4.1 & 3.8 & 12 \end{pmatrix}$ in matrix form.

The numbers within a matrix are called an **array** and the coefficients forming the array are called the **elements** of the matrix. The number of rows in a matrix is usually specified by m and the number of columns by n and a matrix referred to as an 'm by n' matrix. Thus, $\begin{pmatrix} 2 & 3 & 6 \\ 4 & 5 & 7 \end{pmatrix}$ is a '2 by 3' matrix. Matrices cannot be expressed as a single numerical value, but they can often be simplified or combined, and unknown element values can be determined by comparison methods. Just as there are rules for addition, subtraction, multiplication and division of numbers in arithmetic, rules for these operations can be applied to matrices and the rules of matrices are such that they obey most of those governing the algebra of numbers.

22.2 Addition, subtraction and multiplication of matrices

(i) Addition of matrices

Corresponding elements in two matrices may be added to form a single matrix.

Problem 1. Add the matrices

(a) $\begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix}$ and $\begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix}$ and

(b) $\begin{pmatrix} 3 & 1 & -4 \\ 4 & 3 & 1 \\ 1 & 4 & -3 \end{pmatrix}$ and $\begin{pmatrix} 2 & 7 & -5 \\ -2 & 1 & 0 \\ 6 & 3 & 4 \end{pmatrix}$

(a) Adding the corresponding elements gives:

$$\begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix} + \begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix}$$
$$= \begin{pmatrix} 2+(-3) & -1+0 \\ -7+7 & 4+(-4) \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{-1} & \mathbf{-1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

(b) Adding the corresponding elements gives:

$$\begin{pmatrix} 3 & 1 & -4 \\ 4 & 3 & 1 \\ 1 & 4 & -3 \end{pmatrix} + \begin{pmatrix} 2 & 7 & -5 \\ -2 & 1 & 0 \\ 6 & 3 & 4 \end{pmatrix}$$
$$= \begin{pmatrix} 3+2 & 1+7 & -4+(-5) \\ 4+(-2) & 3+1 & 1+0 \\ 1+6 & 4+3 & -3+4 \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{5} & \mathbf{8} & \mathbf{-9} \\ \mathbf{2} & \mathbf{4} & \mathbf{1} \\ \mathbf{7} & \mathbf{7} & \mathbf{1} \end{pmatrix}$$

(ii) Subtraction of matrices

If A is a matrix and B is another matrix, then $(A - B)$ is a single matrix formed by subtracting the elements of B from the corresponding elements of A.

Problem 2. Subtract

(a) $\begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix}$ from $\begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix}$ and

(b) $\begin{pmatrix} 2 & 7 & -5 \\ -2 & 1 & 0 \\ 6 & 3 & 4 \end{pmatrix}$ from $\begin{pmatrix} 3 & 1 & -4 \\ 4 & 3 & 1 \\ 1 & 4 & -3 \end{pmatrix}$

To find matrix A minus matrix B, the elements of B are taken from the corresponding elements of A. Thus:

(a) $\begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix} - \begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix}$
$$= \begin{pmatrix} 2-(-3) & -1-0 \\ -7-7 & 4-(-4) \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{5} & \mathbf{-1} \\ \mathbf{-14} & \mathbf{8} \end{pmatrix}$$

(b) $\begin{pmatrix} 3 & 1 & -4 \\ 4 & 3 & 1 \\ 1 & 4 & -3 \end{pmatrix} - \begin{pmatrix} 2 & 7 & -5 \\ -2 & 1 & 0 \\ 6 & 3 & 4 \end{pmatrix}$
$$= \begin{pmatrix} 3-2 & 1-7 & -4-(-5) \\ 4-(-2) & 3-1 & 1-0 \\ 1-6 & 4-3 & -3-4 \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{1} & \mathbf{-6} & \mathbf{1} \\ \mathbf{6} & \mathbf{2} & \mathbf{1} \\ \mathbf{-5} & \mathbf{1} & \mathbf{-7} \end{pmatrix}$$

Problem 3. If $A = \begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 \\ -2 & -4 \end{pmatrix}$ find $A + B - C$.

$$A + B = \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}$$

(from Problem 1)

Hence, $A + B - C = \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ -2 & -4 \end{pmatrix}$
$$= \begin{pmatrix} -1-1 & -1-0 \\ 0-(-2) & 0-(-4) \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{-2} & \mathbf{-1} \\ \mathbf{2} & \mathbf{4} \end{pmatrix}$$

Alternatively $A + B - C$

$$= \begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ -2 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} -3+2-1 & 0+(-1)-0 \\ 7+(-7)-(-2) & -4+4-(-4) \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{-2} & \mathbf{-1} \\ \mathbf{2} & \mathbf{4} \end{pmatrix}$$ as obtained previously

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(iii) Multiplication

When a matrix is multiplied by a number, called **scalar multiplication**, a single matrix results in which each element of the original matrix has been multiplied by the number.

Problem 4. If $A = \begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 \\ -2 & -4 \end{pmatrix}$ find $2A - 3B + 4C$.

For scalar multiplication, each element is multiplied by the scalar quantity, hence

$$2A = 2\begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix} = \begin{pmatrix} -6 & 0 \\ 14 & -8 \end{pmatrix}$$

$$3B = 3\begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix} = \begin{pmatrix} 6 & -3 \\ -21 & 12 \end{pmatrix}$$

and $$4C = 4\begin{pmatrix} 1 & 0 \\ -2 & -4 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ -8 & -16 \end{pmatrix}$$

Hence $2A - 3B + 4C$

$$= \begin{pmatrix} -6 & 0 \\ 14 & -8 \end{pmatrix} - \begin{pmatrix} 6 & -3 \\ -21 & 12 \end{pmatrix} + \begin{pmatrix} 4 & 0 \\ -8 & -16 \end{pmatrix}$$

$$= \begin{pmatrix} -6-6+4 & 0-(-3)+0 \\ 14-(-21)+(-8) & -8-12+(-16) \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{-8} & \mathbf{3} \\ \mathbf{27} & \mathbf{-36} \end{pmatrix}$$

When a matrix A is multiplied by another matrix B, a single matrix results in which elements are obtained from the sum of the products of the corresponding rows of A and the corresponding columns of B.

Two matrices A and B may be multiplied together, provided the number of elements in the rows of matrix A are equal to the number of elements in the columns of matrix B. In general terms, when multiplying a matrix of dimensions (m by n) by a matrix of dimensions (n by r), the resulting matrix has dimensions (m by r). Thus a 2 by 3 matrix multiplied by a 3 by 1 matrix gives a matrix of dimensions 2 by 1.

Problem 5. If $A = \begin{pmatrix} 2 & 3 \\ 1 & -4 \end{pmatrix}$ and $B = \begin{pmatrix} -5 & 7 \\ -3 & 4 \end{pmatrix}$ find $A \times B$.

Let $A \times B = C$ where $C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$

C_{11} is the sum of the products of the first row elements of A and the first column elements of B taken one at a time,

i.e. $C_{11} = (2 \times (-5)) + (3 \times (-3)) = -19$

C_{12} is the sum of the products of the first row elements of A and the second column elements of B, taken one at a time,

i.e. $C_{12} = (2 \times 7) + (3 \times 4) = 26$

C_{21} is the sum of the products of the second row elements of A and the first column elements of B, taken one at a time,

i.e. $C_{21} = (1 \times (-5)) + (-4 \times (-3)) = 7$

Finally, C_{22} is the sum of the products of the second row elements of A and the second column elements of B, taken one at a time,

i.e. $C_{22} = (1 \times 7) + ((-4) \times 4) = -9$

Thus, $A \times B = \begin{pmatrix} \mathbf{-19} & \mathbf{26} \\ \mathbf{7} & \mathbf{-9} \end{pmatrix}$

Problem 6. Simplify $\begin{pmatrix} 3 & 4 & 0 \\ -2 & 6 & -3 \\ 7 & -4 & 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$

The sum of the products of the elements of each row of the first matrix and the elements of the second matrix, (called a **column matrix**), are taken one at a time. Thus:

$$\begin{pmatrix} 3 & 4 & 0 \\ -2 & 6 & -3 \\ 7 & -4 & 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} (3 \times 2) & + (4 \times 5) & + (0 \times (-1)) \\ (-2 \times 2) & + (6 \times 5) & + (-3 \times (-1)) \\ (7 \times 2) & + (-4 \times 5) & + (1 \times (-1)) \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{26} \\ \mathbf{29} \\ \mathbf{-7} \end{pmatrix}$$

Problem 7. If $A = \begin{pmatrix} 3 & 4 & 0 \\ -2 & 6 & -3 \\ 7 & -4 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -5 \\ 5 & -6 \\ -1 & -7 \end{pmatrix}$, find $A \times B$.

The sum of the products of the elements of each row of the first matrix and the elements of each column of the second matrix are taken one at a time. Thus:

$$\begin{pmatrix} 3 & 4 & 0 \\ -2 & 6 & -3 \\ 7 & -4 & 1 \end{pmatrix} \times \begin{pmatrix} 2 & -5 \\ 5 & -6 \\ -1 & -7 \end{pmatrix}$$

$$= \begin{pmatrix} [(3\times 2) & [(3\times(-5)) \\ +(4\times 5) & +(4\times(-6)) \\ +(0\times(-1))] & +(0\times(-7))] \\ [(-2\times 2) & [(-2\times(-5)) \\ +(6\times 5) & +(6\times(-6)) \\ +(-3\times(-1))] & +(-3\times(-7))] \\ [(7\times 2) & [(7\times(-5)) \\ +(-4\times 5) & +(-4\times(-6)) \\ +(1\times(-1))] & +(1\times(-7))] \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{26} & \mathbf{-39} \\ \mathbf{29} & \mathbf{-5} \\ \mathbf{-7} & \mathbf{-18} \end{pmatrix}$$

Problem 8. Determine

$$\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 2 \\ 1 & 3 & 1 \end{pmatrix} \times \begin{pmatrix} 2 & 2 & 0 \\ 1 & 3 & 2 \\ 3 & 2 & 0 \end{pmatrix}$$

The sum of the products of the elements of each row of the first matrix and the elements of each column of the second matrix are taken one at a time. Thus:

$$\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 2 \\ 1 & 3 & 1 \end{pmatrix} \times \begin{pmatrix} 2 & 2 & 0 \\ 1 & 3 & 2 \\ 3 & 2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} [(1\times 2) & [(1\times 2) & [(1\times 0) \\ +(0\times 1) & +(0\times 3) & +(0\times 2) \\ +(3\times 3)] & +(3\times 2)] & +(3\times 0)] \\ [(2\times 2) & [(2\times 2) & [(2\times 0) \\ +(1\times 1) & +(1\times 3) & +(1\times 2) \\ +(2\times 3)] & +(2\times 2)] & +(2\times 0)] \\ [(1\times 2) & [(1\times 2) & [(1\times 0) \\ +(3\times 1) & +(3\times 3) & +(3\times 2) \\ +(1\times 3)] & +(1\times 2)] & +(1\times 0)] \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{11} & \mathbf{8} & \mathbf{0} \\ \mathbf{11} & \mathbf{11} & \mathbf{2} \\ \mathbf{8} & \mathbf{13} & \mathbf{6} \end{pmatrix}$$

In algebra, the commutative law of multiplication states that $a\times b = b\times a$. For matrices, this law is only true in a few special cases, and in general $A\times B$ is **not** equal to $B\times A$.

Problem 9. If $A = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$ show that $A\times B \neq B\times A$.

$$A\times B = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} [(2\times 2)+(3\times 0)] & [(2\times 3)+(3\times 1)] \\ [(1\times 2)+(0\times 0)] & [(1\times 3)+(0\times 1)] \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 9 \\ 2 & 3 \end{pmatrix}$$

$$B\times A = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} [(2\times 2)+(3\times 1)] & [(2\times 3)+(3\times 0)] \\ [(0\times 2)+(1\times 1)] & [(0\times 3)+(1\times 0)] \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 6 \\ 1 & 0 \end{pmatrix}$$

Since $\begin{pmatrix} 4 & 9 \\ 2 & 3 \end{pmatrix} \neq \begin{pmatrix} 7 & 6 \\ 1 & 0 \end{pmatrix}$, then $\boldsymbol{A\times B \neq B\times A}$

Now try the following exercise

Exercise 93 Further problems on addition, subtraction and multiplication of matrices

In Problems 1 to 13, the matrices A to K are:

$$A = \begin{pmatrix} 3 & -1 \\ -4 & 7 \end{pmatrix} \quad B = \begin{pmatrix} 5 & 2 \\ -1 & 6 \end{pmatrix}$$

$$C = \begin{pmatrix} -1.3 & 7.4 \\ 2.5 & -3.9 \end{pmatrix}$$

$$D = \begin{pmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{pmatrix}$$

$$E = \begin{pmatrix} 3 & 6 & 2 \\ 5 & -3 & 7 \\ -1 & 0 & 2 \end{pmatrix}$$

$$F = \begin{pmatrix} 3.1 & 2.4 & 6.4 \\ -1.6 & 3.8 & -1.9 \\ 5.3 & 3.4 & -4.8 \end{pmatrix} \quad G = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$$

$$H = \begin{pmatrix} -2 \\ 5 \end{pmatrix} \quad J = \begin{pmatrix} 4 \\ -11 \\ 7 \end{pmatrix} \quad K = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Addition, subtraction and multiplication

In Problems 1 to 12, perform the matrix operation stated.

1. $A+B$ $\left[\begin{pmatrix} 8 & 1 \\ -5 & 13 \end{pmatrix}\right]$

2. $D+E$ $\begin{bmatrix}\begin{pmatrix}7 & -1 & 8\\ 3 & 1 & 7\\ 4 & 7 & -2\end{pmatrix}\end{bmatrix}$

3. $A-B$ $\begin{bmatrix}\begin{pmatrix}-2 & -3\\ -3 & 1\end{pmatrix}\end{bmatrix}$

4. $A+B-C$ $\begin{bmatrix}\begin{pmatrix}9.3 & -6.4\\ -7.5 & 16.9\end{pmatrix}\end{bmatrix}$

5. $5A+6B$ $\begin{bmatrix}\begin{pmatrix}45 & 7\\ -26 & 71\end{pmatrix}\end{bmatrix}$

6. $2D+3E-4F$ $\begin{bmatrix}\begin{pmatrix}4.6 & -5.6 & -7.6\\ 17.4 & -16.2 & 28.6\\ -14.2 & 0.4 & 17.2\end{pmatrix}\end{bmatrix}$

7. $A \times H$ $\begin{bmatrix}\begin{pmatrix}-11\\ 43\end{pmatrix}\end{bmatrix}$

8. $A \times B$ $\begin{bmatrix}\begin{pmatrix}16 & 0\\ -27 & 34\end{pmatrix}\end{bmatrix}$

9. $A \times C$ $\begin{bmatrix}\begin{pmatrix}-6.4 & 26.1\\ 22.7 & -56.9\end{pmatrix}\end{bmatrix}$

10. $D \times J$ $\begin{bmatrix}\begin{pmatrix}135\\ -52\\ -85\end{pmatrix}\end{bmatrix}$

11. $E \times K$ $\begin{bmatrix}\begin{pmatrix}5 & 6\\ 12 & -3\\ 1 & 0\end{pmatrix}\end{bmatrix}$

12. $D \times F$ $\begin{bmatrix}\begin{pmatrix}55.4 & 3.4 & 10.1\\ -12.6 & 10.4 & -20.4\\ -16.9 & 25.0 & 37.9\end{pmatrix}\end{bmatrix}$

13. Show that $A \times C \neq C \times A$

$$\begin{bmatrix} A \times C = \begin{pmatrix}-6.4 & 26.1\\ 22.7 & -56.9\end{pmatrix} \\ C \times A = \begin{pmatrix}-33.5 & -53.1\\ 23.1 & -29.8\end{pmatrix} \\ \text{Hence they are not equal}\end{bmatrix}$$

22.3 The unit matrix

A **unit matrix, I**, is one in which all elements of the leading diagonal (\) have a value of 1 and all other elements have a value of 0. Multiplication of a matrix by I is the equivalent of multiplying by 1 in arithmetic.

22.4 The determinant of a 2 by 2 matrix

The **determinant** of a 2 by 2 matrix, $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ is defined as $(ad-bc)$.

The elements of the determinant of a matrix are written between vertical lines. Thus, the determinant of $\begin{pmatrix}3 & -4\\ 1 & 6\end{pmatrix}$ is written as $\begin{vmatrix}3 & -4\\ 1 & 6\end{vmatrix}$ and is equal to $(3\times 6)-(-4\times 1)$, i.e. $18-(-4)$ or 22. Hence the determinant of a matrix can be expressed as a single numerical value, i.e. $\begin{vmatrix}3 & -4\\ 1 & 6\end{vmatrix}=22$.

Problem 10. Determine the value of $\begin{vmatrix}3 & -2\\ 7 & 4\end{vmatrix}$

$$\begin{vmatrix}3 & -2\\ 7 & 4\end{vmatrix} = (3\times 4)-(-2\times 7)$$
$$= 12-(-14) = \mathbf{26}$$

Problem 11. Evaluate $\begin{vmatrix}(1+j) & j2\\ -j3 & (1-j4)\end{vmatrix}$

$$\begin{vmatrix}(1+j) & j2\\ -j3 & (1-j4)\end{vmatrix} = (1+j)(1-j4)-(j2)(-j3)$$
$$= 1-j4+j-j^2 4+j^2 6$$
$$= 1-j4+j-(-4)+(-6)$$

since from Chapter 20, $j^2=-1$

$$= 1-j4+j+4-6$$
$$= \mathbf{-1-j3}$$

Problem 12. Evaluate $\begin{vmatrix}5\angle 30^\circ & 2\angle -60^\circ\\ 3\angle 60^\circ & 4\angle -90^\circ\end{vmatrix}$

$$\begin{vmatrix} 5\angle 30^\circ & 2\angle -60^\circ \\ 3\angle 60^\circ & 4\angle -90^\circ \end{vmatrix} = (5\angle 30^\circ)(4\angle -90^\circ) - (2\angle -60^\circ)(3\angle 60^\circ)$$

$$= (20\angle -60^\circ) - (6\angle 0^\circ)$$

$$= (10 - j17.32) - (6 + j0)$$

$$= \mathbf{(4 - j\,17.32)} \text{ or } \mathbf{17.78\angle -77^\circ}$$

Now try the following exercise

Exercise 94 Further problems on 2 by 2 determinants

1. Calculate the determinant of $\begin{pmatrix} 3 & -1 \\ -4 & 7 \end{pmatrix}$ [17]
2. Calculate the determinant of $\begin{pmatrix} -2 & 5 \\ 3 & -6 \end{pmatrix}$ [−3]
3. Calculate the determinant of $\begin{pmatrix} -1.3 & 7.4 \\ 2.5 & -3.9 \end{pmatrix}$ [−13.43]
4. Evaluate $\begin{vmatrix} j2 & -j3 \\ (1+j) & j \end{vmatrix}$ [−5 + j3]
5. Evaluate $\begin{vmatrix} 2\angle 40^\circ & 5\angle -20^\circ \\ 7\angle -32^\circ & 4\angle -117^\circ \end{vmatrix}$ [(−19.75 + j19.79) or 27.96∠134.94°]

22.5 The inverse or reciprocal of a 2 by 2 matrix

The inverse of matrix A is A^{-1} such that $A \times A^{-1} = I$, the unit matrix.

Let matrix A be $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and let the inverse matrix, A^{-1} be $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Then, since $A \times A^{-1} = I$,

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \times \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Multiplying the matrices on the left hand side, gives

$$\begin{pmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Equating corresponding elements gives:

$$b + 2d = 0, \text{ i.e. } b = -2d$$

and $3a + 4c = 0$, i.e. $a = -\frac{4}{3}c$

Substituting for a and b gives:

$$\begin{pmatrix} -\frac{4}{3}c + 2c & -2d + 2d \\ 3\left(-\frac{4}{3}c\right) + 4c & 3(-2d) + 4d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

i.e. $$\begin{pmatrix} \frac{2}{3}c & 0 \\ 0 & -2d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

showing that $\frac{2}{3}c = 1$, i.e. $c = \frac{3}{2}$ and $-2d = 1$, i.e. $d = -\frac{1}{2}$

Since $b = -2d$, $b = 1$ and since $a = -\frac{4}{3}c$, $a = -2$.

Thus the inverse of matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ that is, $\begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$

There is, however, **a quicker method of obtaining the inverse** of a 2 by 2 matrix.

For any matrix $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ the inverse may be obtained by:

(i) interchanging the positions of p and s,

(ii) changing the signs of q and r, and

(iii) multiplying this new matrix by the reciprocal of the determinant of $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$

Thus the inverse of matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is

$$\frac{1}{4-6}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

as obtained previously.

Problem 13. Determine the inverse of $\begin{pmatrix} 3 & -2 \\ 7 & 4 \end{pmatrix}$

The inverse of matrix $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ is obtained by interchanging the positions of p and s, changing the signs of q and r and multiplying by the reciprocal of the determinant $\begin{vmatrix} p & q \\ r & s \end{vmatrix}$. Thus, the inverse of

$$\begin{pmatrix} 3 & -2 \\ 7 & 4 \end{pmatrix} = \frac{1}{(3\times 4)-(-2\times 7)}\begin{pmatrix} 4 & 2 \\ -7 & 3 \end{pmatrix}$$

$$= \frac{1}{26}\begin{pmatrix} 4 & 2 \\ -7 & 3 \end{pmatrix} = \begin{pmatrix} \mathbf{\frac{2}{13}} & \mathbf{\frac{1}{13}} \\ \mathbf{\frac{-7}{26}} & \mathbf{\frac{3}{26}} \end{pmatrix}$$

Now try the following exercise

Exercise 95 Further problems on the inverse of 2 by 2 matrices

1. Determine the inverse of $\begin{pmatrix} 3 & -1 \\ -4 & 7 \end{pmatrix}$

$$\left[\begin{pmatrix} \frac{7}{17} & \frac{1}{17} \\ \frac{4}{17} & \frac{3}{17} \end{pmatrix}\right]$$

2. Determine the inverse of $\begin{pmatrix} \frac{1}{2} & \frac{2}{3} \\ -\frac{1}{3} & -\frac{3}{5} \end{pmatrix}$

$$\left[\begin{pmatrix} 7\frac{5}{7} & 8\frac{4}{7} \\ -4\frac{2}{7} & -6\frac{3}{7} \end{pmatrix}\right]$$

3. Determine the inverse of $\begin{pmatrix} -1.3 & 7.4 \\ 2.5 & -3.9 \end{pmatrix}$

$$\left[\begin{pmatrix} 0.290 & 0.551 \\ 0.186 & 0.097 \end{pmatrix} \text{ correct to 3 dec. places}\right]$$

22.6 The determinant of a 3 by 3 matrix

(i) The **minor** of an element of a 3 by 3 matrix is the value of the 2 by 2 determinant obtained by covering up the row and column containing that element.

Thus for the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ the minor of element 4 is obtained by covering the row (4 5 6) and the column $\begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}$, leaving the 2 by 2 determinant $\begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix}$, i.e. the minor of element 4 is $(2\times 9)-(3\times 8) = -6$.

(ii) The sign of a minor depends on its position within the matrix, the sign pattern being $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$. Thus the signed-minor of element 4 in the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ is $-\begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} = -(-6) = 6$.

The signed-minor of an element is called the **cofactor** of the element.

(iii) **The value of a 3 by 3 determinant is the sum of the products of the elements and their cofactors of any row or any column of the corresponding 3 by 3 matrix.**

There are thus six different ways of evaluating a 3×3 determinant—and all should give the same value.

Problem 14. Find the value of

$$\begin{vmatrix} 3 & 4 & -1 \\ 2 & 0 & 7 \\ 1 & -3 & -2 \end{vmatrix}$$

The value of this determinant is the sum of the products of the elements and their cofactors, of any row or of any column. If the second row or second column is selected, the element 0 will make the product of the element and its cofactor zero and reduce the amount of arithmetic to be done to a minimum.

Supposing a second row expansion is selected.

The minor of 2 is the value of the determinant remaining when the row and column containing the 2 (i.e. the second row and the first column), is covered up. Thus the cofactor of element 2 is $\begin{vmatrix} 4 & -1 \\ -3 & -2 \end{vmatrix}$ i.e. -11.

The sign of element 2 is minus, (see (ii) above), hence the cofactor of element 2, (the signed-minor) is $+11$.

Similarly the minor of element 7 is $\begin{vmatrix} 3 & 4 \\ 1 & -3 \end{vmatrix}$ i.e. -13, and its cofactor is $+13$. Hence the value of the sum of

the products of the elements and their cofactors is $2 \times 11 + 7 \times 13$, i.e.,

$$\begin{vmatrix} 3 & 4 & -1 \\ 2 & 0 & 7 \\ 1 & -3 & -2 \end{vmatrix} = 2(11) + 0 + 7(13) = \mathbf{113}$$

The same result will be obtained whichever row or column is selected. For example, the third column expansion is

$$(-1)\begin{vmatrix} 2 & 0 \\ 1 & -3 \end{vmatrix} - 7\begin{vmatrix} 3 & 4 \\ 1 & -3 \end{vmatrix} + (-2)\begin{vmatrix} 3 & 4 \\ 2 & 0 \end{vmatrix}$$

$= 6 + 91 + 16 = \mathbf{113}$, as obtained previously.

Problem 15. Evaluate $\begin{vmatrix} 1 & 4 & -3 \\ -5 & 2 & 6 \\ -1 & -4 & 2 \end{vmatrix}$

Using the first row: $\begin{vmatrix} 1 & 4 & -3 \\ -5 & 2 & 6 \\ -1 & -4 & 2 \end{vmatrix}$

$$= 1\begin{vmatrix} 2 & 6 \\ -4 & 2 \end{vmatrix} - 4\begin{vmatrix} -5 & 6 \\ -1 & 2 \end{vmatrix} + (-3)\begin{vmatrix} -5 & 2 \\ -1 & -4 \end{vmatrix}$$

$$= (4 + 24) - 4(-10 + 6) - 3(20 + 2)$$

$$= 28 + 16 - 66 = \mathbf{-22}$$

Using the second column: $\begin{vmatrix} 1 & 4 & -3 \\ -5 & 2 & 6 \\ -1 & -4 & 2 \end{vmatrix}$

$$= -4\begin{vmatrix} -5 & 6 \\ -1 & 2 \end{vmatrix} + 2\begin{vmatrix} 1 & -3 \\ -1 & 2 \end{vmatrix} - (-4)\begin{vmatrix} 1 & -3 \\ -5 & 6 \end{vmatrix}$$

$$= -4(-10 + 6) + 2(2 - 3) + 4(6 - 15)$$

$$= 16 - 2 - 36 = \mathbf{-22}$$

Problem 16. Determine the value of $\begin{vmatrix} j2 & (1+j) & 3 \\ (1-j) & 1 & j \\ 0 & j4 & 5 \end{vmatrix}$

Using the first column, the value of the determinant is:

$$(j2)\begin{vmatrix} 1 & j \\ j4 & 5 \end{vmatrix} - (1-j)\begin{vmatrix} (1+j) & 3 \\ j4 & 5 \end{vmatrix} + (0)\begin{vmatrix} (1+j) & 3 \\ 1 & j \end{vmatrix}$$

$$= j2(5 - j^2 4) - (1-j)(5 + j5 - j12) + 0$$

$$= j2(9) - (1-j)(5 - j7)$$

$$= j18 - [5 - j7 - j5 + j^2 7]$$

$$= j18 - [-2 - j12]$$

$$= j18 + 2 + j12 = \mathbf{2 + j\,30} \text{ or } \mathbf{30.07\angle 86.19°}$$

Now try the following exercise

Exercise 96 Further problems on 3 by 3 determinants

1. Find the matrix of minors of

$$\begin{pmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{pmatrix}$$

$$\left[\begin{pmatrix} -16 & 8 & -34 \\ -14 & -46 & 63 \\ -24 & 12 & 2 \end{pmatrix}\right]$$

2. Find the matrix of cofactors of

$$\begin{pmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{pmatrix}$$

$$\left[\begin{pmatrix} -16 & -8 & -34 \\ 14 & -46 & -63 \\ -24 & -12 & 2 \end{pmatrix}\right]$$

3. Calculate the determinant of

$$\begin{pmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{pmatrix}$$

[−212]

4. Evaluate $\begin{vmatrix} 8 & -2 & -10 \\ 2 & -3 & -2 \\ 6 & 3 & 8 \end{vmatrix}$ [−328]

5. Calculate the determinant of

$$\begin{pmatrix} 3.1 & 2.4 & 6.4 \\ -1.6 & 3.8 & -1.9 \\ 5.3 & 3.4 & -4.8 \end{pmatrix}$$

[−242.83]

6. Evaluate $\begin{vmatrix} j2 & 2 & j \\ (1+j) & 1 & -3 \\ 5 & -j4 & 0 \end{vmatrix}$ [−2 − j]

7. Evaluate $\begin{vmatrix} 3\angle 60^\circ & j2 & 1 \\ 0 & (1+j) & 2\angle 30^\circ \\ 0 & 2 & j5 \end{vmatrix}$

$[26.94\angle -139.52^\circ \text{ or } (-20.49 - j17.49)]$

8. Find the eigenvalues λ that satisfy the following equations:

(a) $\begin{vmatrix} (2-\lambda) & 2 \\ -1 & (5-\lambda) \end{vmatrix} = 0$

(b) $\begin{vmatrix} (5-\lambda) & 7 & -5 \\ 0 & (4-\lambda) & -1 \\ 2 & 8 & (-3-\lambda) \end{vmatrix} = 0$

(You may need to refer to chapter 1, pages 8–12, for the solution of cubic equations).

[(a) $\lambda = 3$ or 4 (b) $\lambda = 1$ or 2 or 3]

22.7 The inverse or reciprocal of a 3 by 3 matrix

The **adjoint** of a matrix A is obtained by:

(i) forming a matrix B of the cofactors of A, and

(ii) **transposing** matrix B to give B^T, where B^T is the matrix obtained by writing the rows of B as the columns of B^T. Then **adj** $\boldsymbol{A = B^T}$.

The **inverse of matrix** A, A^{-1} is given by

$$\boldsymbol{A^{-1} = \frac{\text{adj } A}{|A|}}$$

where adj A is the adjoint of matrix A and $|A|$ is the determinant of matrix A.

Problem 17. Determine the inverse of the matrix $\begin{pmatrix} 3 & 4 & -1 \\ 2 & 0 & 7 \\ 1 & -3 & -2 \end{pmatrix}$

The inverse of matrix A, $A^{-1} = \dfrac{\text{adj } A}{|A|}$

The adjoint of A is found by:

(i) obtaining the matrix of the cofactors of the elements, and

(ii) transposing this matrix.

The cofactor of element 3 is $+\begin{vmatrix} 0 & 7 \\ -3 & -2 \end{vmatrix} = 21$.

The cofactor of element 4 is $-\begin{vmatrix} 2 & 7 \\ 1 & -2 \end{vmatrix} = 11$, and so on.

The matrix of cofactors is $\begin{pmatrix} 21 & 11 & -6 \\ 11 & -5 & 13 \\ 28 & -23 & -8 \end{pmatrix}$

The transpose of the matrix of cofactors, i.e. the adjoint of the matrix, is obtained by writing the rows as columns, and is $\begin{pmatrix} 21 & 11 & 28 \\ 11 & -5 & -23 \\ -6 & 13 & -8 \end{pmatrix}$

From Problem 14, the determinant of $\begin{vmatrix} 3 & 4 & -1 \\ 2 & 0 & 7 \\ 1 & -3 & -2 \end{vmatrix}$ is 113.

Hence the inverse of $\begin{pmatrix} 3 & 4 & -1 \\ 2 & 0 & 7 \\ 1 & -3 & -2 \end{pmatrix}$ is

$$\frac{\begin{pmatrix} 21 & 11 & 28 \\ 11 & -5 & -23 \\ -6 & 13 & -8 \end{pmatrix}}{113} \text{ or } \frac{1}{\mathbf{113}} \begin{pmatrix} \mathbf{21} & \mathbf{11} & \mathbf{28} \\ \mathbf{11} & \mathbf{-5} & \mathbf{-23} \\ \mathbf{-6} & \mathbf{13} & \mathbf{-8} \end{pmatrix}$$

Problem 18. Find the inverse of $\begin{pmatrix} 1 & 5 & -2 \\ 3 & -1 & 4 \\ -3 & 6 & -7 \end{pmatrix}$

$$\text{Inverse} = \frac{\text{adjoint}}{\text{determinant}}$$

The matrix of cofactors is $\begin{pmatrix} -17 & 9 & 15 \\ 23 & -13 & -21 \\ 18 & -10 & -16 \end{pmatrix}$

The transpose of the matrix of cofactors (i.e. the adjoint) is $\begin{pmatrix} -17 & 23 & 18 \\ 9 & -13 & -10 \\ 15 & -21 & -16 \end{pmatrix}$

The determinant of $\begin{pmatrix} 1 & 5 & -2 \\ 3 & -1 & 4 \\ -3 & 6 & -7 \end{pmatrix}$

$$= 1(7-24) - 5(-21+12) - 2(18-3)$$
$$= -17 + 45 - 30 = -2$$

Hence the inverse of $\begin{pmatrix} 1 & 5 & -2 \\ 3 & -1 & 4 \\ -3 & 6 & -7 \end{pmatrix}$

$$= \frac{\begin{pmatrix} -17 & 23 & 18 \\ 9 & -13 & -10 \\ 15 & -21 & -16 \end{pmatrix}}{-2}$$
$$= \begin{pmatrix} \mathbf{8.5} & \mathbf{-11.5} & \mathbf{-9} \\ \mathbf{-4.5} & \mathbf{6.5} & \mathbf{5} \\ \mathbf{-7.5} & \mathbf{10.5} & \mathbf{8} \end{pmatrix}$$

Now try the following exercise

Exercise 97 Further problems on the inverse of a 3 by 3 matrix

1. Write down the transpose of

$$\begin{pmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{pmatrix}$$

$$\left[\begin{pmatrix} 4 & -2 & 5 \\ -7 & 4 & 7 \\ 6 & 0 & -4 \end{pmatrix}\right]$$

2. Write down the transpose of

$$\begin{pmatrix} 3 & 6 & \frac{1}{2} \\ 5 & -\frac{2}{3} & 7 \\ -1 & 0 & \frac{3}{5} \end{pmatrix}$$

$$\left[\begin{pmatrix} 3 & 5 & -1 \\ 6 & -\frac{2}{3} & 0 \\ \frac{1}{2} & 7 & \frac{3}{5} \end{pmatrix}\right]$$

3. Determine the adjoint of

$$\begin{pmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{pmatrix}$$

$$\left[\begin{pmatrix} -16 & 14 & -24 \\ -8 & -46 & -12 \\ -34 & -63 & 2 \end{pmatrix}\right]$$

4. Determine the adjoint of

$$\begin{pmatrix} 3 & 6 & \frac{1}{2} \\ 5 & -\frac{2}{3} & 7 \\ -1 & 0 & \frac{3}{5} \end{pmatrix}$$

$$\left[\begin{pmatrix} -\frac{2}{5} & -3\frac{3}{5} & 42\frac{1}{3} \\ -10 & 2\frac{3}{10} & -18\frac{1}{2} \\ -\frac{2}{3} & -6 & -32 \end{pmatrix}\right]$$

5. Find the inverse of

$$\begin{pmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{pmatrix}$$

$$\left[-\frac{1}{212}\begin{pmatrix} -16 & 14 & -24 \\ -8 & -46 & -12 \\ -34 & -63 & 2 \end{pmatrix}\right]$$

6. Find the inverse of $\begin{pmatrix} 3 & 6 & \frac{1}{2} \\ 5 & -\frac{2}{3} & 7 \\ -1 & 0 & \frac{3}{5} \end{pmatrix}$

$$\left[-\frac{15}{923}\begin{pmatrix} -\frac{2}{5} & -3\frac{3}{5} & 42\frac{1}{3} \\ -10 & 2\frac{3}{10} & -18\frac{1}{2} \\ -\frac{2}{3} & -6 & -32 \end{pmatrix}\right]$$

Chapter 23

The solution of simultaneous equations by matrices and determinants

23.1 Solution of simultaneous equations by matrices

(a) The procedure for solving linear simultaneous equations in **two unknowns using matrices** is:

(i) write the equations in the form

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

(ii) write the matrix equation corresponding to these equations,

i.e. $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

(iii) determine the inverse matrix of $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$

i.e. $\dfrac{1}{a_1b_2 - b_1a_2} \begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix}$

(from Chapter 22)

(iv) multiply each side of (ii) by the inverse matrix, and

(v) solve for x and y by equating corresponding elements.

Problem 1. Use matrices to solve the simultaneous equations:

$$3x + 5y - 7 = 0 \quad (1)$$
$$4x - 3y - 19 = 0 \quad (2)$$

(i) Writing the equations in the $a_1x + b_1y = c$ form gives:

$$3x + 5y = 7$$
$$4x - 3y = 19$$

(ii) The matrix equation is

$$\begin{pmatrix} 3 & 5 \\ 4 & -3 \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 19 \end{pmatrix}$$

(iii) The inverse of matrix $\begin{pmatrix} 3 & 5 \\ 4 & -3 \end{pmatrix}$ is

$$\frac{1}{3 \times (-3) - 5 \times 4} \begin{pmatrix} -3 & -5 \\ -4 & 3 \end{pmatrix}$$

i.e. $\begin{pmatrix} \frac{3}{29} & \frac{5}{29} \\ \frac{4}{29} & \frac{-3}{29} \end{pmatrix}$

(iv) Multiplying each side of (ii) by (iii) and remembering that $A \times A^{-1} = I$, the unit matrix, gives:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{3}{29} & \frac{5}{29} \\ \frac{4}{29} & \frac{-3}{29} \end{pmatrix} \times \begin{pmatrix} 7 \\ 19 \end{pmatrix}$$

Thus $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{21}{29} + \frac{95}{29} \\ \frac{28}{29} - \frac{57}{29} \end{pmatrix}$

i.e. $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

(v) By comparing corresponding elements:

$$\boldsymbol{x=4} \quad \text{and} \quad \boldsymbol{y=-1}$$

Checking:

equation (1),

$$3 \times 4 + 5 \times (-1) - 7 = 0 = \text{RHS}$$

equation (2),

$$4 \times 4 - 3 \times (-1) - 19 = 0 = \text{RHS}$$

(b) The procedure for solving linear simultaneous equations in **three unknowns using matrices** is:

(i) write the equations in the form

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

(ii) write the matrix equation corresponding to these equations, i.e.

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

(iii) determine the inverse matrix of

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$ (see Chapter 22)

(iv) multiply each side of (ii) by the inverse matrix, and

(v) solve for x, y and z by equating the corresponding elements.

Problem 2. Use matrices to solve the simultaneous equations:

$$x + y + z - 4 = 0 \quad (1)$$
$$2x - 3y + 4z - 33 = 0 \quad (2)$$
$$3x - 2y - 2z - 2 = 0 \quad (3)$$

(i) Writing the equations in the $a_1x + b_1y + c_1z = d_1$ form gives:

$$x + y + z = 4$$
$$2x - 3y + 4z = 33$$
$$3x - 2y - 2z = 2$$

(ii) The matrix equation is

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & -2 & -2 \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 33 \\ 2 \end{pmatrix}$$

(iii) The inverse matrix of

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & -2 & -2 \end{pmatrix}$$

is given by

$$A^{-1} = \frac{\text{adj } A}{|A|}$$

The adjoint of A is the transpose of the matrix of the cofactors of the elements (see Chapter 22). The matrix of cofactors is

$$\begin{pmatrix} 14 & 16 & 5 \\ 0 & -5 & 5 \\ 7 & -2 & -5 \end{pmatrix}$$

and the transpose of this matrix gives

$$\text{adj } A = \begin{pmatrix} 14 & 0 & 7 \\ 16 & -5 & -2 \\ 5 & 5 & -5 \end{pmatrix}$$

The determinant of A, i.e. the sum of the products of elements and their cofactors, using a first row expansion is

$$1\begin{vmatrix} -3 & 4 \\ -2 & -2 \end{vmatrix} - 1\begin{vmatrix} 2 & 4 \\ 3 & -2 \end{vmatrix} + 1\begin{vmatrix} 2 & -3 \\ 3 & -2 \end{vmatrix}$$

$$= (1 \times 14) - (1 \times (-16)) + (1 \times 5) = 35$$

Hence the inverse of A,

$$A^{-1} = \frac{1}{35}\begin{pmatrix} 14 & 0 & 7 \\ 16 & -5 & -2 \\ 5 & 5 & -5 \end{pmatrix}$$

(iv) Multiplying each side of (ii) by (iii), and remembering that $A \times A^{-1} = I$, the unit matrix, gives

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\times\begin{pmatrix}x\\y\\z\end{pmatrix}=\frac{1}{35}\times\begin{pmatrix}(14\times4)+(0\times33)+(7\times2)\\(16\times4)+((-5)\times33)+((-2)\times2)\\(5\times4)+(5\times33)+((-5)\times2)\end{pmatrix}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\frac{1}{35}\begin{pmatrix}(14\times4)+(0\times33)+(7\times2)\\(16\times4)+((-5)\times33)+((-2)\times2)\\(5\times4)+(5\times33)+((-5)\times2)\end{pmatrix}=\frac{1}{35}\begin{pmatrix}70\\-105\\175\end{pmatrix}=\begin{pmatrix}2\\-3\\5\end{pmatrix}$$

(v) By comparing corresponding elements, $\boldsymbol{x=2}$, $\boldsymbol{y=-3}$, $\boldsymbol{z=5}$, which can be checked in the original equations.

Now try the following exercise

Exercise 98 Further problems on solving simultaneous equations using matrices

In Problems 1 to 5 use **matrices** to solve the simultaneous equations given.

1. $3x+4y=0$, $2x+5y+7=0$ $[x=4, y=-3]$
2. $2p+5q+14.6=0$, $3.1p+1.7q+2.06=0$ $[p=1.2, q=-3.4]$
3. $x+2y+3z=5$, $2x-3y-z=3$, $-3x+4y+5z=3$ $[x=1, y=-1, z=2]$
4. $3a+4b-3c=2$, $-2a+2b+2c=15$, $7a-5b+4c=26$ $[a=2.5, b=3.5, c=6.5]$
5. $p+2q+3r+7.8=0$, $2p+5q-r-1.4=0$, $5p-q+7r-3.5=0$ $[p=4.1, q=-1.9, r=-2.7]$
6. In two closed loops of an electrical circuit, the currents flowing are given by the simultaneous equations: $I_1+2I_2+4=0$, $5I_1+3I_2-1=0$. Use matrices to solve for I_1 and I_2. $[I_1=2, I_2=-3]$
7. The relationship between the displacement, s, velocity, v, and acceleration, a, of a piston is given by the equations: $s+2v+2a=4$, $3s-v+4a=25$, $3s+2v-a=-4$. Use matrices to determine the values of s, v and a. $[s=2, v=-3, a=4]$
8. In a mechanical system, acceleration $\ddot{x}$, velocity $\dot{x}$ and distance x are related by the simultaneous equations: $3.4\ddot{x}+7.0\dot{x}-13.2x=-11.39$, $-6.0\ddot{x}+4.0\dot{x}+3.5x=4.98$, $2.7\ddot{x}+6.0\dot{x}+7.1x=15.91$. Use matrices to find the values of $\ddot{x}$, $\dot{x}$ and x. $[\ddot{x}=0.5, \dot{x}=0.77, x=1.4]$

23.2 Solution of simultaneous equations by determinants

(a) When solving linear simultaneous equations in **two unknowns using determinants:**

(i) write the equations in the form

$$a_1x+b_1y+c_1=0$$

$$a_2x+b_2y+c_2=0$$

and then

(ii) the solution is given by

$$\frac{x}{D_x}=\frac{-y}{D_y}=\frac{1}{D}$$

where $D_x=\begin{vmatrix}b_1&c_1\\b_2&c_2\end{vmatrix}$

i.e. the determinant of the coefficients left when the x-column is covered up,

$$D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

i.e. the determinant of the coefficients left when the y-column is covered up,

and $$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

i.e. the determinant of the coefficients left when the constants-column is covered up.

Problem 3. Solve the following simultaneous equations using determinants:

$$3x - 4y = 12$$
$$7x + 5y = 6.5$$

Following the above procedure:

(i) $3x - 4y - 12 = 0$
$7x + 5y - 6.5 = 0$

(ii) $$\frac{x}{\begin{vmatrix} -4 & -12 \\ 5 & -6.5 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 3 & -12 \\ 7 & -6.5 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 3 & -4 \\ 7 & 5 \end{vmatrix}}$$

i.e. $$\frac{x}{(-4)(-6.5) - (-12)(5)} = \frac{-y}{(3)(-6.5) - (-12)(7)} = \frac{1}{(3)(5) - (-4)(7)}$$

i.e. $$\frac{x}{26 + 60} = \frac{-y}{-19.5 + 84} = \frac{1}{15 + 28}$$

i.e. $$\frac{x}{86} = \frac{-y}{64.5} = \frac{1}{43}$$

Since $\dfrac{x}{86} = \dfrac{1}{43}$ then $\boldsymbol{x = \dfrac{86}{43} = 2}$

and since

$$\frac{-y}{64.5} = \frac{1}{43} \text{ then } \boldsymbol{y = -\frac{64.5}{43} = -1.5}$$

Problem 4. The velocity of a car, accelerating at uniform acceleration a between two points, is given by $v = u + at$, where u is its velocity when passing the first point and t is the time taken to pass between the two points. If $v = 21$ m/s when $t = 3.5$ s and $v = 33$ m/s when $t = 6.1$ s, use determinants to find the values of u and a, each correct to 4 significant figures.

Substituting the given values in $v = u + at$ gives:

$$21 = u + 3.5a \quad (1)$$
$$33 = u + 6.1a \quad (2)$$

(i) The equations are written in the form

$$a_1x + b_1y + c_1 = 0,$$
i.e. $u + 3.5a - 21 = 0$
and $u + 6.1a - 33 = 0$

(ii) The solution is given by

$$\frac{u}{D_u} = \frac{-a}{D_a} = \frac{1}{D}$$

where D_u is the determinant of coefficients left when the u column is covered up,

i.e. $$D_u = \begin{vmatrix} 3.5 & -21 \\ 6.1 & -33 \end{vmatrix}$$
$$= (3.5)(-33) - (-21)(6.1)$$
$$= 12.6$$

Similarly, $$D_a = \begin{vmatrix} 1 & -21 \\ 1 & -33 \end{vmatrix}$$
$$= (1)(-33) - (-21)(1)$$
$$= -12$$

and $$D = \begin{vmatrix} 1 & 3.5 \\ 1 & 6.1 \end{vmatrix}$$
$$= (1)(6.1) - (3.5)(1) = 2.6$$

Thus $$\frac{u}{12.6} = \frac{-a}{-12} = \frac{1}{26}$$

i.e. $$\boldsymbol{u = \frac{12.6}{2.6} = 4.846\,m/s}$$

and $$\boldsymbol{a = \frac{12}{2.6} = 4.615\,m/s^2},$$

each correct to 4 significant figures.

Problem 5. Applying Kirchhoff's laws to an electric circuit results in the following equations:

$$(9 + j12)I_1 - (6 + j8)I_2 = 5$$
$$-(6 + j8)I_1 + (8 + j3)I_2 = (2 + j4)$$

Solve the equations for I_1 and I_2

Following the procedure:

(i) $(9+j12)I_1-(6+j8)I_2-5=0$

$-(6+j8)I_1+(8+j3)I_2-(2+j4)=0$

(ii) $$\frac{I_1}{\begin{vmatrix}-(6+j8) & -5\\ (8+j3) & -(2+j4)\end{vmatrix}}$$

$$=\frac{-I_2}{\begin{vmatrix}(9+j12) & -5\\ -(6+j8) & -(2+j4)\end{vmatrix}}$$

$$=\frac{1}{\begin{vmatrix}(9+j12) & -(6+j8)\\ -(6+j8) & (8+j3)\end{vmatrix}}$$

$$\frac{I_1}{(-20+j40)+(40+j15)}=\frac{-I_2}{(30-j60)-(30+j40)}=\frac{1}{(36+j123)-(-28+j96)}$$

$$\frac{I_1}{20+j55}=\frac{-I_2}{-j100}=\frac{1}{64+j27}$$

Hence $\boldsymbol{I_1}=\dfrac{20+j55}{64+j27}=\dfrac{58.52\angle 70.02^\circ}{69.46\angle 22.87^\circ}=\mathbf{0.84\angle 47.15^\circ A}$

and $\boldsymbol{I_2}=\dfrac{100\angle 90^\circ}{69.46\angle 22.87^\circ}=\mathbf{1.44\angle 67.13^\circ A}$

(b) When solving simultaneous equations in **three unknowns using determinants:**

(i) Write the equations in the form

$$a_1x+b_1y+c_1z+d_1=0$$
$$a_2x+b_2y+c_2z+d_2=0$$
$$a_3x+b_3y+c_3z+d_3=0$$

and then

(ii) the solution is given by

$$\frac{x}{D_x}=\frac{-y}{D_y}=\frac{z}{D_z}=\frac{-1}{D}$$

where D_x is $\begin{vmatrix}b_1 & c_1 & d_1\\ b_2 & c_2 & d_2\\ b_3 & c_3 & d_3\end{vmatrix}$

i.e. the determinant of the coefficients obtained by covering up the x column.

D_y is $\begin{vmatrix}a_1 & c_1 & d_1\\ a_2 & c_2 & d_2\\ a_3 & c_3 & d_3\end{vmatrix}$

i.e., the determinant of the coefficients obtained by covering up the y column.

D_z is $\begin{vmatrix}a_1 & b_1 & d_1\\ a_2 & b_2 & d_2\\ a_3 & b_3 & d_3\end{vmatrix}$

i.e. the determinant of the coefficients obtained by covering up the z column.

and D is $\begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}$

i.e. the determinant of the coefficients obtained by covering up the constants column.

Problem 6. A d.c. circuit comprises three closed loops. Applying Kirchhoff's laws to the closed loops gives the following equations for current flow in milliamperes:

$$2I_1+3I_2-4I_3=26$$
$$I_1-5I_2-3I_3=-87$$
$$-7I_1+2I_2+6I_3=12$$

Use determinants to solve for I_1, I_2 and I_3.

(i) Writing the equations in the $a_1x+b_1y+c_1z+d_1=0$ form gives:

$$2I_1+3I_2-4I_3-26=0$$
$$I_1-5I_2-3I_3+87=0$$
$$-7I_1+2I_2+6I_3-12=0$$

(ii) the solution is given by

$$\frac{I_1}{D_{I_1}}=\frac{-I_2}{D_{I_2}}=\frac{I_3}{D_{I_3}}=\frac{-1}{D}$$

where D_{I_1} is the determinant of coefficients obtained by covering up the I_1 column, i.e.

$$D_{I_1} = \begin{vmatrix} 3 & -4 & -26 \\ -5 & -3 & 87 \\ 2 & 6 & -12 \end{vmatrix}$$

$$= (3)\begin{vmatrix} -3 & 87 \\ 6 & -12 \end{vmatrix} - (-4)\begin{vmatrix} -5 & 87 \\ 2 & -12 \end{vmatrix} + (-26)\begin{vmatrix} -5 & -3 \\ 2 & 6 \end{vmatrix}$$

$$= 3(-486) + 4(-114) - 26(-24)$$

$$= \mathbf{-1290}$$

$$D_{I_2} = \begin{vmatrix} 2 & -4 & -26 \\ 1 & -3 & 87 \\ -7 & 6 & -12 \end{vmatrix}$$

$$= (2)(36 - 522) - (-4)(-12 + 609) + (-26)(6 - 21)$$

$$= -972 + 2388 + 390$$

$$= \mathbf{1806}$$

$$D_{I_3} = \begin{vmatrix} 2 & 3 & -26 \\ 1 & -5 & 87 \\ -7 & 2 & -12 \end{vmatrix}$$

$$= (2)(60 - 174) - (3)(-12 + 609) + (-26)(2 - 35)$$

$$= -228 - 1791 + 858 = \mathbf{-1161}$$

and $$D = \begin{vmatrix} 2 & 3 & -4 \\ 1 & -5 & -3 \\ -7 & 2 & 6 \end{vmatrix}$$

$$= (2)(-30 + 6) - (3)(6 - 21) + (-4)(2 - 35)$$

$$= -48 + 45 + 132 = \mathbf{129}$$

Thus

$$\frac{I_1}{-1290} = \frac{-I_2}{1806} = \frac{I_3}{-1161} = \frac{-1}{129}$$

giving

$$\boldsymbol{I_1} = \frac{-1290}{-129} = \mathbf{10\,mA},$$

$$\boldsymbol{I_2} = \frac{1806}{129} = \mathbf{14\,mA}$$

and $$\boldsymbol{I_3} = \frac{1161}{129} = \mathbf{9\,mA}$$

Now try the following exercise

Exercise 99 Further problems on solving simultaneous equations using determinants

In Problems 1 to 5 use **determinants** to solve the simultaneous equations given.

1. $3x - 5y = -17.6$
   $7y - 2x - 22 = 0$
   $[x = -1.2, y = 2.8]$

2. $2.3m - 4.4n = 6.84$
   $8.5n - 6.7m = 1.23$
   $[m = -6.4, n = -4.9]$

3. $3x + 4y + z = 10$
   $2x - 3y + 5z + 9 = 0$
   $x + 2y - z = 6$
   $[x = 1, y = 2, z = -1]$

4. $1.2p - 2.3q - 3.1r + 10.1 = 0$
   $4.7p + 3.8q - 5.3r - 21.5 = 0$
   $3.7p - 8.3q + 7.4r + 28.1 = 0$
   $[p = 1.5, q = 4.5, r = 0.5]$

5. $\frac{x}{2} - \frac{y}{3} + \frac{2z}{5} = -\frac{1}{20}$
   $\frac{x}{4} + \frac{2y}{3} - \frac{z}{2} = \frac{19}{40}$
   $x + y - z = \frac{59}{60}$
   $\left[x = \frac{7}{20}, y = \frac{17}{40}, z = -\frac{5}{24}\right]$

6. In a system of forces, the relationship between two forces F_1 and F_2 is given by:
   $5F_1 + 3F_2 + 6 = 0$
   $3F_1 + 5F_2 + 18 = 0$
   Use determinants to solve for F_1 and F_2.
   $[F_1 = 1.5, F_2 = -4.5]$

7. Applying mesh-current analysis to an a.c. circuit results in the following equations:
   $(5 - j4)I_1 - (-j4)I_2 = 100\angle 0°$
   $(4 + j3 - j4)I_2 - (-j4)I_1 = 0$
   Solve the equations for I_1 and I_2.
   $[I_1 = 10.77\angle 19.23°A, I_2 = 10.45\angle -56.73°A]$

8. Kirchhoff's laws are used to determine the current equations in an electrical network and show that

$$i_1 + 8i_2 + 3i_3 = -31$$
$$3i_1 - 2i_2 + i_3 = -5$$
$$2i_1 - 3i_2 + 2i_3 = 6$$

Use determinants to find the values of i_1, i_2 and i_3. $[i_1 = -5, i_2 = -4, i_3 = 2]$

9. The forces in three members of a framework are F_1, F_2 and F_3. They are related by the simultaneous equations shown below.

$$1.4F_1 + 2.8F_2 + 2.8F_3 = 5.6$$
$$4.2F_1 - 1.4F_2 + 5.6F_3 = 35.0$$
$$4.2F_1 + 2.8F_2 - 1.4F_3 = -5.6$$

Find the values of F_1, F_2 and F_3 using determinants.

$[F_1 = 2,\ F_2 = -3,\ F_3 = 4]$

10. Mesh-current analysis produces the following three equations:

$$20\angle 0° = (5 + 3 - j4)I_1 - (3 - j4)I_2$$
$$10\angle 90° = (3 - j4 + 2)I_2 - (3 - j4)I_1 - 2I_3$$
$$-15\angle 0° - 10\angle 90° = (12 + 2)I_3 - 2I_2$$

Solve the equations for the loop currents I_1, I_2 and I_3.

$$\begin{bmatrix} I_1 = 3.317\angle 22.57° \text{ A} \\ I_2 = 1.963\angle 40.97° \text{ A} \\ I_3 = 1.010\angle -148.32° \text{ A} \end{bmatrix}$$

23.3 Solution of simultaneous equations using Cramers rule

Cramers rule states that if

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

then $\boldsymbol{x = \dfrac{D_x}{D}}$, $\boldsymbol{y = \dfrac{D_y}{D}}$ and $\boldsymbol{z = \dfrac{D_z}{D}}$

where $D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$D_x = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$

i.e. the x-column has been replaced by the R.H.S. b column,

$D_y = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$

i.e. the y-column has been replaced by the R.H.S. b column,

$D_z = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$

i.e. the z-column has been replaced by the R.H.S. b column.

Problem 7. Solve the following simultaneous equations using Cramers rule.

$$x + y + z = 4$$
$$2x - 3y + 4z = 33$$
$$3x - 2y - 2z = 2$$

(This is the same as Problem 2 and a comparison of methods may be made). Following the above method:

$$\boldsymbol{D} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & -2 & -2 \end{vmatrix}$$
$$= 1(6 - (-8)) - 1((-4) - 12) + 1((-4) - (-9)) = 14 + 16 + 5 = \mathbf{35}$$

$$\boldsymbol{D_x} = \begin{vmatrix} 4 & 1 & 1 \\ 33 & -3 & 4 \\ 2 & -2 & -2 \end{vmatrix}$$
$$= 4(6 - (-8)) - 1((-66) - 8) + 1((-66) - (-6)) = 56 + 74 - 60 = \mathbf{70}$$

$$\boldsymbol{D_y} = \begin{vmatrix} 1 & 4 & 1 \\ 2 & 33 & 4 \\ 3 & 2 & -2 \end{vmatrix}$$
$$= 1((-66) - 8) - 4((-4) - 12) + 1(4 - 99)$$
$$= -74 + 64 - 95 = \mathbf{-105}$$

$$\boldsymbol{D_z} = \begin{vmatrix} 1 & 1 & 4 \\ 2 & -3 & 33 \\ 3 & -2 & 2 \end{vmatrix}$$
$$= 1((-6) - (-66)) - 1(4 - 99) + 4((-4) - (-9)) = 60 + 95 + 20 = \mathbf{175}$$

Hence

$$x = \frac{D_x}{D} = \frac{70}{35} = \mathbf{2}, \quad y = \frac{D_y}{D} = \frac{-105}{35} = \mathbf{-3}$$

and $z = \dfrac{D_z}{D} = \dfrac{175}{35} = \mathbf{5}$

Now try the following exercise

Exercise 100 Further problems on solving simultaneous equations using Cramers rule

1. Repeat problems 3, 4, 5, 7 and 8 of Exercise 98 on page 241, using Cramers rule.
2. Repeat problems 3, 4, 8 and 9 of Exercise 99 on page 244, using Cramers rule.

23.4 Solution of simultaneous equations using the Gaussian elimination method

Consider the following simultaneous equations:

$$x + y + z = 4 \quad (1)$$
$$2x - 3y + 4z = 33 \quad (2)$$
$$3x - 2y - 2z = 2 \quad (3)$$

Leaving equation (1) as it is gives:

$$x + y + z = 4 \quad (1)$$

Equation (2) $-$ 2 $\times$ equation (1) gives:

$$0 - 5y + 2z = 25 \quad (2')$$

and equation (3) $-$ 3 $\times$ equation (1) gives:

$$0 - 5y - 5z = -10 \quad (3')$$

Leaving equations (1) and (2′) as they are gives:

$$x + y + z = 4 \quad (1)$$
$$0 - 5y + 2z = 25 \quad (2')$$

Equation (3′) $-$ equation (2′) gives:

$$0 + 0 - 7z = -35 \quad (3'')$$

By appropriately manipulating the three original equations we have deliberately obtained zeros in the positions shown in equations (2′) and (3″).

Working backwards, from equation (3″),

$$z = \frac{-35}{-7} = \mathbf{5},$$

from equation (2′),

$$-5y + 2(5) = 25,$$

from which,

$$y = \frac{25 - 10}{-5} = \mathbf{-3}$$

and from equation (1),

$$x + (-3) + 5 = 4,$$

from which,

$$x = 4 + 3 - 5 = \mathbf{2}$$

(This is the same example as Problems 2 and 7, and a comparison of methods can be made). The above method is known as the **Gaussian elimination method**.

We conclude from the above example that if

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

the three-step **procedure** to solve simultaneous equations in three unknowns using the **Gaussian elimination method** is:

1. Equation (2) $-\dfrac{a_{21}}{a_{11}} \times$ equation (1) to form equation (2′) and equation (3) $-\dfrac{a_{31}}{a_{11}} \times$ equation (1) to form equation (3′).
2. Equation (3′) $-\dfrac{a_{32}}{a_{22}} \times$ equation (2′) to form equation (3″).
3. Determine z from equation (3″), then y from equation (2′) and finally, x from equation (1).

Problem 8. A d.c. circuit comprises three closed loops. Applying Kirchhoff's laws to the closed loops gives the following equations for current flow in milliamperes:

$$2I_1 + 3I_2 - 4I_3 = 26 \quad (1)$$
$$I_1 - 5I_2 - 3I_3 = -87 \quad (2)$$
$$-7I_1 + 2I_2 + 6I_3 = 12 \quad (3)$$

Use the Gaussian elimination method to solve for I_1, I_2 and I_3.

(This is the same example as Problem 6 on page 243, and a comparison of methods may be made)

Following the above procedure:

1. $2I_1 + 3I_2 - 4I_3 = 26$ (1)

   Equation (2) $-\frac{1}{2} \times$ equation (1) gives:

   $0 - 6.5I_2 - I_3 = -100$ (2′)

   Equation (3) $-\frac{-7}{2} \times$ equation (1) gives:

   $0 + 12.5I_2 - 8I_3 = 103$ (3′)

2. $2I_1 + 3I_2 - 4I_3 = 26$ (1)

   $0 - 6.5I_2 - I_3 = -100$ (2′)

   Equation (3′) $-\frac{12.5}{-6.5} \times$ equation (2′) gives:

   $0 + 0 - 9.923I_3 = -89.308$ (3″)

3. From equation (3″),

   $$\boldsymbol{I_3} = \frac{-89.308}{-9.923} = \mathbf{9\,mA},$$

   from equation (2′), $-6.5I_2 - 9 = -100$,

   from which, $\boldsymbol{I_2} = \frac{-100+9}{-6.5} = \mathbf{14\,mA}$

   and from equation (1), $2I_1 + 3(14) - 4(9) = 26$,

   from which, $\boldsymbol{I_1} = \frac{26 - 42 + 36}{2} = \frac{20}{2}$

   $$= \mathbf{10\,mA}$$

Now try the following exercise

Exercise 101 Further problems on solving simultaneous equations using Gaussian elimination

1. In a mass-spring-damper system, the acceleration $\ddot{x}$ m/s^2, velocity $\dot{x}$ m/s and displacement x m are related by the following simultaneous equations:

   $$6.2\ddot{x} + 7.9\dot{x} + 12.6x = 18.0$$
   $$7.5\ddot{x} + 4.8\dot{x} + 4.8x = 6.39$$
   $$13.0\ddot{x} + 3.5\dot{x} - 13.0x = -17.4$$

   By using Gaussian elimination, determine the acceleration, velocity and displacement for the system, correct to 2 decimal places.

   $[\ddot{x} = -0.30, \dot{x} = 0.60, x = 1.20]$

2. The tensions, T_1, T_2 and T_3 in a simple framework are given by the equations:

   $$5T_1 + 5T_2 + 5T_3 = 7.0$$
   $$T_1 + 2T_2 + 4T_3 = 2.4$$
   $$4T_1 + 2T_2 \qquad = 4.0$$

   Determine T_1, T_2 and T_3 using Gaussian elimination.

   $[T_1 = 0.8, T_2 = 0.4, T_3 = 0.2]$

3. Repeat problems 3, 4, 5, 7 and 8 of Exercise 98 on page 241, using the Gaussian elimination method.

4. Repeat problems 3, 4, 8 and 9 of Exercise 99 on page 244, using the Gaussian elimination method.

Revision Test 7

This Revision Test covers the material contained in Chapters 20 to 23. *The marks for each question are shown in brackets at the end of each question.*

1. Solve the quadratic equation $x^2 - 2x + 5 = 0$ and show the roots on an Argand diagram. (9)

2. If $Z_1 = 2 + j5$, $Z_2 = 1 - j3$ and $Z_3 = 4 - j$ determine, in both Cartesian and polar forms, the value of $\dfrac{Z_1 Z_2}{Z_1 + Z_2} + Z_3$, correct to 2 decimal places. (9)

3. Three vectors are represented by $\boldsymbol{A}$, $4.2\angle 45°$, $\boldsymbol{B}$, $5.5\angle -32°$ and $\boldsymbol{C}$, $2.8\angle 75°$. Determine in polar form the resultant $\boldsymbol{D}$, where $\boldsymbol{D = B + C - A}$. (8)

4. Two impedances, $Z_1 = (2 + j7)$ ohms and $Z_2 = (3 - j4)$ ohms, are connected in series to a supply voltage V of $150\angle 0°$ V. Determine the magnitude of the current I and its phase angle relative to the voltage. (6)

5. Determine in both polar and rectangular forms:

   (a) $[2.37\angle 35°]^4$ (b) $[3.2 - j4.8]^5$

   (c) $\sqrt{[-1 - j3]}$ (15)

In questions 6 to 10, the matrices stated are:

$$A = \begin{pmatrix} -5 & 2 \\ 7 & -8 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 6 \\ -3 & -4 \end{pmatrix}$$

$$C = \begin{pmatrix} j3 & (1 + j2) \\ (-1 - j4) & -j2 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & -1 & 3 \\ -5 & 1 & 0 \\ 4 & -6 & 2 \end{pmatrix} \quad E = \begin{pmatrix} -1 & 3 & 0 \\ 4 & -9 & 2 \\ -5 & 7 & 1 \end{pmatrix}$$

6. Determine $A \times B$. (4)

7. Calculate the determinant of matrix C. (4)

8. Determine the inverse of matrix A. (4)

9. Determine $E \times D$. (9)

10. Calculate the determinant of matrix D. (6)

11. Solve the following simultaneous equations:

$$4x - 3y = 17$$
$$x + y + 1 = 0$$

using matrices. (6)

12. Use determinants to solve the following simultaneous equations:

$$4x + 9y + 2z = 21$$
$$-8x + 6y - 3z = 41$$
$$3x + y - 5z = -73$$ (10)

13. The simultaneous equations representing the currents flowing in an unbalanced, three-phase, star-connected, electrical network are as follows:

$$2.4I_1 + 3.6I_2 + 4.8I_3 = 1.2$$
$$-3.9I_1 + 1.3I_2 - 6.5I_3 = 2.6$$
$$1.7I_1 + 11.9I_2 + 8.5I_3 = 0$$

Using matrices, solve the equations for I_1, I_2 and I_3. (10)

Chapter 24

Vectors

24.1 Introduction

This chapter initially explains the difference between scalar and vector quantities and shows how a vector is drawn and represented.

Any object that is acted upon by an external force will respond to that force by moving in the line of the force. However, if two or more forces act simultaneously, the result is more difficult to predict; the ability to add two or more vectors then becomes important.

This chapter thus shows how vectors are added and subtracted, both by drawing and by calculation, and finding the resultant of two or more vectors has many uses in engineering. (Resultant means the single vector which would have the same effect as the individual vectors.) Relative velocities and vector $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ notation are also briefly explained.

24.2 Scalars and vectors

The time taken to fill a water tank may be measured as, say, 50 s. Similarly, the temperature in a room may be measured as, say, 16°C, or the mass of a bearing may be measured as, say, 3 kg.

Quantities such as time, temperature and mass are entirely defined by a numerical value and are called **scalars** or **scalar quantities**.

Not all quantities are like this. Some are defined by more than just size; some also have direction. For example, the velocity of a car is 90 km/h due west, or a force of 20 N acts vertically downwards, or an acceleration of $10\,\text{m/s}^2$ acts at 50° to the horizontal.

Quantities such as velocity, force and acceleration, which **have both a magnitude and a direction**, are called **vectors**.

Now try the following exercise

Exercise 102 Further problems on scalar and vector quantities

1. State the difference between scalar and vector quantities.

In problems 2 to 9, state whether the quantities given are scalar (S) or vector (V) – answers below.

2. A temperature of 70°C
3. $5\,\text{m}^3$ volume
4. A downward force of 20 N
5. 500 J of work
6. $30\,\text{cm}^2$ area
7. A south-westerly wind of 10 knots
8. 50 m distance
9. An acceleration of $15\,\text{m/s}^2$ at 60° to the horizontal

[Answers: 2. S 3. S 4. V 5. S 6. S 7. V 8. S 9. V]

24.3 Drawing a vector

A vector quantity can be represented graphically by a line, drawn so that:

(a) the **length** of the line denotes the magnitude of the quantity, and

(b) the **direction** of the line denotes the direction in which the vector quantity acts.

An arrow is used to denote the sense, or direction, of the vector.
The arrow end of a vector is called the 'nose' and the other end the 'tail'.
For example, a force of 9 N acting at 45° to the horizontal is shown in Fig. 24.1.
Note that an angle of **+45°** is drawn from the horizontal and moves **anticlockwise**.

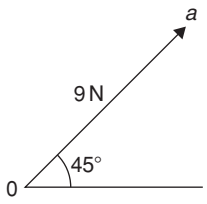


Figure 24.1

A velocity of 20 m/s at **−60°** is shown in Fig. 24.2.
Note that an angle of **−60°** is drawn from the horizontal and moves **clockwise**.

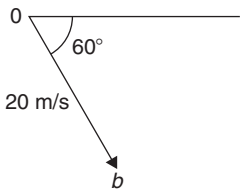


Figure 24.2

Representing a vector

There are a number of ways of representing vector quantities. These include:

1. Using **bold print**
2. $\overrightarrow{AB}$ where an arrow above two capital letters denotes the sense of direction, where A is the starting point and B the end point of the vector
3. $\overline{AB}$ or $\overline{a}$ i.e. a line over the top of letters
4. $\underline{a}$ i.e. an underlined letter

The force of 9 N at 45° shown in Fig. 24.1 may be represented as:

$$\mathbf{0a} \quad \text{or} \quad \overrightarrow{0a} \quad \text{or} \quad \overline{0a}$$

The magnitude of the force is $0a$
Similarly, the velocity of 20 m/s at −60° shown in Fig. 24.2 may be represented as:

$$\mathbf{0b} \quad \text{or} \quad \overrightarrow{0b} \quad \text{or} \quad \overline{0b}$$

The magnitude of the velocity is $0b$

In this chapter a vector quantity is denoted by **bold print**.

24.4 Addition of vectors by drawing

Adding two or more vectors by drawing assumes that a ruler, pencil and protractor are available. Results obtained by drawing are naturally not as accurate as those obtained by calculation.

(a) Nose-to-tail method

Two force vectors, F_1 and F_2, are shown in Fig. 24.3. When an object is subjected to more than one force, the resultant of the forces is found by the addition of vectors.

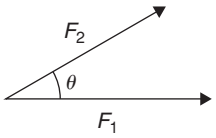


Figure 24.3

To add forces F_1 and F_2:

(i) Force F_1 is drawn to scale horizontally, shown as $\mathbf{0}\boldsymbol{a}$ in Fig. 24.4.

(ii) From the nose of F_1, force F_2 is drawn at angle θ to the horizontal, shown as $\boldsymbol{ab}$.

(iii) The resultant force is given by length $\mathbf{0}\boldsymbol{b}$, which may be measured.

This procedure is called the '**nose-to-tail**' or '**triangle**' method.

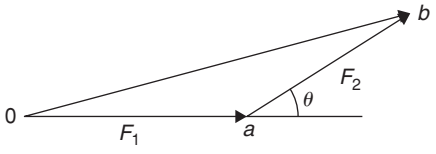


Figure 24.4

(b) Parallelogram method

To add the two force vectors, F_1 and F_2, of Fig. 24.3:

(i) A line cb is constructed which is parallel to and equal in length to $\mathbf{0}\boldsymbol{a}$ (see Fig. 24.5).

(ii) A line ab is constructed which is parallel to and equal in length to $\mathbf{0}\boldsymbol{c}$.

(iii) The resultant force is given by the diagonal of the parallelogram, i.e. length $\mathbf{0}\boldsymbol{b}$.

This procedure is called the '**parallelogram**' method.

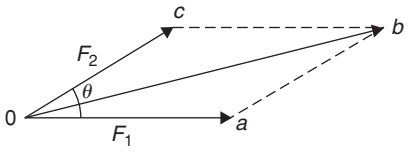


Figure 24.5

Problem 1. A force of 5 N is inclined at an angle of 45° to a second force of 8 N, both forces acting at a point. Find the magnitude of the resultant of these two forces and the direction of the resultant with respect to the 8 N force by:

(a) the 'nose-to-tail' method, and (b) the 'parallelogram' method.

The two forces are shown in Fig. 24.6. (Although the 8 N force is shown horizontal, it could have been drawn in any direction.)

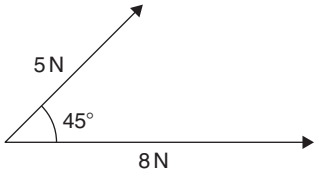


Figure 24.6

(a) 'Nose-to tail' method

(i) The 8 N force is drawn horizontally 8 units long, shown as $\mathbf{0}\boldsymbol{a}$ in Fig. 24.7

(ii) From the nose of the 8 N force, the 5 N force is drawn 5 units long at an angle of 45° to the horizontal, shown as $\boldsymbol{ab}$

(iii) The resultant force is given by length $0b$ and is measured as **12 N** and angle θ is measured as **17°**.

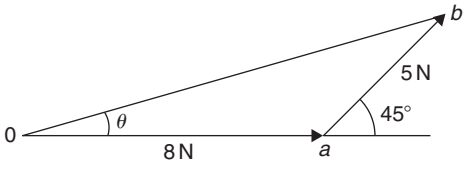


Figure 24.7

(b) 'Parallelogram' method

(i) In Fig. 24.8, a line is constructed which is parallel to and equal in length to the 8 N force

(ii) A line is constructed which is parallel to and equal in length to the 5 N force

(iii) The resultant force is given by the diagonal of the parallelogram, i.e. length $0b$, and is measured as **12 N** and angle θ is measured as **17°**.

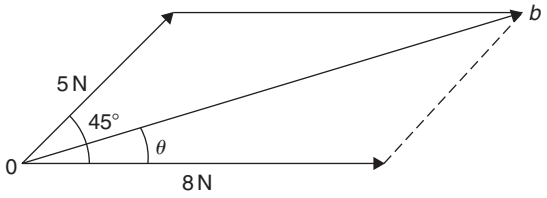


Figure 24.8

Thus, **the resultant of the two force vectors in Fig. 24.6 is 12 N at 17° to the 8 N force.**

Problem 2. Forces of 15 N and 10 N are at an angle of 90° to each other as shown in Fig. 24.9. Find, by drawing, the magnitude of the resultant of these two forces and the direction of the resultant with respect to the 15 N force.

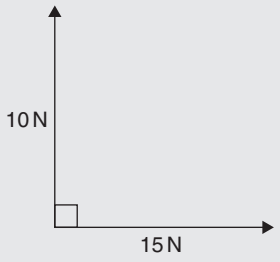


Figure 24.9

Using the 'nose-to-tail' method:

(i) The 15 N force is drawn horizontally 15 units long as shown in Fig. 24.10

(ii) From the nose of the 15 N force, the 10 N force is drawn 10 units long at an angle of 90° to the horizontal as shown

(iii) The resultant force is shown as R and is measured as **18 N** and angle θ is measured as **34°**.

Thus, **the resultant of the two force vectors is 18 N at 34° to the 15 N force.**

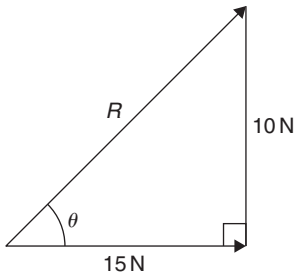


Figure 24.10

Problem 3. Velocities of 10 m/s, 20 m/s and 15 m/s act as shown in Fig. 24.11. Determine, by drawing, the magnitude of the resultant velocity and its direction relative to the horizontal.

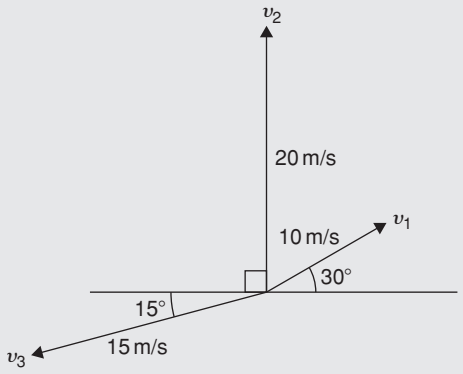


Figure 24.11

When more than two vectors are being added the 'nose-to-tail' method is used.

The order in which the vectors are added does not matter. In this case the order taken is v_1, then v_2, then v_3. However, if a different order is taken the same result will occur.

(i) v_1 is drawn 10 units long at an angle of 30° to the horizontal, shown as **0*a*** in Fig. 24.12

(ii) From the nose of v_1, v_2 is drawn 20 units long at an angle of 90° to the horizontal, shown as ***ab***

(iii) From the nose of v_2, v_3 is drawn 15 units long at an angle of 195° to the horizontal, shown as ***br***

(iv) The resultant velocity is given by length **0*r*** and is measured as **22 m/s** and the angle measured to the horizontal is **105°**.

Thus, **the resultant of the three velocities is 22 m/s at 105° to the horizontal.**

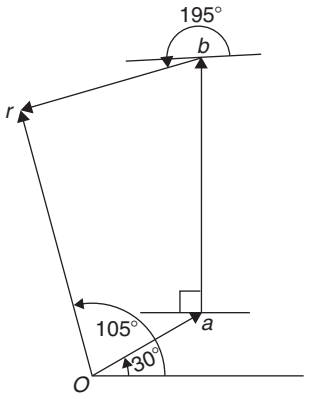


Figure 24.12

Worked Problems 1 to 3 have demonstrated how vectors are added to determine their resultant and their direction. However, drawing to scale is time-consuming and not highly accurate. The following sections demonstrate how to determine resultant vectors by calculation using horizontal and vertical components and, where possible, by Pythagoras's theorem.

24.5 Resolving vectors into horizontal and vertical components

A force vector ***F*** is shown in Fig. 24.13 at angle θ to the horizontal. Such a vector can be resolved into two components such that the vector addition of the components is equal to the original vector.

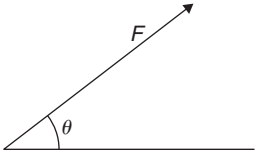


Figure 24.13

The two components usually taken are a **horizontal component** and a **vertical component**.

If a right-angled triangle is constructed as shown in Fig. 24.14, then $0a$ is called the horizontal component of F and ab is called the vertical component of F.

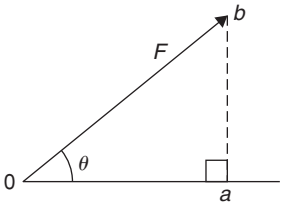


Figure 24.14

From trigonometry (see Chapter 11),

$$\cos\theta = \frac{0a}{0b} \quad \text{from which,} \quad 0a = 0b\cos\theta = F\cos\theta$$

i.e. **the horizontal component of $F = F\cos\theta$**

and $$\sin\theta = \frac{ab}{0b} \quad \text{from which,} \quad ab = 0b\sin\theta = F\sin\theta$$

i.e. **the vertical component of $F = F\sin\theta$**

Problem 4. Resolve the force vector of 50 N at an angle of 35° to the horizontal into its horizontal and vertical components.

The **horizontal component** of the 50 N force, $0a = 50\cos 35° = \mathbf{40.96\,N}$
The **vertical component** of the 50 N force, $ab = 50\sin 35° = \mathbf{28.68\,N}$
The horizontal and vertical components are shown in Fig. 24.15.

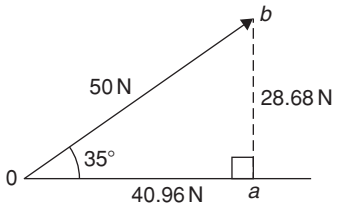


Figure 24.15

(Checking: by Pythagoras, $0b = \sqrt{40.96^2 + 28.68^2}$
$= 50\,\text{N}$
and $\theta = \tan^{-1}\left(\frac{28.68}{40.96}\right) = 35°$
Thus, the vector addition of components 40.96 N and 28.68 N is 50 N at 35°)

Problem 5. Resolve the velocity vector of 20 m/s at an angle of −30° to the horizontal into horizontal and vertical components.

The **horizontal component** of the 20 m/s velocity, $0a = 20\cos(-30°) = \mathbf{17.32\,m/s}$
The **vertical component** of the 20 m/s velocity, $ab = 20\sin(-30°) = \mathbf{-10\,m/s}$
The horizontal and vertical components are shown in Fig. 24.16.

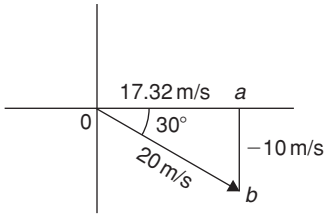


Figure 24.16

Problem 6. Resolve the displacement vector of 40 m at an angle of 120° into horizontal and vertical components.

The **horizontal component** of the 40 m displacement, $0a = 40\cos 120° = \mathbf{-20.0\,m}$
The **vertical component** of the 40 m displacement, $ab = 40\sin 120° = \mathbf{34.64\,m}$
The horizontal and vertical components are shown in Fig. 24.17.

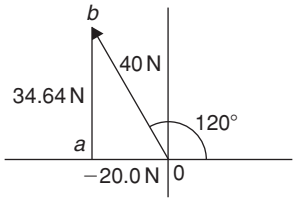


Figure 24.17

24.6 Addition of vectors by calculation

Two force vectors, F_1 and F_2, are shown in Fig. 24.18, F_1 being at an angle of θ_1 and F_2 being at an angle of θ_2.

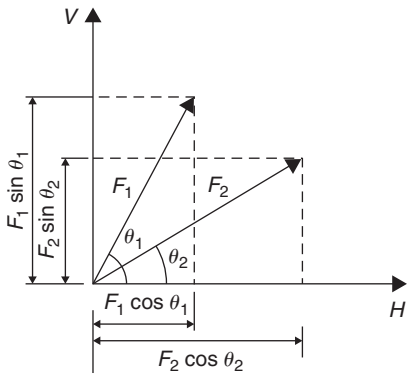


Figure 24.18

A method of adding two vectors together is to use horizontal and vertical components.

The horizontal component of force F_1 is $F_1\cos\theta_1$ and the horizontal component of force F_2 is $F_2\cos\theta_2$

The total horizontal component of the two forces, $\boldsymbol{H = F_1\cos\theta_1 + F_2\cos\theta_2}$

The vertical component of force F_1 is $F_1\sin\theta_1$ and the vertical component of force F_2 is $F_2\sin\theta_2$

The total vertical component of the two forces, $\boldsymbol{V = F_1\sin\theta_1 + F_2\sin\theta_2}$

Since we have H and V, the resultant of F_1 and F_2 is obtained by using the theorem of Pythagoras. From Fig. 24.19, $0b^2 = H^2 + V^2$

i.e. $\textbf{resultant} = \sqrt{\boldsymbol{H^2 + V^2}}$ at an angle given by $\boldsymbol{\theta = \tan^{-1}\left(\frac{V}{H}\right)}$

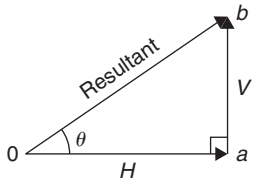


Figure 24.19

Problem 7. A force of 5 N is inclined at an angle of 45° to a second force of 8 N, both forces acting at a point. Calculate the magnitude of the resultant of these two forces and the direction of the resultant with respect to the 8 N force.

The two forces are shown in Fig. 24.20.

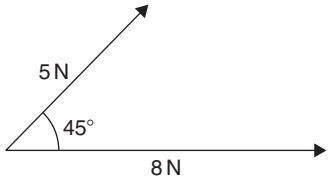


Figure 24.20

The horizontal component of the 8 N force is $8\cos0°$ and the horizontal component of the 5 N force is $5\cos45°$

The total horizontal component of the two forces,

$$\boldsymbol{H} = 8\cos0° + 5\cos45° = 8 + 3.5355 = \mathbf{11.5355}$$

The vertical component of the 8 N force is $8\sin0°$ and the vertical component of the 5 N force is $5\sin45°$

The total vertical component of the two forces,

$$\boldsymbol{V} = 8\sin0° + 5\sin45° = 0 + 3.5355 = \mathbf{3.5355}$$

From Fig. 24.21, magnitude of resultant vector

$$= \sqrt{H^2 + V^2}$$

$$= \sqrt{11.5355^2 + 3.5355^2} = \mathbf{12.07\,N}$$

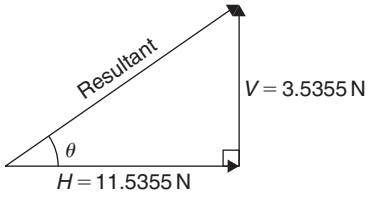


Figure 24.21

The direction of the resultant vector,

$$\theta = \tan^{-1}\left(\frac{V}{H}\right) = \tan^{-1}\left(\frac{3.5355}{11.5355}\right)$$

$$= \tan^{-1}0.30648866\ldots = \mathbf{17.04°}$$

Thus, **the resultant of the two forces is a single vector of 12.07 N at 17.04° to the 8 N vector.**

Perhaps an easier and quicker method of calculating the magnitude and direction of the resultant is to use **complex numbers** (see Chapter 20).

In this example, the **resultant**

$$= 8\angle0° + 5\angle45°$$

$$= (8\cos0° + j8\sin0°) + (5\cos45° + j5\sin45°)$$

$$= (8 + j0) + (3.536 + j3.536)$$

$$= (11.536 + j3.536)\,\text{N} \quad \text{or} \quad \mathbf{12.07\angle17.04°N}$$

as obtained above using horizontal and vertical components.

Problem 8. Forces of 15 N and 10 N are at an angle of 90° to each other as shown in Fig. 24.22. Calculate the magnitude of the resultant of these two forces and its direction with respect to the 15 N force.

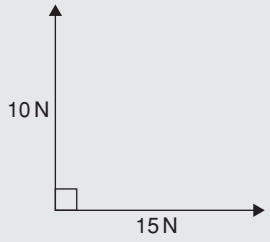


Figure 24.22

The horizontal component of the 15 N force is $15\cos 0°$ and the horizontal component of the 10 N force is $10\cos 90°$
The total horizontal component of the two velocities,

$$\boldsymbol{H} = 15\cos 0° + 10\cos 90° = 15 + 0 = \mathbf{15}$$

The vertical component of the 15 N force is $15\sin 0°$ and the vertical component of the 10 N force is $10\sin 90°$
The total vertical component of the two velocities,

$$\boldsymbol{V} = 15\sin 0° + 10\sin 90° = 0 + 10 = \mathbf{10}$$

Magnitude of resultant vector

$$= \sqrt{H^2 + V^2} = \sqrt{15^2 + 10^2} = \mathbf{18.03\,N}$$

The direction of the resultant vector,

$$\theta = \tan^{-1}\left(\frac{V}{H}\right) = \tan^{-1}\left(\frac{10}{15}\right) = \mathbf{33.69°}$$

Thus, **the resultant of the two forces is a single vector of 18.03 N at 33.69° to the 15 N vector.**

There is an alternative method of calculating the resultant vector in this case.
If we used the triangle method, then the diagram would be as shown in Fig. 24.23.

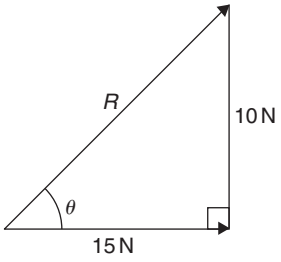


Figure 24.23

Since a right-angled triangle results then we could use **Pythagoras's theorem** without needing to go through the procedure for horizontal and vertical components. In fact, the horizontal and vertical components are 15 N and 10 N respectively.

This is, of course, a special case. Pythagoras **can only be used when there is an angle of 90° between vectors.** This is demonstrated in the next worked problem.

Problem 9. Calculate the magnitude and direction of the resultant of the two acceleration vectors shown in Fig. 24.24.

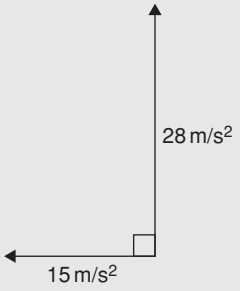


Figure 24.24

The 15 m/s^2 acceleration is drawn horizontally, shown as $\mathbf{0}\boldsymbol{a}$ in Fig. 24.25.
From the nose of the 15 m/s^2 acceleration, the 28 m/s^2 acceleration is drawn at an angle of 90° to the horizontal, shown as $\boldsymbol{ab}$.

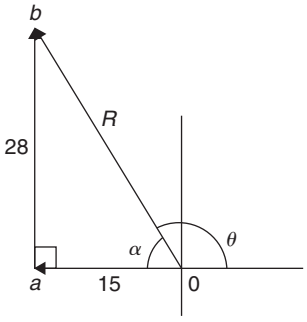


Figure 24.25

The resultant acceleration, R, is given by length $\mathbf{0}\boldsymbol{b}$. Since a right-angled triangle results, the theorem of Pythagoras may be used.

$$0b = \sqrt{15^2 + 28^2} = \mathbf{31.76\,m/s^2}$$

and

$$\alpha = \tan^{-1}\left(\frac{28}{15}\right) = \mathbf{61.82°}$$

Measuring from the horizontal,
$\boldsymbol{\theta} = 180° - 61.82° = \mathbf{118.18°}$

Thus, **the resultant of the two accelerations is a single vector of 31.76 m/s^2 at 118.18° to the horizontal.**

Problem 10. Velocities of 10 m/s, 20 m/s and 15 m/s act as shown in Fig. 24.26. Calculate the magnitude of the resultant velocity and its direction relative to the horizontal.

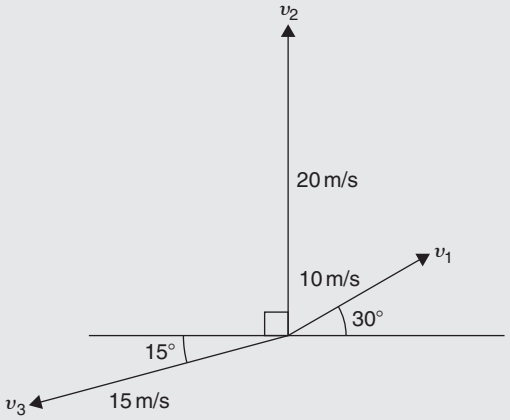


Figure 24.26

The horizontal component of the 10 m/s velocity = $10\cos 30° = 8.660$ m/s,
the horizontal component of the 20 m/s velocity is $20\cos 90° = 0$ m/s,
and the horizontal component of the 15 m/s velocity is $15\cos 195° = -14.489$ m/s.
The total horizontal component of the three velocities,

$$H = 8.660 + 0 - 14.489 = \mathbf{-5.829\,m/s}$$

The vertical component of the 10 m/s velocity = $10\sin 30° = 5$ m/s,
the vertical component of the 20 m/s velocity is $20\sin 90° = 20$ m/s,
and the vertical component of the 15 m/s velocity is $15\sin 195° = -3.882$ m/s.
The total vertical component of the three forces,

$$V = 5 + 20 - 3.882 = \mathbf{21.118\,m/s}$$

From Fig. 24.27, magnitude of resultant vector,

$$R = \sqrt{H^2 + V^2} = \sqrt{5.829^2 + 21.118^2} = \mathbf{21.91\,m/s}$$

The direction of the resultant vector,

$$\alpha = \tan^{-1}\left(\frac{V}{H}\right) = \tan^{-1}\left(\frac{21.118}{5.829}\right) = \mathbf{74.57°}$$

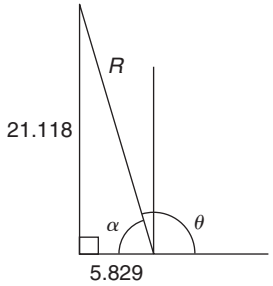


Figure 24.27

Measuring from the horizontal,
$\theta = 180° - 74.57° = \mathbf{105.43°}$
Thus, **the resultant of the three velocities is a single vector of 21.91 m/s at 105.43° to the horizontal.**

Using **complex numbers**, from Fig. 24.26,

$$\begin{aligned}\textbf{resultant} &= 10\angle 30° + 20\angle 90° + 15\angle 195° \\ &= (10\cos 30° + j10\sin 30°) \\ &\quad + (20\cos 90° + j20\sin 90°) \\ &\quad + (15\cos 195° + j15\sin 195°) \\ &= (8.660 + j5.000) + (0 + j20.000) \\ &\quad + (-14.489 - j3.882) \\ &= (-5.829 + j21.118)\,\text{N} \quad \text{or} \\ &\quad \mathbf{21.91\angle 105.43°\,N}\end{aligned}$$

as obtained above using horizontal and vertical components.

The method used to add vectors by calculation will not be specified – the choice is yours, but probably the quickest and easiest method is by using complex numbers.

Now try the following exercise

Exercise 103 Further problems on addition of vectors by calculation

1. A force of 7 N is inclined at an angle of 50° to a second force of 12 N, both forces acting at a point. Calculate magnitude of the

resultant of the two forces, and the direction of the resultant with respect to the 12 N force.

[17.35 N at 18.00° to the 12 N force]

2. Velocities of 5 m/s and 12 m/s act at a point at 90° to each other. Calculate the resultant velocity and its direction relative to the 12 m/s velocity.

[13 m/s at 22.62° to the 12 m/s velocity]

3. Calculate the magnitude and direction of the resultant of the two force vectors shown in Fig. 24.28.

[16.40 N at 37.57° to the 13 N force]

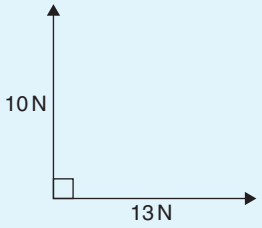


Figure 24.28

4. Calculate the magnitude and direction of the resultant of the two force vectors shown in Fig. 24.29.

[28.43 N at 129.30° to the 18 N force]

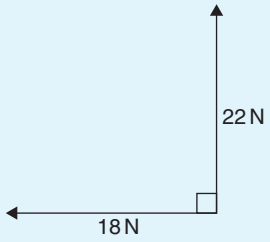


Figure 24.29

5. A displacement vector $\boldsymbol{s_1}$ is 30 m at 0°. A second displacement vector $\boldsymbol{s_2}$ is 12 m at 90°. Calculate magnitude and direction of the resultant vector $\boldsymbol{s_1} + \boldsymbol{s_2}$.

[32.31 m at 21.80° to the 30 m displacement]

6. Three forces of 5 N, 8 N and 13 N act as shown in Fig. 24.30. Calculate the magnitude and direction of the resultant force.

[14.72 N at −14.72° to the 5 N force]

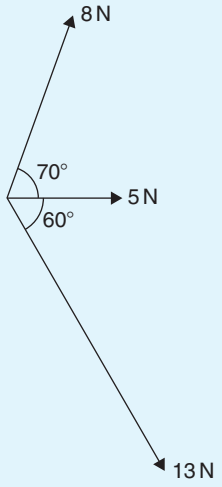


Figure 24.30

7. If velocity $\boldsymbol{v_1} = 25$ m/s at 60° and $\boldsymbol{v_2} = 15$ m/s at −30°, calculate the magnitude and direction of $\boldsymbol{v_1} + \boldsymbol{v_2}$.

[29.15 m/s at 29.04° to the horizontal]

8. Calculate the magnitude and direction of the resultant vector of the force system shown in Fig. 24.31.

[9.28 N at 16.70°]

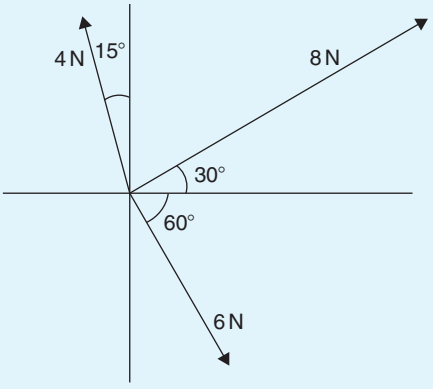


Figure 24.31

9. Calculate the magnitude and direction of the resultant vector of the system shown in Fig. 24.32.

[6.89 m/s at 159.56°]

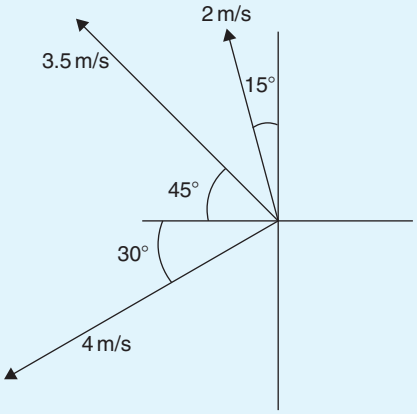


Figure 24.32

10. An object is acted upon by two forces of magnitude 10 N and 8 N at an angle of 60° to each other. Determine the resultant force on the object.
[15.62 N at 26.33° to the 10 N force]

11. A ship heads in a direction of E 20° S at a speed of 20 knots while the current is 4 knots in a direction of N 30° E. Determine the speed and actual direction of the ship.
[21.07 knots, E 9.22° S]

24.7 Vector subtraction

In Fig. 24.33, a force vector $\boldsymbol{F}$ is represented by $\boldsymbol{oa}$. The vector $\boldsymbol{(-oa)}$ can be obtained by drawing a vector from o in the opposite sense to $\boldsymbol{oa}$ but having the same magnitude, shown as $\boldsymbol{ob}$ in Fig. 24.33, i.e. $\boldsymbol{ob = (-oa)}$

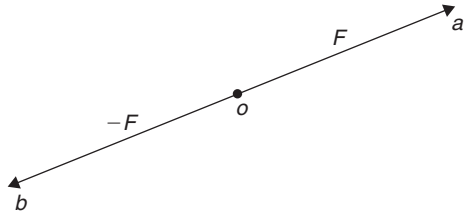


Figure 24.33

For two vectors acting at a point, as shown in Fig. 24.34(a), the resultant of vector addition is: $\boldsymbol{os = oa + ob}$.
Figure 24.33(b) shows vectors $\boldsymbol{ob + (-oa)}$, that is, $\boldsymbol{ob - oa}$ and the vector equation is $\boldsymbol{ob - oa = od}$. Comparing $\boldsymbol{od}$ in Fig. 24.34(b) with the broken line ab in Fig. 24.34(a) shows that the second diagonal of the 'parallelogram' method of vector addition gives the magnitude and direction of vector subtraction of $\boldsymbol{oa}$ from $\boldsymbol{ob}$.

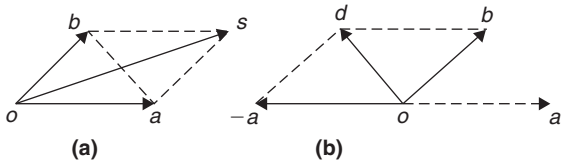


Figure 24.34

Problem 11. Accelerations of $a_1 = 1.5\,\text{m/s}^2$ at 90° and $a_2 = 2.6\,\text{m/s}^2$ at 145° act at a point. Find $\boldsymbol{a_1 + a_2}$ and $\boldsymbol{a_1 - a_2}$ (i) by drawing a scale vector diagram, and (ii) by calculation.

(i) The scale vector diagram is shown in Fig. 24.35. By measurement,

$$\boldsymbol{a_1 + a_2 = 3.7\,\text{m/s}^2 \text{ at } 126°}$$

$$\boldsymbol{a_1 - a_2 = 2.1\,\text{m/s}^2 \text{ at } 0°}$$

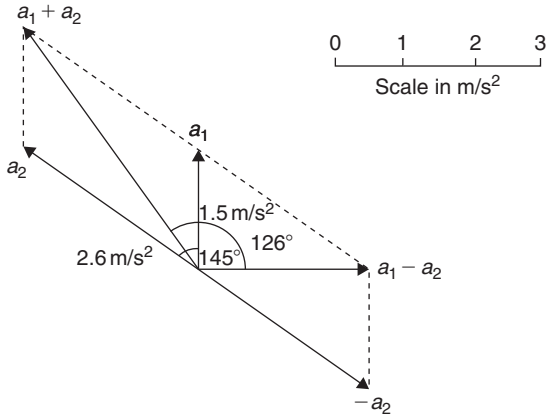


Figure 24.35

(ii) Resolving horizontally and vertically gives:

Horizontal component of $\boldsymbol{a_1 + a_2}$,
$\boldsymbol{H = 1.5\cos 90° + 2.6\cos 145° = -2.13}$

Vertical component of $\boldsymbol{a_1 + a_2}$,
$\boldsymbol{V = 1.5\sin 90° + 2.6\sin 145° = 2.99}$

From Fig. 24.36, magnitude of $\boldsymbol{a_1 + a_2}$,
$\boldsymbol{R = \sqrt{(-2.13)^2 + 2.99^2} = 3.67\,\text{m/s}^2}$

In Fig. 24.36, $\alpha = \tan^{-1}\left(\dfrac{2.99}{2.13}\right) = 54.53°$ and
$\theta = 180° - 54.53° = \boldsymbol{125.47°}$

Thus, $\boldsymbol{a_1 + a_2 = 3.67\,\text{m/s}^2 \text{ at } 125.47°}$

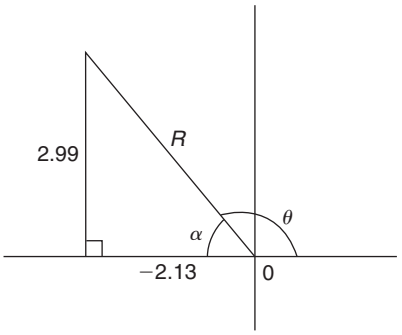


Figure 24.36

Horizontal component of $\boldsymbol{a_1 - a_2}$
$= 1.5\cos 90° - 2.6\cos 145° = \mathbf{2.13}$

Vertical component of $\boldsymbol{a_1 - a_2}$
$= 1.5\sin 90° - 2.6\sin 145° = 0$

Magnitude of $\boldsymbol{a_1 - a_2} = \sqrt{2.13^2 + 0^2}$
$= \mathbf{2.13\,m/s^2}$

Direction of $\boldsymbol{a_1 - a_2} = \tan^{-1}\left(\dfrac{0}{2.13}\right) = \mathbf{0°}$

Thus, $\boldsymbol{a_1 - a_2 = 2.13\,m/s^2}$ **at 0°**

Problem 12. Calculate the resultant of (i) $\boldsymbol{v_1 - v_2 + v_3}$ and (ii) $\boldsymbol{v_2 - v_1 - v_3}$ when $\boldsymbol{v_1} = 22$ units at 140°, $\boldsymbol{v_2} = 40$ units at 190° and $\boldsymbol{v_3} = 15$ units at 290°.

(i) The vectors are shown in Fig. 24.37.

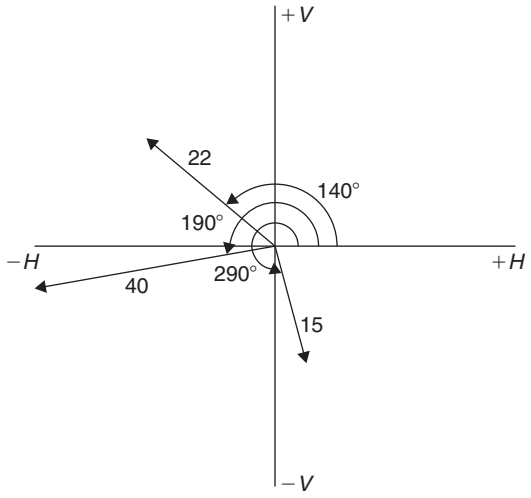


Figure 24.37

The horizontal component of

$$\boldsymbol{v_1 - v_2 + v_3} = (22\cos 140°) - (40\cos 190°) + (15\cos 290°)$$
$$= (-16.85) - (-39.39) + (5.13)$$
$$= \mathbf{27.67\ units}$$

The vertical component of

$$\boldsymbol{v_1 - v_2 + v_3} = (22\sin 140°) - (40\sin 190°) + (15\sin 290°)$$
$$= (14.14) - (-6.95) + (-14.10)$$
$$= \mathbf{6.99\ units}$$

The magnitude of the resultant,

$$\boldsymbol{R = \sqrt{27.67^2 + 6.99^2} = 28.54\ units}$$

The direction of the resultant $R = \tan^{-1}\left(\dfrac{6.99}{27.67}\right)$
$= 14.18°$

Thus, $\boldsymbol{v_1 - v_2 + v_3 = 28.54}$ **units at 14.18°**

Using **complex numbers**,

$$\boldsymbol{v_1 - v_2 + v_3} = 22\angle 140° - 40\angle 190° + 15\angle 290°$$
$$= (-16.853 + j14.141) - (-39.392 - j6.946) + (5.130 - j14.095)$$
$$= 27.669 + j6.992 = \mathbf{28.54\angle 14.18°}$$

(ii) The horizontal component of

$$\boldsymbol{v_2 - v_1 - v_3} = (40\cos 190°) - (22\cos 140°) - (15\cos 290°)$$
$$= (-39.39) - (-16.85) - (5.13)$$
$$= \mathbf{-27.67\ units}$$

The vertical component of

$$\boldsymbol{v_2 - v_1 - v_3} = (40\sin 190°) - (22\sin 140°) - (15\sin 290°)$$
$$= (-6.95) - (14.14) - (-14.10)$$
$$= \mathbf{-6.99\ units}$$

From Fig. 24.38 the magnitude of the resultant,

$$\boldsymbol{R} = \sqrt{(-27.67)^2 + (-6.99)^2} = \mathbf{28.54\ units}$$

and $\alpha = \tan^{-1}\left(\dfrac{6.99}{27.67}\right) = 14.18°$, from which,

$\theta = 180° + 14.18° = 194.18°$

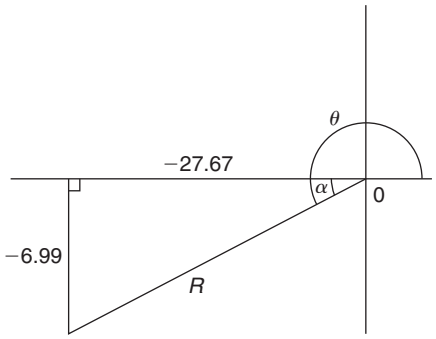


Figure 24.38

Thus, $\boldsymbol{v_2 - v_1 - v_3 = 28.54}$ **units at 194.18°**

This result is as expected, since $\boldsymbol{v_2 - v_1 - v_3 = -(v_1 - v_2 + v_3)}$ and the vector 28.54 units at 194.18° is minus times (i.e. is 180° out of phase with) the vector 28.54 units at 14.18°

Using **complex numbers**,

$$\begin{aligned} v_2 - v_2 - v_3 &= 40\angle 190° - 22\angle 140° - 15\angle 290° \\ &= (-39.392 - j6.946) \\ &\quad - (-16.853 + j14.141) \\ &\quad - (5.130 - j14.095) \\ &= -27.669 - j6.992 \\ &= \mathbf{28.54\angle -165.82°} \text{ or} \\ &\quad \mathbf{28.54\angle 194.18°} \end{aligned}$$

Now try the following exercise

Exercise 104 Further problems on vector subtraction

1. Forces of $F_1 = 40$ N at 45° and $F_2 = 30$ N at 125° act at a point. Determine by drawing and by calculation: (a) $\boldsymbol{F_1 + F_2}$ (b) $\boldsymbol{F_1 - F_2}$.
   [(a) 54.0 N at 78.16° (b) 45.64 N at 4.66°]
2. Calculate the resultant of (a) $\boldsymbol{v_1 + v_2 - v_3}$ (b) $\boldsymbol{v_3 - v_2 + v_1}$ when $\boldsymbol{v_1} = 15$ m/s at 85°, $\boldsymbol{v_2} = 25$ m/s at 175° and $\boldsymbol{v_3} = 12$ m/s at 235°.
   [(a) 31.71 m/s at 121.81° (b) 19.55 m/s at 8.63°]

24.8 Relative velocity

For relative velocity problems, some fixed datum point needs to be selected. This is often a fixed point on the earth's surface. In any vector equation, only the start and finish points affect the resultant vector of a system. Two different systems are shown in Fig. 24.39, but in each of the systems, the resultant vector is $\boldsymbol{ad}$.

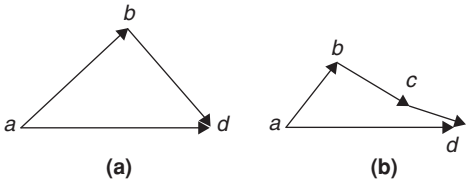


Figure 24.39

The vector equation of the system shown in Fig. 24.39(a) is:

$$\boldsymbol{ad = ab + bd}$$

and that for the system shown in Fig. 24.39(b) is:

$$\boldsymbol{ad = ab + bc + cd}$$

Thus in vector equations of this form, only the first and last letters, 'a' and 'd', respectively, fix the magnitude and direction of the resultant vector. This principle is used in relative velocity problems.

Problem 13. Two cars, P and Q, are travelling towards the junction of two roads which are at right angles to one another. Car P has a velocity of 45 km/h due east and car Q a velocity of 55 km/h due south. Calculate (i) the velocity of car P relative to car Q, and (ii) the velocity of car Q relative to car P.

(i) The directions of the cars are shown in Fig. 24.40(a), called a **space diagram**. The velocity diagram is shown in Fig. 24.40(b), in which $\boldsymbol{pe}$ is taken as the velocity of car P relative to point e on the earth's surface. The velocity of P relative to Q is vector $\boldsymbol{pq}$ and the vector equation is $\boldsymbol{pq = pe + eq}$. Hence the vector directions are as shown, $\boldsymbol{eq}$ being in the opposite direction to $\boldsymbol{qe}$.

From the geometry of the vector triangle, the magnitude of $\boldsymbol{pq} = \sqrt{45^2 + 55^2} = 71.06$ km/h and the direction of $\boldsymbol{pq} = \tan^{-1}\left(\frac{55}{45}\right) = 50.71°$ i.e. **the velocity of car P relative to car Q is 71.06 km/h at 50.71°**

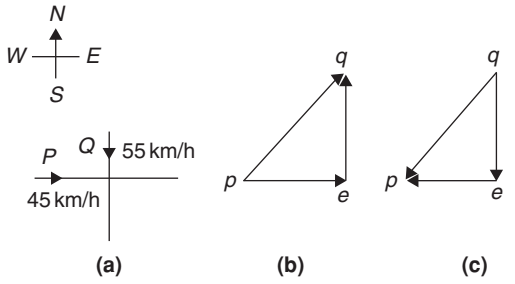


Figure 24.40

(ii) The velocity of car Q relative to car P is given by the vector equation $\boldsymbol{qp = qe + ep}$ and the vector diagram is as shown in Fig. 24.40(c), having $\boldsymbol{ep}$ opposite in direction to $\boldsymbol{pe}$.
From the geometry of this vector triangle, the magnitude of $\boldsymbol{qp} = \sqrt{45^2 + 55^2} = 71.06$ m/s and the direction of $\boldsymbol{qp} = \tan^{-1}\left(\frac{55}{45}\right) = 50.71°$ but must lie in the third quadrant, i.e. the required angle is: $180° + 50.71° = 230.71°$
i.e. **the velocity of car Q relative to car P is 71.06 m/s at 230.71°**

Now try the following exercise

Exercise 105 Further problems on relative velocity

1. A car is moving along a straight horizontal road at 79.2 km/h and rain is falling vertically downwards at 26.4 km/h. Find the velocity of the rain relative to the driver of the car.
[83.5 km/h at 71.6° to the vertical]

2. Calculate the time needed to swim across a river 142 m wide when the swimmer can swim at 2 km/h in still water and the river is flowing at 1 km/h. At what angle to the bank should the swimmer swim?
[4 minutes 55 seconds, 60°]

3. A ship is heading in a direction N 60° E at a speed which in still water would be 20 km/h. It is carried off course by a current of 8 km/h in a direction of E 50° S. Calculate the ship's actual speed and direction.
[22.79 km/h, E 9.78° N]

24.9 i, j and k notation

A method of completely specifying the direction of a vector in space relative to some reference point is to use three unit vectors, $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$, mutually at right angles to each other, as shown in Fig. 24.41.

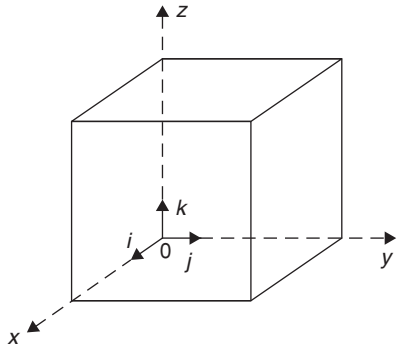


Figure 24.41

Calculations involving vectors given in $\boldsymbol{i}$, $\boldsymbol{j}$ $\boldsymbol{k}$ notation are carried out in exactly the same way as standard algebraic calculations, as shown in the worked example below.

Problem 14. Determine:
$(3\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}) - (4\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k})$

$$(3\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}) - (4\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}) = 3\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k} - 4\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k} = -\boldsymbol{i} + 5\boldsymbol{j}$$

Problem 15. Given $\boldsymbol{p} = 3\boldsymbol{i} + 2\boldsymbol{k}$, $\boldsymbol{q} = 4\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{r} = -3\boldsymbol{i} + 5\boldsymbol{j} - 4\boldsymbol{k}$ determine:

(a) $-\boldsymbol{r}$ (b) $3\boldsymbol{p}$ (c) $2\boldsymbol{p}+3\boldsymbol{q}$ (d) $-\boldsymbol{p}+2\boldsymbol{r}$
(e) $0.2\boldsymbol{p}+0.6\boldsymbol{q}-3.2\boldsymbol{r}$

(a) $-\boldsymbol{r} = -(-3\boldsymbol{i}+5\boldsymbol{j}-4\boldsymbol{k}) = \mathbf{+3\boldsymbol{i}-5\boldsymbol{j}+4\boldsymbol{k}}$

(b) $3\boldsymbol{p} = 3(3\boldsymbol{i}+2\boldsymbol{k}) = \mathbf{9\boldsymbol{i}+6\boldsymbol{k}}$

(c) $2\boldsymbol{p}+3\boldsymbol{q} = 2(3\boldsymbol{i}+2\boldsymbol{k})+3(4\boldsymbol{i}-2\boldsymbol{j}+3\boldsymbol{k})$
$= 6\boldsymbol{i}+4\boldsymbol{k}+12\boldsymbol{i}-6\boldsymbol{j}+9\boldsymbol{k}$
$= \mathbf{18\boldsymbol{i}-6\boldsymbol{j}+13\boldsymbol{k}}$

(d) $-\boldsymbol{p}+2\boldsymbol{r} = -(3\boldsymbol{i}+2\boldsymbol{k})+2(-3\boldsymbol{i}+5\boldsymbol{j}-4\boldsymbol{k})$
$= -3\boldsymbol{i}-2\boldsymbol{k}+(-6\boldsymbol{i}+10\boldsymbol{j}-8\boldsymbol{k})$
$= -3\boldsymbol{i}-2\boldsymbol{k}-6\boldsymbol{i}+10\boldsymbol{j}-8\boldsymbol{k}$
$= \mathbf{-9\boldsymbol{i}+10\boldsymbol{j}-10\boldsymbol{k}}$

(e) $0.2\boldsymbol{p}+0.6\boldsymbol{q}-3.2\boldsymbol{r} = 0.2(3\boldsymbol{i}+2\boldsymbol{k})$
$+0.6(4\boldsymbol{i}-2\boldsymbol{j}+3\boldsymbol{k})-3.2(-3\boldsymbol{i}+5\boldsymbol{j}-4\boldsymbol{k})$
$= 0.6\boldsymbol{i}+0.4\boldsymbol{k}+2.4\boldsymbol{i}-1.2\boldsymbol{j}+1.8\boldsymbol{k}$
$+9.6\boldsymbol{i}-16\boldsymbol{j}+12.8\boldsymbol{k}$
$= \mathbf{12.6\boldsymbol{i}-17.2\boldsymbol{j}+15\boldsymbol{k}}$

Now try the following exercise

Exercise 106 Further problems on i, j, k notation

Given that $\boldsymbol{p} = 2\boldsymbol{i}+0.5\boldsymbol{j}-3\boldsymbol{k}$, $\boldsymbol{q} = -\boldsymbol{i}+\boldsymbol{j}+4\boldsymbol{k}$ and $\boldsymbol{r} = 6\boldsymbol{j}-5\boldsymbol{k}$, evaluate and simplify the following vectors in i, j, k form:

1. $-\boldsymbol{q}$ $[\boldsymbol{i}-\boldsymbol{j}-4\boldsymbol{k}]$
2. $2\boldsymbol{p}$ $[4\boldsymbol{i}+\boldsymbol{j}-6\boldsymbol{k}]$
3. $\boldsymbol{q}+\boldsymbol{r}$ $[-\boldsymbol{i}+7\boldsymbol{j}-\boldsymbol{k}]$
4. $-\boldsymbol{q}+2\boldsymbol{p}$ $[5\boldsymbol{i}-10\boldsymbol{k}]$
5. $3\boldsymbol{q}+4\boldsymbol{r}$ $[-3\boldsymbol{i}+27\boldsymbol{j}-8\boldsymbol{k}]$
6. $\boldsymbol{q}-2\boldsymbol{p}$ $[-5\boldsymbol{i}+10\boldsymbol{k}]$
7. $\boldsymbol{p}+\boldsymbol{q}+\boldsymbol{r}$ $[\boldsymbol{i}+7.5\boldsymbol{j}-4\boldsymbol{k}]$
8. $\boldsymbol{p}+2\boldsymbol{q}+3\boldsymbol{r}$ $[20.5\boldsymbol{j}-10\boldsymbol{k}]$
9. $2\boldsymbol{p}+0.4\boldsymbol{q}+0.5\boldsymbol{r}$ $[3.6\boldsymbol{i}+4.4\boldsymbol{j}-6.9\boldsymbol{k}]$
10. $7\boldsymbol{r}-2\boldsymbol{q}$ $[2\boldsymbol{i}+40\boldsymbol{j}-43\boldsymbol{k}]$

Chapter 25

Methods of adding alternating waveforms

25.1 Combination of two periodic functions

There are a number of instances in engineering and science where waveforms have to be combined and where it is required to determine the single phasor (called the resultant) that could replace two or more separate phasors. Uses are found in electrical alternating current theory, in mechanical vibrations, in the addition of forces and with sound waves.

There are a number of methods of determining the resultant waveform. These include:

(a) by drawing the waveforms and adding graphically

(b) by drawing the phasors and measuring the resultant

(c) by using the cosine and sine rules

(d) by using horizontal and vertical components

(e) by using complex numbers

25.2 Plotting periodic functions

This may be achieved by sketching the separate functions on the same axes and then adding (or subtracting) ordinates at regular intervals. This is demonstrated in the following worked problems.

Problem 1. Plot the graph of $y_1 = 3\sin A$ from $A = 0°$ to $A = 360°$. On the same axes plot $y_2 = 2\cos A$. By adding ordinates, plot $y_R = 3\sin A + 2\cos A$ and obtain a sinusoidal expression for this resultant waveform.

$y_1 = 3\sin A$ and $y_2 = 2\cos A$ are shown plotted in Fig. 25.1. Ordinates may be added at, say, 15° intervals. For example,

at 0°, $y_1 + y_2 = 0 + 2 = 2$
at 15°, $y_1 + y_2 = 0.78 + 1.93 = 2.71$
at 120°, $y_1 + y_2 = 2.60 + -1 = 1.6$
at 210°, $y_1 + y_2 = -1.50 - 1.73 = -3.23$, and so on.

The resultant waveform, shown by the broken line, has the same period, i.e. 360°, and thus the same frequency as the single phasors. The maximum value, or

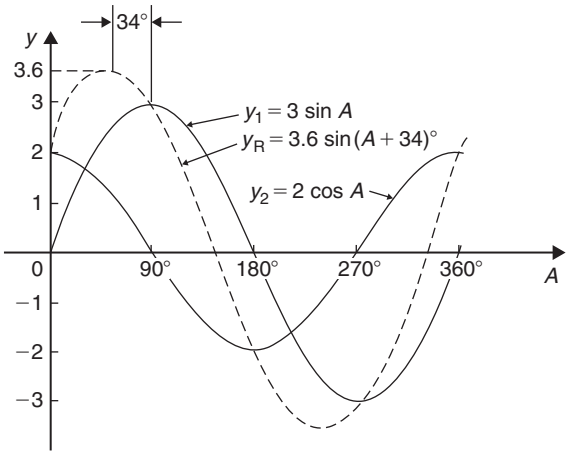


Figure 25.1

amplitude, of the resultant is 3.6. The resultant waveform **leads** $y_1 = 3\sin A$ by 34° or $34 \times \frac{\pi}{180}$ rad $= 0.593$ rad.

The sinusoidal expression for the resultant waveform is:

$$\mathbf{y_R = 3.6\sin(A + 34°)} \quad \text{or}$$

$$\mathbf{y_R = 3.6\sin(A + 0.593)}$$

Problem 2. Plot the graphs of $y_1 = 4\sin\omega t$ and $y_2 = 3\sin(\omega t - \pi/3)$ on the same axes, over one cycle. By adding ordinates at intervals plot $y_R = y_1 + y_2$ and obtain a sinusoidal expression for the resultant waveform.

$y_1 = 4\sin\omega t$ and $y_2 = 3\sin(\omega t - \pi/3)$ are shown plotted in Fig. 25.2.

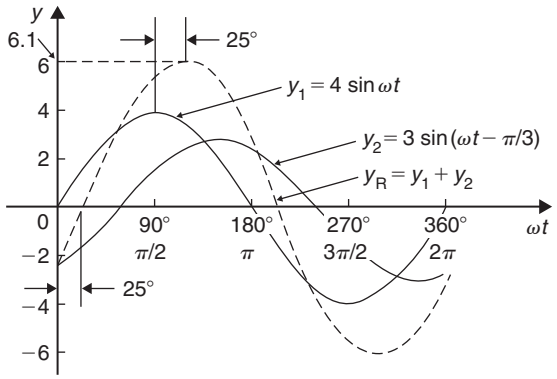


Figure 25.2

Ordinates are added at 15° intervals and the resultant is shown by the broken line. The amplitude of the resultant is 6.1 and it **lags** y_1 by 25° or 0.436 rad.

Hence, the sinusoidal expression for the resultant waveform is:

$$\mathbf{y_R = 6.1\sin(\omega t - 0.436)}$$

Problem 3. Determine a sinusoidal expression for $y_1 - y_2$ when $y_1 = 4\sin\omega t$ and $y_2 = 3\sin(\omega t - \pi/3)$.

y_1 and y_2 are shown plotted in Fig. 25.3. At 15° intervals y_2 is subtracted from y_1. For example:

at 0°, $y_1 - y_2 = 0 - (-2.6) = +2.6$

at 30°, $y_1 - y_2 = 2 - (-1.5) = +3.5$

at 150°, $y_1 - y_2 = 2 - 3 = -1$, and so on.

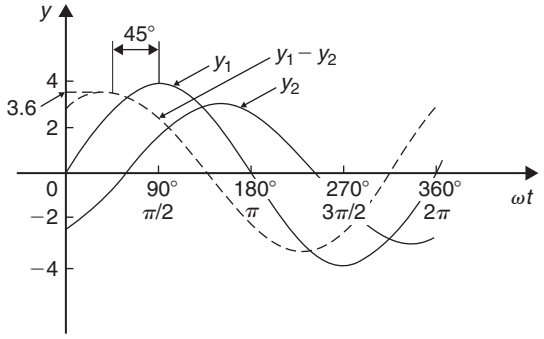


Figure 25.3

The amplitude, or peak value of the resultant (shown by the broken line), is 3.6 and it leads y_1 by 45° or 0.79 rad. Hence,

$$\mathbf{y_1 - y_2 = 3.6\sin(\omega t + 0.79)}$$

Problem 4. Two alternating currents are given by: $i_1 = 20\sin\omega t$ amperes and $i_2 = 10\sin\left(\omega t + \frac{\pi}{3}\right)$ amperes.

By drawing the waveforms on the same axes and adding, determine the sinusoidal expression for the resultant $i_1 + i_2$.

i_1 and i_2 are shown plotted in Fig. 25.4. The resultant waveform for $i_1 + i_2$ is shown by the broken line. It has the same period, and hence frequency, as i_1 and i_2.

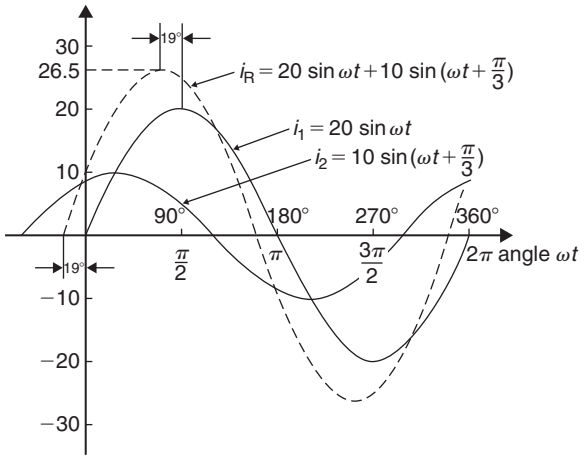


Figure 25.4

The amplitude or peak value is 26.5 A.

The resultant waveform leads the waveform of $i_1 = 20\sin\omega t$ by 19° or 0.33 rad

Hence, the sinusoidal expression for the resultant $i_1 + i_2$ is given by:

$$i_R = i_1 + i_2 = 26.5\sin(\omega t + 0.33)\ \text{A}$$

Now try the following exercise

Exercise 107 Further problems on plotting periodic functions

1. Plot the graph of $y = 2\sin A$ from $A = 0°$ to $A = 360°$. On the same axes plot $y = 4\cos A$. By adding ordinates at intervals plot $y = 2\sin A + 4\cos A$ and obtain a sinusoidal expression for the waveform.
   $[4.5\sin(A + 63.5°)]$

2. Two alternating voltages are given by $v_1 = 10\sin\omega t$ volts and $v_2 = 14\sin(\omega t + \pi/3)$ volts. By plotting v_1 and v_2 on the same axes over one cycle obtain a sinusoidal expression for (a) $v_1 + v_2$ (b) $v_1 - v_2$.
   $\begin{bmatrix}\text{(a) } 20.9\sin(\omega t + 0.63)\text{ volts}\\ \text{(b) } 12.5\sin(\omega t - 1.36)\text{ volts}\end{bmatrix}$

3. Express $12\sin\omega t + 5\cos\omega t$ in the form $A\sin(\omega t \pm \alpha)$ by drawing and measurement.
   $[13\sin(\omega t + 0.395)]$

25.3 Determining resultant phasors by drawing

The resultant of two periodic functions may be found from their relative positions when the time is zero. For example, if $y_1 = 4\sin\omega t$ and $y_2 = 3\sin(\omega t - \pi/3)$ then each may be represented as phasors as shown in Fig. 25.5, y_1 being 4 units long and drawn horizontally and y_2 being 3 units long, lagging y_1 by $\pi/3$ radians or 60°. To determine the resultant of $y_1 + y_2$, y_1 is drawn horizontally as shown in Fig. 25.6 and y_2 is joined to the end of y_1 at 60° to the horizontal. The resultant is given by y_R. This is the same as the diagonal of a parallelogram that is shown completed in Fig. 25.7.

Resultant y_R, in Figs. 25.6 and 25.7, may be determined by drawing the phasors and their directions to scale and measuring using a ruler and protractor.

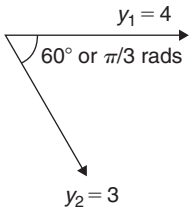


Figure 25.5

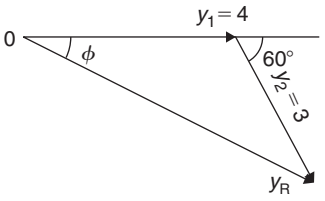


Figure 25.6

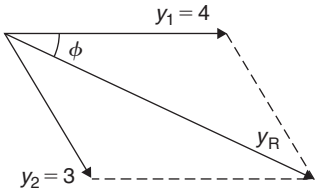


Figure 25.7

In this example, y_R is measured as 6 units long and angle ϕ is measured as 25°.

$$25° = 25 \times \frac{\pi}{180}\ \text{radians} = 0.44\ \text{rad}$$

Hence, summarising, by drawing: $\boldsymbol{y_R = y_1 + y_2} = 4\sin\omega t + 3\sin(\omega t - \pi/3) = \boldsymbol{6\sin(\omega t - 0.44)}$

If the resultant phasor $y_R = y_1 - y_2$ is required, then y_2 is still 3 units long but is drawn in the opposite direction, as shown in Fig. 25.8.

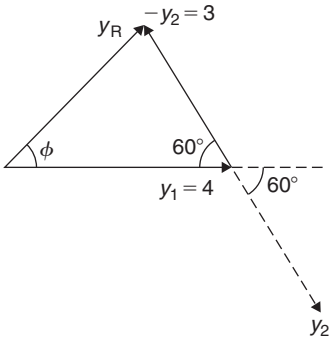


Figure 25.8

Problem 5. Two alternating currents are given by: $i_1 = 20\sin\omega t$ amperes and $i_2 = 10\sin\left(\omega t + \frac{\pi}{3}\right)$ amperes. Determine $i_1 + i_2$ by drawing phasors.

The relative positions of i_1 and i_2 at time $t = 0$ are shown as phasors in Fig. 25.9, where $\frac{\pi}{3}\text{rad} = 60°$.

The phasor diagram in Fig. 25.10 is drawn to scale with a ruler and protractor.

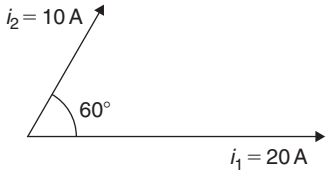


Figure 25.9

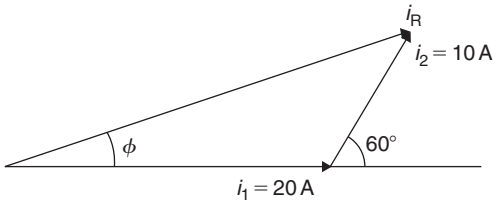


Figure 25.10

The resultant i_R is shown and is measured as 26 A and angle ϕ as 19° or 0.33 rad leading i_1. Hence, by drawing and measuring:

$$\boldsymbol{i_R = i_1 + i_2 = 26\sin(\omega t + 0.33)\text{A}}$$

Problem 6. For the currents in Problem 5, determine $i_1 - i_2$ by drawing phasors.

At time $t = 0$, current i_1 is drawn 20 units long horizontally as shown by $0a$ in Fig. 25.11. Current i_2 is shown, drawn 10 units long in broken line and leading by 60°. The current $-i_2$ is drawn in the opposite direction to the broken line of i_2, shown as ab in Fig. 25.11. The resultant i_R is given by $0b$ lagging by angle ϕ.

By measurement, $i_R = 17$ A and $\phi = 30°$ or 0.52 rad

Hence, by drawing phasors:

$$\boldsymbol{i_R = i_1 - i_2 = 17\sin(\omega t - 0.52)}$$

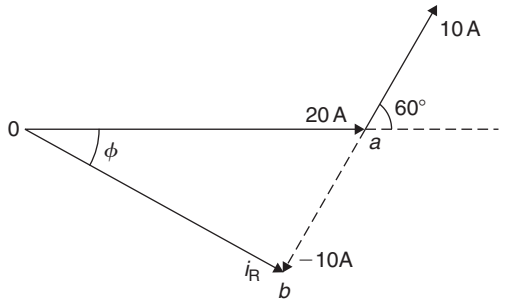


Figure 25.11

Now try the following exercise

Exercise 108 Further problems on determining resultant phasors by drawing

1. Determine a sinusoidal expression for $2\sin\theta + 4\cos\theta$ by drawing phasors.
   [$4.5\sin(A + 63.5°)$]

2. If $v_1 = 10\sin\omega t$ volts and $v_2 = 14\sin(\omega t + \pi/3)$ volts, determine by drawing phasors sinusoidal expressions for (a) $v_1 + v_2$ (b) $v_1 - v_2$.
   [(a) $20.9\sin(\omega t + 0.62)$ volts (b) $12.5\sin(\omega t - 1.33)$ volts]

3. Express $12\sin\omega t + 5\cos\omega t$ in the form $R\sin(\omega t \pm \alpha)$ by drawing phasors.
   [$13\sin(\omega t + 0.40)$]

25.4 Determining resultant phasors by the sine and cosine rules

As stated earlier, the resultant of two periodic functions may be found from their relative positions when the time is zero. For example, if $y_1 = 5\sin\omega t$ and $y_2 = 4\sin(\omega t - \pi/6)$ then each may be represented by phasors as shown in Fig. 25.12, y_1 being 5 units long and drawn horizontally and y_2 being 4 units long, lagging y_1 by $\pi/6$ radians or 30°. To determine the resultant of $y_1 + y_2$, y_1 is drawn horizontally as shown in Fig. 25.13 and y_2 is joined to the end of y_1 at $\pi/6$ radians, i.e. 30° to the horizontal. The resultant is given by y_R.

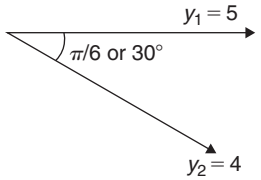


Figure 25.12

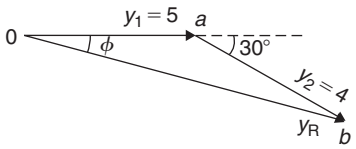


Figure 25.13

Using the cosine rule on triangle $0ab$ of Fig. 25.13 gives:

$$y_R^2 = 5^2 + 4^2 - [2(5)(4)\cos 150°]$$
$$= 25 + 16 - (-34.641)$$
$$= 75.641$$

from which, $y_R = \sqrt{75.641} = 8.697$

Using the sine rule, $\dfrac{8.697}{\sin 150°} = \dfrac{4}{\sin\phi}$

from which, $\sin\phi = \dfrac{4\sin 150°}{8.697}$

$$= 0.22996$$

and $\phi = \sin^{-1} 0.22996$

$$= 13.29° \text{ or } 0.232 \text{ rad}$$

Hence, $\boldsymbol{y_R = y_1 + y_2} = 5\sin\omega t + 4\sin(\omega t - \pi/6)$

$$\boldsymbol{= 8.697\sin(\omega t - 0.232)}$$

Problem 7. Given $y_1 = 2\sin\omega t$ and $y_2 = 3\sin(\omega t + \pi/4)$, obtain an expression, by calculation, for the resultant, $y_R = y_1 + y_2$.

When time $t = 0$, the position of phasors y_1 and y_2 are as shown in Fig. 25.14(a). To obtain the resultant, y_1 is drawn horizontally, 2 units long, y_2 is drawn 3 units long at an angle of $\pi/4$ rads or 45° and joined to the end of y_1 as shown in Fig. 25.14(b).
From Fig. 25.14(b), and using the cosine rule:

$$y_R^2 = 2^2 + 3^2 - [2(2)(3)\cos 135°]$$
$$= 4 + 9 - [-8.485] = 21.49$$

Hence, $y_R = \sqrt{21.49} = 4.6357$

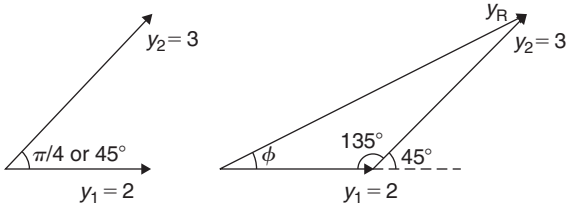


Figure 25.14

Using the sine rule: $\dfrac{3}{\sin\phi} = \dfrac{4.6357}{\sin 135°}$

from which, $\sin\phi = \dfrac{3\sin 135°}{4.6357} = 0.45761$

Hence, $\phi = \sin^{-1} 0.45761$

$$= 27.23° \text{ or } 0.475 \text{ rad.}$$

Thus, by calculation, $\boldsymbol{y_R = 4.635\sin(\omega t + 0.475)}$

Problem 8. Determine $20\sin\omega t + 10\sin\left(\omega t + \dfrac{\pi}{3}\right)$ using the cosine and sine rules.

From the phasor diagram of Fig. 25.15, and using the cosine rule:

$$i_R^2 = 20^2 + 10^2 - [2(20)(10)\cos 120°]$$
$$= 700$$

Hence, $i_R = \sqrt{700} = \mathbf{26.46\ A}$

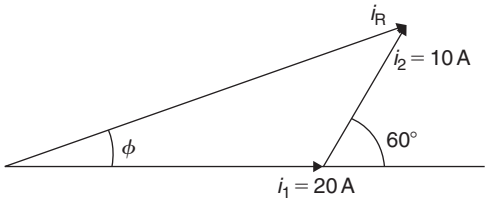


Figure 25.15

Using the sine rule gives : $\dfrac{10}{\sin\phi} = \dfrac{26.46}{\sin 120°}$

from which, $\sin\phi = \dfrac{10\sin 120°}{26.46}$

$$= 0.327296$$

and $\phi = \sin^{-1} 0.327296 = \mathbf{19.10°}$

$$= 19.10 \times \frac{\pi}{180} = \mathbf{0.333\ rad}$$

Hence, by cosine and sine rules,

$$i_R = i_1 + i_2 = 26.46\sin(\omega t + 0.333)\text{ A}$$

Now try the following exercise

Exercise 109 Resultant phasors by the sine and cosine rules

1. Determine, using the cosine and sine rules, a sinusoidal expression for: $y = 2\sin A + 4\cos A$.
   [$4.5\sin(A + 63.5°)$]

2. Given $v_1 = 10\sin\omega t$ volts and $v_2 = 14\sin(\omega t + \pi/3)$ volts use the cosine and sine rules to determine sinusoidal expressions for (a) $v_1 + v_2$ (b) $v_1 - v_2$.
   [(a) $20.88\sin(\omega t + 0.62)$ volts (b) $12.50\sin(\omega t - 1.33)$volts]

In Problems 3 to 5, express the given expressions in the form $A\sin(\omega t \pm \alpha)$ by using the cosine and sine rules.

3. $12\sin\omega t + 5\cos\omega t$
   [$13\sin(\omega t + 0.395)$]

4. $7\sin\omega t + 5\sin\left(\omega t + \frac{\pi}{4}\right)$
   [$11.11\sin(\omega t + 0.324)$]

5. $6\sin\omega t + 3\sin\left(\omega t - \frac{\pi}{6}\right)$
   [$8.73\sin(\omega t - 0.173)$]

25.5 Determining resultant phasors by horizontal and vertical components

If a right-angled triangle is constructed as shown in Fig. 25.16, then $0a$ is called the horizontal component of F and ab is called the vertical component of F.

From trigonometry (see Chapter 11),

$$\cos\theta = \frac{0a}{0b} \quad \text{from which,}$$

$$0a = 0b\cos\theta = F\cos\theta$$

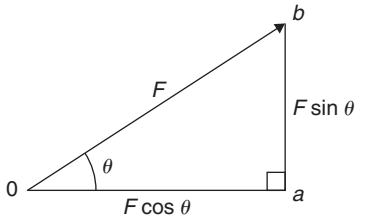


Figure 25.16

i.e. **the horizontal component of F, $H = F\cos\theta$**

$$\text{and } \sin\theta = \frac{ab}{0b} \quad \text{from which } ab = 0b\sin\theta = F\sin\theta$$

i.e. **the vertical component of F, $V = F\sin\theta$**

Determining resultant phasors by horizontal and vertical components is demonstrated in the following worked problems.

Problem 9. Two alternating voltages are given by $v_1 = 15\sin\omega t$ volts and $v_2 = 25\sin(\omega t - \pi/6)$ volts. Determine a sinusoidal expression for the resultant $v_R = v_1 + v_2$ by finding horizontal and vertical components.

The relative positions of v_1 and v_2 at time $t = 0$ are shown in Fig. 25.17(a) and the phasor diagram is shown in Fig. 25.17(b).

The horizontal component of v_R,
$H = 15\cos 0° + 25\cos(-30°) = 0a + ab = \mathbf{36.65\,V}$

The vertical component of v_R,
$V = 15\sin 0° + 25\sin(-30°) = bc = \mathbf{-12.50\,V}$

Hence, $v_R = 0c = \sqrt{36.65^2 + (-12.50)^2}$

by Pythagoras' theorem

$$= \mathbf{38.72\text{ volts}}$$

$$\tan\phi = \frac{V}{H} = \frac{-12.50}{36.65} = -0.3411$$

from which, $\phi = \tan^{-1}(-0.3411) = -18.83°$

or -0.329 radians.

Hence, $\mathbf{v_R = v_1 + v_2 = 38.72\sin(\omega t - 0.329)V}$

Problem 10. For the voltages in Problem 9, determine the resultant $v_R = v_1 - v_2$ using horizontal and vertical components.

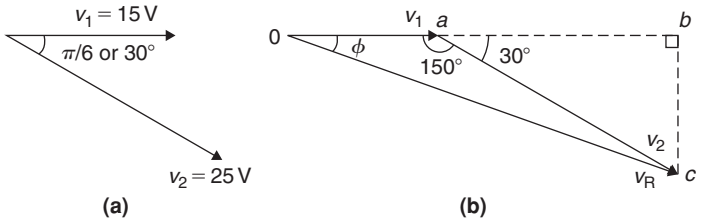


Figure 25.17

The horizontal component of v_R,
$\boldsymbol{H} = 15\cos 0° - 25\cos(-30°) = \mathbf{-6.65V}$

The vertical component of v_R,
$\boldsymbol{V} = 15\sin 0° - 25\sin(-30°) = \mathbf{12.50V}$

Hence, $v_R = \sqrt{(-6.65)^2 + (12.50)^2}$

by Pythagoras' theorem

$= \mathbf{14.16}$ **volts**

$$\tan\phi = \frac{V}{H} = \frac{12.50}{-6.65} = -1.8797$$

from which, $\phi = \tan^{-1}(-1.8797) = 118.01°$

or 2.06 radians.

Hence,

$$\boldsymbol{v_R = v_1 - v_2 = 14.16\sin(\omega t + 2.06)V}$$

The phasor diagram is shown in Fig. 25.18.

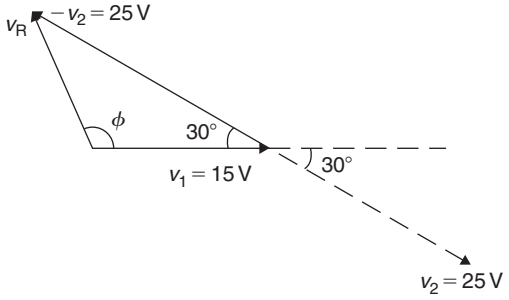


Figure 25.18

Problem 11. Determine $20\sin\omega t + 10\sin\left(\omega t + \frac{\pi}{3}\right)$ using horizontal and vertical components.

From the phasors shown in Fig. 25.19:
Total horizontal component,
$\boldsymbol{H} = 20\cos 0° + 10\cos 60° = \mathbf{25.0}$

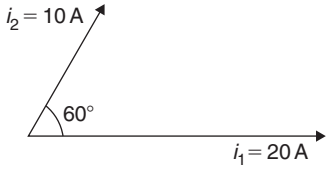


Figure 25.19

Total vertical component,
$\boldsymbol{V} = 20\sin 0° + 10\sin 60° = \mathbf{8.66}$

By Pythagoras, the resultant, $i_R = \sqrt{[25.0^2 + 8.66^2]}$
$= \mathbf{26.46\,A}$

Phase angle, $\phi = \tan^{-1}\left(\frac{8.66}{25.0}\right) = \mathbf{19.11°}$

or **0.333 rad**

Hence, by using horizontal and vertical components,

$$20\sin\omega t + 10\sin\left(\omega t + \frac{\pi}{3}\right) = \mathbf{26.46\sin(\omega t + 0.333)}$$

Now try the following exercise

Exercise 110 Further problems on resultant phasors by horizontal and vertical components

In Problems 1 to 4, express the combination of periodic functions in the form $A\sin(\omega t \pm \alpha)$ by horizontal and vertical components:

1. $7\sin\omega t + 5\sin\left(\omega t + \frac{\pi}{4}\right)$
[$11.11\sin(\omega t + 0.324)$]
2. $6\sin\omega t + 3\sin\left(\omega t - \frac{\pi}{6}\right)$
[$8.73\sin(\omega t - 0.173)$]
3. $i = 25\sin\omega t - 15\sin\left(\omega t + \frac{\pi}{3}\right)$
[$i = 21.79\sin(\omega t - 0.639)$]
4. $x = 9\sin\left(\omega t + \frac{\pi}{3}\right) - 7\sin\left(\omega t - \frac{3\pi}{8}\right)$
[$x = 14.38\sin(\omega t + 1.444)$]

5. The voltage drops across two components when connected in series across an a.c. supply are: $v_1 = 200\sin 314.2t$ and $v_2 = 120\sin(314.2t - \pi/5)$ volts respectively. Determine the:
   (a) voltage of the supply (given by $v_1 + v_2$) in the form $A\sin(\omega t \pm \alpha)$.
   (b) frequency of the supply.
   [(a) $305.3\sin(314.2t - 0.233)$V (b) 50 Hz]

6. If the supply to a circuit is $v = 20\sin 628.3t$ volts and the voltage drop across one of the components is $v_1 = 15\sin(628.3t - 0.52)$ volts, calculate the:
   (a) voltage drop across the remainder of the circuit, given by $v - v_1$, in the form $A\sin(\omega t \pm \alpha)$.
   (b) supply frequency.
   (c) periodic time of the supply.
   [(a) $10.21\sin(628.3t + 0.818)$V (b) 100 Hz (c) 10 ms]

7. The voltages across three components in a series circuit when connected across an a.c. supply are:

$$v_1 = 25\sin\left(300\pi t + \frac{\pi}{6}\right) \text{ volts},$$

$$v_2 = 40\sin\left(300\pi t - \frac{\pi}{4}\right) \text{ volts, and}$$

$$v_3 = 50\sin\left(300\pi t + \frac{\pi}{3}\right) \text{ volts.}$$

   Calculate the:
   (a) supply voltage, in sinusoidal form, in the form $A\sin(\omega t \pm \alpha)$.
   (b) frequency of the supply.
   (c) periodic time.
   [(a) $79.83\sin(300\pi t + 0.352)$V (b) 150 Hz (c) 6.667 ms]

25.6 Determining resultant phasors by complex numbers

As stated earlier, the resultant of two periodic functions may be found from their relative positions when the time is zero. For example, if $y_1 = 5\sin\omega t$ and $y_2 = 4\sin(\omega t - \pi/6)$ then each may be represented by phasors as shown in Fig. 25.20, y_1 being 5 units long and drawn horizontally and y_2 being 4 units long, lagging y_1 by $\pi/6$ radians or 30°. To determine the resultant of $y_1 + y_2$, y_1 is drawn horizontally as shown in Fig. 25.21 and y_2 is joined to the end of y_1 at $\pi/6$ radians, i.e. 30° to the horizontal. The resultant is given by y_R.

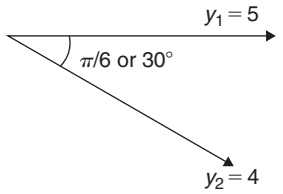


Figure 25.20

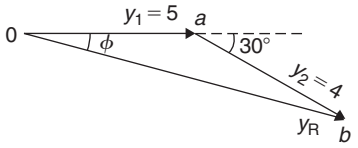


Figure 25.21

In polar form, $y_R = 5\angle 0 + 4\angle -\frac{\pi}{6}$

$= 5\angle 0° + 4\angle -30°$

$= (5 + j0) + (4.33 - j2.0)$

$= 9.33 - j2.0 = 9.54\angle -12.10°$

$= 9.54\angle -0.21\text{rad}$

Hence, by using complex numbers, the resultant in sinusoidal form is:

$$\boldsymbol{y_1 + y_2} = 5\sin\omega t + 4\sin(\omega t - \pi/6)$$
$$= \mathbf{9.54\sin(\omega t - 0.21)}$$

Problem 12. Two alternating voltages are given by $v_1 = 15\sin\omega t$ volts and $v_2 = 25\sin(\omega t - \pi/6)$ volts. Determine a sinusoidal expression for the resultant $v_R = v_1 + v_2$ by using complex numbers.

The relative positions of v_1 and v_2 at time $t = 0$ are shown in Fig. 25.22(a) and the phasor diagram is shown in Fig. 25.22(b).

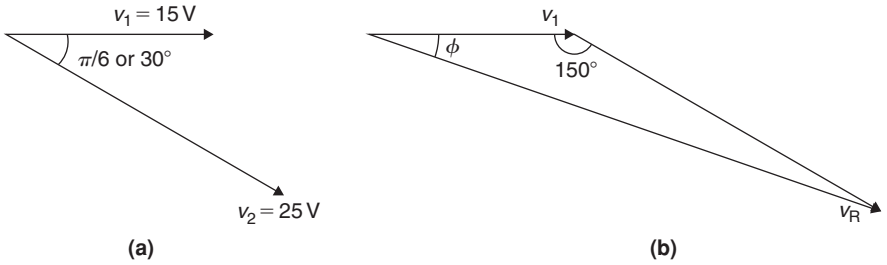


Figure 25.22

In polar form, $v_R = v_1 + v_2 = 15\angle 0 + 25\angle -\dfrac{\pi}{6}$

$= 15\angle 0° + 25\angle -30°$

$= (15 + j0) + (21.65 - j12.5)$

$= 36.65 - j12.5 = 38.72\angle -18.83°$

$= 38.72\angle -0.329$ rad

Hence, by using complex numbers, the resultant in sinusoidal form is:

$$\boldsymbol{v_R = v_1 + v_2} = 15\sin\omega t + 25\sin(\omega t - \pi/6)$$

$$= \mathbf{38.72\sin(\omega t - 0.329)}$$

Problem 13. For the voltages in Problem 12, determine the resultant $v_R = v_1 - v_2$ using complex numbers.

In polar form, $y_R = v_1 - v_2 = 15\angle 0 - 25\angle -\dfrac{\pi}{6}$

$= 15\angle 0° - 25\angle -30°$

$= (15 + j0) - (21.65 - j12.5)$

$= -6.65 + j12.5 = 14.16\angle 118.01°$

$= 14.16\angle 2.06$ rad

Hence, by using complex numbers, the resultant in sinusoidal form is:

$$\boldsymbol{y_1 - y_2} = 15\sin\omega t - 25\sin(\omega t - \pi/6)$$

$$= \mathbf{14.16\sin(\omega t - 2.06)}$$

Problem 14. Determine $20\sin\omega t + 10\sin\left(\omega t + \dfrac{\pi}{3}\right)$ using complex numbers.

From the phasors shown in Fig. 25.23, the resultant may be expressed in polar form as:

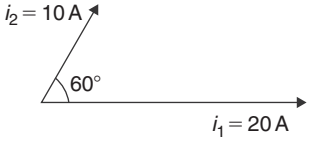


Figure 25.23

$i_R = 20\angle 0° + 10\angle 60°$

i.e. $i_R = (20 + j0) + (5 + j8.66)$

$= (25 + j8.66) = \mathbf{26.46\angle 19.11°A}$ or

$\mathbf{26.46\angle 0.333\,rad\,A}$

Hence, by using complex numbers, the resultant in sinusoidal form is:

$$\boldsymbol{i_R = i_1 + i_2 = 26.46\sin(\omega t + 0.333)A}$$

Problem 15. If the supply to a circuit is $v = 30\sin 100\pi t$ volts and the voltage drop across one of the components is $v_1 = 20\sin(100\pi t - 0.59)$ volts, calculate the:

(a) voltage drop across the remainder of the circuit, given by $v - v_1$, in the form $A\sin(\omega t \pm \alpha)$

(b) supply frequency

(c) periodic time of the supply

(d) r.m.s. value of the supply voltage

(a) Supply voltage, $v = v_1 + v_2$ where v_2 is the voltage across the remainder of the circuit.

Hence, $v_2 = v - v_1 = 30\sin 100\pi t - 20\sin(100\pi t - 0.59)$

$= 30\angle 0 - 20\angle -0.59\,\text{rad}$

$= (30 + j0) - (16.619 - j11.127)$

$= 13.381 + j11.127$

$= 17.40\angle 0.694\,\text{rad}$

Hence, by using complex numbers, the resultant in sinusoidal form is:

$$v - v_1 = 30\sin 100\pi t - 20\sin(100\pi t - 0.59)$$
$$= \mathbf{17.40\sin(\omega t + 0.694)\ volts}$$

(b) **Supply frequency,** $f = \dfrac{\omega}{2\pi} = \dfrac{100\pi}{2\pi} = \mathbf{50\ Hz}$

(c) **Periodic time,** $T = \dfrac{1}{f} = \dfrac{1}{50} = \mathbf{0.02\,s}$ or **20 ms**

(d) **R.m.s. value of supply voltage,** $= 0.707 \times 30 = \mathbf{21.21\ volts}$

Now try the following exercise

Exercise 111 Further problems on resultant phasors by complex numbers

In Problems 1 to 4, express the combination of periodic functions in the form $A\sin(\omega t \pm \alpha)$ by using complex numbers:

1. $8\sin\omega t + 5\sin\left(\omega t + \dfrac{\pi}{4}\right)$
   [$12.07\sin(\omega t + 0.297)$]

2. $6\sin\omega t + 9\sin\left(\omega t - \dfrac{\pi}{6}\right)$
   [$14.51\sin(\omega t - 0.315)$]

3. $v = 12\sin\omega t - 5\sin\left(\omega t - \dfrac{\pi}{4}\right)$
   [$9.173\sin(\omega t + 0.396)$]

4. $x = 10\sin\left(\omega t + \dfrac{\pi}{3}\right) - 8\sin\left(\omega t - \dfrac{3\pi}{8}\right)$
   [$16.168\sin(\omega t + 1.451)$]

5. The voltage drops across two components when connected in series across an a.c. supply are: $v_1 = 240\sin 314.2t$ and $v_2 = 150\sin(314.2t - \pi/5)$ volts respectively. Determine the:
   (a) voltage of the supply (given by $v_1 + v_2$) in the form $A\sin(\omega t \pm \alpha)$.
   (b) frequency of the supply.
   [(a) $371.95\sin(314.2t - 0.239)$V (b) 50 Hz]

6. If the supply to a circuit is $v = 25\sin 200\pi t$ volts and the voltage drop across one of the components is $v_1 = 18\sin(200\pi t - 0.43)$ volts, calculate the:
   (a) voltage drop across the remainder of the circuit, given by $v - v_1$, in the form $A\sin(\omega t \pm \alpha)$.
   (b) supply frequency.
   (c) periodic time of the supply.
   [(a) $11.44\sin(200\pi t + 0.715)$V (b) 100 Hz (c) 10 ms]

7. The voltages across three components in a series circuit when connected across an a.c. supply are:

   $v_1 = 20\sin\left(300\pi t - \dfrac{\pi}{6}\right)$ volts,

   $v_2 = 30\sin\left(300\pi t + \dfrac{\pi}{4}\right)$ volts, and

   $v_3 = 60\sin\left(300\pi t - \dfrac{\pi}{3}\right)$ volts.

   Calculate the:
   (a) supply voltage, in sinusoidal form, in the form $A\sin(\omega t \pm \alpha)$.
   (b) frequency of the supply.
   (c) periodic time.
   (d) r.m.s. value of the supply voltage.
   [(a) $79.73\sin(300\pi - 0.536)$ V (b) 150 Hz (c) 6.667 ms (d) 56.37 V]

Chapter 26

Scalar and vector products

26.1 The unit triad

When a vector x of magnitude x units and direction $\theta°$ is divided by the magnitude of the vector, the result is a vector of unit length at angle $\theta°$. The unit vector for a velocity of 10 m/s at 50° is $\frac{10\,\text{m/s at } 50°}{10\,\text{m/s}}$, i.e. 1 at 50°. In general, the unit vector for $\boldsymbol{oa}$ is $\frac{\boldsymbol{oa}}{|\boldsymbol{oa}|}$, the $\boldsymbol{oa}$ being a vector and having both magnitude and direction and $|\boldsymbol{oa}|$ being the magnitude of the vector only.

One method of completely specifying the direction of a vector in space relative to some reference point is to use three unit vectors, mutually at right angles to each other, as shown in Fig. 26.1. Such a system is called a **unit triad**.

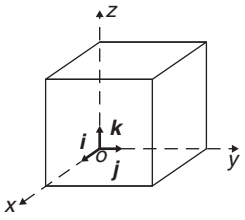


Figure 26.1

In Fig. 26.2, one way to get from o to r is to move x units along $\boldsymbol{i}$ to point a, then y units in direction $\boldsymbol{j}$ to get to b and finally z units in direction $\boldsymbol{k}$ to get to r. The vector $\boldsymbol{or}$ is specified as

$$\boldsymbol{or} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

Problem 1. With reference to three axes drawn mutually at right angles, depict the vectors (i) $\boldsymbol{op} = 4\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}$ and (ii) $\boldsymbol{or} = 5\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$.

The required vectors are depicted in Fig. 26.3, $\boldsymbol{op}$ being shown in Fig. 26.3(a) and $\boldsymbol{or}$ in Fig. 26.3(b).

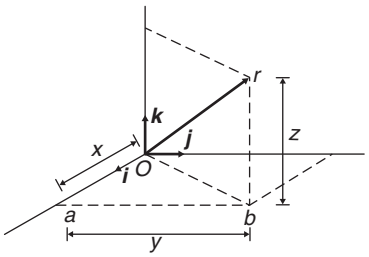
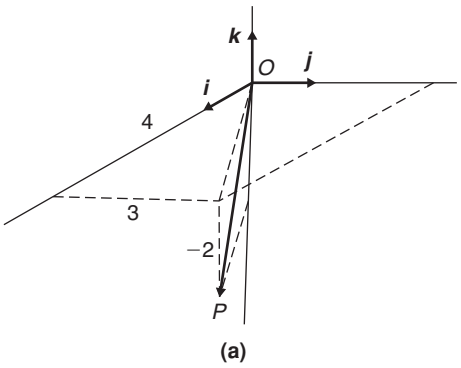


Figure 26.2



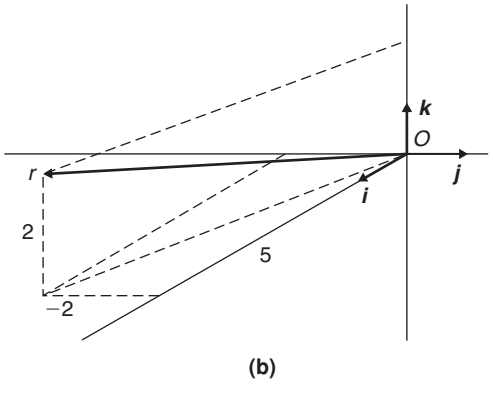


Figure 26.3

26.2 The scalar product of two vectors

When vector $\boldsymbol{oa}$ is multiplied by a scalar quantity, say k, the magnitude of the resultant vector will be k times the magnitude of $\boldsymbol{oa}$ and its direction will remain the same. Thus $2 \times (5\,\text{N at } 20°)$ results in a vector of magnitude 10 N at 20°.

One of the products of two vector quantities is called the **scalar** or **dot product** of two vectors and is defined as the product of their magnitudes multiplied by the cosine of the angle between them. The scalar product of $\boldsymbol{oa}$ and $\boldsymbol{ob}$ is shown as $\boldsymbol{oa} \cdot \boldsymbol{ob}$. For vectors $\boldsymbol{oa} = oa$ at θ_1, and $\boldsymbol{ob} = ob$ at θ_2 where $\theta_2 > \theta_1$, **the scalar product is**:

$$\boldsymbol{oa} \cdot \boldsymbol{ob} = oa\, ob \cos(\theta_2 - \theta_1)$$

For vectors $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ shown in Fig. 26.4, the scalar product is:

$$\boldsymbol{v_1} \cdot \boldsymbol{v_2} = v_1 v_2 \cos\theta$$

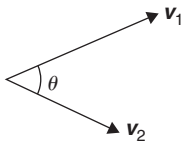


Figure 26.4

The commutative law of algebra, $a \times b = b \times a$ applies to scalar products. This is demonstrated in Fig. 26.5. Let $\boldsymbol{oa}$ represent vector $\boldsymbol{v_1}$ and $\boldsymbol{ob}$ represent vector $\boldsymbol{v_2}$. Then:

$$\boldsymbol{oa} \cdot \boldsymbol{ob} = v_1 v_2 \cos\theta \text{ (by definition of a scalar product)}$$

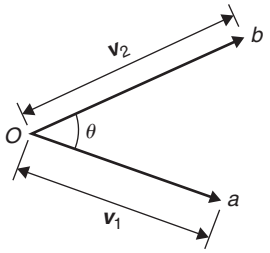


Figure 26.5

Similarly, $\boldsymbol{ob} \cdot \boldsymbol{oa} = v_2 v_1 \cos\theta = v_1 v_2 \cos\theta$ by the commutative law of algebra. Thus $\boldsymbol{oa} \cdot \boldsymbol{ob} = \boldsymbol{ob} \cdot \boldsymbol{oa}$.

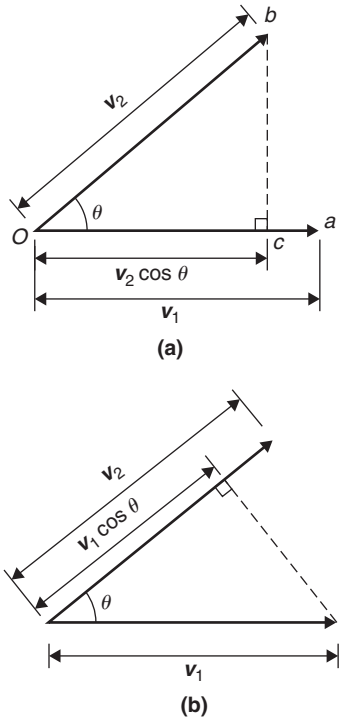


Figure 26.6

The projection of $\boldsymbol{ob}$ on $\boldsymbol{oa}$ is shown in Fig. 26.6(a) and by the geometry of triangle obc, it can be seen that the projection is $v_2 \cos\theta$. Since, by definition

$$\boldsymbol{oa} \cdot \boldsymbol{ob} = v_1 (v_2 \cos\theta),$$

it follows that

$$\boldsymbol{oa} \cdot \boldsymbol{ob} = v_1 \text{ (the projection of } v_2 \text{ on } v_1)$$

Similarly the projection of $\boldsymbol{oa}$ on $\boldsymbol{ob}$ is shown in Fig. 26.6(b) and is $v_1 \cos\theta$. Since by definition

$$\boldsymbol{ob} \cdot \boldsymbol{oa} = v_2 (v_1 \cos\theta),$$

it follows that

$$\boldsymbol{ob} \cdot \boldsymbol{oa} = v_2 (\text{the projection of } v_1 \text{ on } v_2)$$

This shows that the scalar product of two vectors is the product of the magnitude of one vector and the magnitude of the projection of the other vector on it.

The **angle between two vectors** can be expressed in terms of the vector constants as follows:

Because $\boldsymbol{a} \cdot \boldsymbol{b} = a\, b \cos\theta$,

then
$$\cos\theta = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{ab} \qquad (1)$$

Let $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$

and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$

$$\boldsymbol{a}\cdot\boldsymbol{b} = (a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}) \cdot (b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k})$$

Multiplying out the brackets gives:

$$\begin{aligned}\boldsymbol{a}\cdot\boldsymbol{b} = {} & a_1b_1\boldsymbol{i}\cdot\boldsymbol{i} + a_1b_2\boldsymbol{i}\cdot\boldsymbol{j} + a_1b_3\boldsymbol{i}\cdot\boldsymbol{k} \\ & + a_2b_1\boldsymbol{j}\cdot\boldsymbol{i} + a_2b_2\boldsymbol{j}\cdot\boldsymbol{j} + a_2b_3\boldsymbol{j}\cdot\boldsymbol{k} \\ & + a_3b_1\boldsymbol{k}\cdot\boldsymbol{i} + a_3b_2\boldsymbol{k}\cdot\boldsymbol{j} + a_3b_3\boldsymbol{k}\cdot\boldsymbol{k}\end{aligned}$$

However, the unit vectors $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ all have a magnitude of 1 and $\boldsymbol{i}\cdot\boldsymbol{i} = (1)(1)\cos 0° = 1$, $\boldsymbol{i}\cdot\boldsymbol{j} = (1)(1)\cos 90° = 0$, $\boldsymbol{i}\cdot\boldsymbol{k} = (1)(1)\cos 90° = 0$ and similarly $\boldsymbol{j}\cdot\boldsymbol{j} = 1$, $\boldsymbol{j}\cdot\boldsymbol{k} = 0$ and $\boldsymbol{k}\cdot\boldsymbol{k} = 1$. Thus, only terms containing $\boldsymbol{i}\cdot\boldsymbol{i}$, $\boldsymbol{j}\cdot\boldsymbol{j}$ or $\boldsymbol{k}\cdot\boldsymbol{k}$ in the expansion above will not be zero.
Thus, the scalar product

$$\boldsymbol{a}\cdot\boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3 \tag{2}$$

Both a and b in equation (1) can be expressed in terms of a_1, b_1, a_2, b_2, a_3 and b_3.

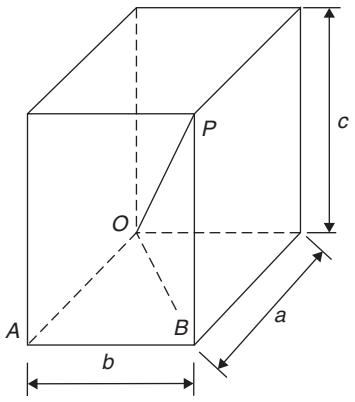


Figure 26.7

From the geometry of Fig. 26.7, the length of diagonal OP in terms of side lengths a, b and c can be obtained from Pythagoras' theorem as follows:

$$OP^2 = OB^2 + BP^2 \text{ and}$$
$$OB^2 = OA^2 + AB^2$$

Thus, $OP^2 = OA^2 + AB^2 + BP^2$
$= a^2 + b^2 + c^2$,
in terms of side lengths

Thus, the **length** or **modulus** or **magnitude** or **norm of vector** $\boldsymbol{OP}$ is given by:

$$\boldsymbol{OP} = \sqrt{(a^2 + b^2 + c^2)} \tag{3}$$

Relating this result to the two vectors $a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$, gives:

$$a = \sqrt{(a_1^2 + a_2^2 + a_3^2)}$$

and $b = \sqrt{(b_1^2 + b_2^2 + b_3^2)}$.

That is, from equation (1),

$$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{(a_1^2 + a_2^2 + a_3^2)}\sqrt{(b_1^2 + b_2^2 + b_3^2)}} \tag{4}$$

Problem 2. Find vector $\boldsymbol{a}$ joining points P and Q where point P has co-ordinates $(4, -1, 3)$ and point Q has co-ordinates $(2, 5, 0)$. Also, find $|a|$, the magnitude or norm of $\boldsymbol{a}$.

Let O be the origin, i.e. its co-ordinates are $(0, 0, 0)$. The position vector of P and Q are given by:

$$\boldsymbol{OP} = 4\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k} \text{ and } \boldsymbol{OQ} = 2\boldsymbol{i} + 5\boldsymbol{j}$$

By the addition law of vectors $\boldsymbol{OP} + \boldsymbol{PQ} = \boldsymbol{OQ}$.

Hence $\boldsymbol{a} = \boldsymbol{PQ} = \boldsymbol{OQ} - \boldsymbol{OP}$

i.e. $\boldsymbol{a} = \boldsymbol{PQ} = (2\boldsymbol{i} + 5\boldsymbol{j}) - (4i - \boldsymbol{j} + 3\boldsymbol{k})$
$= -2\boldsymbol{i} + 6\boldsymbol{j} - 3\boldsymbol{k}$

From equation (3), the magnitude or norm of $\boldsymbol{a}$,

$$|a| = \sqrt{(a^2 + b^2 + c^2)}$$
$$= \sqrt{[(-2)^2 + 6^2 + (-3)^2]} = \sqrt{49} = 7$$

Problem 3. If $\boldsymbol{p} = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{q} = \boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}$ determine:

(i) $\boldsymbol{p}\cdot\boldsymbol{q}$ (ii) $\boldsymbol{p} + \boldsymbol{q}$

(iii) $|\boldsymbol{p} + \boldsymbol{q}|$ (iv) $|\boldsymbol{p}| + |\boldsymbol{q}|$

(i) From equation (2),

if $\boldsymbol{p} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$

and $\boldsymbol{q} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$

then $\boldsymbol{p}\cdot\boldsymbol{q} = a_1b_1 + a_2b_2 + a_3b_3$

When $\boldsymbol{p} = \boldsymbol{2i+j-k}$,

$a_1 = 2,\ a_2 = 1$ and $a_3 = -1$

and when $\boldsymbol{q} = \boldsymbol{i-3j+2k}$,

$b_1 = 1,\ b_2 = -3$ and $b_3 = 2$

Hence $\boldsymbol{p}\cdot\boldsymbol{q} = (2)(1) + (1)(-3) + (-1)(2)$

i.e. $\boldsymbol{p}\cdot\boldsymbol{q} = \boldsymbol{-3}$

(ii) $\boldsymbol{p+q} = (2\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}) + (\boldsymbol{i}-3\boldsymbol{j}+2\boldsymbol{k})$

$= \boldsymbol{3i-2j+k}$

(iii) $|\boldsymbol{p+q}| = |3i - 2j + k|$

From equation (3),

$$|\boldsymbol{p+q}| = \sqrt{[3^2 + (-2)^2 + 1^2]} = \sqrt{\mathbf{14}}$$

(iv) From equation (3),

$$|\boldsymbol{p}| = |2i + j - k| = \sqrt{[2^2 + 1^2 + (-1)^2]} = \sqrt{6}$$

Similarly,

$$|\boldsymbol{q}| = |i - 3j + 2k| = \sqrt{[1^2 + (-3)^2 + 2^2]} = \sqrt{14}$$

Hence $|\boldsymbol{p}| + |\boldsymbol{q}| = \sqrt{6} + \sqrt{14} = \mathbf{6.191}$, correct to 3 decimal places.

Problem 4. Determine the angle between vectors $\boldsymbol{oa}$ and $\boldsymbol{ob}$ when

$\boldsymbol{oa} = \boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}$

and $\boldsymbol{ob} = 2\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}$.

An equation for $\cos\theta$ is given in equation (4)

$$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{(a_1^2 + a_2^2 + a_3^2)}\sqrt{(b_1^2 + b_2^2 + b_3^2)}}$$

Since $\boldsymbol{oa} = \boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}$,

$a_1 = 1, a_2 = 2$ and $a_3 = -3$

Since $\boldsymbol{ob} = 2\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}$,

$b_1 = 2, b_2 = -1$ and $b_3 = 4$

Thus,

$$\cos\theta = \frac{(1\times 2) + (2\times -1) + (-3\times 4)}{\sqrt{(1^2 + 2^2 + (-3)^2)}\sqrt{(2^2 + (-1)^2 + 4^2)}}$$

$$= \frac{-12}{\sqrt{14}\sqrt{21}} = -0.6999$$

i.e. $\theta = 134.4°$ or $225.6°$.

By sketching the position of the two vectors as shown in Problem 1, it will be seen that 225.6° is not an acceptable answer.

Thus the angle between the vectors $\boldsymbol{oa}$ and $\boldsymbol{ob}$, $\boldsymbol{\theta = 134.4°}$

Direction cosines

From Fig. 26.2, $\boldsymbol{or} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ and from equation (3), $|\boldsymbol{or}| = \sqrt{x^2 + y^2 + z^2}$.

If $\boldsymbol{or}$ makes angles of α, β and γ with the co-ordinate axes i, j and k respectively, then:

The direction cosines are:

$$\cos\boldsymbol{\alpha} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

$$\cos\boldsymbol{\beta} = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$

and $$\cos\boldsymbol{\gamma} = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$

such that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

The values of $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called the **direction cosines** of $\boldsymbol{or}$.

Problem 5. Find the direction cosines of $3\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$.

$$\sqrt{x^2 + y^2 + z^2} = \sqrt{3^2 + 2^2 + 1^2} = \sqrt{14}$$

The direction cosines are:

$$\cos\alpha = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{3}{\sqrt{14}} = \mathbf{0.802}$$

$$\cos\beta = \frac{y}{\sqrt{x^2+y^2+z^2}} = \frac{2}{\sqrt{14}} = \mathbf{0.535}$$

and $$\cos\gamma = \frac{y}{\sqrt{x^2+y^2+z^2}} = \frac{1}{\sqrt{14}} = \mathbf{0.267}$$

(and hence $\alpha = \cos^{-1} 0.802 = 36.7°$, $\beta = \cos^{-1} 0.535 = 57.7°$ and $\gamma = \cos^{-1} 0.267 = 74.5°$).
Note that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 0.802^2 + 0.535^2 + 0.267^2 = 1$.

Practical application of scalar product

Problem 6. A constant force of $\boldsymbol{F} = 10\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ newtons displaces an object from $\boldsymbol{A} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ to $\boldsymbol{B} = 2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$ (in metres). Find the work done in newton metres.

One of the applications of scalar products is to the work done by a constant force when moving a body. The work done is the product of the applied force and the distance moved in the direction of the force.

i.e. **work done** $= \boldsymbol{F} \cdot \boldsymbol{d}$

The principles developed in Problem 13, page 262, apply equally to this problem when determining the displacement. From the sketch shown in Fig. 26.8,

$$\boldsymbol{AB} = \boldsymbol{AO} + \boldsymbol{OB} = \boldsymbol{OB} - \boldsymbol{OA}$$

that is $$\boldsymbol{AB} = (2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}) - (\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$$

$$= \boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$$

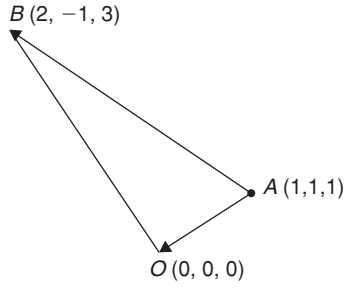


Figure 26.8

The work done is $\boldsymbol{F} \cdot \boldsymbol{d}$, that is $\boldsymbol{F} \cdot \boldsymbol{AB}$ in this case

i.e. **work done** $= (10\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}) \cdot (\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k})$

But from equation (2),

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Hence **work done** $=$
$(10 \times 1) + (2 \times (-2)) + ((-1) \times 2) = \mathbf{4\,Nm}$.
(Theoretically, it is quite possible to get a negative answer to a 'work done' problem. This indicates that the force must be in the opposite sense to that given, in order to give the displacement stated.)

Now try the following exercise

Exercise 112 Further problems on scalar products

1. Find the scalar product $\boldsymbol{a} \cdot \boldsymbol{b}$ when
   (i) $\boldsymbol{a} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$
   (ii) $\boldsymbol{a} = \boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$
   [(i) 7 (ii) 0]

Given $\boldsymbol{p} = 2\boldsymbol{i} - 3\boldsymbol{j}$, $\boldsymbol{q} = 4\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{r} = \boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}$, determine the quantities stated in problems 2 to 8.

2. (a) $\boldsymbol{p} \cdot \boldsymbol{q}$ (b) $\boldsymbol{p} \cdot \boldsymbol{r}$ [(a) −12 (b) −4]
3. (a) $\boldsymbol{q} \cdot \boldsymbol{r}$ (b) $\boldsymbol{r} \cdot \boldsymbol{q}$ [(a) 11 (b) 11]
4. (a) $|\boldsymbol{p}|$ (b) $|\boldsymbol{r}|$ [(a) $\sqrt{13}$ (b) $\sqrt{14}$]
5. (a) $\boldsymbol{p} \cdot (\boldsymbol{q} + \boldsymbol{r})$ (b) $2\boldsymbol{r} \cdot (\boldsymbol{q} - 2\boldsymbol{p})$
   [(a) −16 (b) 38]
6. (a) $|\boldsymbol{p} + \boldsymbol{r}|$ (b) $|\boldsymbol{p}| + |\boldsymbol{r}|$
   [(a) $\sqrt{19}$ (b) 7.347]
7. Find the angle between (a) $\boldsymbol{p}$ and $\boldsymbol{q}$ (b) $\boldsymbol{q}$ and $\boldsymbol{r}$. [(a) 143.82° (b) 44.52°]
8. Determine the direction cosines of (a) $\boldsymbol{p}$ (b) $\boldsymbol{q}$ (c) $\boldsymbol{r}$.
   $$\begin{bmatrix} \text{(a) } 0.555, -0.832, 0 \\ \text{(b) } 0, 0.970, -0.243 \\ \text{(c) } 0.267, 0.535, -0.802 \end{bmatrix}$$
9. Determine the angle between the forces:
   $\boldsymbol{F_1} = 3\boldsymbol{i} + 4\boldsymbol{j} + 5\boldsymbol{k}$ and
   $\boldsymbol{F_2} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ [11.54°]

10. Find the angle between the velocity vectors $\boldsymbol{v_1} = 5\boldsymbol{i} + 2\boldsymbol{j} + 7\boldsymbol{k}$ and $\boldsymbol{v_2} = 4\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$. [66.40°]

11. Calculate the work done by a force $\boldsymbol{F} = (-5\boldsymbol{i} + \boldsymbol{j} + 7\boldsymbol{k})$ N when its point of application moves from point $(-2\boldsymbol{i} - 6\boldsymbol{j} + \boldsymbol{k})$ m to the point $(\boldsymbol{i} - \boldsymbol{j} + 10\boldsymbol{k})$ m. [53 Nm]

26.3 Vector products

A second product of two vectors is called the **vector** or **cross product** and is defined in terms of its modulus and the magnitudes of the two vectors and the sine of the angle between them. The vector product of vectors $\boldsymbol{oa}$ and $\boldsymbol{ob}$ is written as $\boldsymbol{oa} \times \boldsymbol{ob}$ and is defined by:

$$|\boldsymbol{oa} \times \boldsymbol{ob}| = oa\ ob \sin\theta$$

where θ is the angle between the two vectors.
The direction of $\boldsymbol{oa} \times \boldsymbol{ob}$ is perpendicular to both $\boldsymbol{oa}$ and $\boldsymbol{ob}$, as shown in Fig. 26.9.

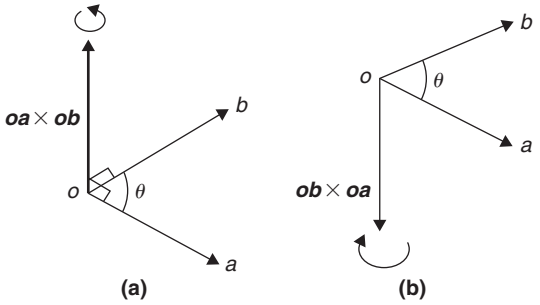


Figure 26.9

The direction is obtained by considering that a right-handed screw is screwed along $\boldsymbol{oa} \times \boldsymbol{ob}$ with its head at the origin and if the direction of $\boldsymbol{oa} \times \boldsymbol{ob}$ is correct, the head should rotate from $\boldsymbol{oa}$ to $\boldsymbol{ob}$, as shown in Fig. 26.9(a). It follows that the direction of $\boldsymbol{ob} \times \boldsymbol{oa}$ is as shown in Fig. 26.9(b). Thus $\boldsymbol{oa} \times \boldsymbol{ob}$ is not equal to $\boldsymbol{ob} \times \boldsymbol{oa}$. The magnitudes of $oa\,ob\sin\theta$ are the same but their directions are 180° displaced, i.e.

$$\boldsymbol{oa} \times \boldsymbol{ob} = -\boldsymbol{ob} \times \boldsymbol{oa}$$

The vector product of two vectors may be expressed in terms of the unit vectors. Let two vectors, a and b, be such that:

$$\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k} \text{ and}$$
$$\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$$

Then,

$$\begin{aligned}\boldsymbol{a} \times \boldsymbol{b} &= (a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}) \times (b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}) \\ &= a_1b_1\boldsymbol{i} \times \boldsymbol{i} + a_1b_2\boldsymbol{i} \times \boldsymbol{j} \\ &\quad + a_1b_3\boldsymbol{i} \times \boldsymbol{k} + a_2b_1\boldsymbol{j} \times \boldsymbol{i} + a_2b_2\boldsymbol{j} \times \boldsymbol{j} \\ &\quad + a_2b_3\boldsymbol{j} \times \boldsymbol{k} + a_3b_1\boldsymbol{k} \times \boldsymbol{i} + a_3b_2\boldsymbol{k} \times \boldsymbol{j} \\ &\quad + a_3b_3\boldsymbol{k} \times \boldsymbol{k}\end{aligned}$$

But by the definition of a vector product,

$$\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k},\ \boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i} \text{ and } \boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j}$$

Also $\boldsymbol{i} \times \boldsymbol{i} = \boldsymbol{j} \times \boldsymbol{j} = \boldsymbol{k} \times \boldsymbol{k} = (1)(1)\sin 0° = 0$.

Remembering that $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$ gives:

$$\begin{aligned}\boldsymbol{a} \times \boldsymbol{b} &= a_1b_2\boldsymbol{k} - a_1b_3\boldsymbol{j} - a_2b_1\boldsymbol{k} + a_2b_3\boldsymbol{i} \\ &\quad + a_3b_1\boldsymbol{j} - a_3b_2\boldsymbol{i}\end{aligned}$$

Grouping the $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ terms together, gives:

$$\begin{aligned}\boldsymbol{a} \times \boldsymbol{b} &= (a_2b_3 - a_3b_2)\boldsymbol{i} + (a_3b_1 - a_1b_3)\boldsymbol{j} \\ &\quad + (a_1b_2 - a_2b_1)\boldsymbol{k}\end{aligned}$$

The vector product can be written in determinant form as:

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \boldsymbol{a_1} & \boldsymbol{a_2} & \boldsymbol{a_3} \\ \boldsymbol{b_1} & \boldsymbol{b_2} & \boldsymbol{b_3} \end{vmatrix} \tag{5}$$

The 3×3 determinant $\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ is evaluated as:

$$\boldsymbol{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

where

$$\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = a_2b_3 - a_3b_2,$$

$$\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} = a_1b_3 - a_3b_1 \text{ and}$$

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

The magnitude of the vector product of two vectors can be found by expressing it in scalar product form and then using the relationship

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Squaring both sides of a vector product equation gives:

$$(|\boldsymbol{a}\times\boldsymbol{b}|)^2 = a^2b^2\sin^2\theta = a^2b^2(1-\cos^2\theta)$$
$$= a^2b^2 - a^2b^2\cos^2\theta \qquad (6)$$

It is stated in Section 26.2 that $\boldsymbol{a}\cdot\boldsymbol{b} = ab\cos\theta$, hence

$$\boldsymbol{a}\cdot\boldsymbol{a} = a^2\cos\theta.$$

But $\theta = 0°$, thus $\boldsymbol{a}\cdot\boldsymbol{a} = a^2$

Also, $\cos\theta = \dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{ab}$.

Multiplying both sides of this equation by a^2b^2 and squaring gives:

$$a^2b^2\cos^2\theta = \frac{a^2b^2(a\cdot b)^2}{a^2b^2} = (\boldsymbol{a}\cdot\boldsymbol{b})^2$$

Substituting in equation (6) above for $a^2 = \boldsymbol{a}\cdot\boldsymbol{a}$, $b^2 = \boldsymbol{b}\cdot\boldsymbol{b}$ and $a^2b^2\cos^2\theta = (\boldsymbol{a}\cdot\boldsymbol{b})^2$ gives:

$$(|\boldsymbol{a}\times\boldsymbol{b}|)^2 = (\boldsymbol{a}\cdot\boldsymbol{a})(\boldsymbol{b}\cdot\boldsymbol{b}) - (\boldsymbol{a}\cdot\boldsymbol{b})^2$$

That is,

$$|\boldsymbol{a}\times\boldsymbol{b}| = \sqrt{[(\boldsymbol{a}\cdot\boldsymbol{a})(\boldsymbol{b}\cdot\boldsymbol{b}) - (\boldsymbol{a}\cdot\boldsymbol{b})^2]} \qquad (7)$$

Problem 7. For the vectors $\boldsymbol{a} = \boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$ find (i) $\boldsymbol{a}\times\boldsymbol{b}$ and (ii) $|\boldsymbol{a}\times\boldsymbol{b}|$.

(i) From equation (5),

$$\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$
$$= \boldsymbol{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Hence

$$\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} i & j & k \\ 1 & 4 & -2 \\ 2 & -1 & 3 \end{vmatrix}$$
$$= \boldsymbol{i}\begin{vmatrix} 4 & -2 \\ -1 & 3 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} 1 & 4 \\ 2 & -1 \end{vmatrix}$$
$$= \boldsymbol{i}(12-2) - \boldsymbol{j}(3+4) + \boldsymbol{k}(-1-8)$$
$$= \mathbf{10}\boldsymbol{i} - \mathbf{7}\boldsymbol{j} - \mathbf{9}\boldsymbol{k}$$

(ii) From equation (7)

$$|\boldsymbol{a}\times\boldsymbol{b}| = \sqrt{[(\boldsymbol{a}\cdot\boldsymbol{a})(\boldsymbol{b}\cdot\boldsymbol{b}) - (\boldsymbol{a}\cdot\boldsymbol{b})^2]}$$

Now $\boldsymbol{a}\cdot\boldsymbol{a} = (1)(1) + (4\times 4) + (-2)(-2)$
$= 21$

$\boldsymbol{b}\cdot\boldsymbol{b} = (2)(2) + (-1)(-1) + (3)(3)$
$= 14$

and $\boldsymbol{a}\cdot\boldsymbol{b} = (1)(2) + (4)(-1) + (-2)(3)$
$= -8$

Thus $|\boldsymbol{a}\times\boldsymbol{b}| = \sqrt{(21\times 14 - 64)}$
$= \sqrt{230} = \mathbf{15.17}$

Problem 8. If $\boldsymbol{p} = 4\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$, $\boldsymbol{q} = 3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{r} = \boldsymbol{i} - 2\boldsymbol{k}$ find (a) $(\boldsymbol{p} - 2\boldsymbol{q})\times\boldsymbol{r}$ (b) $\boldsymbol{p}\times(2\boldsymbol{r}\times 3\boldsymbol{q})$.

(a) $(\boldsymbol{p} - 2\boldsymbol{q})\times\boldsymbol{r} = [4\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$
$- 2(3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k})]\times(\boldsymbol{i} - 2\boldsymbol{k})$
$= (-2\boldsymbol{i} + 5\boldsymbol{j} - 4\boldsymbol{k})\times(\boldsymbol{i} - 2\boldsymbol{k})$
$= \begin{vmatrix} i & j & k \\ -2 & 5 & -4 \\ 1 & 0 & -2 \end{vmatrix}$

from equation (5)

$$= \boldsymbol{i}\begin{vmatrix} 5 & -4 \\ 0 & -2 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} -2 & -4 \\ 1 & -2 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} -2 & 5 \\ 1 & 0 \end{vmatrix}$$
$$= \boldsymbol{i}(-10-0) - \boldsymbol{j}(4+4) + \boldsymbol{k}(0-5), \text{ i.e.}$$
$$(\boldsymbol{p} - 2\boldsymbol{q})\times\boldsymbol{r} = \mathbf{-10}\boldsymbol{i} - \mathbf{8}\boldsymbol{j} - \mathbf{5}\boldsymbol{k}$$

(b) $(2\boldsymbol{r}\times 3\boldsymbol{q}) = (2\boldsymbol{i} - 4\boldsymbol{k})\times(9\boldsymbol{i} - 6\boldsymbol{j} + 3\boldsymbol{k})$
$= \begin{vmatrix} i & j & k \\ 2 & 0 & -4 \\ 9 & -6 & 3 \end{vmatrix}$
$= \boldsymbol{i}(0-24) - \boldsymbol{j}(6+36) + \boldsymbol{k}(-12-0)$
$= \mathbf{-24}\boldsymbol{i} - \mathbf{42}\boldsymbol{j} - \mathbf{12}\boldsymbol{k}$

Hence

$$\begin{aligned}\boldsymbol{p}\times(2\boldsymbol{r}\times 3\boldsymbol{q}) &= (4\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{k})\\ &\quad\times(-24\boldsymbol{i}-42\boldsymbol{j}-12\boldsymbol{k})\\ &= \begin{vmatrix} i & j & k\\ 4 & 1 & -2\\ -24 & -42 & -12\end{vmatrix}\\ &= \boldsymbol{i}(-12-84)-\boldsymbol{j}(-48-48)\\ &\quad+\boldsymbol{k}(-168+24)\\ &= \mathbf{-96\boldsymbol{i}+96\boldsymbol{j}-144\boldsymbol{k}}\\ &\quad\mathbf{or}\ \mathbf{-48(2\boldsymbol{i}-2\boldsymbol{j}+3\boldsymbol{k})}\end{aligned}$$

Practical applications of vector products

Problem 9. Find the moment and the magnitude of the moment of a force of $(\boldsymbol{i}+2\boldsymbol{j}-3\boldsymbol{k})$ newtons about point B having co-ordinates (0, 1, 1), when the force acts on a line through A whose co-ordinates are (1, 3, 4).

The moment $\boldsymbol{M}$ about point B of a force vector $\boldsymbol{F}$ which has a position vector of $\boldsymbol{r}$ from A is given by:

$$\boldsymbol{M}=\boldsymbol{r}\times\boldsymbol{F}$$

$\boldsymbol{r}$ is the vector from B to A, i.e. $\boldsymbol{r}=\boldsymbol{BA}$.
But $\boldsymbol{BA}=\boldsymbol{BO}+\boldsymbol{OA}=\boldsymbol{OA}-\boldsymbol{OB}$ (see Problem 13, page 262), that is:

$$\begin{aligned}\boldsymbol{r} &= (\boldsymbol{i}+3\boldsymbol{j}+4\boldsymbol{k})-(\boldsymbol{j}+\boldsymbol{k})\\ &= \boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k}\end{aligned}$$

Moment,

$$\begin{aligned}\boldsymbol{M}=\boldsymbol{r}\times\boldsymbol{F} &= (\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k})\times(\boldsymbol{i}+2\boldsymbol{j}-3\boldsymbol{k})\\ &= \begin{vmatrix} i & j & k\\ 1 & 2 & 3\\ 1 & 2 & -3\end{vmatrix}\\ &= \boldsymbol{i}(-6-6)-\boldsymbol{j}(-3-3)\\ &\quad+\boldsymbol{k}(2-2)\\ &= \mathbf{-12\boldsymbol{i}+6\boldsymbol{j}\,Nm}\end{aligned}$$

The magnitude of $\boldsymbol{M}$,

$$\begin{aligned}|\boldsymbol{M}| &= |\boldsymbol{r}\times\boldsymbol{F}|\\ &= \sqrt{[(\boldsymbol{r}\cdot\boldsymbol{r})(\boldsymbol{F}\cdot\boldsymbol{F})-(\boldsymbol{r}\cdot\boldsymbol{F})^2]}\\ \boldsymbol{r}\cdot\boldsymbol{r} &= (1)(1)+(2)(2)+(3)(3)=14\\ \boldsymbol{F}\cdot\boldsymbol{F} &= (1)(1)+(2)(2)+(-3)(-3)=14\\ \boldsymbol{r}\cdot\boldsymbol{F} &= (1)(1)+(2)(2)+(3)(-3)=-4\\ |\boldsymbol{M}| &= \sqrt{[14\times 14-(-4)^2]}\\ &= \sqrt{180}\,\mathrm{Nm}=\mathbf{13.42\,Nm}\end{aligned}$$

Problem 10. The axis of a circular cylinder coincides with the z-axis and it rotates with an angular velocity of $(2\boldsymbol{i}-5\boldsymbol{j}+7\boldsymbol{k})$ rad/s. Determine the tangential velocity at a point P on the cylinder, whose co-ordinates are $(\boldsymbol{j}+3\boldsymbol{k})$ metres, and also determine the magnitude of the tangential velocity.

The velocity v of point P on a body rotating with angular velocity ω about a fixed axis is given by:

$$\boldsymbol{v}=\boldsymbol{\omega}\times\boldsymbol{r},$$

where r is the point on vector P.

Thus
$$\begin{aligned}\boldsymbol{v} &= (2\boldsymbol{i}-5\boldsymbol{j}+7\boldsymbol{k})\times(\boldsymbol{j}+3\boldsymbol{k})\\ &= \begin{vmatrix} i & j & k\\ 2 & -5 & 7\\ 0 & 1 & 3\end{vmatrix}\\ &= \boldsymbol{i}(-15-7)-\boldsymbol{j}(6-0)+\boldsymbol{k}(2-0)\\ &= \mathbf{(-22\boldsymbol{i}-6\boldsymbol{j}+2\boldsymbol{k})\,m/s}\end{aligned}$$

The magnitude of $\boldsymbol{v}$,

$$\begin{aligned}|\boldsymbol{v}| &= \sqrt{[(\boldsymbol{\omega}\cdot\boldsymbol{\omega})(\boldsymbol{r}\cdot\boldsymbol{r})-(\boldsymbol{r}\cdot\boldsymbol{\omega})^2]}\\ \boldsymbol{\omega}\cdot\boldsymbol{\omega} &= (2)(2)+(-5)(-5)+(7)(7)=78\\ \boldsymbol{r}\cdot\boldsymbol{r} &= (0)(0)+(1)(1)+(3)(3)=10\\ \boldsymbol{\omega}\cdot\boldsymbol{r} &= (2)(0)+(-5)(1)+(7)(3)=16\end{aligned}$$

Hence,

$$\begin{aligned}|\boldsymbol{v}| &= \sqrt{(78\times 10-16^2)}\\ &= \sqrt{524}\,\mathrm{m/s}=\mathbf{22.89\,m/s}\end{aligned}$$

Now try the following exercise

Exercise 113 Further problems on vector products

In problems 1 to 4, determine the quantities stated when
$\boldsymbol{p}=3\boldsymbol{i}+2\boldsymbol{k}$, $\boldsymbol{q}=\boldsymbol{i}-2\boldsymbol{j}+3\boldsymbol{k}$ and
$\boldsymbol{r}=-4\boldsymbol{i}+3\boldsymbol{j}-\boldsymbol{k}$.

1. (a) $\boldsymbol{p}\times\boldsymbol{q}$ (b) $\boldsymbol{q}\times\boldsymbol{p}$
[(a) $4\boldsymbol{i}-7\boldsymbol{j}-6\boldsymbol{k}$ (b) $-4\boldsymbol{i}+7\boldsymbol{j}+6\boldsymbol{k}$]

2. (a) $|\boldsymbol{p}\times\boldsymbol{r}|$ (b) $|\boldsymbol{r}\times\boldsymbol{q}|$
[(a) 11.92 (b) 13.96]

3. (a) $2\boldsymbol{p}\times 3\boldsymbol{r}$ (b) $(\boldsymbol{p}+\boldsymbol{r})\times\boldsymbol{q}$
$\left[\begin{array}{l}\text{(a) } -36\boldsymbol{i}-30\boldsymbol{j}-54\boldsymbol{k}\\ \text{(b) } 11\boldsymbol{i}+4\boldsymbol{j}-\boldsymbol{k}\end{array}\right]$

4. (a) $\boldsymbol{p}\times(\boldsymbol{r}\times\boldsymbol{q})$ (b) $(3\boldsymbol{p}\times 2\boldsymbol{r})\times\boldsymbol{q}$
$\left[\begin{array}{l}\text{(a) } -22\boldsymbol{i}-\boldsymbol{j}+33\boldsymbol{k}\\ \text{(b) } 18\boldsymbol{i}+162\boldsymbol{j}+102\boldsymbol{k}\end{array}\right]$

5. For vectors $\boldsymbol{p}=4\boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k}$ and $\boldsymbol{q}=-2\boldsymbol{i}+3\boldsymbol{j}-2\boldsymbol{k}$ determine: (i) $\boldsymbol{p}\cdot\boldsymbol{q}$ (ii) $\boldsymbol{p}\times\boldsymbol{q}$ (iii) $|\boldsymbol{p}\times\boldsymbol{q}|$ (iv) $\boldsymbol{q}\times\boldsymbol{p}$ and (v) the angle between the vectors.
$\left[\begin{array}{l}\text{(i) } -15 \text{ (ii) } -4\boldsymbol{i}+4\boldsymbol{j}+10\boldsymbol{k}\\ \text{(iii) } 11.49 \text{ (iv) } 4\boldsymbol{i}-4\boldsymbol{j}-10\boldsymbol{k}\\ \text{(v) } 142.55^\circ\end{array}\right]$

6. For vectors $\boldsymbol{a}=-7\boldsymbol{i}+4\boldsymbol{j}+\frac{1}{2}\boldsymbol{k}$ and $\boldsymbol{b}=6\boldsymbol{i}-5\boldsymbol{j}-\boldsymbol{k}$ find (i) $\boldsymbol{a}\cdot\boldsymbol{b}$ (ii) $\boldsymbol{a}\times\boldsymbol{b}$ (iii) $|\boldsymbol{a}\times\boldsymbol{b}|$ (iv) $\boldsymbol{b}\times\boldsymbol{a}$ and (v) the angle between the vectors.
$\left[\begin{array}{l}\text{(i) } -62\frac{1}{2} \text{ (ii) } -1\frac{1}{2}\boldsymbol{i}-4\boldsymbol{j}+11\boldsymbol{k}\\ \text{(iii) } 11.80 \text{ (iv) } 1\frac{1}{2}\boldsymbol{i}+4\boldsymbol{j}-11\boldsymbol{k}\\ \text{(v) } 169.31^\circ\end{array}\right]$

7. Forces of $(\boldsymbol{i}+3\boldsymbol{j})$, $(-2\boldsymbol{i}-\boldsymbol{j})$, $(\boldsymbol{i}-2\boldsymbol{j})$ newtons act at three points having position vectors of $(2\boldsymbol{i}+5\boldsymbol{j})$, $4\boldsymbol{j}$ and $(-\boldsymbol{i}+\boldsymbol{j})$ metres respectively. Calculate the magnitude of the moment.
[10 Nm]

8. A force of $(2\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k})$ newtons acts on a line through point P having co-ordinates (0, 3, 1) metres. Determine the moment vector and its magnitude about point Q having co-ordinates (4, 0, −1) metres.
$\left[\begin{array}{l}\boldsymbol{M}=(5\boldsymbol{i}+8\boldsymbol{j}-2\boldsymbol{k})\text{ Nm},\\ |\boldsymbol{M}|=9.64\text{ Nm}\end{array}\right]$

9. A sphere is rotating with angular velocity ω about the z-axis of a system, the axis coinciding with the axis of the sphere. Determine the velocity vector and its magnitude at position $(-5\boldsymbol{i}+2\boldsymbol{j}-7\boldsymbol{k})$ m, when the angular velocity is $(\boldsymbol{i}+2\boldsymbol{j})$ rad/s.
$\left[\begin{array}{l}\boldsymbol{v}=-14\boldsymbol{i}+7\boldsymbol{j}+12\boldsymbol{k},\\ |\boldsymbol{v}|=19.72\text{ m/s}\end{array}\right]$

10. Calculate the velocity vector and its magnitude for a particle rotating about the z-axis at an angular velocity of $(3\boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k})$ rad/s when the position vector of the particle is at $(\boldsymbol{i}-5\boldsymbol{j}+4\boldsymbol{k})$ m.
[$6\boldsymbol{i}-10\boldsymbol{j}-14\boldsymbol{k}$, 18.22 m/s]

26.4 Vector equation of a line

The equation of a straight line may be determined, given that it passes through the point A with position vector $\boldsymbol{a}$ relative to O, and is parallel to vector $\boldsymbol{b}$. Let r be the position vector of a point P on the line, as shown in Fig. 26.10.

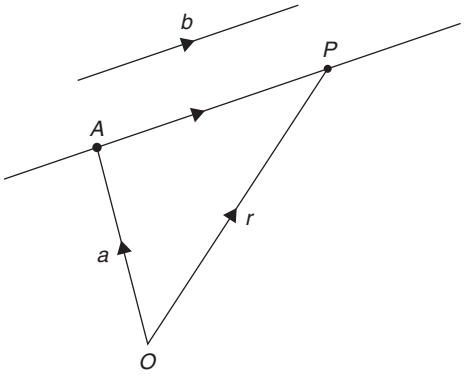


Figure 26.10

By vector addition, $\boldsymbol{OP}=\boldsymbol{OA}+\boldsymbol{AP}$,
i.e. $\boldsymbol{r}=\boldsymbol{a}+\boldsymbol{AP}$.
However, as the straight line through A is parallel to the free vector b (**free vector** means one that has the same

magnitude, direction and sense), then $\boldsymbol{AP}=\lambda\boldsymbol{b}$, where λ is a scalar quantity. Hence, from above,

$$\boldsymbol{r}=\boldsymbol{a}+\lambda\boldsymbol{b} \tag{8}$$

If, say, $\boldsymbol{r}=x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$, $\boldsymbol{a}=a_1\boldsymbol{i}+a_2\boldsymbol{j}+a_3\boldsymbol{k}$ and $\boldsymbol{b}=b_1\boldsymbol{i}+b_2\boldsymbol{j}+b_3\boldsymbol{k}$, then from equation (8),

$$x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}=(a_1\boldsymbol{i}+a_2\boldsymbol{j}+a_3\boldsymbol{k}) + \lambda(b_1\boldsymbol{i}+b_2\boldsymbol{j}+b_3\boldsymbol{k})$$

Hence $x=a_1+\lambda b_1$, $y=a_2+\lambda b_2$ and $z=a_3+\lambda b_3$. Solving for λ gives:

$$\frac{x-a_1}{b_1}=\frac{y-a_2}{b_2}=\frac{z-a_3}{b_3}=\lambda \tag{9}$$

Equation (9) is the standard Cartesian form for the vector equation of a straight line.

Problem 11. (a) Determine the vector equation of the line through the point with position vector $2\boldsymbol{i}+3\boldsymbol{j}-\boldsymbol{k}$ which is parallel to the vector $\boldsymbol{i}-2\boldsymbol{j}+3\boldsymbol{k}$. (b) Find the point on the line corresponding to $\lambda=3$ in the resulting equation of part (a). (c) Express the vector equation of the line in standard Cartesian form.

(a) From equation (8),

$$\boldsymbol{r}=\boldsymbol{a}+\lambda\boldsymbol{b}$$

i.e. $\boldsymbol{r}=(2\boldsymbol{i}+3\boldsymbol{j}-\boldsymbol{k})+\lambda(\boldsymbol{i}-2\boldsymbol{j}+3\boldsymbol{k})$

or $\boldsymbol{r}=(2+\lambda)\boldsymbol{i}+(3-2\lambda)\boldsymbol{j}+(3\lambda-1)\boldsymbol{k}$

which is the vector equation of the line.

(b) When $\lambda=3$, $\boldsymbol{r}=5\boldsymbol{i}-3\boldsymbol{j}+8\boldsymbol{k}$.

(c) From equation (9),

$$\frac{x-a_1}{b_1}=\frac{y-a_2}{b_2}=\frac{z-a_3}{b_3}=\lambda$$

Since $\boldsymbol{a}=2\boldsymbol{i}+3\boldsymbol{j}-\boldsymbol{k}$, then $a_1=2$, $a_2=3$ and $a_3=-1$ and $\boldsymbol{b}=\boldsymbol{i}-2\boldsymbol{j}+3\boldsymbol{k}$, then $b_1=1, b_2=-2$ and $b_3=3$

Hence, the Cartesian equations are:

$$\frac{x-2}{1}=\frac{y-3}{-2}=\frac{z-(-1)}{3}=\lambda$$

i.e. $\boldsymbol{x-2=\dfrac{3-y}{2}=\dfrac{z+1}{3}=\lambda}$

Problem 12. The equation

$$\frac{2x-1}{3}=\frac{y+4}{3}=\frac{-z+5}{2}$$

represents a straight line. Express this in vector form.

Comparing the given equation with equation (9), shows that the coefficients of x, y and z need to be equal to unity.

Thus $\dfrac{2x-1}{3}=\dfrac{y+4}{3}=\dfrac{-z+5}{2}$ becomes:

$$\frac{x-\frac{1}{2}}{\frac{3}{2}}=\frac{y+4}{3}=\frac{z-5}{-2}$$

Again, comparing with equation (9), shows that

$a_1=\dfrac{1}{2}$, $a_2=-4$ and $a_3=5$ and

$b_1=\dfrac{3}{2}$, $b_2=3$ and $b_3=-2$

In vector form the equation is:

$$\boldsymbol{r}=(a_1+\lambda b_1)\boldsymbol{i}+(a_2+\lambda b_2)\boldsymbol{j}+(a_3+\lambda b_3)\boldsymbol{k},$$ from equation (8)

i.e. $\boldsymbol{r}=\left(\dfrac{1}{2}+\dfrac{3}{2}\lambda\right)\boldsymbol{i}+(-4+3\lambda)\boldsymbol{j}+(5-2\lambda)\boldsymbol{k}$

or $\boldsymbol{r}=\dfrac{1}{2}(\boldsymbol{1+3\lambda})\boldsymbol{i}+(\boldsymbol{3\lambda-4})\boldsymbol{j}+(\boldsymbol{5-2\lambda})\boldsymbol{k}$

Now try the following exercise

Exercise 114 Further problems on the vector equation of a line

1. Find the vector equation of the line through the point with position vector $5\boldsymbol{i}-2\boldsymbol{j}+3\boldsymbol{k}$ which

is parallel to the vector $2\boldsymbol{i}+7\boldsymbol{j}-4\boldsymbol{k}$. Determine the point on the line corresponding to $\lambda=2$ in the resulting equation.

$$\left[\begin{array}{c}\boldsymbol{r}=(5+2\lambda)\boldsymbol{i}+(7\lambda-2)\boldsymbol{j}\\ +(3-4\lambda)\boldsymbol{k};\\ \boldsymbol{r}=9\boldsymbol{i}+12\boldsymbol{j}-5\boldsymbol{k}\end{array}\right]$$

2. Express the vector equation of the line in problem 1 in standard Cartesian form.

$$\left[\frac{x-5}{2}=\frac{y+2}{7}=\frac{3-z}{4}=\lambda\right]$$

In problems 3 and 4, express the given straight line equations in vector form.

3. $\dfrac{3x-1}{4}=\dfrac{5y+1}{2}=\dfrac{4-z}{3}$

$$\left[\begin{array}{c}\boldsymbol{r}=\frac{1}{3}(1+4\lambda)\boldsymbol{i}+\frac{1}{5}(2\lambda-1)\boldsymbol{j}\\ +(4-3\lambda)\boldsymbol{k}\end{array}\right]$$

4. $2x+1=\dfrac{1-4y}{5}=\dfrac{3z-1}{4}$

$$\left[\begin{array}{c}\boldsymbol{r}=\frac{1}{2}(\lambda-1)\boldsymbol{i}+\frac{1}{4}(1-5\lambda)\boldsymbol{j}\\ +\frac{1}{3}(1+4\lambda)\boldsymbol{k}\end{array}\right]$$

Revision Test 8

This Revision Test covers the material contained in Chapters 24 to 26. *The marks for each question are shown in brackets at the end of each question.*

1. State whether the following are scalar or vector quantities:
   (a) A temperature of 50°C
   (b) A downward force of 80 N
   (c) 300 J of work
   (d) A south-westerly wind of 15 knots
   (e) 70 m distance
   (f) An acceleration of 25 m/s^2 at 30° to the horizontal (6)

2. Calculate the resultant and direction of the force vectors shown in Fig. RT8.1, correct to 2 decimal places. (7)

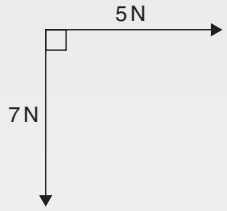


Figure RT8.1

3. Four coplanar forces act at a point A as shown in Fig. RT8.2 Determine the value and direction of the resultant force by (a) drawing (b) by calculation using horizontal and vertical components. (10)

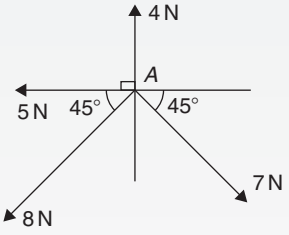


Figure RT8.2

4. The instantaneous values of two alternating voltages are given by:

$$v_1 = 150\sin(\omega t + \pi/3) \text{ volts and}$$

$$v_2 = 90\sin(\omega t - \pi/6) \text{ volts}$$

Plot the two voltages on the same axes to scales of 1 cm = 50 volts and 1 cm = $\frac{\pi}{6}$ rad.

Obtain a sinusoidal expression for the resultant $v_1 + v_2$ in the form $R\sin(\omega t + \alpha)$: (a) by adding ordinates at intervals and (b) by calculation. (13)

5. If velocity $\boldsymbol{v_1} = 26$ m/s at 52° and $\boldsymbol{v_2} = 17$ m/s at −28° calculate the magnitude and direction of $\boldsymbol{v_1} + \boldsymbol{v_2}$, correct to 2 decimal places, using complex numbers. (10)

6. Given $\boldsymbol{a} = -3\boldsymbol{i} + 3\boldsymbol{j} + 5\boldsymbol{k}$, $\boldsymbol{b} = 2\boldsymbol{i} - 5\boldsymbol{j} + 7\boldsymbol{k}$ and $\boldsymbol{c} = 3\boldsymbol{i} + 6\boldsymbol{j} - 4\boldsymbol{k}$, determine the following: (i) $-4\boldsymbol{b}$ (ii) $\boldsymbol{a} + \boldsymbol{b} - \boldsymbol{c}$ (iii) $5\boldsymbol{b} - 3\boldsymbol{c}$. (8)

7. If $\boldsymbol{a} = 2\boldsymbol{i} + 4\boldsymbol{j} - 5\boldsymbol{k}$ and $\boldsymbol{b} = 3\boldsymbol{i} - 2\boldsymbol{j} + 6\boldsymbol{k}$ determine: (i) $\boldsymbol{a} \cdot \boldsymbol{b}$ (ii) $|\boldsymbol{a} + \boldsymbol{b}|$ (iii) $\boldsymbol{a} \times \boldsymbol{b}$ (iv) the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. (14)

8. Determine the work done by a force of F newtons acting at a point A on a body, when A is displaced to point B, the co-ordinates of A and B being (2, 5, −3) and (1, −3, 0) metres respectively, and when $\boldsymbol{F} = 2\boldsymbol{i} - 5\boldsymbol{j} + 4\boldsymbol{k}$ newtons. (4)

9. A force of $\boldsymbol{F} = 3\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}$ newtons acts on a line passing through a point P. Determine moment $\boldsymbol{M}$ and its magnitude of the force $\boldsymbol{F}$ about a point Q when P has co-ordinates (4, −1, 5) metres and Q has co-ordinates (4, 0, −3) metres. (8)

Chapter 27

Methods of differentiation

27.1 Introduction to calculus

Calculus is a branch of mathematics involving or leading to calculations dealing with continuously varying functions – such as velocity and acceleration, rates of change and maximum and minimum values of curves.

Calculus has widespread applications in science and engineering and is used to solve complicated problems for which algebra alone is insufficient.

Calculus is a subject that falls into two parts:

(i) **differential calculus**, or **differentiation**, which is covered in Chapters 27 to 36, and

(ii) **integral calculus**, or **integration**, which is covered in Chapters 37 to 44.

27.2 The gradient of a curve

If a tangent is drawn at a point P on a curve, then the gradient of this tangent is said to be the **gradient of the curve** at P. In Fig. 27.1, the gradient of the curve at P is equal to the gradient of the tangent PQ.

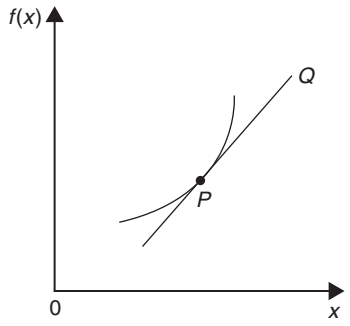


Figure 27.1

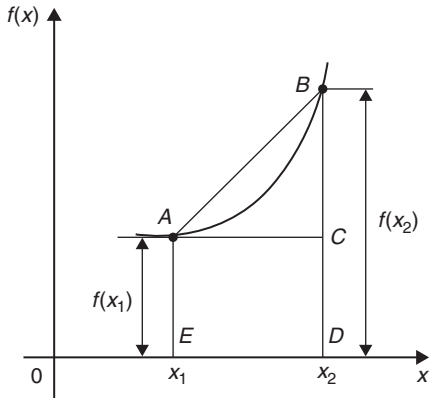


Figure 27.2

For the curve shown in Fig. 27.2, let the points A and B have co-ordinates (x_1, y_1) and (x_2, y_2), respectively. In functional notation, $y_1 = f(x_1)$ and $y_2 = f(x_2)$ as shown.

The gradient of the chord AB

$$= \frac{BC}{AC} = \frac{BD - CD}{ED} = \frac{f(x_2) - f(x_1)}{(x_2 - x_1)}$$

For the curve $f(x) = x^2$ shown in Fig. 27.3.

(i) the gradient of chord AB

$$= \frac{f(3) - f(1)}{3 - 1} = \frac{9 - 1}{2} = \mathbf{4}$$

(ii) the gradient of chord AC

$$= \frac{f(2) - f(1)}{2 - 1} = \frac{4 - 1}{1} = \mathbf{3}$$

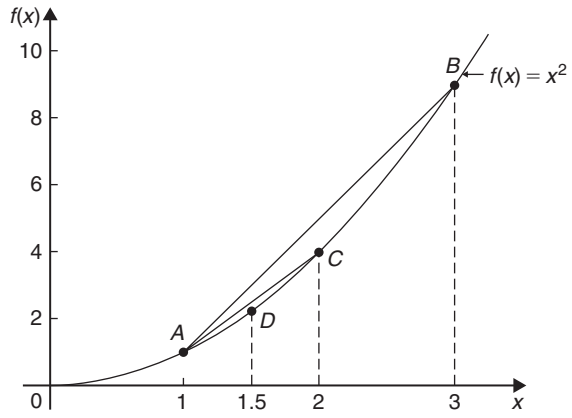


Figure 27.3

(iii) the gradient of chord AD

$$= \frac{f(1.5) - f(1)}{1.5 - 1} = \frac{2.25 - 1}{0.5} = \mathbf{2.5}$$

(iv) if E is the point on the curve $(1.1,\ f(1.1))$ then the gradient of chord AE

$$= \frac{f(1.1) - f(1)}{1.1 - 1} = \frac{1.21 - 1}{0.1} = \mathbf{2.1}$$

(v) if F is the point on the curve $(1.01,\ f(1.01))$ then the gradient of chord AF

$$= \frac{f(1.01) - f(1)}{1.01 - 1} = \frac{1.0201 - 1}{0.01} = \mathbf{2.01}$$

Thus as point B moves closer and closer to point A the gradient of the chord approaches nearer and nearer to the value **2**. This is called the **limiting value** of the gradient of the chord AB and when B coincides with A the chord becomes the tangent to the curve.

27.3 Differentiation from first principles

In Fig. 27.4, A and B are two points very close together on a curve, δx (delta x) and δy (delta y) representing small increments in the x and y directions, respectively.

Gradient of chord $AB = \dfrac{\delta y}{\delta x}$; however, $\delta y = f(x + \delta x) - f(x)$.

Hence $\dfrac{\delta y}{\delta x} = \dfrac{f(x + \delta x) - f(x)}{\delta x}$.

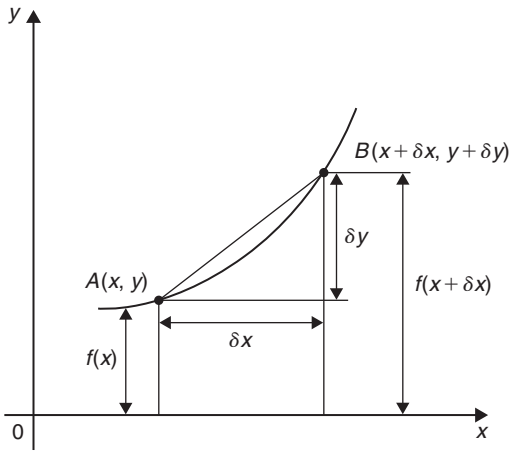


Figure 27.4

As δx approaches zero, $\dfrac{\delta y}{\delta x}$ approaches a limiting value and the gradient of the chord approaches the gradient of the tangent at A.

When determining the gradient of a tangent to a curve there are two notations used. The gradient of the curve at A in Fig. 27.4 can either be written as

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} \quad \text{or} \quad \lim_{\delta x \to 0} \left\{ \frac{f(x + \delta x) - f(x)}{\delta x} \right\}$$

In **Leibniz notation**, $\dfrac{\mathbf{dy}}{\mathbf{dx}} = \lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x}$

In **functional notation**,

$$f'(x) = \lim_{\delta x \to 0} \left\{ \frac{f(x + \delta x) - f(x)}{\delta x} \right\}$$

$\dfrac{dy}{dx}$ is the same as $f'(x)$ and is called the **differential coefficient** or the **derivative**. The process of finding the differential coefficient is called **differentiation**.

Problem 1. Differentiate from first principle $f(x) = x^2$ and determine the value of the gradient of the curve at $x = 2$.

To 'differentiate from first principles' means 'to find $f'(x)$' by using the expression

$$f'(x) = \lim_{\delta x \to 0} \left\{ \frac{f(x + \delta x) - f(x)}{\delta x} \right\}$$

$$f(x) = x^2$$

Substituting $(x+\delta x)$ for x gives
$f(x+\delta x)=(x+\delta x)^2=x^2+2x\delta x+\delta x^2$, hence

$$f'(x)=\lim_{\delta x\to 0}\left\{\frac{(x^2+2x\delta x+\delta x^2)-(x^2)}{\delta x}\right\}$$

$$=\lim_{\delta x\to 0}\left\{\frac{(2x\delta x+\delta x^2)}{\delta x}\right\}$$

$$=\lim_{\delta x\to 0}[2x+\delta x]$$

As $\delta x\to 0$, $[2x+\delta x]\to[2x+0]$. Thus $\boldsymbol{f'(x)=2x}$, i.e. the differential coefficient of x^2 is $2x$. At $x=2$, the gradient of the curve, $f'(x)=2(2)=\mathbf{4}$.

Differentiation from first principles can be a lengthy process and it would not be convenient to go through this procedure every time we want to differentiate a function. In reality we do not have to because a set of general rules have evolved from the above procedure, which we consider in the following section.

27.4 Differentiation of common functions

From differentiation by first principles of a number of examples such as in Problem 1 above, a general rule for differentiating $y=ax^n$ emerges, where a and n are constants.

The rule is: **if $y=ax^n$ then $\dfrac{dy}{dx}=anx^{n-1}$**

(or, **if $f(x)=ax^n$ then $f'(x)=anx^{n-1}$**) and is true for all real values of a and n.

For example, if $y=4x^3$ then $a=4$ and $n=3$, and

$$\frac{dy}{dx}=anx^{n-1}=(4)(3)x^{3-1}=12x^2$$

If $y=ax^n$ and $n=0$ then $y=ax^0$ and

$$\frac{dy}{dx}=(a)(0)x^{0-1}=0,$$

i.e. **the differential coefficient of a constant is zero**.

Figure 27.5(a) shows a graph of $y=\sin x$. The gradient is continually changing as the curve moves from 0 to A to B to C to D. The gradient, given by $\dfrac{dy}{dx}$, may be plotted in a corresponding position below $y=\sin x$, as shown in Fig. 27.5(b).

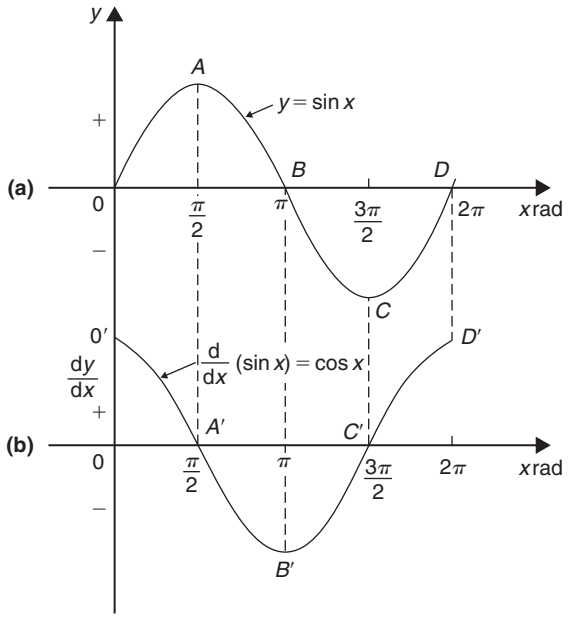


Figure 27.5

(i) At 0, the gradient is positive and is at its steepest. Hence $0'$ is a maximum positive value.

(ii) Between 0 and A the gradient is positive but is decreasing in value until at A the gradient is zero, shown as A'.

(iii) Between A and B the gradient is negative but is increasing in value until at B the gradient is at its steepest negative value. Hence B' is a maximum negative value.

(iv) If the gradient of $y=\sin x$ is further investigated between B and D then the resulting graph of $\dfrac{dy}{dx}$ is seen to be a cosine wave. Hence the rate of change of $\sin x$ is $\cos x$,

i.e. **if $y=\sin x$ then $\dfrac{dy}{dx}=\cos x$**

By a similar construction to that shown in Fig. 27.5 it may be shown that:

if $y=\sin ax$ then $\dfrac{dy}{dx}=a\cos ax$

If graphs of $y=\cos x$, $y=e^x$ and $y=\ln x$ are plotted and their gradients investigated, their differential coefficients may be determined in a similar manner to that shown for $y=\sin x$. The rate of change of a function is a measure of the derivative.

The **standard derivatives** summarized below may be proved theoretically and are true for all real values of x

y or $f(x)$	$\frac{dy}{dx}$ or $f'(x)$
ax^n	anx^{n-1}
$\sin ax$	$a\cos ax$
$\cos ax$	$-a\sin ax$
e^{ax}	ae^{ax}
$\ln ax$	$\frac{1}{x}$

The **differential coefficient of a sum or difference** is the sum or difference of the differential coefficients of the separate terms.

Thus, if $f(x) = p(x) + q(x) - r(x)$,
(where f, p, q and r are functions),
then $f'(x) = p'(x) + q'(x) - r'(x)$

Differentiation of common functions is demonstrated in the following worked problems.

Problem 2. Find the differential coefficients of
(a) $y = 12x^3$ (b) $y = \dfrac{12}{x^3}$

If $y = ax^n$ then $\dfrac{dy}{dx} = anx^{n-1}$

(a) Since $y = 12x^3$, $a = 12$ and $n = 3$ thus $\dfrac{dy}{dx} = (12)(3)x^{3-1} = \mathbf{36x^2}$

(b) $y = \dfrac{12}{x^3}$ is rewritten in the standard ax^n form as $y = 12x^{-3}$ and in the general rule $a = 12$ and $n = -3$.

Thus $\dfrac{dy}{dx} = (12)(-3)x^{-3-1} = -36x^{-4} = \mathbf{-\dfrac{36}{x^4}}$

Problem 3. Differentiate (a) $y = 6$ (b) $y = 6x$.

(a) $y = 6$ may be written as $y = 6x^0$, i.e. in the general rule $a = 6$ and $n = 0$.

Hence $\dfrac{dy}{dx} = (6)(0)x^{0-1} = \mathbf{0}$

In general, **the differential coefficient of a constant is always zero**.

(b) Since $y = 6x$, in the general rule $a = 6$ and $n = 1$.

Hence $\dfrac{dy}{dx} = (6)(1)x^{1-1} = 6x^0 = \mathbf{6}$

In general, the differential coefficient of kx, where k is a constant, is always k.

Problem 4. Find the derivatives of
(a) $y = 3\sqrt{x}$ (b) $y = \dfrac{5}{\sqrt[3]{x^4}}$

(a) $y = 3\sqrt{x}$ is rewritten in the standard differential form as $y = 3x^{\frac{1}{2}}$.
In the general rule, $a = 3$ and $n = \dfrac{1}{2}$

Thus $\dfrac{dy}{dx} = (3)\left(\dfrac{1}{2}\right)x^{\frac{1}{2}-1} = \dfrac{3}{2}x^{-\frac{1}{2}}$

$$= \frac{3}{2x^{\frac{1}{2}}} = \mathbf{\frac{3}{2\sqrt{x}}}$$

(b) $y = \dfrac{5}{\sqrt[3]{x^4}} = \dfrac{5}{x^{\frac{4}{3}}} = 5x^{-\frac{4}{3}}$ in the standard differential form.
In the general rule, $a = 5$ and $n = -\frac{4}{3}$

Thus $\dfrac{dy}{dx} = (5)\left(-\dfrac{4}{3}\right)x^{-\frac{4}{3}-1} = \dfrac{-20}{3}x^{-\frac{7}{3}}$

$$= \frac{-20}{3x^{\frac{7}{3}}} = \mathbf{\frac{-20}{3\sqrt[3]{x^7}}}$$

Problem 5. Differentiate, with respect to x,
$y = 5x^4 + 4x - \dfrac{1}{2x^2} + \dfrac{1}{\sqrt{x}} - 3$.

$y = 5x^4 + 4x - \dfrac{1}{2x^2} + \dfrac{1}{\sqrt{x}} - 3$ is rewritten as

$y = 5x^4 + 4x - \dfrac{1}{2}x^{-2} + x^{-\frac{1}{2}} - 3$

When differentiating a sum, each term is differentiated in turn.

Thus $\frac{dy}{dx} = (5)(4)x^{4-1} + (4)(1)x^{1-1} - \frac{1}{2}(-2)x^{-2-1}$

$$+ (1)\left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1} - 0$$

$$= 20x^3 + 4 + x^{-3} - \frac{1}{2}x^{-\frac{3}{2}}$$

i.e. $\mathbf{\frac{dy}{dx} = 20x^3 + 4 + \frac{1}{x^3} - \frac{1}{2\sqrt{x^3}}}$

Problem 6. Find the differential coefficients of (a) $y = 3\sin 4x$ (b) $f(t) = 2\cos 3t$ with respect to the variable.

(a) When $y = 3\sin 4x$ then $\frac{dy}{dx} = (3)(4\cos 4x)$
$= \mathbf{12\cos 4x}$

(b) When $f(t) = 2\cos 3t$ then
$f'(t) = (2)(-3\sin 3t) = \mathbf{-6\sin 3t}$

Problem 7. Determine the derivatives of (a) $y = 3e^{5x}$ (b) $f(\theta) = \frac{2}{e^{3\theta}}$ (c) $y = 6\ln 2x$.

(a) When $y = 3e^{5x}$ then $\frac{dy}{dx} = (3)(5)e^{5x} = \mathbf{15e^{5x}}$

(b) $f(\theta) = \frac{2}{e^{3\theta}} = 2e^{-3\theta}$, thus

$$f'(\theta) = (2)(-3)e^{-30} = -6e^{-3\theta} = \frac{\mathbf{-6}}{\mathbf{e^{3\theta}}}$$

(c) When $y = 6\ln 2x$ then $\frac{dy}{dx} = 6\left(\frac{1}{x}\right) = \frac{\mathbf{6}}{\mathbf{x}}$

Problem 8. Find the gradient of the curve $y = 3x^4 - 2x^2 + 5x - 2$ at the points (0, −2) and (1, 4).

The gradient of a curve at a given point is given by the corresponding value of the derivative. Thus, since $y = 3x^4 - 2x^2 + 5x - 2$

Then the gradient $= \frac{dy}{dx} = 12x^3 - 4x + 5$

At the point (0, −2), $x = 0$
Thus the gradient $= 12(0)^3 - 4(0) + 5 = \mathbf{5}$

At the point (1, 4), $x = 1$
Thus the gradient $= 12(1)^3 - 4(1) + 5 = \mathbf{13}$.

Problem 9. Determine the co-ordinates of the point on the graph $y = 3x^2 - 7x + 2$ where the gradient is −1.

The gradient of the curve is given by the derivative.

When $y = 3x^2 - 7x + 2$ then $\frac{dy}{dx} = 6x - 7$

Since the gradient is −1 then $6x - 7 = -1$, from which, $x = 1$

When $x = 1$, $y = 3(1)^2 - 7(1) + 2 = -2$

Hence the gradient is −1 at the point (1, −2).

Now try the following exercise

Exercise 115 Further problems on differentiating common functions

In Problems 1 to 6 find the differential coefficients of the given functions with respect to the variable.

1. (a) $5x^5$ (b) $2.4x^{3.5}$ (c) $\frac{1}{x}$

$\left[\text{(a) } 25x^4 \text{ (b) } 8.4x^{2.5} \text{ (c) } -\frac{1}{x^2}\right]$

2. (a) $\frac{-4}{x^2}$ (b) 6 (c) $2x$ $\left[\text{(a) } \frac{8}{x^3} \text{ (b) } 0 \text{ (c) } 2\right]$

3. (a) $2\sqrt{x}$ (b) $3\sqrt[3]{x^5}$ (c) $\frac{4}{\sqrt{x}}$

$\left[\text{(a) } \frac{1}{\sqrt{x}} \text{ (b) } 5\sqrt[3]{x^2} \text{ (c) } -\frac{2}{\sqrt{x^3}}\right]$

4. (a) $\frac{-3}{\sqrt[3]{x}}$ (b) $(x-1)^2$ (c) $2\sin 3x$

$\left[\text{(a) } \frac{1}{\sqrt[3]{x^4}} \text{ (b) } 2(x-1) \text{ (c) } 6\cos 3x\right]$

5. (a) $-4\cos 2x$ (b) $2e^{6x}$ (c) $\frac{3}{e^{5x}}$

$\left[\text{(a) } 8\sin 2x \text{ (b) } 12e^{6x} \text{ (c) } \frac{-15}{e^{5x}}\right]$

6. (a) $4\ln 9x$ (b) $\dfrac{e^x - e^{-x}}{2}$ (c) $\dfrac{1-\sqrt{x}}{x}$

$$\left[\begin{array}{l}\text{(a) } \dfrac{4}{x} \quad \text{(b) } \dfrac{e^x + e^{-x}}{2} \\ \text{(c) } \dfrac{-1}{x^2} + \dfrac{1}{2\sqrt{x^3}}\end{array}\right]$$

7. Find the gradient of the curve $y = 2t^4 + 3t^3 - t + 4$ at the points (0, 4) and (1, 8). $[-1, 16]$

8. Find the co-ordinates of the point on the graph $y = 5x^2 - 3x + 1$ where the gradient is 2. $\left[\left(\frac{1}{2}, \frac{3}{4}\right)\right]$

9. (a) Differentiate $y = \dfrac{2}{\theta^2} + 2\ln 2\theta - 2(\cos 5\theta + 3\sin 2\theta) - \dfrac{2}{e^{3\theta}}$

(b) Evaluate $\dfrac{dy}{d\theta}$ in part (a) when $\theta = \dfrac{\pi}{2}$, correct to 4 significant figures.

$$\left[\begin{array}{l}\text{(a) } \dfrac{-4}{\theta^3} + \dfrac{2}{\theta} + 10\sin 5\theta \\ \qquad -12\cos 2\theta + \dfrac{6}{e^{3\theta}} \\ \text{(b) } 22.30\end{array}\right]$$

10. Evaluate $\dfrac{ds}{dt}$, correct to 3 significant figures, when $t = \dfrac{\pi}{6}$ given $s = 3\sin t - 3 + \sqrt{t}$. [3.29]

27.5 Differentiation of a product

When $y = uv$, and u and v are both functions of x,

then $$\boldsymbol{\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}}$$

This is known as the **product rule**.

Problem 10. Find the differential coefficient of $y = 3x^2\sin 2x$.

$3x^2\sin 2x$ is a product of two terms $3x^2$ and $\sin 2x$
Let $u = 3x^2$ and $v = \sin 2x$

Using the product rule:

$$\frac{dy}{dx} = \underset{\downarrow}{u}\ \underset{\downarrow}{\frac{dv}{dx}} + \underset{\downarrow}{v}\ \underset{\downarrow}{\frac{du}{dx}}$$

gives: $\dfrac{dy}{dx} = (3x^2)(2\cos 2x) + (\sin 2x)(6x)$

i.e. $\dfrac{dy}{dx} = 6x^2\cos 2x + 6x\sin 2x$

$$= \mathbf{6x(x\cos 2x + \sin 2x)}$$

Note that the differential coefficient of a product is **not** obtained by merely differentiating each term and multiplying the two answers together. The product rule formula **must** be used when differentiating products.

Problem 11. Find the rate of change of y with respect to x given $y = 3\sqrt{x}\ln 2x$.

The rate of change of y with respect to x is given by $\dfrac{dy}{dx}$

$y = 3\sqrt{x}\ln 2x = 3x^{\frac{1}{2}}\ln 2x$, which is a product.

Let $u = 3x^{\frac{1}{2}}$ and $v = \ln 2x$

Then $$\frac{dy}{dx} = \underset{\downarrow}{u}\ \underset{\downarrow}{\frac{dv}{dx}} + \underset{\downarrow}{v}\ \underset{\downarrow}{\frac{du}{dx}}$$

$$= \left(3x^{\frac{1}{2}}\right)\left(\frac{1}{x}\right) + (\ln 2x)\left[3\left(\frac{1}{2}\right)x^{\frac{1}{2}-1}\right]$$

$$= 3x^{\frac{1}{2}-1} + (\ln 2x)\left(\frac{3}{2}\right)x^{-\frac{1}{2}}$$

$$= 3x^{-\frac{1}{2}}\left(1 + \frac{1}{2}\ln 2x\right)$$

i.e. $$\boldsymbol{\frac{dy}{dx} = \frac{3}{\sqrt{x}}\left(1 + \frac{1}{2}\ln 2x\right)}$$

Problem 12. Differentiate $y = x^3\cos 3x\ln x$.

Let $u = x^3\cos 3x$ (i.e. a product) and $v = \ln x$

Then $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

where $\dfrac{du}{dx} = (x^3)(-3\sin 3x) + (\cos 3x)(3x^2)$

and $\dfrac{dv}{dx} = \dfrac{1}{x}$

Hence $\dfrac{dy}{dx} = (x^3\cos 3x)\left(\dfrac{1}{x}\right) + (\ln x)[-3x^3\sin 3x + 3x^2\cos 3x]$

$= x^2\cos 3x + 3x^2\ln x(\cos 3x - x\sin 3x)$

i.e. $\dfrac{dy}{dx} = x^2\{\cos 3x + 3\ln x(\cos 3x - x\sin 3x)\}$

Problem 13. Determine the rate of change of voltage, given $v = 5t\sin 2t$ volts when $t = 0.2$ s.

Rate of change of voltage $= \dfrac{dv}{dt}$

$= (5t)(2\cos 2t) + (\sin 2t)(5)$

$= 10t\cos 2t + 5\sin 2t$

When $t = 0.2$, $\dfrac{dv}{dt} = 10(0.2)\cos 2(0.2) + 5\sin 2(0.2)$

$= 2\cos 0.4 + 5\sin 0.4$ (where $\cos 0.4$ means the cosine of 0.4 radians)

Hence $\dfrac{dv}{dt} = 2(0.92106) + 5(0.38942)$

$= 1.8421 + 1.9471 = 3.7892$

i.e. the rate of change of voltage when $t = 0.2$ s is 3.79 volts/s, correct to 3 significant figures.

Now try the following exercise

Exercise 116 Further problems on differentiating products

In Problems 1 to 8 differentiate the given products with respect to the variable.

1. $x\sin x$ $[x\cos x + \sin x]$
2. x^2e^{2x} $[2xe^{2x}(x+1)]$
3. $x^2\ln x$ $[x(1+2\ln x)]$
4. $2x^3\cos 3x$ $[6x^2(\cos 3x - x\sin 3x)]$
5. $\sqrt{x^3}\ln 3x$ $\left[\sqrt{x}\left(1+\frac{3}{2}\ln 3x\right)\right]$
6. $e^{3t}\sin 4t$ $[e^{3t}(4\cos 4t + 3\sin 4t)]$
7. $e^{4\theta}\ln 3\theta$ $\left[e^{4\theta}\left(\dfrac{1}{\theta} + 4\ln 3\theta\right)\right]$
8. $e^t\ln t\cos t$ $\left[e^t\left\{\left(\dfrac{1}{t} + \ln t\right)\cos t - \ln t\sin t\right\}\right]$
9. Evaluate $\dfrac{di}{dt}$, correct to 4 significant figures, when $t = 0.1$, and $i = 15t\sin 3t$. [8.732]
10. Evaluate $\dfrac{dz}{dt}$, correct to 4 significant figures, when $t = 0.5$, given that $z = 2e^{3t}\sin 2t$. [32.31]

27.6 Differentiation of a quotient

When $y = \dfrac{u}{v}$, and u and v are both functions of x

then $$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

This is known as the **quotient rule**.

Problem 14. Find the differential coefficient of $y = \dfrac{4\sin 5x}{5x^4}$

$\dfrac{4\sin 5x}{5x^4}$ is a quotient. Let $u = 4\sin 5x$ and $v = 5x^4$

(Note that v is **always** the denominator and u the numerator.)

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

where $\dfrac{du}{dx} = (4)(5)\cos 5x = 20\cos 5x$

and $\dfrac{dv}{dx} = (5)(4)x^3 = 20x^3$

Hence $\dfrac{dy}{dx} = \dfrac{(5x^4)(20\cos 5x) - (4\sin 5x)(20x^3)}{(5x^4)^2}$

$= \dfrac{100x^4\cos 5x - 80x^3\sin 5x}{25x^8}$

$= \dfrac{20x^3[5x\cos 5x - 4\sin 5x]}{25x^8}$

i.e. $\dfrac{dy}{dx} = \dfrac{4}{5x^5}(5x\cos 5x - 4\sin 5x)$

Note that the differential coefficient is **not** obtained by merely differentiating each term in turn and then dividing the numerator by the denominator. The quotient formula **must** be used when differentiating quotients.

Problem 15. Determine the differential coefficient of $y = \tan ax$.

$y = \tan ax = \dfrac{\sin ax}{\cos ax}$. Differentiation of $\tan ax$ is thus treated as a quotient with $u = \sin ax$ and $v = \cos ax$

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

$$= \frac{(\cos ax)(a\cos ax) - (\sin ax)(-a\sin ax)}{(\cos ax)^2}$$

$$= \frac{a\cos^2 ax + a\sin^2 ax}{(\cos ax)^2} = \frac{a(\cos^2 ax + \sin^2 ax)}{\cos^2 ax}$$

$$= \frac{a}{\cos^2 ax}, \text{ since } \cos^2 ax + \sin^2 ax = 1$$

(see Chapter 15)

Hence $\dfrac{\mathbf{dy}}{\mathbf{dx}} = \boldsymbol{a}\sec^2 \boldsymbol{ax}$ since $\sec^2 ax = \dfrac{1}{\cos^2 ax}$ (see Chapter 11).

Problem 16. Find the derivative of $y = \sec ax$.

$y = \sec ax = \dfrac{1}{\cos ax}$ (i.e. a quotient). Let $u = 1$ and $v = \cos ax$

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

$$= \frac{(\cos ax)(0) - (1)(-a\sin ax)}{(\cos ax)^2}$$

$$= \frac{a\sin ax}{\cos^2 ax} = a\left(\frac{1}{\cos ax}\right)\left(\frac{\sin ax}{\cos ax}\right)$$

i.e. $\dfrac{\mathbf{dy}}{\mathbf{dx}} = \boldsymbol{a}\sec \boldsymbol{ax}\tan \boldsymbol{ax}$

Problem 17. Differentiate $y = \dfrac{te^{2t}}{2\cos t}$

The function $\dfrac{te^{2t}}{2\cos t}$ is a quotient, whose numerator is a product.

Let $u = te^{2t}$ and $v = 2\cos t$ then

$\dfrac{du}{dt} = (t)(2e^{2t}) + (e^{2t})(1)$ and $\dfrac{dv}{dt} = -2\sin t$

Hence $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

$$= \frac{(2\cos t)[2te^{2t} + e^{2t}] - (te^{2t})(-2\sin t)}{(2\cos t)^2}$$

$$= \frac{4te^{2t}\cos t + 2e^{2t}\cos t + 2te^{2t}\sin t}{4\cos^2 t}$$

$$= \frac{2e^{2t}[2t\cos t + \cos t + t\sin t]}{4\cos^2 t}$$

i.e. $\dfrac{\mathbf{dy}}{\mathbf{dx}} = \dfrac{\mathbf{e^{2t}}}{\mathbf{2\cos^2 t}}\boldsymbol{(2t\cos t + \cos t + t\sin t)}$

Problem 18. Determine the gradient of the curve $y = \dfrac{5x}{2x^2+4}$ at the point $\left(\sqrt{3}, \dfrac{\sqrt{3}}{2}\right)$.

Let $y = 5x$ and $v = 2x^2 + 4$

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} = \frac{(2x^2+4)(5) - (5x)(4x)}{(2x^2+4)^2}$$

$$= \frac{10x^2 + 20 - 20x^2}{(2x^2+4)^2} = \frac{20 - 10x^2}{(2x^2+4)^2}$$

At the point $\left(\sqrt{3}, \dfrac{\sqrt{3}}{2}\right)$, $x = \sqrt{3}$,

hence the gradient $= \dfrac{dy}{dx} = \dfrac{20 - 10(\sqrt{3})^2}{[2(\sqrt{3})^2 + 4]^2}$

$$= \frac{20 - 30}{100} = -\frac{\mathbf{1}}{\mathbf{10}}$$

Now try the following exercise

Exercise 117 Further problems on differentiating quotients

In Problems 1 to 7, differentiate the quotients with respect to the variable.

1. $\dfrac{\sin x}{x}$ $\left[\dfrac{x\cos x - \sin x}{x^2}\right]$

2. $\dfrac{2\cos 3x}{x^3}$ $\left[\dfrac{-6}{x^4}(x\sin 3x+\cos 3x)\right]$

3. $\dfrac{2x}{x^2+1}$ $\left[\dfrac{2(1-x^2)}{(x^2+1)^2}\right]$

4. $\dfrac{\sqrt{x}}{\cos x}$ $\left[\dfrac{\frac{\cos x}{2\sqrt{x}}+\sqrt{x}\sin x}{\cos^2 x}\right]$

5. $\dfrac{3\sqrt{\theta^3}}{2\sin 2\theta}$ $\left[\dfrac{3\sqrt{\theta}(3\sin 2\theta-4\theta\cos 2\theta)}{4\sin^2 2\theta}\right]$

6. $\dfrac{\ln 2t}{\sqrt{t}}$ $\left[\dfrac{1-\frac{1}{2}\ln 2t}{\sqrt{t^3}}\right]$

7. $\dfrac{2xe^{4x}}{\sin x}$ $\left[\dfrac{2e^{4x}}{\sin^2 x}\{(1+4x)\sin x-x\cos x\}\right]$

8. Find the gradient of the curve $y=\dfrac{2x}{x^2-5}$ at the point (2, −4). [−18]

9. Evaluate $\dfrac{dy}{dx}$ at $x=2.5$, correct to 3 significant figures, given $y=\dfrac{2x^2+3}{\ln 2x}$. [3.82]

27.7 Function of a function

It is often easier to make a substitution before differentiating.

If y is a function of x then $\dfrac{dy}{dx}=\dfrac{dy}{du}\times\dfrac{du}{dx}$

This is known as the **'function of a function'** rule (or sometimes the **chain rule**).

For example, if $y=(3x-1)^9$ then, by making the substitution $u=(3x-1)$, $y=u^9$, which is of the 'standard' form.

Hence $\dfrac{dy}{du}=9u^8$ and $\dfrac{du}{dx}=3$

Then $\dfrac{dy}{dx}=\dfrac{dy}{du}\times\dfrac{du}{dx}=(9u^8)(3)=27u^8$

Rewriting u as $(3x-1)$ gives: $\dfrac{dy}{dx}=\mathbf{27(3x-1)^8}$

Since y is a function of u, and u is a function of x, then y is a function of a function of x.

Problem 19. Differentiate $y=3\cos(5x^2+2)$.

Let $u=5x^2+2$ then $y=3\cos u$

Hence $\dfrac{du}{dx}=10x$ and $\dfrac{dy}{du}=-3\sin u$.

Using the function of a function rule,

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=(-3\sin u)(10x)=-30x\sin u$$

Rewriting u as $5x^2+2$ gives:

$$\frac{dy}{dx}=\mathbf{-30x\sin(5x^2+2)}$$

Problem 20. Find the derivative of $y=(4t^3-3t)^6$.

Let $u=4t^3-3t$, then $y=u^6$

Hence $\dfrac{du}{dt}=12t^2-3$ and $\dfrac{dy}{du}=6u^5$

Using the function of a function rule,

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=(6u^5)(12t^2-3)$$

Rewriting u as $(4t^3-3t)$ gives:

$$\frac{dy}{dt}=6(4t^3-3t)^5(12t^2-3)$$
$$=\mathbf{18(4t^2-1)(4t^3-3t)^5}$$

Problem 21. Determine the differential coefficient of $y=\sqrt{(3x^2+4x-1)}$.

$y=\sqrt{(3x^2+4x-1)}=(3x^2+4x-1)^{\frac{1}{2}}$

Let $u=3x^2+4x-1$ then $y=u^{\frac{1}{2}}$

Hence $\dfrac{du}{dx}=6x+4$ and $\dfrac{dy}{du}=\dfrac{1}{2}u^{-\frac{1}{2}}=\dfrac{1}{2\sqrt{u}}$

Using the function of a function rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \left(\frac{1}{2\sqrt{u}}\right)(6x+4) = \frac{3x+2}{\sqrt{u}}$$

i.e. $$\mathbf{\frac{dy}{dx} = \frac{3x+2}{\sqrt{(3x^2+4x-1)}}}$$

Problem 22. Differentiate $y = 3\tan^4 3x$.

Let $u = \tan 3x$ then $y = 3u^4$

Hence $\frac{du}{dx} = 3\sec^2 3x$, (from Problem 15), and

$$\frac{dy}{du} = 12u^3$$

Then $$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = (12u^3)(3\sec^2 3x)$$

$$= 12(\tan 3x)^3(3\sec^2 3x)$$

i.e. $$\mathbf{\frac{dy}{dx} = 36\tan^3 3x \sec^2 3x}$$

Problem 23. Find the differential coefficient of $y = \frac{2}{(2t^3-5)^4}$

$y = \frac{2}{(2t^3-5)^4} = 2(2t^3-5)^{-4}$. Let $u = (2t^3-5)$, then $y = 2u^{-4}$

Hence $\frac{du}{dt} = 6t^2$ and $\frac{dy}{du} = -8u^{-5} = \frac{-8}{u^5}$

Then $$\frac{dy}{dt} = \frac{dy}{du} \times \frac{du}{dt} = \left(\frac{-8}{u^5}\right)(6t^2)$$

$$= \mathbf{\frac{-48t^2}{(2t^3-5)^5}}$$

Now try the following exercise

Exercise 118 Further problems on the function of a function

In Problems 1 to 9, find the differential coefficients with respect to the variable.

1. $(2x-1)^6$ [$12(2x-1)^5$]
2. $(2x^3-5x)^5$ [$5(6x^2-5)(2x^3-5x)^4$]
3. $2\sin(3\theta-2)$ [$6\cos(3\theta-2)$]
4. $2\cos^5\alpha$ [$-10\cos^4\alpha\sin\alpha$]
5. $\frac{1}{(x^3-2x+1)^5}$ $\left[\frac{5(2-3x^2)}{(x^3-2x+1)^6}\right]$
6. $5e^{2t+1}$ [$10e^{2t+1}$]
7. $2\cot(5t^2+3)$ [$-20t\ \mathrm{cosec}^2(5t^2+3)$]
8. $6\tan(3y+1)$ [$18\sec^2(3y+1)$]
9. $2e^{\tan\theta}$ [$2\sec^2\theta\, e^{\tan\theta}$]
10. Differentiate $\theta\sin\left(\theta-\frac{\pi}{3}\right)$ with respect to θ, and evaluate, correct to 3 significant figures, when $\theta = \frac{\pi}{2}$. [1.86]

27.8 Successive differentiation

When a function $y = f(x)$ is differentiated with respect to x the differential coefficient is written as $\frac{dy}{dx}$ or $f'(x)$. If the expression is differentiated again, the second differential coefficient is obtained and is written as $\frac{d^2y}{dx^2}$ (pronounced dee two y by dee x squared) or $f''(x)$ (pronounced f double-dash x).

By successive differentiation further higher derivatives such as $\frac{d^3y}{dx^3}$ and $\frac{d^4y}{dx^4}$ may be obtained.

Thus if $y = 3x^4$, $\frac{dy}{dx} = 12x^3$, $\frac{d^2y}{dx^2} = 36x^2$,

$\frac{d^3y}{dx^3} = 72x$, $\frac{d^4y}{dx^4} = 72$ and $\frac{d^5y}{dx^5} = 0$.

Problem 24. If $f(x) = 2x^5 - 4x^3 + 3x - 5$, find $f''(x)$.

$$f(x) = 2x^5 - 4x^3 + 3x - 5$$

$$f'(x) = 10x^4 - 12x^2 + 3$$

$$\mathbf{f''(x) = 40x^3 - 24x = 4x(10x^2 - 6)}$$

Problem 25. If $y = \cos x - \sin x$, evaluate x, in the range $0 \leq x \leq \frac{\pi}{2}$, when $\frac{d^2y}{dx^2}$ is zero.

Since $y = \cos x - \sin x$, $\frac{dy}{dx} = -\sin x - \cos x$ and $\frac{d^2y}{dx^2} = -\cos x + \sin x$.

When $\frac{d^2y}{dx^2}$ is zero, $-\cos x + \sin x = 0$,

i.e. $\sin x = \cos x$ or $\frac{\sin x}{\cos x} = 1$.

Hence $\tan x = 1$ and $\boldsymbol{x = \arctan 1 = 45°}$ **or** $\boldsymbol{\frac{\pi}{4}}$ **rads** in the range $0 \leq x \leq \frac{\pi}{2}$

Problem 26. Given $y = 2xe^{-3x}$ show that

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0.$$

$y = 2xe^{-3x}$ (i.e. a product)

Hence $\frac{dy}{dx} = (2x)(-3e^{-3x}) + (e^{-3x})(2)$

$= -6xe^{-3x} + 2e^{-3x}$

$\frac{d^2y}{dx^2} = [(-6x)(-3e^{-3x}) + (e^{-3x})(-6)] + (-6e^{-3x})$

$= 18xe^{-3x} - 6e^{-3x} - 6e^{-3x}$

i.e. $\frac{d^2y}{dx^2} = 18xe^{-3x} - 12e^{-3x}$

Substituting values into $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y$ gives:

$(18xe^{-3x} - 12e^{-3x}) + 6(-6xe^{-3x} + 2e^{-3x})$

$+ 9(2xe^{-3x}) = 18xe^{-3x} - 12e^{-3x} - 36xe^{-3x}$

$+ 12e^{-3x} + 18xe^{-3x} = 0$

Thus when $y = 2xe^{-3x}$, $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$

Problem 27. Evaluate $\frac{d^2y}{d\theta^2}$ when $\theta = 0$ given $y = 4\sec 2\theta$.

Since $y = 4\sec 2\theta$,

then $\frac{dy}{d\theta} = (4)(2)\sec 2\theta \tan 2\theta$ (from Problem 16)

$= 8\sec 2\theta \tan 2\theta$ (i.e. a product)

$\frac{d^2y}{d\theta^2} = (8\sec 2\theta)(2\sec^2 2\theta) + (\tan 2\theta)[(8)(2)\sec 2\theta \tan 2\theta]$

$= 16\sec^3 2\theta + 16\sec 2\theta \tan^2 2\theta$

When $\theta = 0$, $\frac{d^2y}{d\theta^2} = 16\sec^3 0 + 16\sec 0 \tan^2 0$

$= 16(1) + 16(1)(0) = \mathbf{16}$.

Now try the following exercise

Exercise 119 Further problems on successive differentiation

1. If $y = 3x^4 + 2x^3 - 3x + 2$ find
   (a) $\frac{d^2y}{dx^2}$ (b) $\frac{d^3y}{dx^3}$.
   [(a) $36x^2 + 12x$ (b) $72x + 12$]

2. (a) Given $f(t) = \frac{2}{5}t^2 - \frac{1}{t^3} + \frac{3}{t} - \sqrt{t} + 1$ determine $f''(t)$.
   (b) Evaluate $f''(t)$ when $t = 1$.
   $\left[\begin{array}{l}\text{(a) } \frac{4}{5} - \frac{12}{t^5} + \frac{6}{t^3} + \frac{1}{4\sqrt{t^3}} \\ \text{(b) } -4.95\end{array}\right]$

In Problems 3 and 4, find the second differential coefficient with respect to the variable.

3. (a) $3\sin 2t + \cos t$ (b) $2\ln 4\theta$
   $\left[\text{(a) } -(12\sin 2t + \cos t) \text{ (b) } \frac{-2}{\theta^2}\right]$

4. (a) $2\cos^2 x$ (b) $(2x - 3)^4$
   [(a) $4(\sin^2 x - \cos^2 x)$ (b) $48(2x - 3)^2$]

5. Evaluate $f''(\theta)$ when $\theta = 0$ given $f(\theta) = 2\sec 3\theta$. [18]

6. Show that the differential equation $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 0$ is satisfied when $y = xe^{2x}$.

7. Show that, if P and Q are constants and $y = P\cos(\ln t) + Q\sin(\ln t)$, then

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + y = 0$$

Chapter 28

Some applications of differentiation

28.1 Rates of change

If a quantity y depends on and varies with a quantity x then the rate of change of y with respect to x is $\frac{dy}{dx}$. Thus, for example, the rate of change of pressure p with height h is $\frac{dp}{dh}$.

A rate of change with respect to time is usually just called 'the rate of change', the 'with respect to time' being assumed. Thus, for example, a rate of change of current, i, is $\frac{di}{dt}$ and a rate of change of temperature, θ, is $\frac{d\theta}{dt}$, and so on.

Problem 1. The length l metres of a certain metal rod at temperature θ°C is given by $l = 1 + 0.00005\theta + 0.0000004\theta^2$. Determine the rate of change of length, in mm/°C, when the temperature is (a) 100°C and (b) 400°C.

The rate of change of length means $\frac{dl}{d\theta}$.

Since length $l = 1 + 0.00005\theta + 0.0000004\theta^2$,

then $\frac{dl}{d\theta} = 0.00005 + 0.0000008\theta$

(a) When $\theta = 100$°C,

$$\frac{dl}{d\theta} = 0.00005 + (0.0000008)(100)$$
$$= 0.00013\,\text{m/°C}$$
$$= \mathbf{0.13\,mm/°C}$$

(b) When $\theta = 400$°C,

$$\frac{dl}{d\theta} = 0.00005 + (0.0000008)(400)$$
$$= 0.00037\,\text{m/°C}$$
$$= \mathbf{0.37\,mm/°C}$$

Problem 2. The luminous intensity I candelas of a lamp at varying voltage V is given by $I = 4 \times 10^{-4} V^2$. Determine the voltage at which the light is increasing at a rate of 0.6 candelas per volt.

The rate of change of light with respect to voltage is given by $\frac{dI}{dV}$.

Since $I = 4 \times 10^{-4} V^2$,

$$\frac{dI}{dV} = (4 \times 10^{-4})(2)V = 8 \times 10^{-4} V$$

When the light is increasing at 0.6 candelas per volt then $+0.6 = 8 \times 10^{-4} V$, from which, voltage

$$V = \frac{0.6}{8 \times 10^{-4}} = 0.075 \times 10^{+4}$$
$$= \mathbf{750\,volts}$$

Problem 3. Newtons law of cooling is given by $\theta = \theta_0 e^{-kt}$, where the excess of temperature at zero time is θ_0°C and at time t seconds is θ°C. Determine the rate of change of temperature after 40 s, given that $\theta_0 = 16$°C and $k = -0.03$

The rate of change of temperature is $\frac{d\theta}{dt}$

Since $\theta = \theta_0 e^{-kt}$

then $\frac{d\theta}{dt} = (\theta_0)(-k)e^{-kt} = -k\theta_0 e^{-kt}$

When $\theta_0 = 16, k = -0.03$ and $t = 40$

then $\frac{d\theta}{dt} = -(-0.03)(16)e^{-(-0.03)(40)}$

$= 0.48e^{1.2} = \mathbf{1.594°C/s}$

Problem 4. The displacement s cm of the end of a stiff spring at time t seconds is given by $s = ae^{-kt}\sin 2\pi ft$. Determine the velocity of the end of the spring after 1 s, if $a = 2$, $k = 0.9$ and $f = 5$.

Velocity, $v = \frac{ds}{dt}$ where $s = ae^{-kt}\sin 2\pi ft$ (i.e. a product).

Using the product rule,

$$\frac{ds}{dt} = (ae^{-kt})(2\pi f\cos 2\pi ft) + (\sin 2\pi ft)(-ake^{-kt})$$

When $a = 2$, $k = 0.9$, $f = 5$ and $t = 1$,

$$\textbf{velocity}, v = (2e^{-0.9})(2\pi 5\cos 2\pi 5) + (\sin 2\pi 5)(-2)(0.9)e^{-0.9}$$

$$= 25.5455\cos 10\pi - 0.7318\sin 10\pi$$

$$= 25.5455(1) - 0.7318(0)$$

$$= \mathbf{25.55\,cm/s}$$

(Note that $\cos 10\pi$ means 'the cosine of 10π radians', *not* degrees, and $\cos 10\pi \equiv \cos 2\pi = 1$.)

Now try the following exercise

Exercise 120 Further problems on rates of change

1. An alternating current, i amperes, is given by $i = 10\sin 2\pi ft$, where f is the frequency in hertz and t the time in seconds. Determine the rate of change of current when $t = 20$ ms, given that $f = 150$ Hz. [3000π A/s]

2. The luminous intensity, I candelas, of a lamp is given by $I = 6 \times 10^{-4} V^2$, where V is the voltage. Find (a) the rate of change of luminous intensity with voltage when $V = 200$ volts, and (b) the voltage at which the light is increasing at a rate of 0.3 candelas per volt. [(a) 0.24 cd/V (b) 250 V]

3. The voltage across the plates of a capacitor at any time t seconds is given by $v = Ve^{-t/CR}$, where V, C and R are constants.

   Given $V = 300$ volts, $C = 0.12 \times 10^{-6}$ F and $R = 4 \times 10^6\ \Omega$ find (a) the initial rate of change of voltage, and (b) the rate of change of voltage after 0.5 s. [(a) −625 V/s (b) −220.5 V/s]

4. The pressure p of the atmosphere at height h above ground level is given by $p = p_0e^{-h/c}$, where p_0 is the pressure at ground level and c is a constant. Determine the rate of change of pressure with height when $p_0 = 1.013 \times 10^5$ pascals and $c = 6.05 \times 10^4$ at 1450 metres. [−1.635 Pa/m]

28.2 Velocity and acceleration

When a car moves a distance x metres in a time t seconds along a straight road, if the velocity v is constant then $v = \frac{x}{t}$ m/s, i.e. the gradient of the distance/time graph shown in Fig. 28.1 is constant.

If, however, the velocity of the car is not constant then the distance/time graph will not be a straight line. It may be as shown in Fig. 28.2.

The average velocity over a small time δt and distance δx is given by the gradient of the chord AB, i.e. the average velocity over time δt is $\frac{\delta x}{\delta t}$.

As $\delta t \to 0$, the chord AB becomes a tangent, such that at point A, the velocity is given by:

$$v = \frac{dx}{dt}$$

Hence the velocity of the car at any instant is given by the gradient of the distance/time graph. If an expression

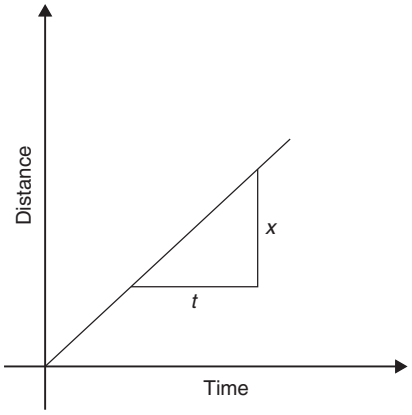


Figure 28.1

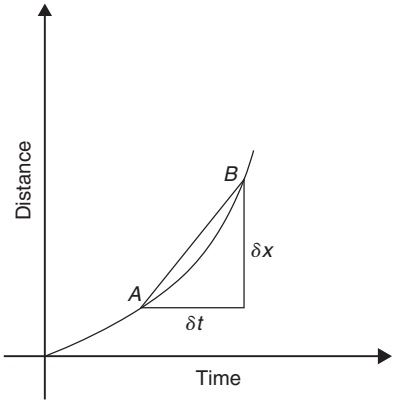


Figure 28.2

for the distance x is known in terms of time t then the velocity is obtained by differentiating the expression.

The acceleration a of the car is defined as the rate of change of velocity. A velocity/time graph is shown in Fig. 28.3. If δv is the change in v and δt the corresponding change in time, then $a = \dfrac{\delta v}{\delta t}$.

As $\delta t \to 0$, the chord CD becomes a tangent, such that at point C, the acceleration is given by:

$$a = \frac{dv}{dt}$$

Hence the acceleration of the car at any instant is given by the gradient of the velocity/time graph. If an expression for velocity is known in terms of time t then the acceleration is obtained by differentiating the expression.

Acceleration $a = \dfrac{dv}{dt}$. However, $v = \dfrac{dx}{dt}$. Hence $a = \dfrac{d}{dt}\left(\dfrac{dx}{dt}\right) = \dfrac{d^2x}{dx^2}$

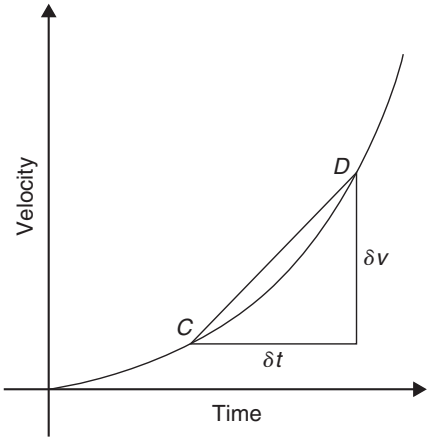


Figure 28.3

The acceleration is given by the second differential coefficient of distance x with respect to time t.

Summarizing, if a body moves a distance x metres in a time t seconds then:

(i) **distance $x = f(t)$.**

(ii) **velocity $v = f'(t)$ or $\dfrac{dx}{dt}$**, which is the gradient of the distance/time graph.

(iii) **acceleration $a = \dfrac{dv}{dt} = f''(t)$ or $\dfrac{d^2x}{dt^2}$**, which is the gradient of the velocity/time graph.

Problem 5. The distance x metres moved by a car in a time t seconds is given by $x = 3t^3 - 2t^2 + 4t - 1$. Determine the velocity and acceleration when (a) $t = 0$ and (b) $t = 1.5$ s.

Distance $x = 3t^3 - 2t^2 + 4t - 1\,\text{m}$

Velocity $v = \dfrac{dx}{dt} = 9t^2 - 4t + 4\,\text{m/s}$

Acceleration $a = \dfrac{d^2x}{dx^2} = 18t - 4\,\text{m/s}^2$

(a) When time $t = 0$,
velocity $v = 9(0)^2 - 4(0) + 4 = \mathbf{4\,m/s}$ and
acceleration $a = 18(0) - 4 = \mathbf{-4\,m/s^2}$ (i.e. a deceleration)

(b) When time $t = 1.5$ s,
velocity $v = 9(1.5)^2 - 4(1.5) + 4 = \mathbf{18.25\,m/s}$
and acceleration $a = 18(1.5) - 4 = \mathbf{23\,m/s^2}$

Problem 6. Supplies are dropped from a helicopter and the distance fallen in a time t seconds is given by $x = \frac{1}{2}gt^2$, where $g = 9.8\,\text{m/s}^2$. Determine the velocity and acceleration of the supplies after it has fallen for 2 seconds.

Distance $x = \frac{1}{2}gt^2 = \frac{1}{2}(9.8)t^2 = 4.9t^2\,\text{m}$

Velocity $v = \frac{dx}{dt} = 9.8t\,\text{m/s}$

and acceleration $a = \frac{d^2x}{dt^2} = 9.8\,\text{m/s}^2$

When time $t = 2\,\text{s}$,

velocity, $\boldsymbol{v} = (9.8)(2) = \mathbf{19.6\,m/s}$

and **acceleration** $\boldsymbol{a = 9.8\,m/s^2}$

(which is acceleration due to gravity).

Problem 7. The distance x metres travelled by a vehicle in time t seconds after the brakes are applied is given by $x = 20t - \frac{5}{3}t^2$. Determine (a) the speed of the vehicle (in km/h) at the instant the brakes are applied, and (b) the distance the car travels before it stops.

(a) Distance, $x = 20t - \frac{5}{3}t^2$.

Hence velocity $v = \frac{dx}{dt} = 20 - \frac{10}{3}t$.

At the instant the brakes are applied, time $= 0$.

Hence **velocity**, $\boldsymbol{v} = 20\,\text{m/s}$

$$= \frac{20 \times 60 \times 60}{1000}\,\text{km/h}$$

$$= \mathbf{72\,km/h}$$

(Note: changing from m/s to km/h merely involves multiplying by 3.6.)

(b) When the car finally stops, the velocity is zero, i.e. $v = 20 - \frac{10}{3}t = 0$, from which, $20 = \frac{10}{3}t$, giving $t = 6\,\text{s}$.

Hence the distance travelled before the car stops is given by:

$$x = 20t - \tfrac{5}{3}t^2 = 20(6) - \tfrac{5}{3}(6)^2$$

$$= 120 - 60 = \mathbf{60\,m}$$

Problem 8. The angular displacement θ radians of a flywheel varies with time t seconds and follows the equation $\theta = 9t^2 - 2t^3$. Determine (a) the angular velocity and acceleration of the flywheel when time, $t = 1\,\text{s}$, and (b) the time when the angular acceleration is zero.

(a) Angular displacement $\theta = 9t^2 - 2t^3\,\text{rad}$

Angular velocity $\omega = \frac{d\theta}{dt} = 18t - 6t^2\,\text{rad/s}$

When time $t = 1\,\text{s}$,

$$\omega = 18(1) - 6(1)^2 = \mathbf{12\,rad/s}$$

Angular acceleration $\alpha = \frac{d^2\theta}{dt^2} = 18 - 12t\,\text{rad/s}^2$

When time $t = 1\,\text{s}$,

$$\alpha = 18 - 12(1) = \mathbf{6\,rad/s^2}$$

(b) When the angular acceleration is zero, $18 - 12t = 0$, from which, $18 = 12t$, giving time, $\boldsymbol{t = 1.5\,s}$.

Problem 9. The displacement x cm of the slide valve of an engine is given by $x = 2.2\cos 5\pi t + 3.6\sin 5\pi t$. Evaluate the velocity (in m/s) when time $t = 30\,\text{ms}$.

Displacement $x = 2.2\cos 5\pi t + 3.6\sin 5\pi t$

Velocity $v = \frac{dx}{dt}$

$$= (2.2)(-5\pi)\sin 5\pi t + (3.6)(5\pi)\cos 5\pi t$$

$$= -11\pi\sin 5\pi t + 18\pi\cos 5\pi t\,\text{cm/s}$$

When time $t = 30\,\text{ms}$, velocity

$$= -11\pi\sin\left(5\pi \cdot \frac{30}{10^3}\right) + 18\pi\cos\left(5\pi \cdot \frac{30}{10^3}\right)$$

$$= -11\pi\sin 0.4712 + 18\pi\cos 0.4712$$

$$= -11\pi\sin 27^\circ + 18\pi\cos 27^\circ$$

$$= -15.69 + 50.39 = 34.7\,\text{cm/s}$$

$$= \mathbf{0.347\,m/s}$$

Now try the following exercise

Exercise 121 Further problems on velocity and acceleration

1. A missile fired from ground level rises x metres vertically upwards in t seconds and $x = 100t - \frac{25}{2}t^2$. Find (a) the initial velocity of the missile, (b) the time when the height of the missile is a maximum, (c) the maximum height reached, (d) the velocity with which the missile strikes the ground.

$$\begin{bmatrix} \text{(a) } 100\,\text{m/s} & \text{(b) } 4\,\text{s} \\ \text{(c) } 200\,\text{m} & \text{(d) } -100\,\text{m/s} \end{bmatrix}$$

2. The distance s metres travelled by a car in t seconds after the brakes are applied is given by $s = 25t - 2.5t^2$. Find (a) the speed of the car (in km/h) when the brakes are applied, (b) the distance the car travels before it stops. [(a) 90 km/h (b) 62.5 m]

3. The equation $\theta = 10\pi + 24t - 3t^2$ gives the angle θ, in radians, through which a wheel turns in t seconds. Determine (a) the time the wheel takes to come to rest, (b) the angle turned through in the last second of movement. [(a) 4 s (b) 3 rads]

4. At any time t seconds the distance x metres of a particle moving in a straight line from a fixed point is given by $x = 4t + \ln(1 - t)$. Determine (a) the initial velocity and acceleration (b) the velocity and acceleration after 1.5 s (c) the time when the velocity is zero.

$$\begin{bmatrix} \text{(a) } 3\,\text{m/s};\ -1\,\text{m/s}^2 \\ \text{(b) } 6\,\text{m/s};\ -4\,\text{m/s}^2 \\ \text{(c) } \frac{3}{4}\,\text{s} \end{bmatrix}$$

5. The angular displacement θ of a rotating disc is given by $\theta = 6\sin\frac{t}{4}$, where t is the time in seconds. Determine (a) the angular velocity of the disc when t is 1.5 s, (b) the angular acceleration when t is 5.5 s, and (c) the first time when the angular velocity is zero.

$$\begin{bmatrix} \text{(a) } \omega = 1.40\,\text{rad/s} \\ \text{(b) } \alpha = -0.37\,\text{rad/s}^2 \\ \text{(c) } t = 6.28\,\text{s} \end{bmatrix}$$

6. $x = \frac{20t^3}{3} - \frac{23t^2}{2} + 6t + 5$ represents the distance, x metres, moved by a body in t seconds. Determine (a) the velocity and acceleration at the start, (b) the velocity and acceleration when $t = 3$ s, (c) the values of t when the body is at rest, (d) the value of t when the acceleration is 37 m/s^2 and (e) the distance travelled in the third second.

$$\begin{bmatrix} \text{(a) } 6\,\text{m/s};\ -23\,\text{m/s}^2 \\ \text{(b) } 117\,\text{m/s};\ 97\,\text{m/s}^2 \\ \text{(c) } \frac{3}{4}\,\text{s or } \frac{2}{5}\,\text{s} \\ \text{(d) } 1\frac{1}{2}\,\text{s} \\ \text{(e) } 75\frac{1}{6}\,\text{m} \end{bmatrix}$$

28.3 Turning points

In Fig. 28.4, the gradient (or rate of change) of the curve changes from positive between O and P to negative between P and Q, and then positive again between Q and R. At point P, the gradient is zero and, as x increases, the gradient of the curve changes from positive just before P to negative just after. Such a point is called a **maximum point** and appears as the 'crest of a wave'. At point Q, the gradient is also zero and, as x increases, the gradient of the curve changes from negative just before Q to positive just after. Such a point is called a **minimum point**, and appears as the 'bottom of a valley'. Points such as P and Q are given the general name of **turning points**.

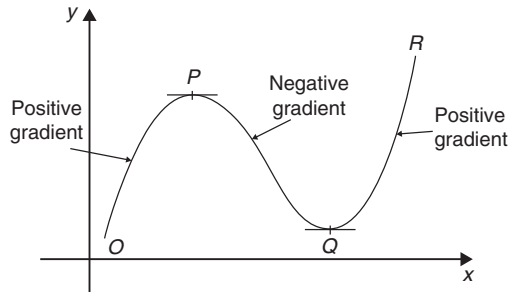


Figure 28.4

It is possible to have a turning point, the gradient on either side of which is the same. Such a point is given the special name of a **point of inflexion**, and examples are shown in Fig. 28.5.

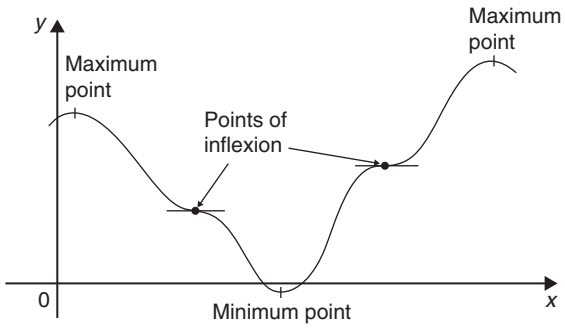


Figure 28.5

Maximum and minimum points and points of inflexion are given the general term of **stationary points**.

Procedure for finding and distinguishing between stationary points:

(i) Given $y = f(x)$, determine $\frac{dy}{dx}$ (i.e. $f'(x)$)

(ii) Let $\frac{dy}{dx} = 0$ and solve for the values of x.

(iii) Substitute the values of x into the original equation, $y = f(x)$, to find the corresponding y-ordinate values. This establishes the co-ordinates of the stationary points.

To determine the nature of the stationary points:
Either

(iv) Find $\frac{d^2y}{dx^2}$ and substitute into it the values of x found in (ii).

If the result is:

(a) positive—the point is a minimum one,
(b) negative—the point is a maximum one,
(c) zero—the point is a point of inflexion,

or

(v) Determine the sign of the gradient of the curve just before and just after the stationary points. If the sign change for the gradient of the curve is:

(a) positive to negative—the point is a maximum one,

(b) negative to positive—the point is a minimum one,

(c) positive to positive or negative to negative—the point is a point of inflexion.

Problem 10. Locate the turning point on the curve $y = 3x^2 - 6x$ and determine its nature by examining the sign of the gradient on either side.

Following the above procedure:

(i) Since $y = 3x^2 - 6x$, $\frac{dy}{dx} = 6x - 6$.

(ii) At a turning point, $\frac{dy}{dx} = 0$. Hence $6x - 6 = 0$, from which, $x = 1$.

(iii) When $x = 1$, $y = 3(1)^2 - 6(1) = -3$.

Hence the co-ordinates of the turning point are (1, −3).

(iv) If x is slightly less than 1, say, 0.9, then

$$\frac{dy}{dx} = 6(0.9) - 6 = -0.6,$$

i.e. negative.
If x is slightly greater than 1, say, 1.1, then

$$\frac{dy}{dx} = 6(1.1) - 6 = 0.6,$$

i.e. positive.

Since the gradient of the curve is negative just before the turning point and positive just after (i.e. $-\vee+$), **(1, −3) is a minimum point**.

Problem 11. Find the maximum and minimum values of the curve $y = x^3 - 3x + 5$ by

(a) examining the gradient on either side of the turning points, and

(b) determining the sign of the second derivative.

Since $y = x^3 - 3x + 5$ then $\frac{dy}{dx} = 3x^2 - 3$

For a maximum or minimum value $\frac{dy}{dx} = 0$

Hence $3x^2 - 3 = 0$, from which, $3x^2 = 3$ and $x = \pm 1$

When $x = 1$, $y = (1)^3 - 3(1) + 5 = 3$

When $x = -1$, $y = (-1)^3 - 3(-1) + 5 = 7$

Hence (1, 3) and (−1, 7) are the co-ordinates of the turning points.

(a) Considering the point (1, 3):
If x is slightly less than 1, say 0.9, then

$$\frac{dy}{dx} = 3(0.9)^2 - 3,$$

which is negative.
If x is slightly more than 1, say 1.1, then

$$\frac{dy}{dx} = 3(1.1)^2 - 3,$$

which is positive.
Since the gradient changes from negative to positive, **the point (1, 3) is a minimum point**.

Considering the point $(-1, 7)$:

If x is slightly less than -1, say -1.1, then

$$\frac{dy}{dx} = 3(-1.1)^2 - 3,$$

which is positive.

If x is slightly more than -1, say -0.9, then

$$\frac{dy}{dx} = 3(-0.9)^2 - 3,$$

which is negative.

Since the gradient changes from positive to negative, **the point $(-1, 7)$ is a maximum point**.

(b) Since $\frac{dy}{dx} = 3x^2 - 3$, then $\frac{d^2y}{dx^2} = 6x$

When $x = 1$, $\frac{d^2y}{dx^2}$ is positive, hence (1, 3) is a **minimum value**.

When $x = -1$, $\frac{d^2y}{dx^2}$ is negative, hence $(-1, 7)$ is a **maximum value**.

Thus the maximum value is 7 and the minimum value is 3.

It can be seen that the second differential method of determining the nature of the turning points is, in this case, quicker than investigating the gradient.

Problem 12. Locate the turning point on the following curve and determine whether it is a maximum or minimum point: $y = 4\theta + e^{-\theta}$.

Since $y = 4\theta + e^{-\theta}$

then $\frac{dy}{d\theta} = 4 - e^{-\theta} = 0$

for a maximum or minimum value.

Hence $4 = e^{-\theta}$, $\frac{1}{4} = e^{\theta}$, giving $\theta = \ln\frac{1}{4} = -1.3863$ (see Chapter 4).

When $\theta = -1.3863$, $y = 4(-1.3863) + e^{-(-1.3863)}$
$= 5.5452 + 4.0000 = -1.5452$

Thus $(-1.3863, -1.5452)$ are the co-ordinates of the turning point.

$$\frac{d^2y}{d\theta^2} = e^{-\theta}.$$

When $\theta = -1.3863$,

$$\frac{d^2y}{d\theta^2} = e^{+1.3863} = 4.0,$$

which is positive, hence **$(-1.3863, -1.5452)$ is a minimum point**.

Problem 13. Determine the co-ordinates of the maximum and minimum values of the graph $y = \frac{x^3}{3} - \frac{x^2}{2} - 6x + \frac{5}{3}$ and distinguish between them. Sketch the graph.

Following the given procedure:

(i) Since $y = \frac{x^3}{3} - \frac{x^2}{2} - 6x + \frac{5}{3}$ then

$$\frac{dy}{dx} = x^2 - x - 6$$

(ii) At a turning point, $\frac{dy}{dx} = 0$. Hence

$x^2 - x - 6 = 0$, i.e. $(x + 2)(x - 3) = 0$,

from which $x = -2$ or $x = 3$.

(iii) When $x = -2$,

$$y = \frac{(-2)^3}{3} - \frac{(-2)^2}{2} - 6(-2) + \frac{5}{3} = 9$$

When $x = 3$,

$$y = \frac{(3)^3}{3} - \frac{(3)^2}{2} - 6(3) + \frac{5}{3} = -11\frac{5}{6}$$

Thus the co-ordinates of the turning points are $(-2, 9)$ and $\left(3, -11\frac{5}{6}\right)$.

(iv) Since $\frac{dy}{dx} = x^2 - x - 6$ then $\frac{d^2y}{dx^2} = 2x - 1$.

When $x = -2$,

$$\frac{d^2y}{dx^2} = 2(-2) - 1 = -5,$$

which is negative.

Hence (−2, 9) is a maximum point.

When $x = 3$,

$$\frac{d^2y}{dx^2} = 2(3) - 1 = 5,$$

which is positive.

Hence $\left(3, -11\frac{5}{6}\right)$ is a minimum point.

Knowing (−2, 9) is a maximum point (i.e. crest of a wave), and $\left(3, -11\frac{5}{6}\right)$ is a minimum point (i.e. bottom of a valley) and that when $x = 0$, $y = \frac{5}{3}$, a sketch may be drawn as shown in Fig. 28.6.

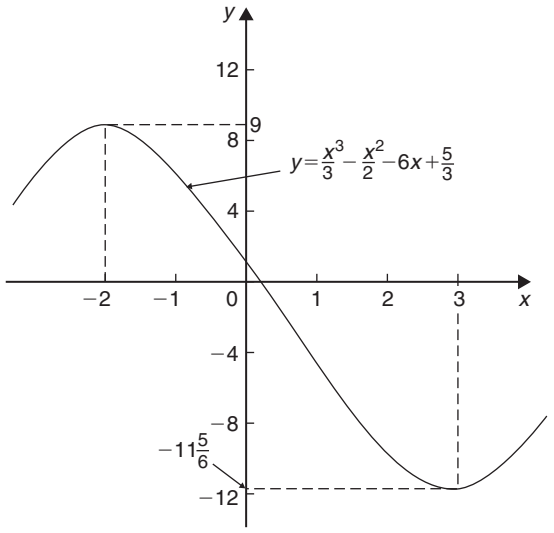


Figure 28.6

Problem 14. Determine the turning points on the curve $y = 4\sin x - 3\cos x$ in the range $x = 0$ to $x = 2\pi$ radians, and distinguish between them. Sketch the curve over one cycle.

Since $y = 4\sin x - 3\cos x$

then $\frac{dy}{dx} = 4\cos x + 3\sin x = 0,$

for a turning point, from which,

$4\cos x = -3\sin x$ and

$$\frac{-4}{3} = \frac{\sin x}{\cos x} = \tan x$$

Hence $x = \tan^{-1}\left(\frac{-4}{3}\right) = 126.87°$ or $306.87°$, since tangent is negative in the second and fourth quadrants.

When $x = 126.87°$,

$$y = 4\sin 126.87° - 3\cos 126.87° = 5$$

When $x = 306.87°$,

$$y = 4\sin 306.87° - 3\cos 306.87° = -5$$

$$126.87° = \left(126.87° \times \frac{\pi}{180}\right) \text{ radians}$$

$$= 2.214 \text{ rad}$$

$$306.87° = \left(306.87° \times \frac{\pi}{180}\right) \text{ radians}$$

$$= 5.356 \text{ rad}$$

Hence (2.214, 5) and (5.356, −5) are the co-ordinates of the turning points.

$$\frac{d^2y}{dx^2} = -4\sin x + 3\cos x$$

When $x = 2.214$ rad,

$$\frac{d^2y}{dx^2} = -4\sin 2.214 + 3\cos 2.214,$$

which is negative.

Hence (2.214, 5) is a maximum point.

When $x = 5.356$ rad,

$$\frac{d^2y}{dx^2} = -4\sin 5.356 + 3\cos 5.356,$$

which is positive.

Hence (5.356, −5) is a minimum point.

A sketch of $y = 4\sin x - 3\cos x$ is shown in Fig. 28.7.

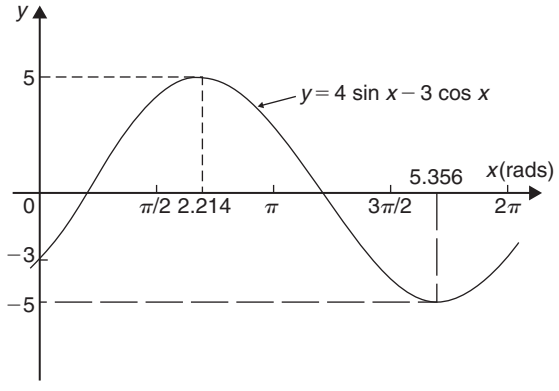


Figure 28.7

Now try the following exercise

Exercise 122 Further problems on turning points

In Problems 1 to 11, find the turning points and distinguish between them.

1. $y = x^2 - 6x$ [(3, −9) Minimum]
2. $y = 8 + 2x - x^2$ [(1, 9) Maximum]
3. $y = x^2 - 4x + 3$ [(2, −1) Minimum]
4. $y = 3 + 3x^2 - x^3$ [(0, 3) Minimum, (2, 7) Maximum]
5. $y = 3x^2 - 4x + 2$ [Minimum at $(\frac{2}{3}, \frac{2}{3})$]
6. $x = \theta(6 - \theta)$ [Maximum at (3, 9)]
7. $y = 4x^3 + 3x^2 - 60x - 12$ [Minimum (2, −88); Maximum(−2.5, 94.25)]
8. $y = 5x - 2\ln x$ [Minimum at (0.4000, 3.8326)]
9. $y = 2x - e^x$ [Maximum at (0.6931, −0.6136)]
10. $y = t^3 - \frac{t^2}{2} - 2t + 4$ [Minimum at (1, 2.5); Maximum at $\left(-\frac{2}{3}, 4\frac{22}{27}\right)$]
11. $x = 8t + \frac{1}{2t^2}$ [Minimum at (0.5, 6)]
12. Determine the maximum and minimum values on the graph $y = 12\cos\theta - 5\sin\theta$ in the range $\theta = 0$ to $\theta = 360°$. Sketch the graph over one cycle showing relevant points. [Maximum of 13 at 337.38°, Minimum of −13 at 157.38°]
13. Show that the curve $y = \frac{2}{3}(t-1)^3 + 2t(t-2)$ has a maximum value of $\frac{2}{3}$ and a minimum value of −2.

28.4 Practical problems involving maximum and minimum values

There are many **practical problems** involving maximum and minimum values which occur in science and engineering. Usually, an equation has to be determined from given data, and rearranged where necessary, so that it contains only one variable. Some examples are demonstrated in Problems 15 to 20.

Problem 15. A rectangular area is formed having a perimeter of 40 cm. Determine the length and breadth of the rectangle if it is to enclose the maximum possible area.

Let the dimensions of the rectangle be x and y. Then the perimeter of the rectangle is $(2x + 2y)$. Hence

$$2x + 2y = 40,$$

or
$$x + y = 20 \qquad (1)$$

Since the rectangle is to enclose the maximum possible area, a formula for area A must be obtained in terms of one variable only.
Area $A = xy$. From equation (1), $x = 20 - y$
Hence, area $A = (20 - y)y = 20y - y^2$

$$\frac{dA}{dy} = 20 - 2y = 0$$

for a turning point, from which, $y = 10$ cm

$$\frac{d^2A}{dy^2} = -2,$$

which is negative, giving a maximum point.
When $y = 10$ cm, $x = 10$ cm, from equation (1).

Hence the length and breadth of the rectangle are each 10 cm, i.e. a square gives the maximum possible area. When the perimeter of a rectangle is 40 cm, the maximum possible area is $10 \times 10 =$ **100 cm^2**.

Problem 16. A rectangular sheet of metal having dimensions 20 cm by 12 cm has squares removed from each of the four corners and the sides bent

upwards to form an open box. Determine the maximum possible volume of the box.

The squares to be removed from each corner are shown in Fig. 28.8, having sides x cm. When the sides are bent upwards the dimensions of the box will be:
length $(20-2x)$ cm, breadth $(12-2x)$ cm and height, x cm.

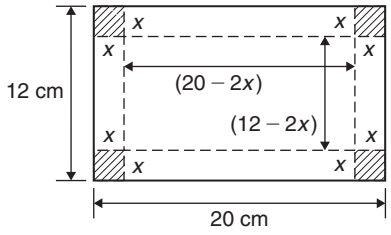


Figure 28.8

Volume of box,

$$V = (20-2x)(12-2x)(x)$$
$$= 240x - 64x^2 + 4x^3$$

$$\frac{dV}{dx} = 240 - 128x + 12x^2 = 0$$

for a turning point.

Hence $4(60 - 32x + 3x^2) = 0$,

i.e. $3x^2 - 32x + 60 = 0$

Using the quadratic formula,

$$x = \frac{32 \pm \sqrt{(-32)^2 - 4(3)(60)}}{2(3)}$$
$$= 8.239\,\text{cm or } 2.427\,\text{cm}.$$

Since the breadth is $(12-2x)$ cm then $x = 8.239$ cm is not possible and is neglected. Hence $x = 2.427$ cm

$$\frac{d^2V}{dx^2} = -128 + 24x.$$

When $x = 2.427$, $\dfrac{d^2V}{dx^2}$ is negative, giving a maximum value.

The dimensions of the box are:

$$\text{length} = 20 - 2(2.427) = 15.146\,\text{cm},$$
$$\text{breadth} = 12 - 2(2.427) = 7.146\,\text{cm},$$
$$\text{and height} = 2.427\,\text{cm}$$

$$\textbf{Maximum volume} = (15.146)(7.146)(2.427)$$
$$= \mathbf{262.7\,cm^3}$$

Problem 17. Determine the height and radius of a cylinder of volume 200 cm^3 which has the least surface area.

Let the cylinder have radius r and perpendicular height h.
Volume of cylinder,

$$V = \pi r^2 h = 200 \qquad (1)$$

Surface area of cylinder,

$$A = 2\pi rh + 2\pi r^2$$

Least surface area means minimum surface area and a formula for the surface area in terms of one variable only is required.
From equation (1),

$$h = \frac{200}{\pi r^2} \qquad (2)$$

Hence surface area,

$$A = 2\pi r\left(\frac{200}{\pi r^2}\right) + 2\pi r^2$$
$$= \frac{400}{r} + 2\pi r^2 = 400r^{-1} + 2\pi r^2$$
$$\frac{dA}{dr} = \frac{-400}{r^2} + 4\pi r = 0,$$

for a turning point.

Hence $4\pi r = \dfrac{400}{r^2}$ and $r^3 = \dfrac{400}{4\pi}$,

from which,

$$r = \sqrt[3]{\left(\frac{100}{\pi}\right)} = 3.169\,\text{cm}$$

$$\frac{d^2A}{dr^2} = \frac{800}{r^3} + 4\pi.$$

When $r = 3.169$ cm, $\dfrac{d^2A}{dr^2}$ is positive, giving a minimum value.
From equation (2),

when $r = 3.169$ cm,

$$h = \frac{200}{\pi(3.169)^2} = 6.339\,\text{cm}$$

Hence for the least surface area, a cylinder of volume 200 cm^3 has a radius of 3.169 cm and height of 6.339 cm.

Problem 18. Determine the area of the largest piece of rectangular ground that can be enclosed by 100 m of fencing, if part of an existing straight wall is used as one side.

Let the dimensions of the rectangle be x and y as shown in Fig. 28.9, where PQ represents the straight wall.

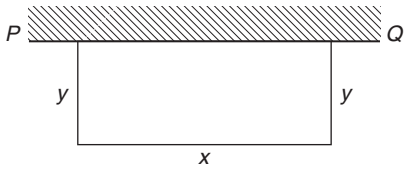


Figure 28.9

From Fig. 28.9,

$$x + 2y = 100 \qquad (1)$$

Area of rectangle,

$$A = xy \qquad (2)$$

Since the maximum area is required, a formula for area A is needed in terms of one variable only.
From equation (1), $x = 100 - 2y$
Hence area $A = xy = (100 - 2y)y = 100y - 2y^2$

$$\frac{dA}{dy} = 100 - 4y = 0,$$

for a turning point, from which, $y = 25$ m

$$\frac{d^2A}{dy^2} = -4,$$

which is negative, giving a maximum value.
When $y = 25$ m, $x = 50$ m from equation (1).
Hence the **maximum possible area** $= xy = (50)(25) =$ **1250 m^2**.

Problem 19. An open rectangular box with square ends is fitted with an overlapping lid which covers the top and the front face. Determine the maximum volume of the box if 6 m^2 of metal are used in its construction.

A rectangular box having square ends of side x and length y is shown in Fig. 28.10.

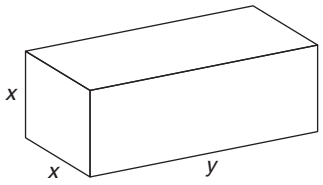


Figure 28.10

Surface area of box, A, consists of two ends and five faces (since the lid also covers the front face.)
Hence

$$A = 2x^2 + 5xy = 6 \qquad (1)$$

Since it is the maximum volume required, a formula for the volume in terms of one variable only is needed.
Volume of box, $V = x^2 y$.
From equation (1),

$$y = \frac{6 - 2x^2}{5x} = \frac{6}{5x} - \frac{2x}{5} \qquad (2)$$

Hence volume

$$V = x^2 y = x^2\left(\frac{6}{5x} - \frac{2x}{5}\right) = \frac{6x}{5} - \frac{2x^3}{5}$$

$$\frac{dV}{dx} = \frac{6}{5} - \frac{6x^2}{5} = 0$$

for a maximum or minimum value.
Hence $6 = 6x^2$, giving $x = 1$ m ($x = -1$ is not possible, and is thus neglected).

$$\frac{d^2V}{dx^2} = \frac{-12x}{5}$$

When $x = 1$, $\dfrac{d^2V}{dx^2}$ is negative, giving a maximum value.
From equation (2), when $x = 1$,

$$y = \frac{6}{5(1)} - \frac{2(1)}{5} = \frac{4}{5}$$

Hence the maximum volume of the box is given by

$$V = x^2 y = (1)^2\left(\tfrac{4}{5}\right) = \mathbf{\tfrac{4}{5}\ m^3}$$

Problem 20. Find the diameter and height of a cylinder of maximum volume which can be cut from a sphere of radius 12 cm.

A cylinder of radius r and height h is shown enclosed in a sphere of radius $R = 12$ cm in Fig. 28.11.

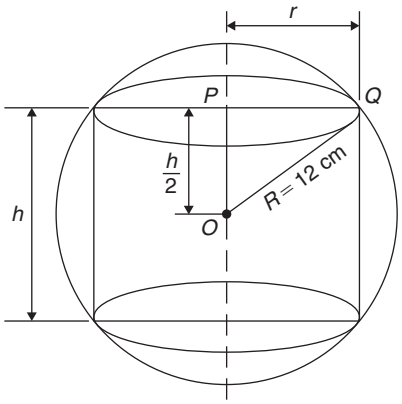


Figure 28.11

Volume of cylinder,

$$V = \pi r^2 h \qquad (1)$$

Using the right-angled triangle OPQ shown in Fig. 28.11,

$$r^2 + \left(\frac{h}{2}\right)^2 = R^2 \quad \text{by Pythagoras' theorem,}$$

i.e. $$r^2 + \frac{h^2}{4} = 144 \qquad (2)$$

Since the maximum volume is required, a formula for the volume V is needed in terms of one variable only. From equation (2),

$$r^2 = 144 - \frac{h^2}{4}$$

Substituting into equation (1) gives:

$$V = \pi\left(144 - \frac{h^2}{4}\right)h = 144\pi h - \frac{\pi h^3}{4}$$

$$\frac{dV}{dh} = 144\pi - \frac{3\pi h^2}{4} = 0,$$

for a maximum or minimum value.
Hence

$$144\pi = \frac{3\pi h^2}{4}$$

from which, $$h = \sqrt{\frac{(144)(4)}{3}} = 13.86\,\text{cm}$$

$$\frac{d^2V}{dh^2} = \frac{-6\pi h}{4}$$

When $h = 13.86$, $\dfrac{d^2V}{dh^2}$ is negative, giving a maximum value.
From equation (2),

$$r^2 = 144 - \frac{h^2}{4} = 144 - \frac{13.86^2}{4}$$

from which, radius $r = 9.80\,\text{cm}$

Diameter of cylinder $= 2r = 2(9.80) = 19.60\,\text{cm}$.

Hence the cylinder having the maximum volume that can be cut from a sphere of radius 12 cm is one in which the diameter is 19.60 cm and the height is 13.86 cm.

Now try the following exercise

Exercise 123 Further problems on practical maximum and minimum problems

1. The speed, v, of a car (in m/s) is related to time t s by the equation $v = 3 + 12t - 3t^2$. Determine the maximum speed of the car in km/h. [54 km/h]

2. Determine the maximum area of a rectangular piece of land that can be enclosed by 1200 m of fencing. [90000 m^2]

3. A shell is fired vertically upwards and its vertical height, x metres, is given by $x = 24t - 3t^2$, where t is the time in seconds. Determine the maximum height reached. [48 m]

4. A lidless box with square ends is to be made from a thin sheet of metal. Determine the least area of the metal for which the volume of the box is 3.5 m^3. [11.42 m^2]

5. A closed cylindrical container has a surface area of 400 cm^2. Determine the dimensions for maximum volume.
[radius = 4.607 cm; height = 9.212 cm]

6. Calculate the height of a cylinder of maximum volume which can be cut from a cone of height 20 cm and base radius 80 cm. [6.67 cm]

7. The power developed in a resistor R by a battery of emf E and internal resistance r is given by $P=\dfrac{E^2R}{(R+r)^2}$. Differentiate P with respect to R and show that the power is a maximum when $R=r$.

8. Find the height and radius of a closed cylinder of volume $125\,\text{cm}^3$ which has the least surface area.

$$\begin{bmatrix}\text{height} = 5.42\,\text{cm};\\ \text{radius} = 2.71\,\text{cm}\end{bmatrix}$$

9. Resistance to motion, F, of a moving vehicle, is given by $F=\frac{5}{x}+100x$. Determine the minimum value of resistance. [44.72]

10. An electrical voltage E is given by $E=(15\sin 50\pi t+40\cos 50\pi t)$ volts, where t is the time in seconds. Determine the maximum value of voltage.

[42.72 volts]

11. The fuel economy E of a car, in miles per gallon, is given by:

$$E = 21 + 2.10\times10^{-2}v^2 - 3.80\times10^{-6}v^4$$

where v is the speed of the car in miles per hour.
Determine, correct to 3 significant figures, the most economical fuel consumption, and the speed at which it is achieved.

[50.0 miles/gallon, 52.6 miles/hour]

28.5 Tangents and normals

Tangents

The equation of the tangent to a curve $y=f(x)$ at the point (x_1, y_1) is given by:

$$y - y_1 = m(x - x_1)$$

where $m=\dfrac{dy}{dx}=$ gradient of the curve at (x_1, y_1).

Problem 21. Find the equation of the tangent to the curve $y=x^2-x-2$ at the point $(1, -2)$.

Gradient, m

$$=\frac{dy}{dx}=2x-1$$

At the point $(1, -2)$, $x=1$ and $m=2(1)-1=1$.
Hence the equation of the tangent is:

$$y - y_1 = m(x - x_1)$$

i.e. $y-(-2)=1(x-1)$

i.e. $y+2=x-1$

or $\boldsymbol{y=x-3}$

The graph of $y=x^2-x-2$ is shown in Fig. 28.12. The line AB is the tangent to the curve at the point C, i.e. $(1, -2)$, and the equation of this line is $y=x-3$.

Normals

The normal at any point on a curve is the line which passes through the point and is at right angles to the tangent. Hence, in Fig. 28.12, the line CD is the normal.

It may be shown that if two lines are at right angles then the product of their gradients is -1. Thus if m is the gradient of the tangent, then the gradient of the normal is $-\dfrac{1}{m}$

Hence the equation of the normal at the point (x_1, y_1) is given by:

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

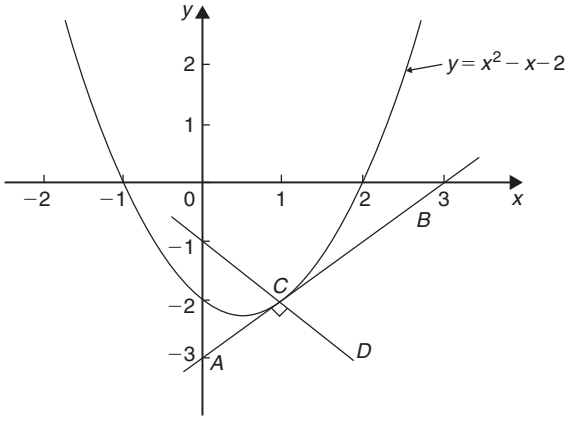


Figure 28.12

Problem 22. Find the equation of the normal to the curve $y = x^2 - x - 2$ at the point $(1, -2)$.

$m = 1$ from Problem 21, hence the equation of the normal is

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

i.e. $y - (-2) = -\frac{1}{1}(x - 1)$

i.e. $y + 2 = -x + 1$

or $\boldsymbol{y = -x - 1}$

Thus the line CD in Fig. 28.12 has the equation $y = -x - 1$.

Problem 23. Determine the equations of the tangent and normal to the curve $y = \frac{x^3}{5}$ at the point $\left(-1, -\frac{1}{5}\right)$

Gradient m of curve $y = \frac{x^3}{5}$ is given by

$$m = \frac{dy}{dx} = \frac{3x^2}{5}$$

At the point $\left(-1, -\frac{1}{5}\right)$, $x = -1$ and $m = \frac{3(-1)^2}{5} = \frac{3}{5}$

Equation of the tangent is:

$$y - y_1 = m(x - x_1)$$

i.e. $y - \left(-\frac{1}{5}\right) = \frac{3}{5}(x - (-1))$

i.e. $y + \frac{1}{5} = \frac{3}{5}(x + 1)$

or $5y + 1 = 3x + 3$

or $\boldsymbol{5y - 3x = 2}$

Equation of the normal is:

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

i.e. $y - \left(-\frac{1}{5}\right) = \frac{-1}{(3/5)}(x - (-1))$

i.e. $y + \frac{1}{5} = -\frac{5}{3}(x + 1)$

i.e. $y + \frac{1}{5} = -\frac{5}{3}x - \frac{5}{3}$

Multiplying each term by 15 gives:

$$15y + 3 = -25x - 25$$

Hence **equation of the normal** is:

$$\boldsymbol{15y + 25x + 28 = 0}$$

Now try the following exercise

Exercise 124 Further problems on tangents and normals

For the curves in problems 1 to 5, at the points given, find (a) the equation of the tangent, and (b) the equation of the normal.

1. $y = 2x^2$ at the point $(1, 2)$ $\begin{bmatrix}\text{(a) } y = 4x - 2\\ \text{(b) } 4y + x = 9\end{bmatrix}$
2. $y = 3x^2 - 2x$ at the point $(2, 8)$ $\begin{bmatrix}\text{(a) } y = 10x - 12\\ \text{(b) } 10y + x = 82\end{bmatrix}$
3. $y = \frac{x^3}{2}$ at the point $\left(-1, -\frac{1}{2}\right)$ $\begin{bmatrix}\text{(a) } y = \frac{3}{2}x + 1\\ \text{(b) } 6y + 4x + 7 = 0\end{bmatrix}$
4. $y = 1 + x - x^2$ at the point $(-2, -5)$ $\begin{bmatrix}\text{(a) } y = 5x + 5\\ \text{(b) } 5y + x + 27 = 0\end{bmatrix}$
5. $\theta = \frac{1}{t}$ at the point $\left(3, \frac{1}{3}\right)$ $\begin{bmatrix}\text{(a) } 9\theta + t = 6\\ \text{(b) } \theta = 9t - 26\frac{2}{3} \text{ or } 3\theta = 27t - 80\end{bmatrix}$

28.6 Small changes

If y is a function of x, i.e. $y = f(x)$, and the approximate change in y corresponding to a small change δx in x is required, then:

$$\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$$

and $\boldsymbol{\delta y \approx \frac{dy}{dx} \cdot \delta x}$ or $\boldsymbol{\delta y \approx f'(x) \cdot \delta x}$

Problem 24. Given $y=4x^2-x$, determine the approximate change in y if x changes from 1 to 1.02.

Since $y=4x^2-x$, then

$$\frac{dy}{dx}=8x-1$$

Approximate change in y,

$$\delta y \approx \frac{dy}{dx}\cdot\delta x \approx (8x-1)\delta x$$

When $x=1$ and $\delta x=0.02$, $\boldsymbol{\delta y} \approx [8(1)-1](0.02)$

$$\approx \mathbf{0.14}$$

[Obviously, in this case, the exact value of dy may be obtained by evaluating y when $x=1.02$, i.e. $y=4(1.02)^2-1.02=3.1416$ and then subtracting from it the value of y when $x=1$, i.e. $y=4(1)^2-1=3$, giving $\boldsymbol{\delta y}=3.1416-3=\mathbf{0.1416}$.

Using $\delta y=\frac{dy}{dx}\cdot\delta x$ above gave 0.14, which shows that the formula gives the approximate change in y for a small change in x.]

Problem 25. The time of swing T of a pendulum is given by $T=k\sqrt{l}$, where k is a constant. Determine the percentage change in the time of swing if the length of the pendulum l changes from 32.1 cm to 32.0 cm.

If $T=k\sqrt{l}=kl^{\frac{1}{2}}$, then

$$\frac{dT}{dl}=k\left(\frac{1}{2}l^{\frac{-1}{2}}\right)=\frac{k}{2\sqrt{l}}$$

Approximate change in T,

$$\delta t \approx \frac{dT}{dl}\delta l \approx \left(\frac{k}{2\sqrt{l}}\right)\delta l$$

$$\approx \left(\frac{k}{2\sqrt{l}}\right)(-0.1)$$

(negative since l decreases)

Percentage error

$$=\left(\frac{\text{approximate change in } T}{\text{original value of } T}\right)100\%$$

$$=\frac{\left(\frac{k}{2\sqrt{l}}\right)(-0.1)}{k\sqrt{l}}\times 100\%$$

$$=\left(\frac{-0.1}{2l}\right)100\%=\left(\frac{-0.1}{2(32.1)}\right)100\%$$

$$=\mathbf{-0.156\%}$$

Hence the change in the time of swing is a decrease of 0.156%.

Problem 26. A circular template has a radius of 10 cm ($\pm$0.02). Determine the possible error in calculating the area of the template. Find also the percentage error.

Area of circular template, $A=\pi r^2$, hence

$$\frac{dA}{dr}=2\pi r$$

Approximate change in area,

$$\delta A \approx \frac{dA}{dr}\cdot\delta r \approx (2\pi r)\delta r$$

When $r=10$ cm and $\delta r=0.02$,

$\delta A=(2\pi 10)(0.02)\approx 0.4\pi$ cm^2

i.e. **the possible error in calculating the template area is approximately 1.257 cm^2.**

$$\textbf{Percentage error} \approx \left(\frac{0.4\pi}{\pi(10)^2}\right)100\%$$

$$=\mathbf{0.40\%}$$

Now try the following exercise

Exercise 125 Further problems on small changes

1. Determine the change in y if x changes from 2.50 to 2.51 when

   (a) $y=2x-x^2$ (b) $y=\frac{5}{x}$

   [(a) -0.03 (b) -0.008]

2. The pressure p and volume v of a mass of gas are related by the equation $pv = 50$. If the pressure increases from 25.0 to 25.4, determine the approximate change in the volume of the gas. Find also the percentage change in the volume of the gas. [-0.032, -1.6%]

3. Determine the approximate increase in (a) the volume, and (b) the surface area of a cube of side x cm if x increases from 20.0 cm to 20.05 cm. [(a) $60\,\text{cm}^3$ (b) $12\,\text{cm}^2$]

4. The radius of a sphere decreases from 6.0 cm to 5.96 cm. Determine the approximate change in (a) the surface area, and (b) the volume. [(a) $-6.03\,\text{cm}^2$ (b) $-18.10\,\text{cm}^3$]

5. The rate of flow of a liquid through a tube is given by Poiseuilles's equation as: $Q = \dfrac{p\pi r^4}{8\eta L}$ where Q is the rate of flow, p is the pressure difference between the ends of the tube, r is the radius of the tube, L is the length of the tube and η is the coefficient of viscosity of the liquid. η is obtained by measuring Q, p, r and L. If Q can be measured accurate to $\pm 0.5\%$, p accurate to $\pm 3\%$, r accurate to $\pm 2\%$ and L accurate to $\pm 1\%$, calculate the maximum possible percentage error in the value of η. [12.5%]

Chapter 29

Differentiation of parametric equations

29.1 Introduction to parametric equations

Certain mathematical functions can be expressed more simply by expressing, say, x and y separately in terms of a third variable. For example, $y = r\sin\theta$, $x = r\cos\theta$. Then, any value given to θ will produce a pair of values for x and y, which may be plotted to provide a curve of $y = f(x)$.

The third variable, θ, is called a **parameter** and the two expressions for y and x are called **parametric equations**.

The above example of $y = r\sin\theta$ and $x = r\cos\theta$ are the parametric equations for a circle. The equation of any point on a circle, centre at the origin and of radius r is given by: $x^2 + y^2 = r^2$, as shown in Chapter 13.

To show that $y = r\sin\theta$ and $x = r\cos\theta$ are suitable parametric equations for such a circle:

Left hand side of equation

$$= x^2 + y^2$$
$$= (r\cos\theta)^2 + (r\sin\theta)^2$$
$$= r^2\cos^2\theta + r^2\sin^2\theta$$
$$= r^2\left(\cos^2\theta + \sin^2\theta\right)$$
$$= r^2 = \text{right hand side}$$

(since $\cos^2\theta + \sin^2\theta = 1$, as shown in Chapter 15)

29.2 Some common parametric equations

The following are some of the most common parametric equations, and Fig. 29.1 shows typical shapes of these curves.

(a) Ellipse $x = a\cos\theta,\ y = b\sin\theta$

(b) Parabola $x = a\,t^2,\ y = 2at$

(c) Hyperbola $x = a\sec\theta,\ y = b\tan\theta$

(d) Rectangular hyperbola $x = ct,\ y = \dfrac{c}{t}$

(e) Cardioid $x = a(2\cos\theta - \cos 2\theta)$, $y = a(2\sin\theta - \sin 2\theta)$

(f) Astroid $x = a\cos^3\theta,\ y = a\sin^3\theta$

(g) Cycloid $x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta)$

29.3 Differentiation in parameters

When x and y are given in terms of a parameter, say θ, then by the function of a function rule of differentiation (from Chapter 27):

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

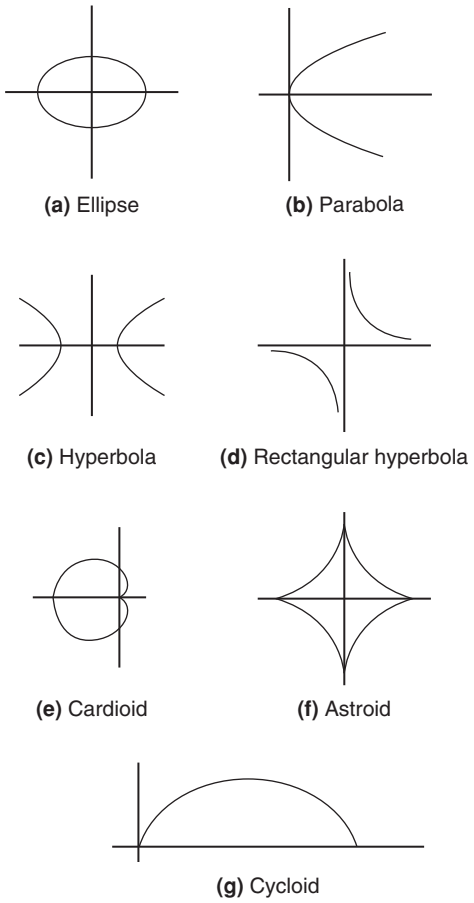


Figure 29.1

It may be shown that this can be written as:

$$\frac{dy}{dx} = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}} \qquad (1)$$

For the second differential,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\cdot\frac{d\theta}{dx}$$

or

$$\frac{d^2y}{dx^2} = \frac{\dfrac{d}{d\theta}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{d\theta}} \qquad (2)$$

Problem 1. Given $x = 5\theta - 1$ and $y = 2\theta\,(\theta - 1)$, determine $\dfrac{dy}{dx}$ in terms of θ.

$x = 5\theta - 1$, hence $\dfrac{dy}{d\theta} = 5$

$y = 2\theta(\theta - 1) = 2\theta^2 - 2\theta$,

hence $\dfrac{dy}{d\theta} = 4\theta - 2 = 2\,(2\theta - 1)$

From equation (1),

$$\frac{dy}{dx} = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}} = \frac{\mathbf{2(2\theta - 1)}}{\mathbf{5}} \text{ or } \frac{\mathbf{2}}{\mathbf{5}}\mathbf{(2\theta - 1)}$$

Problem 2. The parametric equations of a function are given by $y = 3\cos 2t$, $x = 2\sin t$. Determine expressions for (a) $\dfrac{dy}{dx}$ (b) $\dfrac{d^2y}{dx^2}$.

(a) $y = 3\cos 2t$, hence $\dfrac{dy}{dt} = -6\sin 2t$

$x = 2\sin t$, hence $\dfrac{dx}{dt} = 2\cos t$

From equation (1),

$$\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \frac{-6\sin 2t}{2\cos t} = \frac{-6(2\sin t\cos t)}{2\cos t}$$

from double angles, Chapter 17

i.e. $\mathbf{\dfrac{dy}{dx} = -6\sin t}$

(b) From equation (2),

$$\frac{d^2y}{dx^2} = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}} = \frac{\dfrac{d}{dt}(-6\sin t)}{2\cos t} = \frac{-6\cos t}{2\cos t}$$

i.e. $\mathbf{\dfrac{d^2y}{dx^2} = -3}$

Problem 3. The equation of a tangent drawn to a curve at point (x_1, y_1) is given by:

$$y - y_1 = \frac{dy_1}{dx_1}(x - x_1)$$

Determine the equation of the tangent drawn to the parabola $x=2t^2$, $y=4t$ at the point t.

At point t, $x_1=2t^2$, hence $\dfrac{dx_1}{dt}=4t$

and $y_1=4t$, hence $\dfrac{dy_1}{dt}=4$

From equation (1),

$$\frac{dy}{dx}=\frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}=\frac{4}{4t}=\frac{1}{t}$$

Hence, the equation of the tangent is:

$$\boldsymbol{y-4t=\frac{1}{t}\left(x-2t^2\right)}$$

Problem 4. The parametric equations of a cycloid are $x=4(\theta-\sin\theta)$, $y=4(1-\cos\theta)$. Determine (a) $\dfrac{dy}{dx}$ (b) $\dfrac{d^2y}{dx^2}$

(a) $x=4(\theta-\sin\theta)$,

hence $\dfrac{dx}{d\theta}=4-4\cos\theta=4(1-\cos\theta)$

$y=4(1-\cos\theta)$, hence $\dfrac{dy}{d\theta}=4\sin\theta$

From equation (1),

$$\frac{dy}{dx}=\frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}}=\frac{4\sin\theta}{4(1-\cos\theta)}=\boldsymbol{\frac{\sin\theta}{(1-\cos\theta)}}$$

(b) From equation (2),

$$\frac{d^2y}{dx^2}=\frac{\dfrac{d}{d\theta}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{d\theta}}=\frac{\dfrac{d}{d\theta}\left(\dfrac{\sin\theta}{1-\cos\theta}\right)}{4(1-\cos\theta)}$$

$$=\frac{(1-\cos\theta)(\cos\theta)-(\sin\theta)(\sin\theta)}{(1-\cos\theta)^2}}{4(1-\cos\theta)}$$

$$=\frac{\cos\theta-\cos^2\theta-\sin^2\theta}{4(1-\cos\theta)^3}$$

$$=\frac{\cos\theta-\left(\cos^2\theta+\sin^2\theta\right)}{4(1-\cos\theta)^3}$$

$$=\frac{\cos\theta-1}{4(1-\cos\theta)^3}$$

$$=\frac{-(1-\cos\theta)}{4(1-\cos\theta)^3}=\boldsymbol{\frac{-1}{4(1-\cos\theta)^2}}$$

Now try the following exercise

Exercise 126 Further problems on differentiation of parametric equations

1. Given $x=3t-1$ and $y=t(t-1)$, determine $\dfrac{dy}{dx}$ in terms of t. $\left[\frac{1}{3}(2t-1)\right]$
2. A parabola has parametric equations: $x=t^2$, $y=2t$. Evaluate $\dfrac{dy}{dx}$ when $t=0.5$. [2]
3. The parametric equations for an ellipse are $x=4\cos\theta$, $y=\sin\theta$. Determine (a) $\dfrac{dy}{dx}$ (b) $\dfrac{d^2y}{dx^2}$. $\left[(a)\ -\frac{1}{4}\cot\theta\ \ (b)-\frac{1}{16}\operatorname{cosec}^3\theta\right]$
4. Evaluate $\dfrac{dy}{dx}$ at $\theta=\dfrac{\pi}{6}$ radians for the hyperbola whose parametric equations are $x=3\sec\theta$, $y=6\tan\theta$. [4]
5. The parametric equations for a rectangular hyperbola are $x=2t$, $y=\dfrac{2}{t}$. Evaluate $\dfrac{dy}{dx}$ when $t=0.40$. [−6.25]

   The equation of a tangent drawn to a curve at point (x_1, y_1) is given by:

   $$y-y_1=\frac{dy_1}{dx_1}(x-x_1)$$

   Use this in Problems 6 and 7.
6. Determine the equation of the tangent drawn to the ellipse $x=3\cos\theta$, $y=2\sin\theta$ at $\theta=\dfrac{\pi}{6}$. [$y=-1.155x+4$]
7. Determine the equation of the tangent drawn to the rectangular hyperbola $x=5t$, $y=\dfrac{5}{t}$ at $t=2$. $\left[y=-\frac{1}{4}x+5\right]$

29.4 Further worked problems on differentiation of parametric equations

Problem 5. The equation of the normal drawn to a curve at point (x_1, y_1) is given by:

$$y - y_1 = -\frac{1}{\dfrac{dy_1}{dx_1}}(x - x_1)$$

Determine the equation of the normal drawn to the astroid $x = 2\cos^3\theta$, $y = 2\sin^3\theta$ at the point $\theta = \dfrac{\pi}{4}$

$x = 2\cos^3\theta$, hence $\dfrac{dx}{d\theta} = -6\cos^2\theta\sin\theta$

$y = 2\sin^3\theta$, hence $\dfrac{dy}{d\theta} = 6\sin^2\theta\cos\theta$

From equation (1),

$$\frac{dy}{dx} = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}} = \frac{6\sin^2\theta\cos\theta}{-6\cos^2\theta\sin\theta} = -\frac{\sin\theta}{\cos\theta} = -\tan\theta$$

When $\theta = \dfrac{\pi}{4}$, $\dfrac{dy}{dx} = -\tan\dfrac{\pi}{4} = -1$

$x_1 = 2\cos^3\dfrac{\pi}{4} = 0.7071$ and $y_1 = 2\sin^3\dfrac{\pi}{4} = 0.7071$

Hence, **the equation of the normal is**:

$$y - 0.7071 = -\frac{1}{-1}(x - 0.7071)$$

i.e. $y - 0.7071 = x - 0.7071$

i.e. $\boldsymbol{y = x}$

Problem 6. The parametric equations for a hyperbola are $x = 2\sec\theta$, $y = 4\tan\theta$. Evaluate (a) $\dfrac{dy}{dx}$ (b) $\dfrac{d^2y}{dx^2}$, correct to 4 significant figures, when $\theta = 1$ radian.

(a) $x = 2\sec\theta$, hence $\dfrac{dx}{d\theta} = 2\sec\theta\tan\theta$

$y = 4\tan\theta$, hence $\dfrac{dy}{d\theta} = 4\sec^2\theta$

From equation (1),

$$\frac{dy}{dx} = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}} = \frac{4\sec^2\theta}{2\sec\theta\tan\theta} = \frac{2\sec\theta}{\tan\theta}$$

$$= \frac{2\left(\dfrac{1}{\cos\theta}\right)}{\left(\dfrac{\sin\theta}{\cos\theta}\right)} = \frac{2}{\sin\theta} \text{ or } 2\operatorname{cosec}\theta$$

When $\theta = 1$ rad, $\dfrac{dy}{dx} = \dfrac{2}{\sin 1} = \mathbf{2.377}$, correct to 4 significant figures.

(b) From equation (2),

$$\frac{d^2y}{dx^2} = \frac{\dfrac{d}{d\theta}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{d\theta}} = \frac{\dfrac{d}{d\theta}(2\operatorname{cosec}\theta)}{2\sec\theta\tan\theta}$$

$$= \frac{-2\operatorname{cosec}\theta\cot\theta}{2\sec\theta\tan\theta}$$

$$= \frac{-\left(\dfrac{1}{\sin\theta}\right)\left(\dfrac{\cos\theta}{\sin\theta}\right)}{\left(\dfrac{1}{\cos\theta}\right)\left(\dfrac{\sin\theta}{\cos\theta}\right)}$$

$$= -\left(\frac{\cos\theta}{\sin^2\theta}\right)\left(\frac{\cos^2\theta}{\sin\theta}\right)$$

$$= -\frac{\cos^3\theta}{\sin^3\theta} = -\cot^3\theta$$

When $\theta = 1$ rad, $\dfrac{d^2y}{dx^2} = -\cot^3 1 = -\dfrac{1}{(\tan 1)^3}$ $= \mathbf{-0.2647}$, correct to 4 significant figures.

Problem 7. When determining the surface tension of a liquid, the radius of curvature, ρ, of part of the surface is given by:

$$\rho = \frac{\sqrt{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^3}}{\dfrac{d^2y}{dx^2}}$$

Find the radius of curvature of the part of the surface having the parametric equations $x = 3t^2$, $y = 6t$ at the point $t = 2$.

$x = 3t^2$, hence $\dfrac{dx}{dt} = 6t$

$y = 6t$, hence $\dfrac{dy}{dt} = 6$

From equation (1), $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{6}{6t} = \dfrac{1}{t}$

From equation (2),

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{1}{t}\right)}{6t} = \frac{-\frac{1}{t^2}}{6t} = -\frac{1}{6t^3}$$

Hence, radius of curvature, $\rho = \dfrac{\sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]^3}}{\frac{d^2y}{dx^2}}$

$$= \frac{\sqrt{\left[1+\left(\frac{1}{t}\right)^2\right]^3}}{-\frac{1}{6t^3}}$$

When $t = 2$, $\rho = \dfrac{\sqrt{\left[1+\left(\frac{1}{2}\right)^2\right]^3}}{-\frac{1}{6(2)^3}} = \dfrac{\sqrt{(1.25)^3}}{-\frac{1}{48}}$

$$= -48\sqrt{(1.25)^3} = \mathbf{-67.08}$$

Now try the following exercise

Exercise 127 Further problems on differentiation of parametric equations

1. A cycloid has parametric equations $x = 2(\theta - \sin\theta)$, $y = 2(1 - \cos\theta)$. Evaluate, at $\theta = 0.62$ rad, correct to 4 significant figures, (a) $\dfrac{dy}{dx}$ (b) $\dfrac{d^2y}{dx^2}$.

[(a) 3.122 (b) −14.43]

The equation of the normal drawn to a curve at point (x_1, y_1) is given by: $y - y_1 = -\dfrac{1}{\frac{dy_1}{dx_1}}(x - x_1)$

Use this in Problems 2 and 3.

2. Determine the equation of the normal drawn to the parabola $x = \frac{1}{4}t^2$, $y = \frac{1}{2}t$ at $t = 2$.

[$y = -2x + 3$]

3. Find the equation of the normal drawn to the cycloid $x = 2(\theta - \sin\theta)$, $y = 2(1 - \cos\theta)$ at $\theta = \dfrac{\pi}{2}$ rad. [$y = -x + \pi$]

4. Determine the value of $\dfrac{d^2y}{dx^2}$, correct to 4 significant figures, at $\theta = \dfrac{\pi}{6}$ rad for the cardioid $x = 5(2\theta - \cos 2\theta)$, $y = 5(2\sin\theta - \sin 2\theta)$.

[0.02975]

5. The radius of curvature, ρ, of part of a surface when determining the surface tension of a liquid is given by:

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

Find the radius of curvature (correct to 4 significant figures) of the part of the surface having parametric equations

(a) $x = 3t$, $y = \dfrac{3}{t}$ at the point $t = \dfrac{1}{2}$

(b) $x = 4\cos^3 t$, $y = 4\sin^3 t$ at $t = \dfrac{\pi}{6}$ rad.

[(a) 13.14 (b) 5.196]

Chapter 30

Differentiation of implicit functions

30.1 Implicit functions

When an equation can be written in the form $y = f(x)$ it is said to be an **explicit function** of x. Examples of explicit functions include

$$y = 2x^3 - 3x + 4, \quad y = 2x \ln x$$

and $y = \dfrac{3e^x}{\cos x}$

In these examples y may be differentiated with respect to x by using standard derivatives, the product rule and the quotient rule of differentiation respectively.

Sometimes with equations involving, say, y and x, it is impossible to make y the subject of the formula. The equation is then called an **implicit function** and examples of such functions include
$y^3 + 2x^2 = y^2 - x$ and $\sin y = x^2 + 2xy$.

30.2 Differentiating implicit functions

It is possible to **differentiate an implicit function** by using the **function of a function rule**, which may be stated as

$$\frac{du}{dx} = \frac{du}{dy} \times \frac{dy}{dx}$$

Thus, to differentiate y^3 with respect to x, the substitution $u = y^3$ is made, from which, $\dfrac{du}{dy} = 3y^2$. Hence, $\dfrac{d}{dx}(y^3) = (3y^2) \times \dfrac{dy}{dx}$, by the function of a function rule.

A simple rule for differentiating an implicit function is summarised as:

$$\frac{d}{dx}[f(y)] = \frac{d}{dy}[f(y)] \times \frac{dy}{dx} \qquad (1)$$

Problem 1. Differentiate the following functions with respect to x:

(a) $2y^4$ (b) $\sin 3t$.

(a) Let $u = 2y^4$, then, by the function of a function rule:

$$\frac{du}{dx} = \frac{du}{dy} \times \frac{dy}{dx} = \frac{d}{dy}(2y^4) \times \frac{dy}{dx}$$

$$= \mathbf{8y^3 \frac{dy}{dx}}$$

(b) Let $u = \sin 3t$, then, by the function of a function rule:

$$\frac{du}{dx} = \frac{du}{dt} \times \frac{dt}{dx} = \frac{d}{dt}(\sin 3t) \times \frac{dt}{dx}$$

$$= \mathbf{3 \cos 3t \frac{dt}{dx}}$$

Problem 2. Differentiate the following functions with respect to x:

(a) $4 \ln 5y$ (b) $\dfrac{1}{5}e^{3\theta - 2}$

(a) Let $u = 4 \ln 5y$, then, by the function of a function rule:

$$\frac{du}{dx} = \frac{du}{dy} \times \frac{dy}{dx} = \frac{d}{dy}(4\ln 5y) \times \frac{dy}{dx}$$

$$= \frac{\mathbf{4}}{\boldsymbol{y}}\frac{\mathbf{dy}}{\mathbf{dx}}$$

(b) Let $u = \frac{1}{5}e^{3\theta-2}$, then, by the function of a function rule:

$$\frac{du}{dx} = \frac{du}{d\theta} \times \frac{d\theta}{dx} = \frac{d}{d\theta}\left(\frac{1}{5}e^{3\theta-2}\right) \times \frac{d\theta}{dx}$$

$$= \frac{\mathbf{3}}{\mathbf{5}}\mathbf{e}^{\boldsymbol{3\theta-2}}\frac{\mathbf{d\theta}}{\mathbf{dx}}$$

Now try the following exercise

Exercise 128 Further problems on differentiating implicit functions

In Problems 1 and 2 differentiate the given functions with respect to x.

1. (a) $3y^5$ (b) $2\cos 4\theta$ (c) $\sqrt{k}$

$$\left[\begin{array}{l} \text{(a) } 15y^4\dfrac{dy}{dx} \quad \text{(b) } -8\sin 4\theta\dfrac{d\theta}{dx} \\ \text{(c) } \dfrac{1}{2\sqrt{k}}\dfrac{dk}{dx} \end{array}\right]$$

2. (a) $\frac{5}{2}\ln 3t$ (b) $\frac{3}{4}e^{2y+1}$ (c) $2\tan 3y$

$$\left[\begin{array}{ll} \text{(a) } \dfrac{5}{2t}\dfrac{dt}{dx} & \text{(b) } \dfrac{3}{2}e^{2y+1}\dfrac{dy}{dx} \\ \text{(c) } 6\sec^2 3y\dfrac{dy}{dx} & \end{array}\right]$$

3. Differentiate the following with respect to y:
(a) $3\sin 2\theta$ (b) $4\sqrt{x^3}$ (c) $\frac{2}{e^t}$

$$\left[\begin{array}{ll} \text{(a) } 6\cos 2\theta\dfrac{d\theta}{dy} & \text{(b) } 6\sqrt{x}\dfrac{dx}{dy} \\ \text{(c) } \dfrac{-2}{e^t}\dfrac{dt}{dy} & \end{array}\right]$$

4. Differentiate the following with respect to u:
(a) $\frac{2}{(3x+1)}$ (b) $3\sec 2\theta$ (c) $\frac{2}{\sqrt{y}}$

$$\left[\begin{array}{l} \text{(a) } \dfrac{-6}{(3x+1)^2}\dfrac{dx}{du} \\ \text{(b) } 6\sec 2\theta\tan 2\theta\dfrac{d\theta}{du} \\ \text{(c) } \dfrac{-1}{\sqrt{y^3}}\dfrac{dy}{du} \end{array}\right]$$

30.3 Differentiating implicit functions containing products and quotients

The product and quotient rules of differentiation must be applied when differentiating functions containing products and quotients of two variables.

For example, $\frac{d}{dx}(x^2y) = (x^2)\frac{d}{dx}(y) + (y)\frac{d}{dx}(x^2)$,

by the product rule

$$= (x^2)\left(1\frac{dy}{dx}\right) + y(2x),$$

by using equation (1)

$$= \mathbf{x^2}\frac{\mathbf{dy}}{\mathbf{dx}} + \mathbf{2xy}$$

Problem 3. Determine $\frac{d}{dx}(2x^3y^2)$.

In the product rule of differentiation let $u = 2x^3$ and $v = y^2$.

Thus $\frac{d}{dx}(2x^3y^2) = (2x^3)\frac{d}{dx}(y^2) + (y^2)\frac{d}{dx}(2x^3)$

$$= (2x^3)\left(2y\frac{dy}{dx}\right) + (y^2)(6x^2)$$

$$= 4x^3y\frac{dy}{dx} + 6x^2y^2$$

$$= \mathbf{2x^2y}\left(\mathbf{2x}\frac{\mathbf{dy}}{\mathbf{dx}} + \mathbf{3y}\right)$$

Problem 4. Find $\frac{d}{dx}\left(\frac{3y}{2x}\right)$.

In the quotient rule of differentiation let $u = 3y$ and $v = 2x$.

$$\text{Thus } \frac{d}{dx}\left(\frac{3y}{2x}\right) = \frac{(2x)\frac{d}{dx}(3y) - (3y)\frac{d}{dx}(2x)}{(2x)^2}$$

$$= \frac{(2x)\left(3\frac{dy}{dx}\right) - (3y)(2)}{4x^2}$$

$$= \frac{6x\frac{dy}{dx} - 6y}{4x^2} = \mathbf{\frac{3}{2x^2}\left(x\frac{dy}{dx} - y\right)}$$

Problem 5. Differentiate $z = x^2 + 3x\cos 3y$ with respect to y.

$$\frac{dz}{dy} = \frac{d}{dy}(x^2) + \frac{d}{dy}(3x\cos 3y)$$

$$= 2x\frac{dx}{dy} + \left[(3x)(-3\sin 3y) + (\cos 3y)\left(3\frac{dx}{dy}\right)\right]$$

$$= \mathbf{2x\frac{dx}{dy} - 9x\sin 3y + 3\cos 3y\frac{dx}{dy}}$$

Now try the following exercise

Exercise 129 Further problems on differentiating implicit functions involving products and quotients

1. Determine $\frac{d}{dx}(3x^2y^3)$. $\left[3xy^2\left(3x\frac{dy}{dx} + 2y\right)\right]$
2. Find $\frac{d}{dx}\left(\frac{2y}{5x}\right)$. $\left[\frac{2}{5x^2}\left(x\frac{dy}{dx} - y\right)\right]$
3. Determine $\frac{d}{du}\left(\frac{3u}{4v}\right)$. $\left[\frac{3}{4v^2}\left(v - u\frac{dv}{du}\right)\right]$
4. Given $z = 3\sqrt{y}\cos 3x$ find $\frac{dz}{dx}$. $\left[3\left(\frac{\cos 3x}{2\sqrt{y}}\right)\frac{dy}{dx} - 9\sqrt{y}\sin 3x\right]$
5. Determine $\frac{dz}{dy}$ given $z = 2x^3\ln y$. $\left[2x^2\left(\frac{x}{y} + 3\ln y\frac{dx}{dy}\right)\right]$

30.4 Further implicit differentiation

An implicit function such as $3x^2 + y^2 - 5x + y = 2$, may be differentiated term by term with respect to x. This gives:

$$\frac{d}{dx}(3x^2) + \frac{d}{dx}(y^2) - \frac{d}{dx}(5x) + \frac{d}{dx}(y) = \frac{d}{dx}(2)$$

i.e. $$6x + 2y\frac{dy}{dx} - 5 + 1\frac{dy}{dx} = 0,$$

using equation (1) and standard derivatives.

An expression for the derivative $\frac{dy}{dx}$ in terms of x and y may be obtained by rearranging this latter equation. Thus:

$$(2y + 1)\frac{dy}{dx} = 5 - 6x$$

from which, $$\mathbf{\frac{dy}{dx} = \frac{5 - 6x}{2y + 1}}$$

Problem 6. Given $2y^2 - 5x^4 - 2 - 7y^3 = 0$, determine $\frac{dy}{dx}$

Each term in turn is differentiated with respect to x:

$$\text{Hence } \frac{d}{dx}(2y^2) - \frac{d}{dx}(5x^4) - \frac{d}{dx}(2) - \frac{d}{dx}(7y^3) = \frac{d}{dx}(0)$$

i.e. $$4y\frac{dy}{dx} - 20x^3 - 0 - 21y^2\frac{dy}{dx} = 0$$

Rearranging gives:

$$(4y - 21y^2)\frac{dy}{dx} = 20x^3$$

i.e. $$\mathbf{\frac{dy}{dx} = \frac{20x^3}{(4y - 21y^2)}}$$

Problem 7. Determine the values of $\frac{dy}{dx}$ when $x = 4$ given that $x^2 + y^2 = 25$.

Differentiating each term in turn with respect to x gives:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$$

i.e. $2x + 2y\dfrac{dy}{dx} = 0$

Hence $\dfrac{dy}{dx} = -\dfrac{2x}{2y} = -\dfrac{x}{y}$

Since $x^2+y^2 = 25$, when $x=4$, $y=\sqrt{(25-4^2)}=\pm 3$

Thus when $x=4$ and $y=\pm 3$, $\dfrac{\mathbf{dy}}{\mathbf{dx}} = -\dfrac{4}{\pm 3} = \pm\dfrac{\mathbf{4}}{\mathbf{3}}$

$x^2+y^2=25$ is the equation of a circle, centre at the origin and radius 5, as shown in Fig. 30.1. At $x=4$, the two gradients are shown.

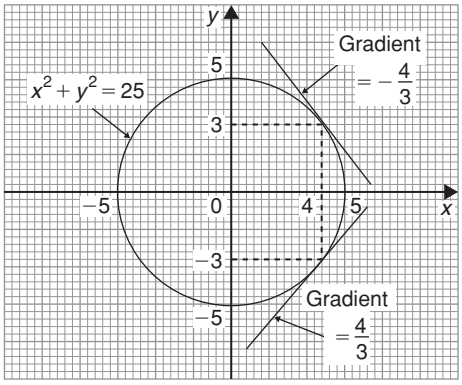


Figure 30.1

Above, $x^2+y^2=25$ was differentiated implicitly; actually, the equation could be transposed to $y=\sqrt{(25-x^2)}$ and differentiated using the function of a function rule. This gives

$$\frac{dy}{dx} = \frac{1}{2}(25-x^2)^{\frac{-1}{2}}(-2x) = -\frac{x}{\sqrt{(25-x^2)}}$$

and when $x=4$, $\dfrac{dy}{dx} = -\dfrac{4}{\sqrt{(25-4^2)}} = \pm\dfrac{4}{3}$ as obtained above.

Problem 8.

(a) Find $\dfrac{dy}{dx}$ in terms of x and y given $4x^2+2xy^3-5y^2=0$.

(b) Evaluate $\dfrac{dy}{dx}$ when $x=1$ and $y=2$.

(a) Differentiating each term in turn with respect to x gives:

$$\frac{d}{dx}(4x^2) + \frac{d}{dx}(2xy^3) - \frac{d}{dx}(5y^2) = \frac{d}{dx}(0)$$

i.e. $8x + \left[(2x)\left(3y^2\dfrac{dy}{dx}\right) + (y^3)(2)\right] - 10y\dfrac{dy}{dx} = 0$

i.e. $8x + 6xy^2\dfrac{dy}{dx} + 2y^3 - 10y\dfrac{dy}{dx} = 0$

Rearranging gives:

$$8x + 2y^3 = (10y - 6xy^2)\frac{dy}{dx}$$

and $$\frac{\mathbf{dy}}{\mathbf{dx}} = \frac{8x+2y^3}{10y-6xy^2} = \frac{\mathbf{4x+y^3}}{\mathbf{y(5-3xy)}}$$

(b) When $x=1$ and $y=2$,

$$\frac{\mathbf{dy}}{\mathbf{dx}} = \frac{4(1)+(2)^3}{2[5-(3)(1)(2)]} = \frac{12}{-2} = \mathbf{-6}$$

Problem 9. Find the gradients of the tangents drawn to the circle $x^2+y^2-2x-2y=3$ at $x=2$.

The gradient of the tangent is given by $\dfrac{dy}{dx}$

Differentiating each term in turn with respect to x gives:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) - \frac{d}{dx}(2x) - \frac{d}{dx}(2y) = \frac{d}{dx}(3)$$

i.e. $2x + 2y\dfrac{dy}{dx} - 2 - 2\dfrac{dy}{dx} = 0$

Hence $(2y-2)\dfrac{dy}{dx} = 2-2x,$

from which $\dfrac{dy}{dx} = \dfrac{2-2x}{2y-2} = \dfrac{1-x}{y-1}$

The value of y when $x=2$ is determined from the original equation.

Hence $(2)^2 + y^2 - 2(2) - 2y = 3$

i.e. $4 + y^2 - 4 - 2y = 3$

or $y^2 - 2y - 3 = 0$

Factorizing gives: $(y+1)(y-3)=0$, from which $y=-1$ or $y=3$.

When $x=2$ and $y=-1$,

$$\frac{dy}{dx} = \frac{1-x}{y-1} = \frac{1-2}{-1-1} = \frac{-1}{-2} = \frac{1}{2}$$

When $x=2$ and $y=3$,

$$\frac{dy}{dx}=\frac{1-2}{3-1}=\frac{-1}{2}$$

Hence the gradients of the tangents are $\pm\frac{1}{2}$

The circle having the given equation has its centre at (1, 1) and radius $\sqrt{5}$ (see Chapter 13) and is shown in Fig. 30.2 with the two gradients of the tangents.

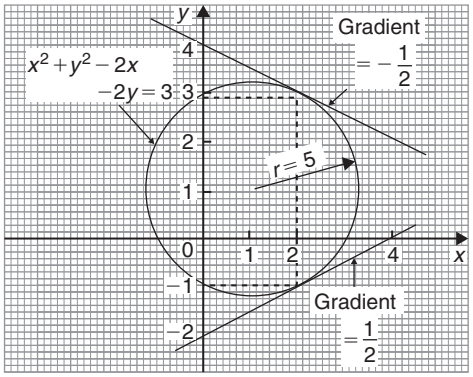


Figure 30.2

Problem 10. Pressure p and volume v of a gas are related by the law $pv^{\gamma}=k$, where γ and k are constants. Show that the rate of change of pressure $\frac{dp}{dt}=-\gamma\frac{p}{v}\frac{dv}{dt}$

Since $pv^{\gamma}=k$, then $p=\frac{k}{v^{\gamma}}=kv^{-\gamma}$

$$\frac{dp}{dt}=\frac{dp}{dv}\times\frac{dv}{dt}$$

by the function of a function rule

$$\frac{dp}{dv}=\frac{d}{dv}(kv^{-\gamma})$$
$$=-\gamma kv^{-\gamma-1}=\frac{-\gamma k}{v^{\gamma+1}}$$
$$\frac{dp}{dt}=\frac{-\gamma k}{v^{\gamma+1}}\times\frac{dv}{dt}$$

Since $k=pv^{\gamma}$,

$$\frac{dp}{dt}=\frac{-\gamma(pv^{\gamma})}{v^{\gamma+1}}\frac{dv}{dt}=\frac{-\gamma pv^{\gamma}}{v^{\gamma}v^{1}}\frac{dv}{dt}$$

i.e. $$\boldsymbol{\frac{dp}{dt}=-\gamma\frac{p}{v}\frac{dv}{dt}}$$

Now try the following exercise

Exercise 130 Further problems on implicit differentiation

In Problems 1 and 2 determine $\frac{dy}{dx}$

1. $x^2+y^2+4x-3y+1=0$ $\left[\frac{2x+4}{3-2y}\right]$

2. $2y^3-y+3x-2=0$ $\left[\frac{3}{1-6y^2}\right]$

3. Given $x^2+y^2=9$ evaluate $\frac{dy}{dx}$ when $x=\sqrt{5}$ and $y=2$. $\left[-\frac{\sqrt{5}}{2}\right]$

In Problems 4 to 7, determine $\frac{dy}{dx}$

4. $x^2+2x\sin 4y=0$ $\left[\frac{-(x+\sin 4y)}{4x\cos 4y}\right]$

5. $3y^2+2xy-4x^2=0$ $\left[\frac{4x-y}{3y+x}\right]$

6. $2x^2y+3x^3=\sin y$ $\left[\frac{x(4y+9x)}{\cos y-2x^2}\right]$

7. $3y+2x\ln y=y^4+x$ $\left[\frac{1-2\ln y}{3+(2x/y)-4y^3}\right]$

8. If $3x^2+2x^2y^3-\frac{5}{4}y^2=0$ evaluate $\frac{dy}{dx}$ when $x=\frac{1}{2}$ and $y=1$. [5]

9. Determine the gradients of the tangents drawn to the circle $x^2+y^2=16$ at the point where $x=2$. Give the answer correct to 4 significant figures. [±0.5774]

10. Find the gradients of the tangents drawn to the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=2$ at the point where $x=2$. [±1.5]

11. Determine the gradient of the curve $3xy+y^2=-2$ at the point (1,−2). [−6]

Chapter 31

Logarithmic Differentiation

31.1 Introduction to logarithmic differentiation

With certain functions containing more complicated products and quotients, differentiation is often made easier if the logarithm of the function is taken before differentiating. This technique, called **'logarithmic differentiation'** is achieved with a knowledge of (i) the laws of logarithms, (ii) the differential coefficients of logarithmic functions, and (iii) the differentiation of implicit functions.

31.2 Laws of logarithms

Three laws of logarithms may be expressed as:

(i) $\log(A \times B) = \log A + \log B$

(ii) $\log\left(\frac{A}{B}\right) = \log A - \log B$

(iii) $\log A^n = n \log A$

In calculus, Napierian logarithms (i.e. logarithms to a base of 'e') are invariably used. Thus for two functions $f(x)$ and $g(x)$ the laws of logarithms may be expressed as:

(i) $\ln[f(x) \cdot g(x)] = \ln f(x) + \ln g(x)$

(ii) $\ln\left(\frac{f(x)}{g(x)}\right) = \ln f(x) - \ln g(x)$

(iii) $\ln[f(x)]^n = n \ln f(x)$

Taking Napierian logarithms of both sides of the equation $y = \frac{f(x) \cdot g(x)}{h(x)}$ gives:

$$\ln y = \ln\left(\frac{f(x) \cdot g(x)}{h(x)}\right)$$

which may be simplified using the above laws of logarithms, giving:

$$\ln y = \ln f(x) + \ln g(x) - \ln h(x)$$

This latter form of the equation is often easier to differentiate.

31.3 Differentiation of logarithmic functions

The differential coefficient of the logarithmic function $\ln x$ is given by:

$$\mathbf{\frac{d}{dx}(\ln x) = \frac{1}{x}}$$

More generally, it may be shown that:

$$\mathbf{\frac{d}{dx}[\ln f(x)] = \frac{f'(x)}{f(x)}} \qquad (1)$$

For example, if $y = \ln(3x^2 + 2x - 1)$ then,

$$\mathbf{\frac{dy}{dx} = \frac{6x + 2}{3x^2 + 2x - 1}}$$

Similarly, if $y = \ln(\sin 3x)$ then

$$\mathbf{\frac{dy}{dx}} = \frac{3\cos 3x}{\sin 3x} = \mathbf{3 \cot 3x}.$$

Now try the following exercise

Exercise 131 Further problems on differentiating logarithmic functions

Differentiate the following using the laws for logarithms.

1. $\ln(4x - 10)$ $\left[\frac{4}{4x - 10}\right]$

2. $\ln(\cos 3x)$ $[-3\tan 3x]$

3. $\ln(3x^3+x)$ $\left[\dfrac{9x^2+1}{3x^3+x}\right]$

4. $\ln(5x^2+10x-7)$ $\left[\dfrac{10x+10}{5x^2+10x-7}\right]$

5. $\ln 8x$ $\left[\dfrac{1}{x}\right]$

6. $\ln(x^2-1)$ $\left[\dfrac{2x}{x^2-1}\right]$

7. $3\ln 4x$ $\left[\dfrac{3}{x}\right]$

8. $2\ln(\sin x)$ $[2\cot x]$

9. $\ln(4x^3-6x^2+3x)$ $\left[\dfrac{12x^2-12x+3}{4x^3-6x^2+3x}\right]$

31.4 Differentiation of further logarithmic functions

As explained in Chapter 30, by using the function of a function rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln y)=\left(\frac{1}{y}\right)\frac{\mathrm{d}y}{\mathrm{d}x} \qquad (2)$$

Differentiation of an expression such as $y=\dfrac{(1+x)^2\sqrt{(x-1)}}{x\sqrt{(x+2)}}$ may be achieved by using the product and quotient rules of differentiation; however the working would be rather complicated. With logarithmic differentiation the following procedure is adopted:

(i) Take Napierian logarithms of both sides of the equation.

$$\text{Thus } \ln y=\ln\left\{\frac{(1+x)^2\sqrt{(x-1)}}{x\sqrt{(x+2)}}\right\}$$

$$=\ln\left\{\frac{(1+x)^2(x-1)^{\frac{1}{2}}}{x(x+2)^{\frac{1}{2}}}\right\}$$

(ii) Apply the laws of logarithms.

$$\text{Thus } \ln y=\ln(1+x)^2+\ln(x-1)^{\frac{1}{2}}-\ln x-\ln(x+2)^{\frac{1}{2}},$$

by laws (i) and (ii) of Section 31.2

$$\text{i.e. } \ln y=2\ln(1+x)+\tfrac{1}{2}\ln(x-1)-\ln x-\tfrac{1}{2}\ln(x+2),$$

by law (iii) of Section 31.2

(iii) Differentiate each term in turn with respect to x using equations (1) and (2).

$$\text{Thus } \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2}{(1+x)}+\frac{\frac{1}{2}}{(x-1)}-\frac{1}{x}-\frac{\frac{1}{2}}{(x+2)}$$

(iv) Rearrange the equation to make $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ the subject.

$$\text{Thus } \frac{\mathrm{d}y}{\mathrm{d}x}=y\left\{\frac{2}{(1+x)}+\frac{1}{2(x-1)}-\frac{1}{x}-\frac{1}{2(x+2)}\right\}$$

(v) Substitute for y in terms of x.

$$\text{Thus } \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{(1+x)^2\sqrt{(x-1)}}{x\sqrt{(x+2)}}\left\{\frac{2}{(1+x)}+\frac{1}{2(x-1)}-\frac{1}{x}-\frac{1}{2(x+2)}\right\}$$

Problem 1. Use logarithmic differentiation to differentiate $y=\dfrac{(x+1)(x-2)^3}{(x-3)}$

Following the above procedure:

(i) Since $y=\dfrac{(x+1)(x-2)^3}{(x-3)}$

$$\text{then } \ln y=\ln\left\{\frac{(x+1)(x-2)^3}{(x-3)}\right\}$$

(ii) $\ln y=\ln(x+1)+\ln(x-2)^3-\ln(x-3)$, by laws (i) and (ii) of Section 31.2,

$$\text{i.e. } \ln y=\ln(x+1)+3\ln(x-2)-\ln(x-3),$$

by law (iii) of Section 31.2.

(iii) Differentiating with respect to x gives:

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{(x+1)}+\frac{3}{(x-2)}-\frac{1}{(x-3)},$$

by using equations (1) and (2)

(iv) Rearranging gives:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=y\left\{\frac{1}{(x+1)}+\frac{3}{(x-2)}-\frac{1}{(x-3)}\right\}$$

(v) Substituting for y gives:

$$\frac{dy}{dx} = \frac{(x+1)(x-2)^3}{(x-3)}\left\{\frac{1}{(x+1)} + \frac{3}{(x-2)} - \frac{1}{(x-3)}\right\}$$

Problem 2. Differentiate $y = \dfrac{\sqrt{(x-2)^3}}{(x+1)^2(2x-1)}$ with respect to x and evaluate $\dfrac{dy}{dx}$ when $x = 3$.

Using logarithmic differentiation and following the above procedure:

(i) Since $y = \dfrac{\sqrt{(x-2)^3}}{(x+1)^2(2x-1)}$

then $\ln y = \ln\left\{\dfrac{\sqrt{(x-2)^3}}{(x+1)^2(2x-1)}\right\}$

$= \ln\left\{\dfrac{(x-2)^{\frac{3}{2}}}{(x+1)^2(2x-1)}\right\}$

(ii) $\ln y = \ln(x-2)^{\frac{3}{2}} - \ln(x+1)^2 - \ln(2x-1)$

i.e. $\ln y = \frac{3}{2}\ln(x-2) - 2\ln(x+1) - \ln(2x-1)$

(iii) $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{\frac{3}{2}}{(x-2)} - \dfrac{2}{(x+1)} - \dfrac{2}{(2x-1)}$

(iv) $\dfrac{dy}{dx} = y\left\{\dfrac{3}{2(x-2)} - \dfrac{2}{(x+1)} - \dfrac{2}{(2x-1)}\right\}$

(v) $$\frac{dy}{dx} = \frac{\sqrt{(x-2)^3}}{(x+1)^2(2x-1)}\left\{\frac{3}{2(x-2)} - \frac{2}{(x+1)} - \frac{2}{(2x-1)}\right\}$$

When $x = 3$, $\dfrac{dy}{dx} = \dfrac{\sqrt{(1)^3}}{(4)^2(5)}\left(\dfrac{3}{2} - \dfrac{2}{4} - \dfrac{2}{5}\right)$

$= \pm\dfrac{1}{80}\left(\dfrac{3}{5}\right) = \pm\dfrac{3}{400}$ or ± 0.0075

Problem 3. Given $y = \dfrac{3e^{2\theta}\sec 2\theta}{\sqrt{(\theta-2)}}$ determine $\dfrac{dy}{d\theta}$

Using logarithmic differentiation and following the procedure gives:

(i) Since $y = \dfrac{3e^{2\theta}\sec 2\theta}{\sqrt{(\theta-2)}}$

then $\ln y = \ln\left\{\dfrac{3e^{2\theta}\sec 2\theta}{\sqrt{(\theta-2)}}\right\}$

$= \ln\left\{\dfrac{3e^{2\theta}\sec 2\theta}{(\theta-2)^{\frac{1}{2}}}\right\}$

(ii) $\ln y = \ln 3e^{2\theta} + \ln\sec 2\theta - \ln(\theta-2)^{\frac{1}{2}}$

i.e. $\ln y = \ln 3 + \ln e^{2\theta} + \ln\sec 2\theta - \frac{1}{2}\ln(\theta-2)$

i.e. $\ln y = \ln 3 + 2\theta + \ln\sec 2\theta - \frac{1}{2}\ln(\theta-2)$

(iii) Differentiating with respect to θ gives:

$\dfrac{1}{y}\dfrac{dy}{d\theta} = 0 + 2 + \dfrac{2\sec 2\theta\tan 2\theta}{\sec 2\theta} - \dfrac{\frac{1}{2}}{(\theta-2)}$ from equations (1) and (2)

(iv) Rearranging gives:

$\dfrac{dy}{d\theta} = y\left\{2 + 2\tan 2\theta - \dfrac{1}{2(\theta-2)}\right\}$

(v) Substituting for y gives:

$$\frac{dy}{d\theta} = \frac{3e^{2\theta}\sec 2\theta}{\sqrt{(\theta-2)}}\left\{2 + 2\tan 2\theta - \frac{1}{2(\theta-2)}\right\}$$

Problem 4. Differentiate $y = \dfrac{x^3\ln 2x}{e^x\sin x}$ with respect to x.

Using logarithmic differentiation and following the procedure gives:

(i) $\ln y = \ln\left\{\dfrac{x^3\ln 2x}{e^x\sin x}\right\}$

(ii) $\ln y = \ln x^3 + \ln(\ln 2x) - \ln(e^x) - \ln(\sin x)$

i.e. $\ln y = 3\ln x + \ln(\ln 2x) - x - \ln(\sin x)$

(iii) $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{3}{x} + \dfrac{\frac{1}{x}}{\ln 2x} - 1 - \dfrac{\cos x}{\sin x}$

(iv) $\dfrac{dy}{dx} = y\left\{\dfrac{3}{x} + \dfrac{1}{x\ln 2x} - 1 - \cot x\right\}$

(v) $\dfrac{\mathbf{dy}}{\mathbf{dx}} = \dfrac{\boldsymbol{x^3 \ln 2x}}{\boldsymbol{e^x \sin x}}\left\{\dfrac{\mathbf{3}}{\boldsymbol{x}} + \dfrac{\mathbf{1}}{\boldsymbol{x \ln 2x}} - \mathbf{1} - \boldsymbol{\cot x}\right\}$

Now try the following exercise

Exercise 132 Further problems on differentiating logarithmic functions

In Problems 1 to 6, use logarithmic differentiation to differentiate the given functions with respect to the variable.

1. $y = \dfrac{(x-2)(x+1)}{(x-1)(x+3)}$

$$\left[\frac{(x-2)(x+1)}{(x-1)(x+3)}\left\{\frac{1}{(x-2)} + \frac{1}{(x+1)} - \frac{1}{(x-1)} - \frac{1}{(x+3)}\right\}\right]$$

2. $y = \dfrac{(x+1)(2x+1)^3}{(x-3)^2(x+2)^4}$

$$\left[\frac{(x+1)(2x+1)^3}{(x-3)^2(x+2)^4}\left\{\frac{1}{(x+1)} + \frac{6}{(2x+1)} - \frac{2}{(x-3)} - \frac{4}{(x+2)}\right\}\right]$$

3. $y = \dfrac{(2x-1)\sqrt{(x+2)}}{(x-3)\sqrt{(x+1)^3}}$

$$\left[\frac{(2x-1)\sqrt{(x+2)}}{(x-3)\sqrt{(x+1)^3}}\left\{\frac{2}{(2x-1)} + \frac{1}{2(x+2)} - \frac{1}{(x-3)} - \frac{3}{2(x+1)}\right\}\right]$$

4. $y = \dfrac{e^{2x}\cos 3x}{\sqrt{(x-4)}}$

$$\left[\frac{e^{2x}\cos 3x}{\sqrt{(x-4)}}\left\{2 - 3\tan 3x - \frac{1}{2(x-4)}\right\}\right]$$

5. $y = 3\theta \sin\theta \cos\theta$

$$\left[3\theta\sin\theta\cos\theta\left\{\frac{1}{\theta} + \cot\theta - \tan\theta\right\}\right]$$

6. $y = \dfrac{2x^4 \tan x}{e^{2x}\ln 2x}$ $\left[\dfrac{2x^4\tan x}{e^{2x}\ln 2x}\left\{\dfrac{4}{x} + \dfrac{1}{\sin x\cos x} - 2 - \dfrac{1}{x\ln 2x}\right\}\right]$

7. Evaluate $\dfrac{dy}{dx}$ when $x = 1$ given $y = \dfrac{(x+1)^2\sqrt{(2x-1)}}{\sqrt{(x+3)^3}}$ $\left[\dfrac{13}{16}\right]$

8. Evaluate $\dfrac{dy}{d\theta}$, correct to 3 significant figures, when $\theta = \dfrac{\pi}{4}$ given $y = \dfrac{2e^{\theta}\sin\theta}{\sqrt{\theta^5}}$ [−6.71]

31.5 Differentiation of $[f(x)]^x$

Whenever an expression to be differentiated contains a term raised to a power which is itself a function of the variable, then logarithmic differentiation must be used. For example, the differentiation of expressions such as x^x, $(x+2)^x$, $\sqrt[x]{(x-1)}$ and x^{3x+2} can only be achieved using logarithmic differentiation.

Problem 5. Determine $\dfrac{dy}{dx}$ given $y = x^x$.

Taking Napierian logarithms of both sides of $y = x^x$ gives:

$\ln y = \ln x^x = x\ln x$, by law (iii) of Section 31.2

Differentiating both sides with respect to x gives:

$$\frac{1}{y}\frac{dy}{dx} = (x)\left(\frac{1}{x}\right) + (\ln x)(1), \text{ using the product rule}$$

i.e. $$\frac{1}{y}\frac{dy}{dx} = 1 + \ln x,$$

from which, $$\frac{dy}{dx} = y(1 + \ln x)$$

i.e. $$\boldsymbol{\frac{dy}{dx} = x^x(1 + \ln x)}$$

Problem 6. Evaluate $\dfrac{dy}{dx}$ when $x = -1$ given $y = (x+2)^x$.

Taking Napierian logarithms of both sides of $y = (x+2)^x$ gives:

$\ln y = \ln(x+2)^x = x\ln(x+2)$, by law (iii) of Section 31.2

Differentiating both sides with respect to x gives:

$$\frac{1}{y}\frac{dy}{dx} = (x)\left(\frac{1}{x+2}\right) + [\ln(x+2)](1),$$

by the product rule.

Hence $$\frac{dy}{dx} = y\left(\frac{x}{x+2} + \ln(x+2)\right)$$

$$= \mathbf{(x+2)^x\left\{\frac{x}{x+2} + \ln(x+2)\right\}}$$

When $x = -1$, $$\frac{dy}{dx} = (1)^{-1}\left(\frac{-1}{1} + \ln 1\right)$$

$$= (+1)(-1) = \mathbf{-1}$$

Problem 7. Determine (a) the differential coefficient of $y = \sqrt[x]{(x-1)}$ and (b) evaluate $\frac{dy}{dx}$ when $x = 2$.

(a) $y = \sqrt[x]{(x-1)} = (x-1)^{\frac{1}{x}}$, since by the laws of indices $\sqrt[n]{a^m} = a^{\frac{m}{n}}$

Taking Napierian logarithms of both sides gives:

$$\ln y = \ln(x-1)^{\frac{1}{x}} = \frac{1}{x}\ln(x-1),$$

by law (iii) of Section 31.2.

Differentiating each side with respect to x gives:

$$\frac{1}{y}\frac{dy}{dx} = \left(\frac{1}{x}\right)\left(\frac{1}{x-1}\right) + [\ln(x-1)]\left(\frac{-1}{x^2}\right),$$

by the product rule.

Hence $$\frac{dy}{dx} = y\left\{\frac{1}{x(x-1)} - \frac{\ln(x-1)}{x^2}\right\}$$

i.e. $$\mathbf{\frac{dy}{dx} = \sqrt[x]{(x-1)}\left\{\frac{1}{x(x-1)} - \frac{\ln(x-1)}{x^2}\right\}}$$

(b) When $x = 2$, $$\frac{dy}{dx} = \sqrt[2]{(1)}\left\{\frac{1}{2(1)} - \frac{\ln(1)}{4}\right\}$$

$$= \pm 1\left\{\frac{1}{2} - 0\right\} = \mathbf{\pm\frac{1}{2}}$$

Problem 8. Differentiate x^{3x+2} with respect to x.

Let $y = x^{3x+2}$

Taking Napierian logarithms of both sides gives:

$$\ln y = \ln x^{3x+2}$$

i.e. $\ln y = (3x+2)\ln x$, by law (iii) of Section 31.2.
Differentiating each term with respect to x gives:

$$\frac{1}{y}\frac{dy}{dx} = (3x+2)\left(\frac{1}{x}\right) + (\ln x)(3),$$

by the product rule.

Hence $$\frac{dy}{dx} = y\left\{\frac{3x+2}{x} + 3\ln x\right\}$$

$$= x^{3x+2}\left\{\frac{3x+2}{x} + 3\ln x\right\}$$

$$= \mathbf{x^{3x+2}\left\{3 + \frac{2}{x} + 3\ln x\right\}}$$

Now try the following exercise

Exercise 133 Further problems on differentiating $[f(x)]^x$ type functions

In Problems 1 to 4, differentiate with respect to x.

1. $y = x^{2x}$ $[2x^{2x}(1+\ln x)]$
2. $y = (2x-1)^x$ $\left[(2x-1)^x\left\{\frac{2x}{2x-1} + \ln(2x-1)\right\}\right]$
3. $y = \sqrt[x]{(x+3)}$ $\left[\sqrt[x]{(x+3)}\left\{\frac{1}{x(x+3)} - \frac{\ln(x+3)}{x^2}\right\}\right]$
4. $y = 3x^{4x+1}$ $\left[3x^{4x+1}\left\{4 + \frac{1}{x} + 4\ln x\right\}\right]$
5. Show that when $y = 2x^x$ and $x = 1$, $\frac{dy}{dx} = 2$.
6. Evaluate $\frac{d}{dx}\left\{\sqrt[x]{(x-2)}\right\}$ when $x = 3$. $\left[\frac{1}{3}\right]$
7. Show that if $y = \theta^\theta$ and $\theta = 2$, $\frac{dy}{d\theta} = 6.77$, correct to 3 significant figures.

Revision Test 9

This Revision Test covers the material contained in Chapters 27 to 31. *The marks for each question are shown in brackets at the end of each question.*

1. Differentiate the following with respect to the variable:

   (a) $y = 5 + 2\sqrt{x^3} - \dfrac{1}{x^2}$ (b) $s = 4e^{2\theta}\sin 3\theta$

   (c) $y = \dfrac{3\ln 5t}{\cos 2t}$

   (d) $x = \dfrac{2}{\sqrt{(t^2 - 3t + 5)}}$ (13)

2. If $f(x) = 2.5x^2 - 6x + 2$ find the co-ordinates at the point at which the gradient is -1. (5)

3. The displacement s cm of the end of a stiff spring at time t seconds is given by: $s = ae^{-kt}\sin 2\pi ft$. Determine the velocity and acceleration of the end of the spring after 2 seconds if $a = 3$, $k = 0.75$ and $f = 20$. (10)

4. Find the co-ordinates of the turning points on the curve $y = 3x^3 + 6x^2 + 3x - 1$ and distinguish between them. (7)

5. The heat capacity C of a gas varies with absolute temperature θ as shown:

   $$C = 26.50 + 7.20 \times 10^{-3}\theta - 1.20 \times 10^{-6}\theta^2$$

   Determine the maximum value of C and the temperature at which it occurs. (5)

6. Determine for the curve $y = 2x^2 - 3x$ at the point (2, 2): (a) the equation of the tangent (b) the equation of the normal. (6)

7. A rectangular block of metal with a square cross-section has a total surface area of 250 cm^2. Find the maximum volume of the block of metal. (7)

8. A cycloid has parametric equations given by: $x = 5(\theta - \sin\theta)$ and $y = 5(1 - \cos\theta)$. Evaluate

   (a) $\dfrac{dy}{dx}$ (b) $\dfrac{d^2y}{dx^2}$ when $\theta = 1.5$ radians. Give answers correct to 3 decimal places. (8)

9. Determine the equation of (a) the tangent, and (b) the normal, drawn to an ellipse $x = 4\cos\theta$, $y = \sin\theta$ at $\theta = \dfrac{\pi}{3}$. (8)

10. Determine expressions for $\dfrac{dz}{dy}$ for each of the following functions:

    (a) $z = 5y^2\cos x$ (b) $z = x^2 + 4xy - y^2$. (5)

11. If $x^2 + y^2 + 6x + 8y + 1 = 0$, find $\dfrac{dy}{dx}$ in terms of x and y. (3)

12. Determine the gradient of the tangents drawn to the hyperbola $x^2 - y^2 = 8$ at $x = 3$. (3)

13. Use logarithmic differentiation to differentiate $y = \dfrac{(x+1)^2\sqrt{(x-2)}}{(2x-1)\sqrt[3]{(x-3)^4}}$ with respect to x. (6)

14. Differentiate $y = \dfrac{3e^{\theta}\sin 2\theta}{\sqrt{\theta^5}}$ and hence evaluate $\dfrac{dy}{d\theta}$, correct to 2 decimal places, when $\theta = \dfrac{\pi}{3}$. (9)

15. Evaluate $\dfrac{d}{dt}\left[\sqrt[t]{(2t+1)}\right]$ when $t = 2$, correct to 4 significant figures. (5)

Chapter 32

Differentiation of hyperbolic functions

32.1 Standard differential coefficients of hyperbolic functions

From Chapter 5,

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \left[\frac{e^x - (-e^{-x})}{2}\right]$$

$$= \left(\frac{e^x + e^{-x}}{2}\right) = \mathbf{\cosh x}$$

If $\boldsymbol{y = \sinh ax}$, where '$a$' is a constant, then $\boldsymbol{\frac{dy}{dx} = a\cosh ax}$

$$\frac{d}{dx}(\cosh x) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \left[\frac{e^x + (-e^{-x})}{2}\right]$$

$$= \left(\frac{e^x - e^{-x}}{2}\right) = \mathbf{\sinh x}$$

If $\boldsymbol{y = \cosh ax}$, where '$a$' is a constant, then $\boldsymbol{\frac{dy}{dx} = a\sinh ax}$

Using the quotient rule of differentiation the derivatives of $\tanh x$, $\operatorname{sech} x$, $\operatorname{cosech} x$ and $\coth x$ may be determined using the above results.

Problem 1. Determine the differential coefficient of: (a) $\operatorname{th} x$ (b) $\operatorname{sech} x$.

(a) $$\frac{d}{dx}(\operatorname{th} x) = \frac{d}{dx}\left(\frac{\operatorname{sh} x}{\operatorname{ch} x}\right)$$

$$= \frac{(\operatorname{ch} x)(\operatorname{ch} x) - (\operatorname{sh} x)(\operatorname{sh} x)}{\operatorname{ch}^2 x}$$

using the quotient rule

$$= \frac{\operatorname{ch}^2 x - \operatorname{sh}^2 x}{\operatorname{ch}^2 x} = \frac{1}{\operatorname{ch}^2 x} = \mathbf{\operatorname{sech}^2 x}$$

(b) $$\frac{d}{dx}(\operatorname{sech} x) = \frac{d}{dx}\left(\frac{1}{\operatorname{ch} x}\right)$$

$$= \frac{(\operatorname{ch} x)(0) - (1)(\operatorname{sh} x)}{\operatorname{ch}^2 x}$$

$$= \frac{-\operatorname{sh} x}{\operatorname{ch}^2 x} = -\left(\frac{1}{\operatorname{ch} x}\right)\left(\frac{\operatorname{sh} x}{\operatorname{ch} x}\right)$$

$$= \mathbf{-\operatorname{sech} x \operatorname{th} x}$$

Problem 2. Determine $\frac{dy}{d\theta}$ given (a) $y = \operatorname{cosech}\theta$ (b) $y = \coth\theta$.

(a) $$\frac{d}{d\theta}(\operatorname{cosec}\theta) = \frac{d}{d\theta}\left(\frac{1}{\operatorname{sh}\theta}\right)$$

$$= \frac{(\operatorname{sh}\theta)(0) - (1)(\operatorname{ch}\theta)}{\operatorname{sh}^2\theta}$$

$$= \frac{-\operatorname{ch}\theta}{\operatorname{sh}^2\theta} = -\left(\frac{1}{\operatorname{sh}\theta}\right)\left(\frac{\operatorname{ch}\theta}{\operatorname{sh}\theta}\right)$$

$$= \mathbf{-\operatorname{cosech}\theta \coth\theta}$$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}\theta}(\coth\theta) = \dfrac{\mathrm{d}}{\mathrm{d}\theta}\left(\dfrac{\mathrm{ch}\,\theta}{\mathrm{sh}\,\theta}\right)$

$$= \frac{(\mathrm{sh}\,\theta)(\mathrm{sh}\,\theta) - (\mathrm{ch}\,\theta)(\mathrm{ch}\,\theta)}{\mathrm{sh}^2\theta}$$

$$= \frac{\mathrm{sh}^2\theta - \mathrm{ch}^2\theta}{\mathrm{sh}^2\theta} = \frac{-(\mathrm{ch}^2\theta - \mathrm{sh}^2\theta)}{\mathrm{sh}^2\theta}$$

$$= \frac{-1}{\mathrm{sh}^2\theta} = \mathbf{-\mathrm{cosech}^2\,\theta}$$

Summary of differential coefficients

y or $f(x)$	$\dfrac{\mathrm{d}y}{\mathrm{d}x}$ or $f'(x)$
$\sinh ax$	$a\cosh ax$
$\cosh ax$	$a\sinh ax$
$\tanh ax$	$a\,\mathrm{sech}^2 ax$
$\mathrm{sech}\,ax$	$-a\,\mathrm{sech}\,ax\tanh ax$
$\mathrm{cosech}\,ax$	$-a\,\mathrm{cosech}\,ax\coth ax$
$\coth ax$	$-a\,\mathrm{cosech}^2 ax$

32.2 Further worked problems on differentiation of hyperbolic functions

Problem 3. Differentiate the following with respect to x:

(a) $y = 4\,\mathrm{sh}\,2x - \dfrac{3}{7}\mathrm{ch}\,3x$

(b) $y = 5\,\mathrm{th}\,\dfrac{x}{2} - 2\coth 4x$.

(a) $y = 4\,\mathrm{sh}\,2x - \dfrac{3}{7}\mathrm{ch}\,3x$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 4(2\cosh 2x) - \frac{3}{7}(3\sinh 3x)$$

$$= \mathbf{8\cosh 2x - \frac{9}{7}\sinh 3x}$$

(b) $y = 5\,\mathrm{th}\,\dfrac{x}{2} - 2\coth 4x$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 5\left(\frac{1}{2}\,\mathrm{sech}^2\frac{x}{2}\right) - 2(-4\,\mathrm{cosech}^2 4x)$$

$$= \mathbf{\frac{5}{2}\,\mathrm{sech}^2\frac{x}{2} + 8\,\mathrm{cosech}^2 4x}$$

Problem 4. Differentiate the following with respect to the variable: (a) $y = 4\sin 3t\,\mathrm{ch}\,4t$ (b) $y = \ln(\mathrm{sh}\,3\theta) - 4\,\mathrm{ch}^2 3\theta$.

(a) $y = 4\sin 3t\,\mathrm{ch}\,4t$ (i.e. a product)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (4\sin 3t)(4\,\mathrm{sh}\,4t) + (\mathrm{ch}\,4t)(4)(3\cos 3t)$$

$$= 16\sin 3t\,\mathrm{sh}\,4t + 12\,\mathrm{ch}\,4t\cos 3t$$

$$= \mathbf{4(4\sin 3t\,\mathrm{sh}\,4t + 3\cos 3t\,\mathrm{ch}\,4t)}$$

(b) $y = \ln(\mathrm{sh}\,3\theta) - 4\,\mathrm{ch}^2 3\theta$

(i.e. a function of a function)

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = \left(\frac{1}{\mathrm{sh}\,3\theta}\right)(3\,\mathrm{ch}\,3\theta) - (4)(2\,\mathrm{ch}\,3\theta)(3\,\mathrm{sh}\,3\theta)$$

$$= 3\coth 3\theta - 24\,\mathrm{ch}\,3\theta\,\mathrm{sh}\,3\theta$$

$$= \mathbf{3(\coth 3\theta - 8\,\mathrm{ch}\,3\theta\,\mathrm{sh}\,3\theta)}$$

Problem 5. Show that the differential coefficient of $y = \dfrac{3x^2}{\mathrm{ch}\,4x}$ is: $6x\,\mathrm{sech}\,4x\,(1 - 2x\,\mathrm{th}\,4x)$.

$y = \dfrac{3x^2}{\mathrm{ch}\,4x}$ (i.e. a quotient)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(\mathrm{ch}\,4x)(6x) - (3x^2)(4\,\mathrm{sh}\,4x)}{(\mathrm{ch}\,4x)^2}$$

$$= \frac{6x(\mathrm{ch}\,4x - 2x\,\mathrm{sh}\,4x)}{\mathrm{ch}^2 4x}$$

$$= 6x\left[\frac{\mathrm{ch}\,4x}{\mathrm{ch}^2 4x} - \frac{2x\,\mathrm{sh}\,4x}{\mathrm{ch}^2 4x}\right]$$

$$= 6x\left[\frac{1}{\mathrm{ch}\,4x} - 2x\left(\frac{\mathrm{sh}\,4x}{\mathrm{ch}\,4x}\right)\left(\frac{1}{\mathrm{ch}\,4x}\right)\right]$$

$$= 6x[\mathrm{sech}\,4x - 2x\,\mathrm{th}\,4x\,\mathrm{sech}\,4x]$$

$$= \mathbf{6x\,\mathrm{sech}\,4x\,(1 - 2x\,\mathrm{th}\,4x)}$$

Now try the following exercise

Exercise 134 Further problems on differentiation of hyperbolic functions

In Problems 1 to 5 differentiate the given functions with respect to the variable:

1. (a) $3\,\text{sh}\,2x$ (b) $2\,\text{ch}\,5\theta$ (c) $4\,\text{th}\,9t$

$$\left[\text{(a) } 6\,\text{ch}\,2x \text{ (b) } 10\,\text{sh}\,5\theta \text{ (c) } 36\,\text{sech}^2 9\text{t}\right]$$

2. (a) $\frac{2}{3}\text{sech}\,5x$ (b) $\frac{5}{8}\text{cosech}\,\frac{t}{2}$ (c) $2\coth 7\theta$

$$\left[\begin{array}{l}\text{(a) } -\frac{10}{3}\,\text{sech}\,5x\,\text{th}\,5x \\ \text{(b) } -\frac{5}{16}\,\text{cosech}\,\frac{t}{2}\,\coth\frac{t}{2} \\ \text{(c) } -14\,\text{cosech}^2 7\theta\end{array}\right]$$

3. (a) $2\ln(\text{sh}\,x)$ (b) $\frac{3}{4}\ln\left(\text{th}\left(\frac{\theta}{2}\right)\right)$

$$\left[\text{(a) } 2\coth x \text{ (b) } \frac{3}{8}\,\text{sech}\,\frac{\theta}{2}\,\text{cosech}\,\frac{\theta}{2}\right]$$

4. (a) $\text{sh}\,2x\,\text{ch}\,2x$ (b) $3\text{e}^{2x}\,\text{th}\,2x$

$$\left[\begin{array}{l}\text{(a) } 2(\text{sh}^2 2x + \text{ch}^2 2x) \\ \text{(b) } 6\text{e}^{2x}(\text{sech}^2 2x + \text{th}\,2x)\end{array}\right]$$

5. (a) $\frac{3\,\text{sh}\,4x}{2x^3}$ (b) $\frac{\text{ch}\,2t}{\cos 2t}$

$$\left[\begin{array}{l}\text{(a) } \frac{12x\,\text{ch}\,4x - 9\,\text{sh}\,4x}{2x^4} \\ \text{(b) } \frac{2(\cos 2t\,\text{sh}\,2t + \text{ch}\,2t\sin 2t)}{\cos^2 2t}\end{array}\right]$$

Chapter 33

Differentiation of inverse trigonometric and hyperbolic functions

33.1 Inverse functions

If $y = 3x - 2$, then by transposition, $x = \dfrac{y+2}{3}$. The function $x = \dfrac{y+2}{3}$ is called the **inverse function** of $y = 3x - 2$ (see page 188).

Inverse trigonometric functions are denoted by prefixing the function with 'arc' or, more commonly, by using the $^{-1}$ notation. For example, if $y = \sin x$, then $x = \arcsin y$ or $x = \sin^{-1} y$. Similarly, if $y = \cos x$, then $x = \arccos y$ or $x = \cos^{-1} y$, and so on. In this chapter the $^{-1}$ notation will be used. A sketch of each of the inverse trigonometric functions is shown in Fig. 33.1.

Inverse hyperbolic functions are denoted by prefixing the function with 'ar' or, more commonly, by using the $^{-1}$ notation. For example, if $y = \sinh x$, then $x = \operatorname{arsinh} y$ or $x = \sinh^{-1} y$. Similarly, if $y = \operatorname{sech} x$, then $x = \operatorname{arsech} y$ or $x = \operatorname{sech}^{-1} y$, and so on. In this chapter the $^{-1}$ notation will be used. A sketch of each of the inverse hyperbolic functions is shown in Fig. 33.2.

33.2 Differentiation of inverse trigonometric functions

(i) If $y = \sin^{-1} x$, then $x = \sin y$.

Differentiating both sides with respect to y gives:

$$\frac{dx}{dy} = \cos y = \sqrt{1 - \sin^2 y}$$

since $\cos^2 y + \sin^2 y = 1$, i.e. $\dfrac{dx}{dy} = \sqrt{1 - x^2}$

However $\dfrac{dy}{dx} = \dfrac{1}{\dfrac{dx}{dy}}$

Hence, when $y = \sin^{-1} x$ then

$$\boldsymbol{\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}}$$

(ii) A sketch of part of the curve of $y = \sin^{-1} x$ is shown in Fig. 33.1(a). The principal value of $\sin^{-1} x$ is defined as the value lying between $-\pi/2$ and $\pi/2$. The gradient of the curve between points A and B is positive for all values of x and thus only the positive value is taken when evaluating $\dfrac{1}{\sqrt{1 - x^2}}$

(iii) Given $y = \sin^{-1} \dfrac{x}{a}$ then $\dfrac{x}{a} = \sin y$ and $x = a \sin y$

Hence $\dfrac{dx}{dy} = a \cos y = a\sqrt{1 - \sin^2 y}$

$$= a\sqrt{\left[1 - \left(\frac{x}{a}\right)^2\right]} = a\sqrt{\left(\frac{a^2 - x^2}{a^2}\right)}$$

$$= \frac{a\sqrt{a^2 - x^2}}{a} = \sqrt{a^2 - x^2}$$

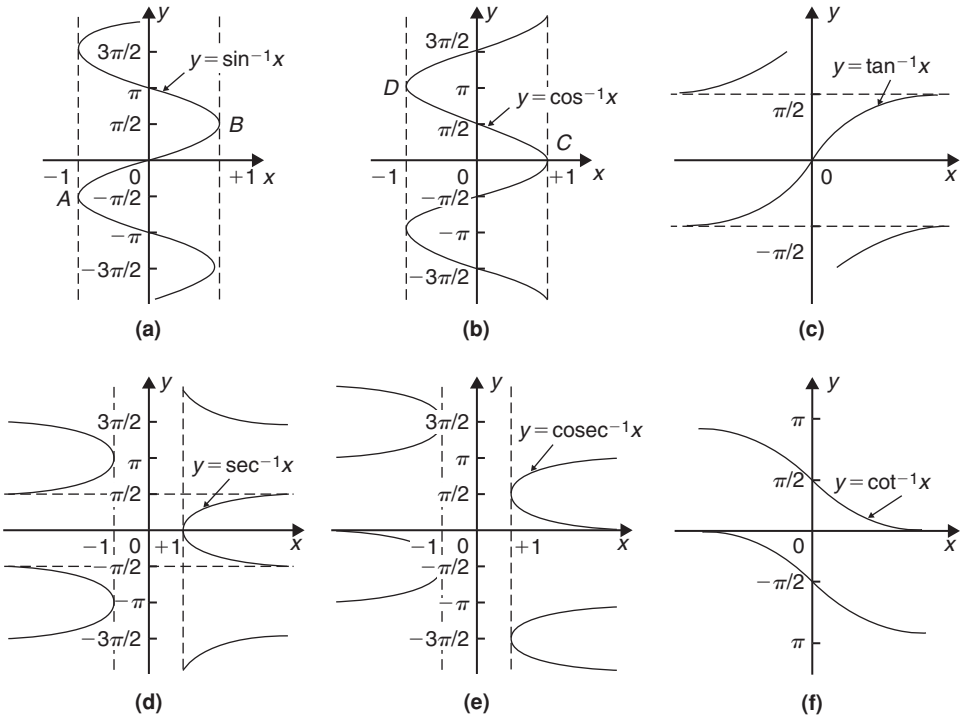


Figure 33.1

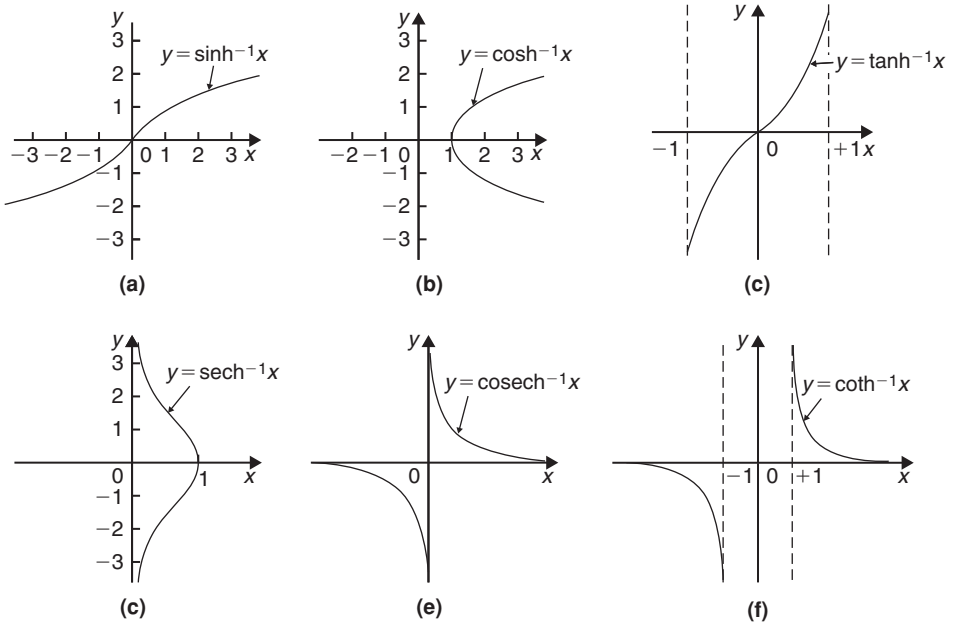


Figure 33.2

Thus $\dfrac{dy}{dx} = \dfrac{1}{\dfrac{dx}{dy}} = \dfrac{1}{\sqrt{a^2 - x^2}}$

i.e. **when** $\mathbf{y = \sin^{-1}\frac{x}{a}}$ **then** $\mathbf{\dfrac{dy}{dx} = \dfrac{1}{\sqrt{a^2 - x^2}}}$

Since integration is the reverse process of differentiation then:

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a} + c$$

(iv) Given $y = \sin^{-1} f(x)$ the function of a function rule may be used to find $\dfrac{dy}{dx}$

Let $u = f(x)$ then $y = \sin^{-1} u$

Then $\dfrac{du}{dx} = f'(x)$ and $\dfrac{dy}{du} = \dfrac{1}{\sqrt{1 - u^2}}$

(see para. (i))

Thus $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx} = \dfrac{1}{\sqrt{1 - u^2}} f'(x)$

$$= \frac{f'(x)}{\sqrt{1 - [f(x)]^2}}$$

(v) The differential coefficients of the remaining inverse trigonometric functions are obtained in a similar manner to that shown above and a summary of the results is shown in Table 33.1.

Problem 1. Find $\dfrac{dy}{dx}$ given $y = \sin^{-1} 5x^2$.

From Table 33.1(i), if

$$y = \sin^{-1} f(x) \text{ then } \frac{dy}{dx} = \frac{f'(x)}{\sqrt{1 - [f(x)]^2}}$$

Hence, if $y = \sin^{-1} 5x^2$ then $f(x) = 5x^2$ and $f'(x) = 10x$.

Thus $\dfrac{dy}{dx} = \dfrac{10x}{\sqrt{1 - (5x^2)^2}} = \dfrac{\mathbf{10x}}{\sqrt{\mathbf{1 - 25x^4}}}$

Problem 2.

(a) Show that if $y = \cos^{-1} x$ then $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1 - x^2}}$

(b) Hence obtain the differential coefficient of $y = \cos^{-1}(1 - 2x^2)$.

Table 33.1 Differential coefficients of inverse trigonometric functions

	y or $f(x)$	$\dfrac{dy}{dx}$ or $f'(x)$
(i)	$\sin^{-1}\dfrac{x}{a}$	$\dfrac{1}{\sqrt{a^2 - x^2}}$
	$\sin^{-1} f(x)$	$\dfrac{f'(x)}{\sqrt{1 - [f(x)]^2}}$
(ii)	$\cos^{-1}\dfrac{x}{a}$	$\dfrac{-1}{\sqrt{a^2 - x^2}}$
	$\cos^{-1} f(x)$	$\dfrac{-f'(x)}{\sqrt{1 - [f(x)]^2}}$
(iii)	$\tan^{-1}\dfrac{x}{a}$	$\dfrac{a}{a^2 + x^2}$
	$\tan^{-1} f(x)$	$\dfrac{f'(x)}{1 + [f(x)]^2}$
(iv)	$\sec^{-1}\dfrac{x}{a}$	$\dfrac{a}{x\sqrt{x^2 - a^2}}$
	$\sec^{-1} f(x)$	$\dfrac{f'(x)}{f(x)\sqrt{[f(x)]^2 - 1}}$
(v)	$\text{cosec}^{-1}\dfrac{x}{a}$	$\dfrac{-a}{x\sqrt{x^2 - a^2}}$
	$\text{cosec}^{-1} f(x)$	$\dfrac{-f'(x)}{f(x)\sqrt{[f(x)]^2 - 1}}$
(vi)	$\cot^{-1}\dfrac{x}{a}$	$\dfrac{-a}{a^2 + x^2}$
	$\cot^{-1} f(x)$	$\dfrac{-f'(x)}{1 + [f(x)]^2}$

(a) If $y = \cos^{-1} x$ then $x = \cos y$.

Differentiating with respect to y gives:

$$\frac{dx}{dy} = -\sin y = -\sqrt{1 - \cos^2 y}$$

$$= -\sqrt{1 - x^2}$$

Hence $\dfrac{dy}{dx} = \dfrac{1}{\dfrac{dx}{dy}} = \mathbf{-\dfrac{1}{\sqrt{1 - x^2}}}$

The principal value of $y = \cos^{-1} x$ is defined as the angle lying between 0 and π, i.e. between points C and D shown in Fig. 33.1(b). The gradient of the curve

is negative between C and D and thus the differential coefficient $\frac{dy}{dx}$ is negative as shown above.

(b) If $y = \cos^{-1} f(x)$ then by letting $u = f(x)$, $y = \cos^{-1} u$

Then $\frac{dy}{du} = -\frac{1}{\sqrt{1-u^2}}$ (from part (a))

and $\frac{du}{dx} = f'(x)$

From the function of a function rule,

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = -\frac{1}{\sqrt{1-u^2}}\,f'(x)$$

$$= \frac{-f'(x)}{\sqrt{1-[f(x)]^2}}$$

Hence, when $y = \cos^{-1}(1-2x^2)$

then $\frac{dy}{dx} = \frac{-(-4x)}{\sqrt{1-[1-2x^2]^2}}$

$$= \frac{4x}{\sqrt{1-(1-4x^2+4x^4)}} = \frac{4x}{\sqrt{(4x^2-4x^4)}}$$

$$= \frac{4x}{\sqrt{[4x^2(1-x^2)]}} = \frac{4x}{2x\sqrt{1-x^2}} = \mathbf{\frac{2}{\sqrt{1-x^2}}}$$

Problem 3. Determine the differential coefficient of $y = \tan^{-1}\frac{x}{a}$ and show that the differential coefficient of $\tan^{-1}\frac{2x}{3}$ is $\frac{6}{9+4x^2}$

If $y = \tan^{-1}\frac{x}{a}$ then $\frac{x}{a} = \tan y$ and $x = a\tan y$

$\frac{dx}{dy} = a\sec^2 y = a(1+\tan^2 y)$ since

$$\sec^2 y = 1+\tan^2 y$$

$$= a\left[1+\left(\frac{x}{a}\right)^2\right] = a\left(\frac{a^2+x^2}{a^2}\right)$$

$$= \frac{a^2+x^2}{a}$$

Hence $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{a}{a^2+x^2}$

The principal value of $y = \tan^{-1}x$ is defined as the angle lying between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ and the gradient $\left(\text{i.e. } \frac{dy}{dx}\right)$ between these two values is always positive (see Fig. 33.1(c)).

Comparing $\tan^{-1}\frac{2x}{3}$ with $\tan^{-1}\frac{x}{a}$ shows that $a = \frac{3}{2}$

Hence if $y = \tan^{-1}\frac{2x}{3}$ then

$$\frac{dy}{dx} = \frac{\frac{3}{2}}{\left(\frac{3}{2}\right)^2+x^2} = \frac{\frac{3}{2}}{\frac{9}{4}+x^2} = \frac{\frac{3}{2}}{\frac{9+4x^2}{4}}$$

$$= \frac{\frac{3}{2}(4)}{9+4x^2} = \mathbf{\frac{6}{9+4x^2}}$$

Problem 4. Find the differential coefficient of $y = \ln(\cos^{-1}3x)$.

Let $u = \cos^{-1}3x$ then $y = \ln u$.

By the function of a function rule,

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{u}\times\frac{d}{dx}(\cos^{-1}3x)$$

$$= \frac{1}{\cos^{-1}3x}\left\{\frac{-3}{\sqrt{1-(3x)^2}}\right\}$$

i.e. $\mathbf{\frac{d}{dx}[\ln(\cos^{-1}3x)] = \frac{-3}{\sqrt{1-9x^2}\cos^{-1}3x}}$

Problem 5. If $y = \tan^{-1}\frac{3}{t^2}$ find $\frac{dy}{dt}$

Using the general form from Table 33.1(iii),

$$f(t) = \frac{3}{t^2} = 3t^{-2},$$

from which $f'(t) = \frac{-6}{t^3}$

Hence $\frac{d}{dt}\left(\tan^{-1}\frac{3}{t^2}\right) = \frac{f'(t)}{1+[f(t)]^2}$

$$= \frac{-\frac{6}{t^3}}{\left\{1+\left(\frac{3}{t^2}\right)^2\right\}} = \frac{-\frac{6}{t^3}}{\frac{t^4+9}{t^4}}$$

$$= \left(-\frac{6}{t^3}\right)\left(\frac{t^4}{t^4+9}\right) = \mathbf{-\frac{6t}{t^4+9}}$$

Problem 6. Differentiate $y = \dfrac{\cot^{-1} 2x}{1+4x^2}$

Using the quotient rule:

$$\frac{dy}{dx} = \frac{(1+4x^2)\left(\dfrac{-2}{1+(2x)^2}\right) - (\cot^{-1}2x)(8x)}{(1+4x^2)^2}$$

from Table 33.1(vi)

$$= \frac{\mathbf{-2(1+4x\cot^{-1}2x)}}{\mathbf{(1+4x^2)^2}}$$

Problem 7. Differentiate $y = x \operatorname{cosec}^{-1} x$.

Using the product rule:

$$\frac{dy}{dx} = (x)\left[\frac{-1}{x\sqrt{x^2-1}}\right] + (\operatorname{cosec}^{-1} x)\ (1)$$

from Table 33.1(v)

$$= \frac{\mathbf{-1}}{\mathbf{\sqrt{x^2-1}}} \mathbf{+ \operatorname{cosec}^{-1} x}$$

Problem 8. Show that if $y = \tan^{-1}\left(\dfrac{\sin t}{\cos t - 1}\right)$ then $\dfrac{dy}{dt} = \dfrac{1}{2}$

If $f(t) = \left(\dfrac{\sin t}{\cos t - 1}\right)$

then $f'(t) = \dfrac{(\cos t - 1)(\cos t) - (\sin t)(-\sin t)}{(\cos t - 1)^2}$

$$= \frac{\cos^2 t - \cos t + \sin^2 t}{(\cos t - 1)^2} = \frac{1 - \cos t}{(\cos t - 1)^2}$$

since $\sin^2 t + \cos^2 t = 1$

$$= \frac{-(\cos t - 1)}{(\cos t - 1)^2} = \frac{-1}{\cos t - 1}$$

Using Table 33.1(iii), when

$$y = \tan^{-1}\left(\frac{\sin t}{\cos t - 1}\right)$$

then $\dfrac{dy}{dt} = \dfrac{\dfrac{-1}{\cos t - 1}}{1 + \left(\dfrac{\sin t}{\cos t - 1}\right)^2}$

$$= \frac{\dfrac{-1}{\cos t - 1}}{\dfrac{(\cos t - 1)^2 + (\sin t)^2}{(\cos t - 1)^2}}$$

$$= \left(\frac{-1}{\cos t - 1}\right)\left(\frac{(\cos t - 1)^2}{\cos^2 t - 2\cos t + 1 + \sin^2 t}\right)$$

$$= \frac{-(\cos t - 1)}{2 - 2\cos t} = \frac{1 - \cos t}{2(1 - \cos t)} = \mathbf{\frac{1}{2}}$$

Now try the following exercise

Exercise 135 Further problems on differentiating inverse trigonometric functions

In Problems 1 to 6, differentiate with respect to the variable.

1. (a) $\sin^{-1} 4x$ (b) $\sin^{-1}\dfrac{x}{2}$

$$\left[\text{(a) } \frac{4}{\sqrt{1-16x^2}} \text{ (b) } \frac{1}{\sqrt{4-x^2}}\right]$$

2. (a) $\cos^{-1} 3x$ (b) $\dfrac{2}{3}\cos^{-1}\dfrac{x}{3}$

$$\left[\text{(a) } \frac{-3}{\sqrt{1-9x^2}} \text{ (b) } \frac{-2}{3\sqrt{9-x^2}}\right]$$

3. (a) $3\tan^{-1} 2x$ (b) $\dfrac{1}{2}\tan^{-1}\sqrt{x}$

$$\left[\text{(a) } \frac{6}{1+4x^2} \text{ (b) } \frac{1}{4\sqrt{x}(1+x)}\right]$$

4. (a) $2\sec^{-1} 2t$ (b) $\sec^{-1}\dfrac{3}{4}x$

$$\left[\text{(a) } \frac{2}{t\sqrt{4t^2-1}} \text{ (b) } \frac{4}{x\sqrt{9x^2-16}}\right]$$

5. (a) $\dfrac{5}{2}\operatorname{cosec}^{-1}\dfrac{\theta}{2}$ (b) $\operatorname{cosec}^{-1} x^2$

$$\left[\text{(a) } \frac{-5}{\theta\sqrt{\theta^2-4}} \text{ (b) } \frac{-2}{x\sqrt{x^4-1}}\right]$$

6. (a) $3\cot^{-1} 2t$ (b) $\cot^{-1}\sqrt{\theta^2-1}$

$$\left[\text{(a) } \frac{-6}{1+4t^2} \text{ (b) } \frac{-1}{\theta\sqrt{\theta^2-1}}\right]$$

7. Show that the differential coefficient of $\tan^{-1}\left(\frac{x}{1-x^2}\right)$ is $\frac{1+x^2}{1-x^2+x^4}$.

In Problems 8 to 11 differentiate with respect to the variable.

8. (a) $2x\sin^{-1}3x$ (b) $t^2\sec^{-1}2t$

$$\left[\begin{array}{l}\text{(a) } \dfrac{6x}{\sqrt{1-9x^2}}+2\sin^{-1}3x \\ \text{(b) } \dfrac{t}{\sqrt{4t^2-1}}+2t\sec^{-1}2t\end{array}\right]$$

9. (a) $\theta^2\cos^{-1}(\theta^2-1)$ (b) $(1-x^2)\tan^{-1}x$

$$\left[\begin{array}{l}\text{(a) } 2\theta\cos^{-1}(\theta^2-1)-\dfrac{2\theta^2}{\sqrt{2-\theta^2}} \\ \text{(b) } \left(\dfrac{1-x^2}{1+x^2}\right)-2x\tan^{-1}x\end{array}\right]$$

10. (a) $2\sqrt{t}\cot^{-1}t$ (b) $x\,\text{cosec}^{-1}\sqrt{x}$

$$\left[\begin{array}{l}\text{(a) } \dfrac{-2\sqrt{t}}{1+t^2}+\dfrac{1}{\sqrt{t}}\cot^{-1}t \\ \text{(b) } \text{cosec}^{-1}\sqrt{x}-\dfrac{1}{2\sqrt{(x-1)}}\end{array}\right]$$

11. (a) $\dfrac{\sin^{-1}3x}{x^2}$ (b) $\dfrac{\cos^{-1}x}{\sqrt{1-x^2}}$

$$\left[\begin{array}{l}\text{(a) } \dfrac{1}{x^3}\left\{\dfrac{3x}{\sqrt{1-9x^2}}-2\sin^{-1}3x\right\} \\ \text{(b) } \dfrac{-1+\dfrac{x}{\sqrt{1-x^2}}\cos^{-1}x}{(1-x^2)}\end{array}\right]$$

33.3 Logarithmic forms of inverse hyperbolic functions

Inverse hyperbolic functions may be evaluated most conveniently when expressed in a **logarithmic form**.

For example, if $y=\sinh^{-1}\frac{x}{a}$ then $\frac{x}{a}=\sinh y$.

From Chapter 5, $e^y=\cosh y+\sinh y$ and $\cosh^2 y-\sinh^2 y=1$, from which,

$\cosh y=\sqrt{1+\sinh^2 y}$ which is positive since $\cosh y$ is always positive (see Fig. 5.2, page 43).

Hence $e^y=\sqrt{1+\sinh^2 y}+\sinh y$

$$=\sqrt{\left[1+\left(\frac{x}{a}\right)^2\right]}+\frac{x}{a}=\sqrt{\left(\frac{a^2+x^2}{a^2}\right)}+\frac{x}{a}$$

$$=\frac{\sqrt{a^2+x^2}}{a}+\frac{x}{a}\quad\text{or}\quad\frac{x+\sqrt{a^2+x^2}}{a}$$

Taking Napierian logarithms of both sides gives:

$$y=\ln\left\{\frac{x+\sqrt{a^2+x^2}}{a}\right\}$$

Hence, $$\boldsymbol{\sinh^{-1}\frac{x}{a}=\ln\left\{\frac{x+\sqrt{a^2+x^2}}{a}\right\}} \qquad (1)$$

Thus to evaluate $\sinh^{-1}\frac{3}{4}$, let $x=3$ and $a=4$ in equation (1).

Then $\sin h^{-1}\frac{3}{4}=\ln\left\{\frac{3+\sqrt{4^2+3^2}}{4}\right\}$

$$=\ln\left(\frac{3+5}{4}\right)=\ln 2=0.6931$$

By similar reasoning to the above it may be shown that:

$$\boldsymbol{\cosh^{-1}\frac{x}{a}=\ln\left\{\frac{x+\sqrt{x^2-a^2}}{a}\right\}}$$

and $$\boldsymbol{\tanh^{-1}\frac{x}{a}=\frac{1}{2}\ln\left(\frac{a+x}{a-x}\right)}$$

Problem 9. Evaluate, correct to 4 decimal places, $\sinh^{-1}2$.

From above, $\sinh^{-1}\frac{x}{a}=\ln\left\{\frac{x+\sqrt{a^2+x^2}}{a}\right\}$

With $x=2$ and $a=1$,

$$\sinh^{-1}2=\ln\left\{\frac{2+\sqrt{1^2+2^2}}{1}\right\}$$

$$=\ln(2+\sqrt{5})=\ln 4.2361$$

$$=\mathbf{1.4436}\text{, correct to 4 decimal places}$$

Using a calculator,

(i) press hyp

(ii) press 4 and $\sinh^{-1}($ appears

(iii) type in 2

(iv) press) to close the brackets

(v) press = and 1.443635475 appears

Hence, $\mathbf{sinh^{-1}2 = 1.4436}$, correct to 4 decimal places.

Problem 10. Show that $\tanh^{-1}\frac{x}{a} = \frac{1}{2}\ln\left(\frac{a+x}{a-x}\right)$ and evaluate, correct to 4 decimal places, $\tanh^{-1}\frac{3}{5}$

If $y = \tanh^{-1}\frac{x}{a}$ then $\frac{x}{a} = \tanh y$.

From Chapter 5,

$$\tanh y = \frac{\sinh x}{\cosh x} = \frac{\frac{1}{2}(e^y - e^{-y})}{\frac{1}{2}(e^y + e^{-y})} = \frac{e^{2y}-1}{e^{2y}+1}$$

by dividing each term by e^{-y}

Thus, $\frac{x}{a} = \frac{e^{2y}-1}{e^{2y}+1}$

from which, $x(e^{2y}+1) = a(e^{2y}-1)$

Hence $x + a = ae^{2y} - xe^{2y} = e^{2y}(a-x)$

from which $e^{2y} = \left(\frac{a+x}{a-x}\right)$

Taking Napierian logarithms of both sides gives:

$$2y = \ln\left(\frac{a+x}{a-x}\right)$$

and $$y = \frac{1}{2}\ln\left(\frac{a+x}{a-x}\right)$$

Hence, $\mathbf{\tanh^{-1}\frac{x}{a} = \frac{1}{2}\ln\left(\frac{a+x}{a-x}\right)}$

Substituting $x = 3$ and $a = 5$ gives:

$$\tanh^{-1}\frac{3}{5} = \frac{1}{2}\ln\left(\frac{5+3}{5-3}\right) = \frac{1}{2}\ln 4$$

$= \mathbf{0.6931}$, correct to 4 decimal places.

Problem 11. Prove that $\cosh^{-1}\frac{x}{a} = \ln\left\{\frac{x+\sqrt{x^2-a^2}}{a}\right\}$ and hence evaluate $\cosh^{-1}1.4$ correct to 4 decimal places.

If $y = \cosh^{-1}\frac{x}{a}$ then $\frac{x}{a} = \cos y$

$$e^y = \cosh y + \sinh y = \cosh y \pm \sqrt{\cosh^2 y - 1}$$

$$= \frac{x}{a} \pm \sqrt{\left[\left(\frac{x}{a}\right)^2 - 1\right]} = \frac{x}{a} \pm \frac{\sqrt{x^2-a^2}}{a}$$

$$= \frac{x \pm \sqrt{x^2-a^2}}{a}$$

Taking Napierian logarithms of both sides gives:

$$y = \ln\left\{\frac{x \pm \sqrt{x^2-a^2}}{a}\right\}$$

Thus, assuming the principal value,

$$\mathbf{\cosh^{-1}\frac{x}{a} = \ln\left\{\frac{x+\sqrt{x^2-a^2}}{a}\right\}}$$

$\cosh^{-1}1.4 = \cosh^{-1}\frac{14}{10} = \cosh^{-1}\frac{7}{5}$

In the equation for $\cosh^{-1}\frac{x}{a}$, let $x = 7$ and $a = 5$

Then $\cosh^{-1}\frac{7}{5} = \ln\left\{\frac{7+\sqrt{7^2-5^2}}{5}\right\}$

$= \ln 2.3798 = \mathbf{0.8670}$,

correct to 4 decimal places.

Now try the following exercise

Exercise 136 Further problems on logarithmic forms of the inverse hyperbolic functions

In Problems 1 to 3 use logarithmic equivalents of inverse hyperbolic functions to evaluate correct to 4 decimal places.

1. (a) $\sinh^{-1}\frac{1}{2}$ (b) $\sinh^{-1}4$ (c) $\sinh^{-1}0.9$
   [(a) 0.4812 (b) 2.0947 (c) 0.8089]
2. (a) $\cosh^{-1}\frac{5}{4}$ (b) $\cosh^{-1}3$ (c) $\cosh^{-1}4.3$
   [(a) 0.6931 (b) 1.7627 (c) 2.1380]
3. (a) $\tanh^{-1}\frac{1}{4}$ (b) $\tanh^{-1}\frac{5}{8}$ (c) $\tanh^{-1}0.7$
   [(a) 0.2554 (b) 0.7332 (c) 0.8673]

33.4 Differentiation of inverse hyperbolic functions

If $y = \sinh^{-1}\frac{x}{a}$ then $\frac{x}{a} = \sinh y$ and $x = a\sinh y$

$\frac{dx}{dy} = a\cosh y$ (from Chapter 32).

Also $\cosh^2 y - \sinh^2 y = 1$, from which,

$$\cosh y = \sqrt{1+\sinh^2 y} = \sqrt{\left[1+\left(\frac{x}{a}\right)^2\right]}$$

$$= \frac{\sqrt{a^2+x^2}}{a}$$

Hence $\frac{dx}{dy} = a\cosh y = \frac{a\sqrt{a^2+x^2}}{a} = \sqrt{a^2+x^2}$

Then $\mathbf{\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\sqrt{a^2+x^2}}}$

[An alternative method of differentiating $\sinh^{-1}\frac{x}{a}$ is to differentiate the logarithmic form $\ln\left\{\frac{x+\sqrt{a^2+x^2}}{a}\right\}$ with respect to x.]

From the sketch of $y = \sinh^{-1} x$ shown in Fig. 33.2(a) it is seen that the gradient $\left(\text{i.e. } \frac{dy}{dx}\right)$ is always positive.

It follows from above that

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \sinh^{-1}\frac{x}{a} + c$$

or $$\ln\left\{\frac{x+\sqrt{a^2+x^2}}{a}\right\} + c$$

It may be shown that

$$\mathbf{\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}}}$$

or more generally

$$\mathbf{\frac{d}{dx}[\sinh^{-1} f(x)] = \frac{f'(x)}{\sqrt{[f(x)]^2+1}}}$$

by using the function of a function rule as in Section 33.2(iv).

The remaining inverse hyperbolic functions are differentiated in a similar manner to that shown above and the results are summarized in Table 33.2.

Table 33.2 Differential coefficients of inverse hyperbolic functions

	y or $f(x)$	$\frac{dy}{dx}$ or $f'(x)$
(i)	$\sinh^{-1}\frac{x}{a}$	$\frac{1}{\sqrt{x^2+a^2}}$
	$\sinh^{-1} f(x)$	$\frac{f'(x)}{\sqrt{[f(x)]^2+1}}$
(ii)	$\cosh^{-1}\frac{x}{a}$	$\frac{1}{\sqrt{x^2-a^2}}$
	$\cosh^{-1} f(x)$	$\frac{f'(x)}{\sqrt{[f(x)]^2-1}}$
(iii)	$\tanh^{-1}\frac{x}{a}$	$\frac{a}{a^2-x^2}$
	$\tanh^{-1} f(x)$	$\frac{f'(x)}{1-[f(x)]^2}$
(iv)	$\text{sech}^{-1}\frac{x}{a}$	$\frac{-a}{x\sqrt{a^2-x^2}}$
	$\text{sech}^{-1} f(x)$	$\frac{-f'(x)}{f(x)\sqrt{1-[f(x)]^2}}$
(v)	$\text{cosech}^{-1}\frac{x}{a}$	$\frac{-a}{x\sqrt{x^2+a^2}}$
	$\text{cosech}^{-1} f(x)$	$\frac{-f'(x)}{f(x)\sqrt{[f(x)]^2+1}}$
(vi)	$\coth^{-1}\frac{x}{a}$	$\frac{a}{a^2-x^2}$
	$\coth^{-1} f(x)$	$\frac{f'(x)}{1-[f(x)]^2}$

Problem 12. Find the differential coefficient of $y = \sinh^{-1} 2x$.

From Table 33.2(i),

$$\frac{d}{dx}[\sinh^{-1} f(x)] = \frac{f'(x)}{\sqrt{[f(x)]^2+1}}$$

Hence $$\frac{d}{dx}(\sinh^{-1} 2x) = \frac{2}{\sqrt{[(2x)^2+1]}}$$

$$= \mathbf{\frac{2}{\sqrt{[4x^2+1]}}}$$

Problem 13. Determine $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\cosh^{-1}\sqrt{(x^2+1)}\right]$

If $y=\cosh^{-1} f(x)$, $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{f'(x)}{\sqrt{[f(x)]^2-1}}$

If $y=\cosh^{-1}\sqrt{(x^2+1)}$, then $f(x)=\sqrt{(x^2+1)}$ and $f'(x)=\dfrac{1}{2}(x+1)^{-1/2}(2x)=\dfrac{x}{\sqrt{(x^2+1)}}$

Hence, $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\cosh^{-1}\sqrt{(x^2+1)}\right]$

$$=\frac{\dfrac{x}{\sqrt{(x^2+1)}}}{\sqrt{\left[\left(\sqrt{(x^2+1)}\right)^2-1\right]}}=\frac{\dfrac{x}{\sqrt{(x^2+1)}}}{\sqrt{(x^2+1-1)}}$$

$$=\frac{\dfrac{x}{\sqrt{(x^2+1)}}}{x}=\frac{\mathbf{1}}{\sqrt{(\boldsymbol{x^2+1})}}$$

Problem 14. Show that $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\tanh^{-1}\dfrac{x}{a}\right]=\dfrac{a}{a^2-x^2}$ and hence determine the differential coefficient of $\tanh^{-1}\dfrac{4x}{3}$

If $y=\tanh^{-1}\dfrac{x}{a}$ then $\dfrac{x}{a}=\tanh y$ and $x=a\tanh y$

$\dfrac{\mathrm{d}x}{\mathrm{d}y}=a\,\mathrm{sech}^2 y=a(1-\tanh^2 y)$, since $1-\mathrm{sech}^2 y=\tanh^2 y$

$$=a\left[1-\left(\frac{x}{a}\right)^2\right]=a\left(\frac{a^2-x^2}{a^2}\right)=\frac{a^2-x^2}{a}$$

Hence $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1}{\dfrac{\mathrm{d}x}{\mathrm{d}y}}=\dfrac{a}{a^2-x^2}$

Comparing $\tanh^{-1}\dfrac{4x}{3}$ with $\tanh^{-1}\dfrac{x}{a}$ shows that $a=\dfrac{3}{4}$

Hence $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\tanh^{-1}\dfrac{4x}{3}\right]=\dfrac{\frac{3}{4}}{\left(\frac{3}{4}\right)^2-x^2}=\dfrac{\frac{3}{4}}{\frac{9}{16}-x^2}$

$$=\frac{\frac{3}{4}}{\frac{9-16x^2}{16}}=\frac{3}{4}\cdot\frac{16}{(9-16x^2)}=\frac{\mathbf{12}}{\boldsymbol{9-16x^2}}$$

Problem 15. Differentiate $\mathrm{cosech}^{-1}(\sinh\theta)$.

From Table 33.2(v),

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{cosech}^{-1} f(x)\right]=\frac{-f'(x)}{f(x)\sqrt{[f(x)]^2+1}}$$

Hence $\dfrac{\mathrm{d}}{\mathrm{d}\theta}[\mathrm{cosech}^{-1}(\sinh\theta)]$

$$=\frac{-\cosh\theta}{\sinh\theta\sqrt{[\sinh^2\theta+1]}}$$

$$=\frac{-\cosh\theta}{\sinh\theta\sqrt{\cosh^2\theta}}\ \text{since}\ \cosh^2\theta-\sinh^2\theta=1$$

$$=\frac{-\cosh\theta}{\sinh\theta\cosh\theta}=\frac{-1}{\sinh\theta}=\boldsymbol{-\mathrm{cosech}\,\theta}$$

Problem 16. Find the differential coefficient of $y=\mathrm{sech}^{-1}(2x-1)$.

From Table 33.2(iv),

$$\frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{sech}^{-1} f(x)]=\frac{-f'(x)}{f(x)\sqrt{1-[f(x)]^2}}$$

Hence, $\dfrac{\mathrm{d}}{\mathrm{d}x}[\mathrm{sech}^{-1}(2x-1)]$

$$=\frac{-2}{(2x-1)\sqrt{[1-(2x-1)^2]}}$$

$$=\frac{-2}{(2x-1)\sqrt{[1-(4x^2-4x+1)]}}$$

$$=\frac{-2}{(2x-1)\sqrt{(4x-4x^2)}}=\frac{-2}{(2x-1)\sqrt{[4x(1-x)]}}$$

$$=\frac{-2}{(2x-1)2\sqrt{[x(1-x)]}}=\frac{\boldsymbol{-1}}{\boldsymbol{(2x-1)\sqrt{[x(1-x)]}}}$$

Problem 17. Show that $\dfrac{\mathrm{d}}{\mathrm{d}x}[\coth^{-1}(\sin x)]=\sec x$.

From Table 33.2(vi),

$$\frac{\mathrm{d}}{\mathrm{d}x}[\coth^{-1} f(x)]=\frac{f'(x)}{1-[f(x)]^2}$$

Hence $\dfrac{d}{dx}[\coth^{-1}(\sin x)] = \dfrac{\cos x}{[1-(\sin x)^2]}$

$= \dfrac{\cos x}{\cos^2 x}$ since $\cos^2 x + \sin^2 x = 1$

$= \dfrac{1}{\cos x} = \mathbf{\sec x}$

Problem 18. Differentiate $y = (x^2-1)\tanh^{-1} x$.

Using the product rule,

$$\frac{dy}{dx} = (x^2-1)\left(\frac{1}{1-x^2}\right) + (\tanh^{-1} x)(2x)$$

$$= \frac{-(1-x^2)}{(1-x^2)} + 2x\ \tanh^{-1} x = \mathbf{2x\ \tanh^{-1} x - 1}$$

Problem 19. Determine $\displaystyle\int \frac{dx}{\sqrt{(x^2+4)}}$.

Since $\dfrac{d}{dx}\left(\sinh^{-1}\dfrac{x}{a}\right) = \dfrac{1}{\sqrt{(x^2+a^2)}}$

then $\displaystyle\int \frac{dx}{\sqrt{(x^2+a^2)}} = \sinh^{-1}\frac{x}{a} + c$

Hence $\displaystyle\int \frac{1}{\sqrt{(x^2+4)}}\,dx = \int \frac{1}{\sqrt{(x^2+2^2)}}\,dx$

$$= \mathbf{\sinh^{-1}\frac{x}{2} + c}$$

Problem 20. Determine $\displaystyle\int \frac{4}{\sqrt{(x^2-3)}}\,dx$.

Since $\dfrac{d}{dx}\left(\cosh^{-1}\dfrac{x}{a}\right) = \dfrac{1}{\sqrt{(x^2-a^2)}}$

then $\displaystyle\int \frac{1}{\sqrt{(x^2-a^2)}}\,dx = \cosh^{-1}\frac{x}{a} + c$

Hence $\displaystyle\int \frac{4}{\sqrt{(x^2-3)}}\,dx = 4\int \frac{1}{\sqrt{[x^2-(\sqrt{3})^2]}}\,dx$

$$= \mathbf{4\ \cosh^{-1}\frac{x}{\sqrt{3}} + c}$$

Problem 21. Find $\displaystyle\int \frac{2}{(9-4x^2)}\,dx$.

Since $\tanh^{-1}\dfrac{x}{a} = \dfrac{a}{a^2-x^2}$

then $\displaystyle\int \frac{a}{a^2-x^2}\,dx = \tanh^{-1}\frac{x}{a} + c$

i.e. $\displaystyle\int \frac{1}{a^2-x^2}\,dx = \frac{1}{a}\tanh^{-1}\frac{x}{a} + c$

Hence $\displaystyle\int \frac{2}{(9-4x^2)}\,dx = 2\int \frac{1}{4\left(\frac{9}{4}-x^2\right)}\,dx$

$$= \frac{1}{2}\int \frac{1}{\left[\left(\frac{3}{2}\right)^2 - x^2\right]}\,dx$$

$$= \frac{1}{2}\left[\frac{1}{\left(\frac{3}{2}\right)}\tanh^{-1}\frac{x}{\left(\frac{3}{2}\right)} + c\right]$$

i.e. $\displaystyle\int \frac{2}{(9-4x^2)}\,dx = \mathbf{\frac{1}{3}\tanh^{-1}\frac{2x}{3} + c}$

Now try the following exercise

Exercise 137 Further problems on differentiation of inverse hyperbolic functions

In Problems 1 to 11, differentiate with respect to the variable.

1. (a) $\sinh^{-1}\dfrac{x}{3}$ (b) $\sinh^{-1} 4x$

$$\left[\text{(a) } \frac{1}{\sqrt{(x^2+9)}} \text{ (b) } \frac{4}{\sqrt{(16x^2+1)}}\right]$$

2. (a) $2\cosh^{-1}\dfrac{t}{3}$ (b) $\dfrac{1}{2}\cosh^{-1} 2\theta$

$$\left[\text{(a) } \frac{2}{\sqrt{(t^2-9)}} \text{ (b) } \frac{1}{\sqrt{(4\theta^2-1)}}\right]$$

3. (a) $\tanh^{-1}\dfrac{2x}{5}$ (b) $3\tanh^{-1} 3x$

$$\left[\text{(a) } \frac{10}{25-4x^2} \text{ (b) } \frac{9}{(1-9x^2)}\right]$$

4. (a) $\operatorname{sech}^{-1}\dfrac{3x}{4}$ (b) $-\dfrac{1}{2}\operatorname{sech}^{-1} 2x$

$$\left[\text{(a) } \frac{-4}{x\sqrt{(16-9x^2)}} \text{ (b) } \frac{1}{2x\sqrt{(1-4x^2)}}\right]$$

5. (a) $\text{cosech}^{-1}\frac{x}{4}$ (b) $\frac{1}{2}\text{cosech}^{-1}\,4x$

$$\left[\text{(a)}\frac{-4}{x\sqrt{(x^2+16)}}\ \text{(b)}\ \frac{-1}{2x\sqrt{(16x^2+1)}}\right]$$

6. (a) $\coth^{-1}\frac{2x}{7}$ (b) $\frac{1}{4}\coth^{-1}\,3t$

$$\left[\text{(a)}\ \frac{14}{49-4x^2}\ \text{(b)}\ \frac{3}{4(1-9t^2)}\right]$$

7. (a) $2\sinh^{-1}\sqrt{(x^2-1)}$

(b) $\frac{1}{2}\cosh^{-1}\sqrt{(x^2+1)}$

$$\left[\text{(a)}\ \frac{2}{\sqrt{(x^2-1)}}\ \text{(b)}\ \frac{1}{2\sqrt{(x^2+1)}}\right]$$

8. (a) $\text{sech}^{-1}(x-1)$ (b) $\tanh^{-1}(\tanh x)$

$$\left[\text{(a)}\ \frac{-1}{(x-1)\sqrt{[x(2-x)]}}\ \text{(b)}\ 1\right]$$

9. (a) $\cosh^{-1}\left(\frac{t}{t-1}\right)$ (b) $\coth^{-1}(\cos x)$

$$\left[\text{(a)}\ \frac{-1}{(t-1)\sqrt{(2t-1)}}\ \text{(b)}\ -\text{cosec}\,x\right]$$

10. (a) $\theta\ \sinh^{-1}\theta$ (b) $\sqrt{x}\ \cosh^{-1}\,x$

$$\left[\begin{array}{l}\text{(a)}\ \dfrac{\theta}{\sqrt{(\theta^2+1)}}+\sinh^{-1}\theta\\[2ex]\text{(b)}\ \dfrac{\sqrt{x}}{\sqrt{(x^2-1)}}+\dfrac{\cosh^{-1}x}{2\sqrt{x}}\end{array}\right]$$

11. (a) $\frac{2\,\text{sech}^{-1}\sqrt{t}}{t^2}$ (b) $\frac{\tanh^{-1}x}{(1-x^2)}$

$$\left[\begin{array}{l}\text{(a)}\ \dfrac{-1}{t^3}\left\{\dfrac{1}{\sqrt{(1-t)}}+4\,\text{sech}^{-1}\sqrt{t}\right\}\\[2ex]\text{(b)}\ \dfrac{1+2x\tanh^{-1}x}{(1-x^2)^2}\end{array}\right]$$

12. Show that $\frac{\mathrm{d}}{\mathrm{d}x}[x\cosh^{-1}(\cosh x)]=2x$.

In Problems 13 to 15, determine the given integrals.

13. (a) $\int\frac{1}{\sqrt{(x^2+9)}}\,\mathrm{d}x$

(b) $\int\frac{3}{\sqrt{(4x^2+25)}}\,\mathrm{d}x$

$$\left[\text{(a)}\ \sinh^{-1}\frac{x}{3}+c\ \text{(b)}\ \frac{3}{2}\sinh^{-1}\frac{2x}{5}+c\right]$$

14. (a) $\int\frac{1}{\sqrt{(x^2-16)}}\,\mathrm{d}x$

(b) $\int\frac{1}{\sqrt{(t^2-5)}}\,\mathrm{d}t$

$$\left[\text{(a)}\ \cosh^{-1}\frac{x}{4}+c\ \text{(b)}\ \cosh^{-1}\frac{t}{\sqrt{5}}+c\right]$$

15. (a) $\int\frac{\mathrm{d}\theta}{\sqrt{(36+\theta^2)}}$ (b) $\int\frac{3}{(16-2x^2)}\,\mathrm{d}x$

$$\left[\begin{array}{l}\text{(a)}\ \dfrac{1}{6}\tan^{-1}\dfrac{\theta}{6}+c\\[2ex]\text{(b)}\ \dfrac{3}{2\sqrt{8}}\tanh^{-1}\dfrac{x}{\sqrt{8}}+c\end{array}\right]$$

Chapter 34

Partial differentiation

34.1 Introduction to partial derivatives

In engineering, it sometimes happens that the variation of one quantity depends on changes taking place in two, or more, other quantities. For example, the volume V of a cylinder is given by $V = \pi r^2 h$. The volume will change if either radius r or height h is changed. The formula for volume may be stated mathematically as $V = f(r, h)$ which means 'V is some function of r and h'. Some other practical examples include:

(i) time of oscillation, $t = 2\pi\sqrt{\dfrac{l}{g}}$ i.e. $t = f(l, g)$.

(ii) torque $T = I\alpha$, i.e. $T = f(I, \alpha)$.

(iii) pressure of an ideal gas $p = \dfrac{mRT}{V}$ i.e. $p = f(T, V)$.

(iv) resonant frequency $f_r = \dfrac{1}{2\pi\sqrt{LC}}$ i.e. $f_r = f(L, C)$, and so on.

When differentiating a function having two variables, **one variable is kept constant and the differential coefficient of the other variable is found with respect to that variable**. The differential coefficient obtained is called a **partial derivative** of the function.

34.2 First order partial derivatives

A 'curly dee', ∂, is used to denote a differential coefficient in an expression containing more than one variable.

Hence if $V = \pi r^2 h$ then $\dfrac{\partial V}{\partial r}$ means 'the partial derivative of V with respect to r, with h remaining constant'. Thus,

$$\frac{\partial V}{\partial r} = (\pi h)\frac{\mathrm{d}}{\mathrm{d}r}(r^2) = (\pi h)(2r) = 2\pi rh.$$

Similarly, $\dfrac{\partial V}{\partial h}$ means 'the partial derivative of V with respect to h, with r remaining constant'. Thus,

$$\frac{\partial V}{\partial h} = (\pi r^2)\frac{\mathrm{d}}{\mathrm{d}h}(h) = (\pi r^2)(1) = \pi r^2.$$

$\dfrac{\partial V}{\partial r}$ and $\dfrac{\partial V}{\partial h}$ are examples of **first order partial derivatives**, since $n = 1$ when written in the form $\dfrac{\partial^n V}{\partial r^n}$.

First order partial derivatives are used when finding the total differential, rates of change and errors for functions of two or more variables (see Chapter 35), when finding maxima, minima and saddle points for functions of two variables (see Chapter 36), and with partial differential equations (see Chapter 53).

Problem 1. If $z = 5x^4 + 2x^3y^2 - 3y$ find (a) $\dfrac{\partial z}{\partial x}$ and (b) $\dfrac{\partial z}{\partial y}$.

(a) To find $\dfrac{\partial z}{\partial x}$, y is kept constant.

Since $z = 5x^4 + (2y^2)x^3 - (3y)$

then,

$$\frac{\partial z}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}x}(5x^4) + (2y^2)\frac{\mathrm{d}}{\mathrm{d}x}(x^3) - (3y)\frac{\mathrm{d}}{\mathrm{d}x}(1)$$

$$= 20x^3 + (2y^2)(3x^2) - 0.$$

Hence $\dfrac{\partial z}{\partial x} = 20x^3 + 6x^2y^2$.

(b) To find $\frac{\partial z}{\partial y}$, x is kept constant.

Since $z=(5x^4)+(2x^3)y^2-3y$

then,

$$\frac{\partial z}{\partial y}=(5x^4)\frac{\mathrm{d}}{\mathrm{d}y}(1)+(2x^3)\frac{\mathrm{d}}{\mathrm{d}y}(y^2)-3\frac{\mathrm{d}}{\mathrm{d}y}(y)$$

$$=0+(2x^3)(2y)-3$$

Hence $\boldsymbol{\frac{\partial z}{\partial y}=4x^3y-3}$.

Problem 2. Given $y=4\sin 3x\cos 2t$, find $\frac{\partial y}{\partial x}$ and $\frac{\partial y}{\partial t}$

To find $\frac{\partial y}{\partial x}$, t is kept constant.

Hence $\frac{\partial y}{\partial x}=(4\cos 2t)\frac{\mathrm{d}}{\mathrm{d}x}(\sin 3x)$

$=(4\cos 2t)(3\cos 3x)$

i.e. $\boldsymbol{\frac{\partial y}{\partial x}=12\cos 3x\cos 2t}$

To find $\frac{\partial y}{\partial t}$, x is kept constant.

Hence $\frac{\partial y}{\partial t}=(4\sin 3x)\frac{\mathrm{d}}{\mathrm{d}t}(\cos 2t)$

$=(4\sin 3x)(-2\sin 2t)$

i.e. $\boldsymbol{\frac{\partial y}{\partial t}=-8\sin 3x\sin 2t}$

Problem 3. If $z=\sin xy$ show that

$$\frac{1}{y}\frac{\partial z}{\partial x}=\frac{1}{x}\frac{\partial z}{\partial y}$$

$\frac{\partial z}{\partial x}=y\cos xy$, since y is kept constant.

$\frac{\partial z}{\partial y}=x\cos xy$, since x is kept constant.

$$\frac{1}{y}\frac{\partial z}{\partial x}=\left(\frac{1}{y}\right)(y\cos xy)=\cos xy$$

and $\frac{1}{x}\frac{\partial z}{\partial y}=\left(\frac{1}{x}\right)(x\cos xy)=\cos xy.$

Hence $\boldsymbol{\frac{1}{y}\frac{\partial z}{\partial x}=\frac{1}{x}\frac{\partial z}{\partial y}}$

Problem 4. Determine $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ when $z=\frac{1}{\sqrt{(x^2+y^2)}}$

$$z=\frac{1}{\sqrt{(x^2+y^2)}}=(x^2+y^2)^{\frac{-1}{2}}$$

$\frac{\partial z}{\partial x}=-\frac{1}{2}(x^2+y^2)^{\frac{-3}{2}}(2x)$, by the function of a function rule (keeping y constant)

$$=\frac{-x}{(x^2+y^2)^{\frac{3}{2}}}=\frac{\boldsymbol{-x}}{\boldsymbol{\sqrt{(x^2+y^2)^3}}}$$

$\frac{\partial z}{\partial y}=-\frac{1}{2}(x^2+y^2)^{\frac{-3}{2}}(2y)$, (keeping x constant)

$$=\frac{\boldsymbol{-y}}{\boldsymbol{\sqrt{(x^2+y^2)^3}}}$$

Problem 5. Pressure p of a mass of gas is given by $pV=mRT$, where m and R are constants, V is the volume and T the temperature. Find expressions for $\frac{\partial p}{\partial T}$ and $\frac{\partial p}{\partial V}$.

Since $pV=mRT$ then $p=\frac{mRT}{V}$

To find $\frac{\partial p}{\partial T}$, V is kept constant.

Hence $\frac{\partial p}{\partial T}=\left(\frac{mR}{V}\right)\frac{\mathrm{d}}{\mathrm{d}T}(T)=\boldsymbol{\frac{mR}{V}}$

To find $\frac{\partial p}{\partial V}$, T is kept constant.

Hence $\frac{\partial p}{\partial V}=(mRT)\frac{\mathrm{d}}{\mathrm{d}V}\left(\frac{1}{V}\right)$

$$=(mRT)(-V^{-2})=\boldsymbol{\frac{-mRT}{V^2}}$$

Problem 6. The time of oscillation, t, of a pendulum is given by $t=2\pi\sqrt{\frac{l}{g}}$ where l is the length of the pendulum and g the free fall acceleration due to gravity. Determine $\frac{\partial t}{\partial l}$ and $\frac{\partial t}{\partial g}$

To find $\frac{\partial t}{\partial l}$, g is kept constant.

$$t = 2\pi\sqrt{\frac{l}{g}} = \left(\frac{2\pi}{\sqrt{g}}\right)\sqrt{l} = \left(\frac{2\pi}{\sqrt{g}}\right)l^{\frac{1}{2}}$$

Hence $\frac{\partial t}{\partial l} = \left(\frac{2\pi}{\sqrt{g}}\right)\frac{\mathrm{d}}{\mathrm{d}l}(l^{\frac{1}{2}}) = \left(\frac{2\pi}{\sqrt{g}}\right)\left(\frac{1}{2}l^{\frac{-1}{2}}\right)$

$$= \left(\frac{2\pi}{\sqrt{g}}\right)\left(\frac{1}{2\sqrt{l}}\right) = \frac{\pi}{\sqrt{lg}}$$

To find $\frac{\partial t}{\partial g}$, l is kept constant.

$$t = 2\pi\sqrt{\frac{l}{g}} = (2\pi\sqrt{l})\left(\frac{1}{\sqrt{g}}\right)$$

$$= (2\pi\sqrt{l})g^{\frac{-1}{2}}$$

Hence $\frac{\partial t}{\partial g} = (2\pi\sqrt{l})\left(-\frac{1}{2}g^{\frac{-3}{2}}\right)$

$$= (2\pi\sqrt{l})\left(\frac{-1}{2\sqrt{g^3}}\right)$$

$$= \frac{-\pi\sqrt{l}}{\sqrt{g^3}} = -\pi\sqrt{\frac{l}{g^3}}$$

Now try the following exercise

Exercise 138 Further problems on first order partial derivatives

In Problems 1 to 6, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

1. $z = 2xy$ $\left[\frac{\partial z}{\partial x} = 2y \quad \frac{\partial z}{\partial y} = 2x\right]$

2. $z = x^3 - 2xy + y^2$ $\left[\begin{array}{l}\frac{\partial z}{\partial x} = 3x^2 - 2y \\ \frac{\partial z}{\partial y} = -2x + 2y\end{array}\right]$

3. $z = \frac{x}{y}$ $\left[\begin{array}{l}\frac{\partial z}{\partial x} = \frac{1}{y} \\ \frac{\partial z}{\partial y} = \frac{-x}{y^2}\end{array}\right]$

4. $z = \sin(4x + 3y)$ $\left[\begin{array}{l}\frac{\partial z}{\partial x} = 4\cos(4x + 3y) \\ \frac{\partial z}{\partial y} = 3\cos(4x + 3y)\end{array}\right]$

5. $z = x^3y^2 - \frac{y}{x^2} + \frac{1}{y}$ $\left[\begin{array}{l}\frac{\partial z}{\partial x} = 3x^2y^2 + \frac{2y}{x^3} \\ \frac{\partial z}{\partial y} = 2x^3y - \frac{1}{x^2} - \frac{1}{y^2}\end{array}\right]$

6. $z = \cos 3x \sin 4y$ $\left[\begin{array}{l}\frac{\partial z}{\partial x} = -3\sin 3x \sin 4y \\ \frac{\partial z}{\partial y} = 4\cos 3x \cos 4y\end{array}\right]$

7. The volume of a cone of height h and base radius r is given by $V = \frac{1}{3}\pi r^2 h$. Determine $\frac{\partial V}{\partial h}$ and $\frac{\partial V}{\partial r}$

$$\left[\frac{\partial V}{\partial h} = \frac{1}{3}\pi r^2 \quad \frac{\partial V}{\partial r} = \frac{2}{3}\pi rh\right]$$

8. The resonant frequency f_r in a series electrical circuit is given by $f_r = \frac{1}{2\pi\sqrt{LC}}$. Show that $\frac{\partial f_r}{\partial L} = \frac{-1}{4\pi\sqrt{CL^3}}$

9. An equation resulting from plucking a string is:
$y = \sin\left(\frac{n\pi}{L}\right)x\left\{k\cos\left(\frac{n\pi b}{L}\right)t + c\sin\left(\frac{n\pi b}{L}\right)t\right\}$
Determine $\frac{\partial y}{\partial t}$ and $\frac{\partial y}{\partial x}$

$$\left[\begin{array}{l}\frac{\partial y}{\partial t} = \frac{n\pi b}{L}\sin\left(\frac{n\pi}{L}\right)x\left\{c\cos\left(\frac{n\pi b}{L}\right)t - k\sin\left(\frac{n\pi b}{L}\right)t\right\} \\ \frac{\partial y}{\partial x} = \frac{n\pi}{L}\cos\left(\frac{n\pi}{L}\right)x\left\{k\cos\left(\frac{n\pi b}{L}\right)t + c\sin\left(\frac{n\pi b}{L}\right)t\right\}\end{array}\right]$$

10. In a thermodynamic system, $k = Ae^{\frac{T\Delta S - \Delta H}{RT}}$, where R, k and A are constants.

Find (a) $\frac{\partial k}{\partial T}$ (b) $\frac{\partial A}{\partial T}$ (c) $\frac{\partial(\Delta S)}{\partial T}$ (d) $\frac{\partial(\Delta H)}{\partial T}$

$$\left[\begin{array}{ll} \text{(a)} & \dfrac{\partial k}{\partial T} = \dfrac{A\Delta H}{RT^2} e^{\frac{T\Delta S - \Delta H}{RT}} \\ \text{(b)} & \dfrac{\partial A}{\partial T} = -\dfrac{k\Delta H}{RT^2} e^{\frac{\Delta H - T\Delta S}{RT}} \\ \text{(c)} & \dfrac{\partial(\Delta S)}{\partial T} = -\dfrac{\Delta H}{T^2} \\ \text{(d)} & \dfrac{\partial(\Delta H)}{\partial T} = \Delta S - R\ln\left(\dfrac{k}{A}\right) \end{array}\right]$$

34.3 Second order partial derivatives

As with ordinary differentiation, where a differential coefficient may be differentiated again, a partial derivative may be differentiated partially again to give higher order partial derivatives.

(i) Differentiating $\frac{\partial V}{\partial r}$ of Section 34.2 with respect to r, keeping h constant, gives $\frac{\partial}{\partial r}\left(\frac{\partial V}{\partial r}\right)$ which is written as $\frac{\partial^2 V}{\partial r^2}$

Thus if $V = \pi r^2 h$,

then $\frac{\partial^2 V}{\partial r^2} = \frac{\partial}{\partial r}(2\pi rh) = \mathbf{2\pi h}$.

(ii) Differentiating $\frac{\partial V}{\partial h}$ with respect to h, keeping r constant, gives $\frac{\partial}{\partial h}\left(\frac{\partial V}{\partial h}\right)$ which is written as $\frac{\partial^2 V}{\partial h^2}$

Thus $\frac{\partial^2 V}{\partial h^2} = \frac{\partial}{\partial h}(\pi r^2) = \mathbf{0}$.

(iii) Differentiating $\frac{\partial V}{\partial h}$ with respect to r, keeping h constant, gives $\frac{\partial}{\partial r}\left(\frac{\partial V}{\partial h}\right)$ which is written as $\frac{\partial^2 V}{\partial r \partial h}$. Thus,

$$\frac{\partial^2 V}{\partial r \partial h} = \frac{\partial}{\partial r}\left(\frac{\partial V}{\partial h}\right) = \frac{\partial}{\partial r}(\pi r^2) = \mathbf{2\pi r}.$$

(iv) Differentiating $\frac{\partial V}{\partial r}$ with respect to h, keeping r constant, gives $\frac{\partial}{\partial h}\left(\frac{\partial V}{\partial r}\right)$, which is written as $\frac{\partial^2 V}{\partial h \partial r}$. Thus,

$$\frac{\partial^2 V}{\partial h \partial r} = \frac{\partial}{\partial h}\left(\frac{\partial V}{\partial r}\right) = \frac{\partial}{\partial h}(2\pi rh) = \mathbf{2\pi r}.$$

(v) $\frac{\partial^2 V}{\partial r^2}$, $\frac{\partial^2 V}{\partial h^2}$, $\frac{\partial^2 V}{\partial r \partial h}$ and $\frac{\partial^2 V}{\partial h \partial r}$ are examples of **second order partial derivatives**.

(vi) It is seen from (iii) and (iv) that $\frac{\partial^2 V}{\partial r \partial h} = \frac{\partial^2 V}{\partial h \partial r}$ and such a result is always true for continuous functions (i.e. a graph of the function which has no sudden jumps or breaks).

Second order partial derivatives are used in the solution of partial differential equations, in waveguide theory, in such areas of thermodynamics covering entropy and the continuity theorem, and when finding maxima, minima and saddle points for functions of two variables (see Chapter 36).

Problem 7. Given $z = 4x^2y^3 - 2x^3 + 7y^2$ find (a) $\frac{\partial^2 z}{\partial x^2}$ (b) $\frac{\partial^2 z}{\partial y^2}$ (c) $\frac{\partial^2 z}{\partial x \partial y}$ (d) $\frac{\partial^2 z}{\partial y \partial x}$

(a) $\frac{\partial z}{\partial x} = 8xy^3 - 6x^2$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}(8xy^3 - 6x^2)$$

$$= \mathbf{8y^3 - 12x}$$

(b) $\frac{\partial z}{\partial y} = 12x^2y^2 + 14y$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}(12x^2y^2 + 14y)$$

$$= \mathbf{24x^2y + 14}$$

(c) $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{\partial}{\partial x}(12x^2y^2 + 14y) = \mathbf{24xy^2}$

(d) $\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial x}\right) = \dfrac{\partial}{\partial y}(8xy^3 - 6x^2) = \mathbf{24xy^2}$

$\left[\text{It is noted that } \dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x}\right]$

Problem 8. Show that when $z = e^{-t}\sin\theta$, (a) $\dfrac{\partial^2 z}{\partial t^2} = -\dfrac{\partial^2 z}{\partial \theta^2}$, and (b) $\dfrac{\partial^2 z}{\partial t \partial \theta} = \dfrac{\partial^2 z}{\partial \theta \partial t}$

(a) $\dfrac{\partial z}{\partial t} = -e^{-t}\sin\theta$ and $\dfrac{\partial^2 z}{\partial t^2} = e^{-t}\sin\theta$

$\dfrac{\partial z}{\partial \theta} = e^{-t}\cos\theta$ and $\dfrac{\partial^2 z}{\partial \theta^2} = -e^{-t}\sin\theta$

Hence $\dfrac{\partial^2 z}{\partial t^2} = -\dfrac{\partial^2 z}{\partial \theta^2}$

(b) $\dfrac{\partial^2 z}{\partial t \partial \theta} = \dfrac{\partial}{\partial t}\left(\dfrac{\partial z}{\partial \theta}\right) = \dfrac{\partial}{\partial t}(e^{-t}\cos\theta)$

$= -e^{-t}\cos\theta$

$\dfrac{\partial^2 z}{\partial \theta \partial t} = \dfrac{\partial}{\partial \theta}\left(\dfrac{\partial z}{\partial t}\right) = \dfrac{\partial}{\partial \theta}(-e^{-t}\sin\theta)$

$= -e^{-t}\cos\theta$

Hence $\dfrac{\partial^2 z}{\partial t \partial \theta} = \dfrac{\partial^2 z}{\partial \theta \partial t}$

Problem 9. Show that if $z = \dfrac{x}{y}\ln y$, then (a) $\dfrac{\partial z}{\partial y} = x\dfrac{\partial^2 z}{\partial y \partial x}$ and (b) evaluate $\dfrac{\partial^2 z}{\partial y^2}$ when $x = -3$ and $y = 1$.

(a) To find $\dfrac{\partial z}{\partial x}$, y is kept constant.

Hence $\dfrac{\partial z}{\partial x} = \left(\dfrac{1}{y}\ln y\right)\dfrac{\mathrm{d}}{\mathrm{d}x}(x) = \dfrac{1}{y}\ln y$

To find $\dfrac{\partial z}{\partial y}$, x is kept constant.

Hence

$$\frac{\partial z}{\partial y} = (x)\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\ln y}{y}\right) = (x)\left\{\frac{(y)\left(\frac{1}{y}\right) - (\ln y)(1)}{y^2}\right\}$$

using the quotient rule

$$= x\left(\frac{1-\ln y}{y^2}\right) = \frac{x}{y^2}(1-\ln y)$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\ln y}{y}\right)$$

$$= \frac{(y)\left(\frac{1}{y}\right) - (\ln y)(1)}{y^2}$$

using the quotient rule

$$= \frac{1}{y^2}(1-\ln y)$$

Hence $\mathbf{x\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{x}{y^2}(1-\ln y) = \dfrac{\partial z}{\partial y}}$

(b) $\dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{\partial}{\partial y}\left\{\dfrac{x}{y^2}(1-\ln y)\right\}$

$$= (x)\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{1-\ln y}{y^2}\right)$$

$$= (x)\left\{\frac{(y^2)\left(-\frac{1}{y}\right) - (1-\ln y)(2y)}{y^4}\right\}$$

using the quotient rule

$$= \frac{x}{y^4}[-y - 2y + 2y\ln y]$$

$$= \frac{xy}{y^4}[-3 + 2\ln y] = \frac{x}{y^3}(2\ln y - 3)$$

When $x = -3$ and $y = 1$,

$$\frac{\partial^2 z}{\partial y^2} = \frac{(-3)}{(1)^3}(2\ln 1 - 3) = (-3)(-3) = \mathbf{9}$$

Now try the following exercise

Exercise 139 Further problems on second order partial derivatives

In Problems 1 to 4, find (a) $\dfrac{\partial^2 z}{\partial x^2}$ (b) $\dfrac{\partial^2 z}{\partial y^2}$ (c) $\dfrac{\partial^2 z}{\partial x \partial y}$ (d) $\dfrac{\partial^2 z}{\partial y \partial x}$

1. $z = (2x - 3y)^2$ $\quad\left[\begin{matrix}\text{(a)} & 8 & \text{(b)} & 18\\ \text{(c)} & -12 & \text{(d)} & -12\end{matrix}\right]$

2. $z = 2\ln xy$

$$\begin{bmatrix} (a)\ \dfrac{-2}{x^2} & (b)\ \dfrac{-2}{y^2} \\ (c)\ 0 & (d)\ 0 \end{bmatrix}$$

3. $z = \dfrac{(x-y)}{(x+y)}$

$$\begin{bmatrix} (a)\ \dfrac{-4y}{(x+y)^3} & (b)\ \dfrac{4x}{(x+y)^3} \\ (c)\ \dfrac{2(x-y)}{(x+y)^3} & (d)\ \dfrac{2(x-y)}{(x+y)^3} \end{bmatrix}$$

4. $z = \sinh x \cosh 2y$

$$\begin{bmatrix} (a)\ \sinh x \cosh 2y \\ (b)\ 4\sinh x \cosh 2y \\ (c)\ 2\cosh x \sinh 2y \\ (d)\ 2\cosh x \sinh 2y \end{bmatrix}$$

5. Given $z = x^2 \sin(x-2y)$ find (a) $\dfrac{\partial^2 z}{\partial x^2}$ and (b) $\dfrac{\partial^2 z}{\partial y^2}$

Show also that $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x}$
$= 2x^2 \sin(x-2y) - 4x\cos(x-2y)$.

$$\begin{bmatrix} (a)\ (2-x^2)\sin(x-2y) \\ \quad + 4x\cos(x-2y) \\ (b)\ -4x^2\sin(x-2y) \end{bmatrix}$$

6. Find $\dfrac{\partial^2 z}{\partial x^2}$, $\dfrac{\partial^2 z}{\partial y^2}$ and show that $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x}$ when $z = \cos^{-1}\dfrac{x}{y}$

$$\begin{bmatrix} (a)\ \dfrac{\partial^2 z}{\partial x^2} = \dfrac{-x}{\sqrt{(y^2-x^2)^3}}, \\ (b)\ \dfrac{\partial^2 z}{\partial y^2} = \dfrac{-x}{\sqrt{(y^2-x^2)}}\left\{\dfrac{1}{y^2} + \dfrac{1}{(y^2-x^2)}\right\} \\ (c)\ \dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{y}{\sqrt{(y^2-x^2)^3}} \end{bmatrix}$$

7. Given $z = \sqrt{\left(\dfrac{3x}{y}\right)}$ show that $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x}$ and evaluate $\dfrac{\partial^2 z}{\partial x^2}$ when $x = \dfrac{1}{2}$ and $y = 3$. $\left[-\dfrac{1}{\sqrt{2}}\right]$

8. An equation used in thermodynamics is the Benedict-Webb-Rubine equation of state for the expansion of a gas. The equation is:

$$p = \frac{RT}{V} + \left(B_0 RT - A_0 - \frac{C_0}{T^2}\right)\frac{1}{V^2} + (bRT - a)\frac{1}{V^3} + \frac{A\alpha}{V^6} + \frac{C\left(1+\frac{\gamma}{V^2}\right)}{T^2}\left(\frac{1}{V^3}\right)e^{-\frac{\gamma}{V^2}}$$

Show that $\dfrac{\partial^2 p}{\partial T^2}$
$= \dfrac{6}{V^2 T^4}\left\{\dfrac{C}{V}\left(1+\dfrac{\gamma}{V^2}\right)e^{-\frac{\gamma}{V^2}} - C_0\right\}$.

Chapter 35

Total differential, rates of change and small changes

35.1 Total differential

In Chapter 34, partial differentiation is introduced for the case where only one variable changes at a time, the other variables being kept constant. In practice, variables may all be changing at the same time.

If $z = f(u, v, w, \ldots)$, then the **total differential**, $\mathbf{dz}$, is given by the sum of the separate partial differentials of z,

$$\text{i.e. } \mathbf{dz} = \frac{\partial z}{\partial u}\mathbf{d}u + \frac{\partial z}{\partial v}\mathbf{d}v + \frac{\partial z}{\partial w}\mathbf{d}w + \cdots \qquad (1)$$

Problem 1. If $z = f(x, y)$ and $z = x^2y^3 + \dfrac{2x}{y} + 1$, determine the total differential, $\mathrm{d}z$.

The total differential is the sum of the partial differentials,

$$\text{i.e.} \quad \mathrm{d}z = \frac{\partial z}{\partial x}\mathrm{d}x + \frac{\partial z}{\partial y}\mathrm{d}y$$

$$\frac{\partial z}{\partial x} = 2xy^3 + \frac{2}{y} \quad \text{(i.e. } y \text{ is kept constant)}$$

$$\frac{\partial z}{\partial y} = 3x^2y^2\frac{2x}{y^2} \quad \text{(i.e. } x \text{ is kept constant)}$$

$$\text{Hence} \quad \mathbf{dz} = \left(\mathbf{2xy^3 + \frac{2}{y}}\right)\mathbf{d}x + \left(\mathbf{3x^2y^2 - \frac{2x}{y^2}}\right)\mathbf{d}y$$

Problem 2. If $z = f(u, v, w)$ and $z = 3u^2 - 2v + 4w^3v^2$ find the total differential, $\mathrm{d}z$.

The total differential

$$\mathrm{d}z = \frac{\partial z}{\partial u}\mathrm{d}u + \frac{\partial z}{\partial v}\mathrm{d}v + \frac{\partial z}{\partial w}\mathrm{d}w$$

$$\frac{\partial z}{\partial u} = 6u \text{ (i.e. } v \text{ and } w \text{ are kept constant)}$$

$$\frac{\partial z}{\partial v} = -2 + 8w^3v$$

(i.e. u and w are kept constant)

$$\frac{\partial z}{\partial w} = 12w^2v^2 \text{ (i.e. } u \text{ and } v \text{ are kept constant)}$$

Hence

$$\mathbf{dz = 6u\,du + (8vw^3 - 2)\,dv + (12v^2w^2)\,dw}$$

Problem 3. The pressure p, volume V and temperature T of a gas are related by $pV = kT$, where k is a constant. Determine the total differentials (a) $\mathrm{d}p$ and (b) $\mathrm{d}T$ in terms of p, V and T.

(a) Total differential $\mathrm{d}p = \dfrac{\partial p}{\partial T}\mathrm{d}T + \dfrac{\partial p}{\partial V}\mathrm{d}V$.

Since $pV = kT$ then $p = \dfrac{kT}{V}$

hence $\dfrac{\partial p}{\partial T} = \dfrac{k}{V}$ and $\dfrac{\partial p}{\partial V} = -\dfrac{kT}{V^2}$

Thus $\mathrm{d}p = \dfrac{k}{V}\mathrm{d}T - \dfrac{kT}{V^2}\mathrm{d}V$

Since $pV = kT, k = \dfrac{pV}{T}$

Hence $\mathrm{d}p = \dfrac{\left(\frac{pV}{T}\right)}{V}\mathrm{d}T - \dfrac{\left(\frac{pV}{T}\right)T}{V^2}\mathrm{d}V$

i.e. $\mathbf{d}p = \dfrac{p}{T}\mathbf{d}T - \dfrac{p}{V}\mathbf{d}V$

(b) Total differential $\mathrm{d}T = \dfrac{\partial T}{\partial p}\mathrm{d}p + \dfrac{\partial T}{\partial V}\mathrm{d}V$

Since $pV = kT, T = \dfrac{pV}{k}$

hence $\dfrac{\partial T}{\partial p} = \dfrac{V}{k}$ and $\dfrac{\partial T}{\partial V} = \dfrac{p}{k}$

Thus $\mathrm{d}T = \dfrac{V}{k}\mathrm{d}p + \dfrac{p}{k}\mathrm{d}V$ and substituting $k = \dfrac{pV}{T}$ gives:

$$\mathrm{d}T = \frac{V}{\left(\frac{pV}{T}\right)}\mathrm{d}p + \frac{p}{\left(\frac{pV}{T}\right)}\mathrm{d}V$$

i.e. $\mathbf{d}T = \dfrac{T}{p}\mathbf{d}p + \dfrac{T}{V}\mathbf{d}V$

Now try the following exercise

Exercise 140 Further problems on the total differential

In Problems 1 to 5, find the total differential $\mathrm{d}z$.

1. $z = x^3 + y^2$ $[3x^2\,\mathrm{d}x + 2y\,\mathrm{d}y]$

2. $z = 2xy - \cos x$ $[(2y + \sin x)\,\mathrm{d}x + 2x\,\mathrm{d}y]$

3. $z = \dfrac{x-y}{x+y}$ $\left[\dfrac{2y}{(x+y)^2}\mathrm{d}x - \dfrac{2x}{(x+y)^2}\mathrm{d}y\right]$

4. $z = x \ln y$ $\left[\ln y\,\mathrm{d}x + \dfrac{x}{y}\mathrm{d}y\right]$

5. $z = xy + \dfrac{\sqrt{x}}{y} - 4$ $\left[\left(y + \dfrac{1}{2y\sqrt{x}}\right)\mathrm{d}x + \left(x - \dfrac{\sqrt{x}}{y^2}\right)\mathrm{d}y\right]$

6. If $z = f(a,b,c)$ and $z = 2ab - 3b^2c + abc$, find the total differential, $\mathrm{d}z$.
$[b(2+c)\,\mathrm{d}a + (2a - 6bc + ac)\,\mathrm{d}b + b(a-3b)\,\mathrm{d}c]$

7. Given $u = \ln\sin(xy)$ show that $\mathrm{d}u = \cot(xy)(y\,\mathrm{d}x + x\,\mathrm{d}y)$.

35.2 Rates of change

Sometimes it is necessary to solve problems in which different quantities have different rates of change. From equation (1), the rate of change of z, $\dfrac{\mathrm{d}z}{\mathrm{d}t}$ is given by:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial z}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{\partial z}{\partial w}\frac{\mathrm{d}w}{\mathrm{d}t} + \cdots \qquad (2)$$

Problem 4. If $z = f(x, y)$ and $z = 2x^3 \sin 2y$ find the rate of change of z, correct to 4 significant figures, when x is 2 units and y is $\pi/6$ radians and when x is increasing at 4 units/s and y is decreasing at 0.5 units/s.

Using equation (2), the rate of change of z,

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

Since $z = 2x^3 \sin 2y$, then

$$\frac{\partial z}{\partial x} = 6x^2 \sin 2y \text{ and } \frac{\partial z}{\partial y} = 4x^3 \cos 2y$$

Since x is increasing at 4 units/s, $\dfrac{\mathrm{d}x}{\mathrm{d}t} = +4$

and since y is decreasing at 0.5 units/s, $\dfrac{\mathrm{d}y}{\mathrm{d}t} = -0.5$

Hence $\dfrac{\mathrm{d}z}{\mathrm{d}t} = (6x^2 \sin 2y)(+4) + (4x^3 \cos 2y)(-0.5)$
$= 24x^2 \sin 2y - 2x^3 \cos 2y$

When $x = 2$ units and $y = \dfrac{\pi}{6}$ radians, then

$$\frac{\mathrm{d}z}{\mathrm{d}t} = 24(2)^2 \sin[2(\pi/6)] - 2(2)^3 \cos[2(\pi/6)]$$
$$= 83.138 - 8.0$$

Hence the rate of change of z, $\dfrac{dz}{dt} = \mathbf{75.14\,units/s}$, correct to 4 significant figures.

Problem 5. The height of a right circular cone is increasing at 3 mm/s and its radius is decreasing at 2 mm/s. Determine, correct to 3 significant figures, the rate at which the volume is changing (in cm^3/s) when the height is 3.2 cm and the radius is 1.5 cm.

Volume of a right circular cone, $V = \frac{1}{3}\pi r^2 h$

Using equation (2), the rate of change of volume,

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}$$

$$\frac{\partial V}{\partial r} = \frac{2}{3}\pi rh \text{ and } \frac{\partial V}{\partial h} = \frac{1}{3}\pi r^2$$

Since the height is increasing at 3 mm/s,

i.e. 0.3 cm/s, then $\dfrac{dh}{dt} = +0.3$

and since the radius is decreasing at 2 mm/s,

i.e. 0.2 cm/s, then $\dfrac{dr}{dt} = -0.2$

Hence
$$\frac{dV}{dt} = \left(\frac{2}{3}\pi rh\right)(-0.2) + \left(\frac{1}{3}\pi r^2\right)(+0.3)$$
$$= \frac{-0.4}{3}\pi rh + 0.1\pi r^2$$

However, $h = 3.2$ cm and $r = 1.5$ cm.

Hence
$$\frac{dV}{dt} = \frac{-0.4}{3}\pi(1.5)(3.2) + (0.1)\pi(1.5)^2$$
$$= -2.011 + 0.707 = -1.304\,cm^3/s$$

Thus the rate of change of volume is $\mathbf{1.30\,cm^3/s}$ decreasing.

Problem 6. The area A of a triangle is given by $A = \frac{1}{2}ac\sin B$, where B is the angle between sides a and c. If a is increasing at 0.4 units/s, c is decreasing at 0.8 units/s and B is increasing at 0.2 units/s, find the rate of change of the area of the triangle, correct to 3 significant figures, when a is 3 units, c is 4 units and B is $\pi/6$ radians.

Using equation (2), the rate of change of area,

$$\frac{dA}{dt} = \frac{\partial A}{\partial a}\frac{da}{dt} + \frac{\partial A}{\partial c}\frac{dc}{dt} + \frac{\partial A}{\partial B}\frac{dB}{dt}$$

Since $A = \frac{1}{2}ac\sin B$, $\dfrac{\partial A}{\partial a} = \frac{1}{2}c\sin B$,

$$\frac{\partial A}{\partial c} = \frac{1}{2}a\sin B \text{ and } \frac{\partial A}{\partial B} = \frac{1}{2}ac\cos B$$

$$\frac{da}{dt} = 0.4 \text{ units/s}, \frac{dc}{dt} = -0.8 \text{ units/s}$$

and $\dfrac{dB}{dt} = 0.2$ units/s

Hence
$$\frac{dA}{dt} = \left(\frac{1}{2}c\sin B\right)(0.4) + \left(\frac{1}{2}a\sin B\right)(-0.8) + \left(\frac{1}{2}ac\cos B\right)(0.2)$$

When $a = 3$, $c = 4$ and $B = \dfrac{\pi}{6}$ then:

$$\frac{\mathbf{dA}}{\mathbf{dt}} = \left(\frac{1}{2}(4)\sin\frac{\pi}{6}\right)(0.4) + \left(\frac{1}{2}(3)\sin\frac{\pi}{6}\right)(-0.8) + \left(\frac{1}{2}(3)(4)\cos\frac{\pi}{6}\right)(0.2)$$

$= 0.4 - 0.6 + 1.039 = \mathbf{0.839\,units^2/s}$, correct to 3 significant figures.

Problem 7. Determine the rate of increase of diagonal AC of the rectangular solid, shown in Fig. 35.1, correct to 2 significant figures, if the sides x, y and z increase at 6 mm/s, 5 mm/s and 4 mm/s when these three sides are 5 cm, 4 cm and 3 cm respectively.

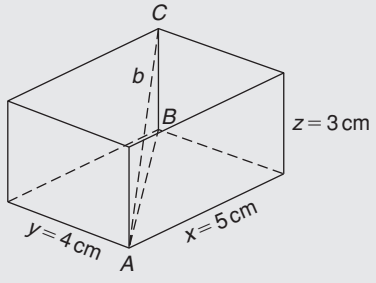


Figure 35.1

Diagonal $AB = \sqrt{(x^2 + y^2)}$

Diagonal $AC = \sqrt{(BC^2 + AB^2)}$

$$= \sqrt{[z^2 + \{\sqrt{(x^2 + y^2)}\}^2}$$

$$= \sqrt{(z^2 + x^2 + y^2)}$$

Let $AC = b$, then $b = \sqrt{(x^2 + y^2 + z^2)}$

Using equation (2), the rate of change of diagonal b is given by:

$$\frac{db}{dt} = \frac{\partial b}{\partial x}\frac{dx}{dt} + \frac{\partial b}{\partial y}\frac{dy}{dt} + \frac{\partial b}{\partial z}\frac{dz}{dt}$$

Since $b = \sqrt{(x^2+y^2+z^2)}$

$$\frac{\partial b}{\partial x} = \frac{1}{2}(x^2+y^2+z^2)^{\frac{-1}{2}}(2x) = \frac{x}{\sqrt{(x^2+y^2+z^2)}}$$

Similarly, $\dfrac{\partial b}{\partial y} = \dfrac{y}{\sqrt{(x^2+y^2+z^2)}}$

and $\dfrac{\partial b}{\partial z} = \dfrac{z}{\sqrt{(x^2+y^2+z^2)}}$

$\dfrac{dx}{dt} = 6\,\text{mm/s} = 0.6\,\text{cm/s}$,

$\dfrac{dy}{dt} = 5\,\text{mm/s} = 0.5\,\text{cm/s}$,

and $\dfrac{dz}{dt} = 4\,\text{mm/s} = 0.4\,\text{cm/s}$

Hence $$\frac{db}{dt} = \left[\frac{x}{\sqrt{(x^2+y^2+z^2)}}\right](0.6) + \left[\frac{y}{\sqrt{(x^2+y^2+z^2)}}\right](0.5) + \left[\frac{z}{\sqrt{(x^2+y^2+z^2)}}\right](0.4)$$

When $x = 5\,\text{cm}$, $y = 4\,\text{cm}$ and $z = 3\,\text{cm}$, then:

$$\frac{db}{dt} = \left[\frac{5}{\sqrt{(5^2+4^2+3^2)}}\right](0.6) + \left[\frac{4}{\sqrt{(5^2+4^2+3^2)}}\right](0.5) + \left[\frac{3}{\sqrt{(5^2+4^2+3^2)}}\right](0.4)$$

$$= 0.4243 + 0.2828 + 0.1697 = 0.8768\,\text{cm/s}$$

Hence the rate of increase of diagonal *AC* is 0.88 cm/s or 8.8 mm/s, correct to 2 significant figures.

Now try the following exercise

Exercise 141 Further problems on rates of change

1. The radius of a right cylinder is increasing at a rate of 8 mm/s and the height is decreasing at a rate of 15 mm/s. Find the rate at which the volume is changing in cm^3/s when the radius is 40 mm and the height is 150 mm. [$+226.2\,\text{cm}^3/\text{s}$]
2. If $z = f(x, y)$ and $z = 3x^2y^5$, find the rate of change of z when x is 3 units and y is 2 units when x is decreasing at 5 units/s and y is increasing at 2.5 units/s. [2520 units/s]
3. Find the rate of change of k, correct to 4 significant figures, given the following data: $k = f(a, b, c)$; $k = 2b\ln a + c^2e^a$; a is increasing at 2 cm/s; b is decreasing at 3 cm/s; c is decreasing at 1 cm/s; $a = 1.5$ cm, $b = 6$ cm and $c = 8$ cm. [515.5 cm/s]
4. A rectangular box has sides of length x cm, y cm and z cm. Sides x and z are expanding at rates of 3 mm/s and 5 mm/s respectively and side y is contracting at a rate of 2 mm/s. Determine the rate of change of volume when x is 3 cm, y is 1.5 cm and z is 6 cm. [$1.35\,\text{cm}^3/\text{s}$]
5. Find the rate of change of the total surface area of a right circular cone at the instant when the base radius is 5 cm and the height is 12 cm if the radius is increasing at 5 mm/s and the height is decreasing at 15 mm/s. [$17.4\,\text{cm}^2/\text{s}$]

35.3 Small changes

It is often useful to find an approximate value for the change (or error) of a quantity caused by small changes (or errors) in the variables associated with the quantity. If $z = f(u, v, w, \ldots)$ and $\delta u, \delta v, \delta w, \ldots$ denote **small changes** in $u, v, w, \ldots$ respectively, then the corresponding approximate change δz in z is obtained from equation (1) by replacing the differentials by the small changes.

Thus $$\delta z \approx \frac{\partial z}{\partial u}\delta u + \frac{\partial z}{\partial v}\delta v + \frac{\partial z}{\partial w}\delta w + \cdots \qquad (3)$$

Problem 8. Pressure p and volume V of a gas are connected by the equation $pV^{1.4}=k$. Determine the approximate percentage error in k when the pressure is increased by 4% and the volume is decreased by 1.5%.

Using equation (3), the approximate error in k,

$$\delta k \approx \frac{\partial k}{\partial p}\delta p + \frac{\partial k}{\partial V}\delta V$$

Let p, V and k refer to the initial values.

Since $k = pV^{1.4}$ then $\dfrac{\partial k}{\partial p} = V^{1.4}$

and $\dfrac{\partial k}{\partial V} = 1.4pV^{0.4}$

Since the pressure is increased by 4%, the change in pressure $\delta p = \dfrac{4}{100}\times p = 0.04p$.

Since the volume is decreased by 1.5%, the change in volume $\delta V = \dfrac{-1.5}{100}\times V = -0.015V$.

Hence the approximate error in k,

$$\delta k \approx (V)^{1.4}(0.04p) + (1.4pV^{0.4})(-0.015V)$$

$$\approx pV^{1.4}[0.04 - 1.4(0.015)]$$

$$\approx pV^{1.4}[0.019] \approx \frac{1.9}{100}pV^{1.4} \approx \frac{1.9}{100}k$$

i.e. **the approximate error in k is a 1.9% increase.**

Problem 9. Modulus of rigidity $G=(R^4\theta)/L$, where R is the radius, θ the angle of twist and L the length. Determine the approximate percentage error in G when R is increased by 2%, θ is reduced by 5% and L is increased by 4%.

Using $\delta G \approx \dfrac{\partial G}{\partial R}\delta R + \dfrac{\partial G}{\partial \theta}\delta\theta + \dfrac{\partial G}{\partial L}\delta L$

Since $G = \dfrac{R^4\theta}{L}$, $\dfrac{\partial G}{\partial R} = \dfrac{4R^3\theta}{L}$, $\dfrac{\partial G}{\partial \theta} = \dfrac{R^4}{L}$

and $\dfrac{\partial G}{\partial L} = \dfrac{-R^4\theta}{L^2}$

Since R is increased by 2%, $\delta R = \dfrac{2}{100}R = 0.02R$

Similarly, $\delta\theta = -0.05\theta$ and $\delta L = 0.04L$

Hence $\delta G \approx \left(\dfrac{4R^3\theta}{L}\right)(0.02R) + \left(\dfrac{R^4}{L}\right)(-0.05\theta) + \left(-\dfrac{R^4\theta}{L^2}\right)(0.04L)$

$$\approx \frac{R^4\theta}{L}[0.08 - 0.05 - 0.04] \approx -0.01\frac{R^4\theta}{L},$$

i.e. $\delta G \approx -\dfrac{1}{100}G$

Hence the approximate percentage error in G is a 1% decrease.

Problem 10. The second moment of area of a rectangle is given by $I=(bl^3)/3$. If b and l are measured as 40 mm and 90 mm respectively and the measurement errors are -5 mm in b and $+8$ mm in l, find the approximate error in the calculated value of I.

Using equation (3), the approximate error in I,

$$\delta I \approx \frac{\partial I}{\partial b}\delta b + \frac{\partial I}{\partial l}\delta l$$

$$\frac{\partial I}{\partial b} = \frac{l^3}{3} \text{ and } \frac{\partial I}{\partial l} = \frac{3bl^2}{3} = bl^2$$

$$\delta b = -5\,\text{mm and } \delta l = +8\,\text{mm}$$

Hence $\delta I \approx \left(\dfrac{l^3}{3}\right)(-5) + (bl^2)(+8)$

Since $b=40$ mm and $l=90$ mm then

$$\delta I \approx \left(\frac{90^3}{3}\right)(-5) + 40(90)^2(8)$$

$$\approx -1215000 + 2592000$$

$$\approx 1377000\,\text{mm}^4 \approx 137.7\,\text{cm}^4$$

Hence the approximate error in the calculated value of I is a 137.7 cm^4 increase.

Problem 11. The time of oscillation t of a pendulum is given by $t=2\pi\sqrt{\dfrac{l}{g}}$. Determine the approximate percentage error in t when l has an error of 0.2% too large and g 0.1% too small.

Using equation (3), the approximate change in t,

$$\delta t \approx \frac{\partial t}{\partial l}\delta l + \frac{\partial t}{\partial g}\delta g$$

Since $t = 2\pi\sqrt{\frac{l}{g}}$, $\frac{\partial t}{\partial l} = \frac{\pi}{\sqrt{lg}}$

and $\frac{\partial t}{\partial g} = -\pi\sqrt{\frac{l}{g^3}}$ (from Problem 6, Chapter 34)

$$\delta l = \frac{0.2}{100}l = 0.002l \text{ and } \delta g = -0.001g$$

hence $\delta t \approx \frac{\pi}{\sqrt{lg}}(0.002l) + -\pi\sqrt{\frac{l}{g^3}}(-0.001\,g)$

$$\approx 0.002\pi\sqrt{\frac{l}{g}} + 0.001\pi\sqrt{\frac{l}{g}}$$

$$\approx (0.001)\left[2\pi\sqrt{\frac{l}{g}}\right] + 0.0005\left[2\pi\sqrt{\frac{l}{g}}\right]$$

$$\approx 0.0015t \approx \frac{0.15}{100}t$$

Hence the approximate error in t is a 0.15% increase.

Now try the following exercise

Exercise 142 Further problems on small changes

1. The power P consumed in a resistor is given by $P = V^2/R$ watts. Determine the approximate change in power when V increases by 5% and R decreases by 0.5% if the original values of V and R are 50 volts and 12.5 ohms respectively. [+21 watts]
2. An equation for heat generated H is $H = i^2Rt$. Determine the error in the calculated value of H if the error in measuring current i is +2%, the error in measuring resistance R is −3% and the error in measuring time t is +1%. [+2%]
3. $f_r = \frac{1}{2\pi\sqrt{LC}}$ represents the resonant frequency of a series connected circuit containing inductance L and capacitance C. Determine the approximate percentage change in f_r when L is decreased by 3% and C is increased by 5%. [−1%]
4. The second moment of area of a rectangle about its centroid parallel to side b is given by $I = bd^3/12$. If b and d are measured as 15 cm and 6 cm respectively and the measurement errors are +12 mm in b and −1.5 mm in d, find the error in the calculated value of I. [+1.35 cm^4]
5. The side b of a triangle is calculated using $b^2 = a^2 + c^2 - 2ac\cos B$. If a, c and B are measured as 3 cm, 4 cm and $\pi/4$ radians respectively and the measurement errors which occur are +0.8 cm, −0.5 cm and $+\pi/90$ radians respectively, determine the error in the calculated value of b. [−0.179 cm]
6. Q factor in a resonant electrical circuit is given by: $Q = \frac{1}{R}\sqrt{\frac{L}{C}}$. Find the percentage change in Q when L increases by 4%, R decreases by 3% and C decreases by 2%. [+6%]
7. The rate of flow of gas in a pipe is given by: $v = \frac{C\sqrt{d}}{\sqrt[6]{T^5}}$, where C is a constant, d is the diameter of the pipe and T is the thermodynamic temperature of the gas. When determining the rate of flow experimentally, d is measured and subsequently found to be in error by +1.4%, and T has an error of −1.8%. Determine the percentage error in the rate of flow based on the measured values of d and T. [+2.2%]

Chapter 36

Maxima, minima and saddle points for functions of two variables

36.1 Functions of two independent variables

If a relation between two real variables, x and y, is such that when x is given, y is determined, then y is said to be a function of x and is denoted by $y = f(x)$; x is called the independent variable and y the dependent variable. If $y = f(u, v)$, then y is a function of two independent variables u and v. For example, if, say, $y = f(u, v) = 3u^2 - 2v$ then when $u = 2$ and $v = 1$, $y = 3(2)^2 - 2(1) = 10$. This may be written as $f(2, 1) = 10$. Similarly, if $u = 1$ and $v = 4$, $f(1, 4) = -5$.

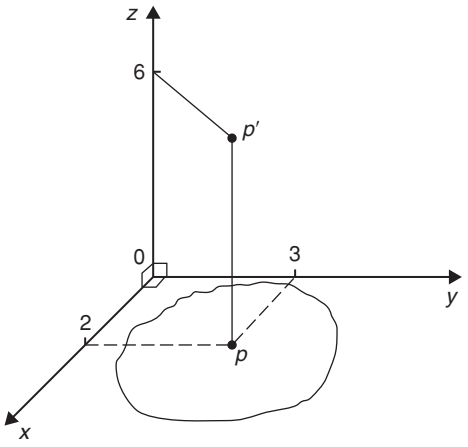


Figure 36.1

Consider a function of two variables x and y defined by $z = f(x, y) = 3x^2 - 2y$. If $(x, y) = (0, 0)$, then $f(0, 0) = 0$ and if $(x, y) = (2, 1)$, then $f(2, 1) = 10$. Each pair of numbers, (x, y), may be represented by a point P in the (x, y) plane of a rectangular Cartesian co-ordinate system as shown in Fig. 36.1. The corresponding value of $z = f(x, y)$ may be represented by a line PP' drawn parallel to the z-axis. Thus, if, for example, $z = 3x^2 - 2y$, as above, and P is the co-ordinate $(2, 3)$ then the length of PP' is $3(2)^2 - 2(3) = 6$. Figure 36.2 shows that when a large number of (x, y) co-ordinates are taken for a function

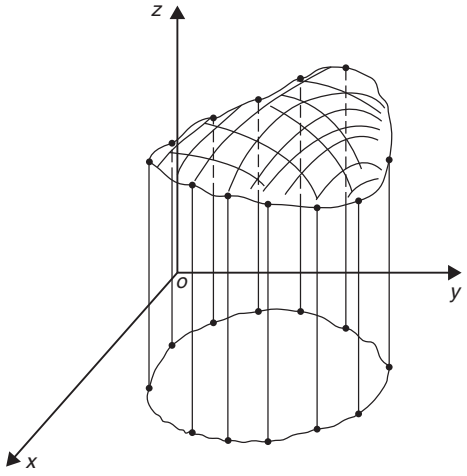


Figure 36.2

$f(x, y)$, and then $f(x, y)$ calculated for each, a large number of lines such as PP' can be constructed, and in the limit when all points in the (x, y) plane are considered, a surface is seen to result as shown in Fig. 36.2. Thus the function $z = f(x, y)$ represents a surface and not a curve.

36.2 Maxima, minima and saddle points

Partial differentiation is used when determining stationary points for functions of two variables. A function $f(x, y)$ is said to be a maximum at a point (x, y) if the value of the function there is greater than at all points in the immediate vicinity, and is a minimum if less than at all points in the immediate vicinity. Figure 36.3 shows geometrically a maximum value of a function of two variables and it is seen that the surface $z = f(x, y)$ is higher at $(x, y) = (a, b)$ than at any point in the immediate vicinity. Figure 36.4 shows a minimum value of a function of two variables and it is seen that the surface $z = f(x, y)$ is lower at $(x, y) = (p, q)$ than at any point in the immediate vicinity.

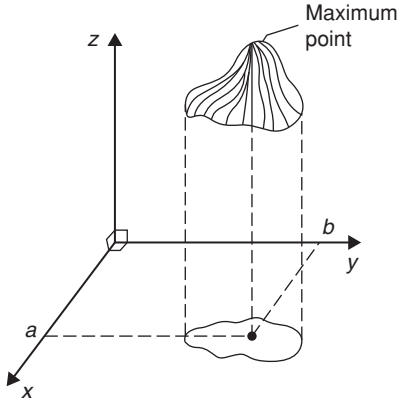


Figure 36.3

If $z = f(x, y)$ and a maximum occurs at (a, b), the curve lying in the two planes $x = a$ and $y = b$ must also have a maximum point (a, b) as shown in Fig. 36.5. Consequently, the tangents (shown as t_1 and t_2) to the curves at (a, b) must be parallel to Ox and Oy respectively. This requires that $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$ at all maximum and minimum values, and the solution of these equations gives the stationary (or critical) points of z.

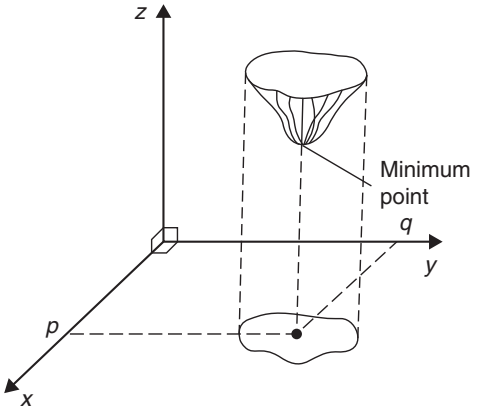


Figure 36.4

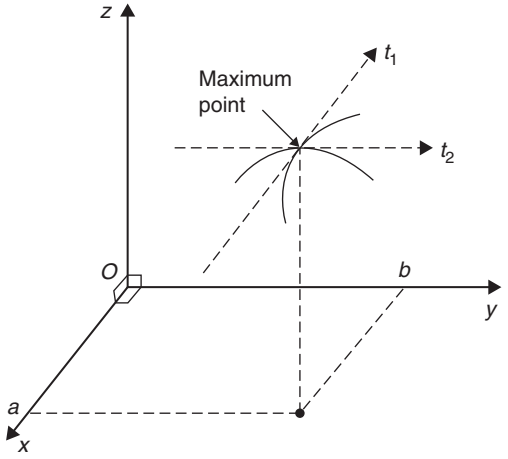


Figure 36.5

With functions of two variables there are three types of stationary points possible, these being a maximum point, a minimum point, and a **saddle point**. A saddle point Q is shown in Fig. 36.6 and is such that a point Q is a maximum for curve 1 and a minimum for curve 2.

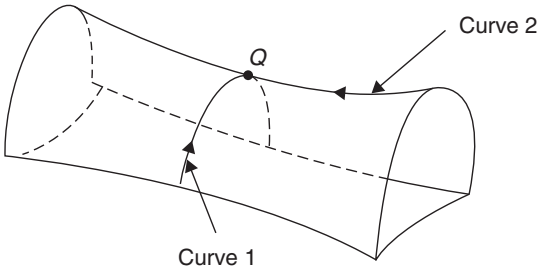


Figure 36.6

36.3 Procedure to determine maxima, minima and saddle points for functions of two variables

Given $z = f(x, y)$:

(i) determine $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$

(ii) for stationary points, $\dfrac{\partial z}{\partial x} = 0$ and $\dfrac{\partial z}{\partial y} = 0$,

(iii) solve the simultaneous equations $\dfrac{\partial z}{\partial x} = 0$ and $\dfrac{\partial z}{\partial y} = 0$ for x and y, which gives the co-ordinates of the stationary points,

(iv) determine $\dfrac{\partial^2 z}{\partial x^2}$, $\dfrac{\partial^2 z}{\partial y^2}$ and $\dfrac{\partial^2 z}{\partial x \partial y}$

(v) for each of the co-ordinates of the stationary points, substitute values of x and y into $\dfrac{\partial^2 z}{\partial x^2}$, $\dfrac{\partial^2 z}{\partial y^2}$ and $\dfrac{\partial^2 z}{\partial x \partial y}$ and evaluate each,

(vi) evaluate $\left(\dfrac{\partial^2 z}{\partial x \partial y}\right)^2$ for each stationary point,

(vii) substitute the values of $\dfrac{\partial^2 z}{\partial x^2}$, $\dfrac{\partial^2 z}{\partial y^2}$ and $\dfrac{\partial^2 z}{\partial x \partial y}$ into the equation

$$\Delta = \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right)$$

and evaluate,

(viii) (a) if $\boldsymbol{\Delta > 0}$ then the stationary point is a **saddle point.**

(b) if $\boldsymbol{\Delta < 0}$ and $\boldsymbol{\dfrac{\partial^2 z}{\partial x^2} < 0}$, then the stationary point is a **maximum point**,

and

(c) if $\boldsymbol{\Delta < 0}$ and $\boldsymbol{\dfrac{\partial^2 z}{\partial x^2} > 0}$, then the stationary point is a **minimum point**.

36.4 Worked problems on maxima, minima and saddle points for functions of two variables

Problem 1. Show that the function $z = (x-1)^2 + (y-2)^2$ has one stationary point only and determine its nature. Sketch the surface represented by z and produce a contour map in the x-y plane.

Following the above procedure:

(i) $\dfrac{\partial z}{\partial x} = 2(x-1)$ and $\dfrac{\partial z}{\partial y} = 2(y-2)$

(ii) $2(x-1) = 0$ (1)

$2(y-2) = 0$ (2)

(iii) From equations (1) and (2), $x = 1$ and $y = 2$, thus the only stationary point exists at (1, 2).

(iv) Since $\dfrac{\partial z}{\partial x} = 2(x-1) = 2x - 2$, $\dfrac{\partial^2 z}{\partial x^2} = 2$

and since $\dfrac{\partial z}{\partial y} = 2(y-2) = 2y - 4$, $\dfrac{\partial^2 z}{\partial y^2} = 2$

and $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{\partial}{\partial x}(2y-4) = 0$

(v) $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial^2 z}{\partial y^2} = 2$ and $\dfrac{\partial^2 z}{\partial x \partial y} = 0$

(vi) $\left(\dfrac{\partial^2 z}{\partial x \partial y}\right)^2 = 0$

(vii) $\Delta = (0)^2 - (2)(2) = -4$

(viii) Since $\Delta < 0$ and $\dfrac{\partial^2 z}{\partial x^2} > 0$, **the stationary point (1, 2) is a minimum**.

The surface $z = (x-1)^2 + (y-2)^2$ is shown in three dimensions in Fig. 36.7. Looking down towards the x-y plane from above, it is possible to produce a **contour map**. A contour is a line on a map which gives places having the same vertical height above a datum line (usually the mean sea-level on a geographical map).

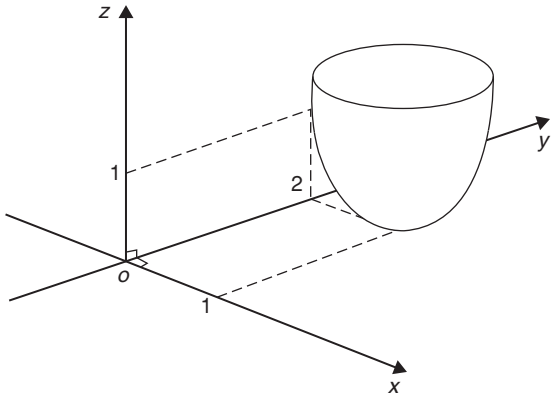


Figure 36.7

A contour map for $z=(x-1)^2+(y-2)^2$ is shown in Fig. 36.8. The values of z are shown on the map and these give an indication of the rise and fall to a stationary point.

Problem 2. Find the stationary points of the surface $f(x, y)=x^3-6xy+y^3$ and determine their nature.

Let $z=f(x,\ y)=x^3-6xy+y^3$

Following the procedure:

(i) $\dfrac{\partial z}{\partial x}=3x^2-6y$ and $\dfrac{\partial z}{\partial y}=-6x+3y^2$

(ii) for stationary points, $3x^2-6y=0$ (1)

and $-6x+3y^2=0$ (2)

(iii) from equation (1), $3x^2=6y$

and $y=\dfrac{3x^2}{6}=\dfrac{1}{2}x^2$

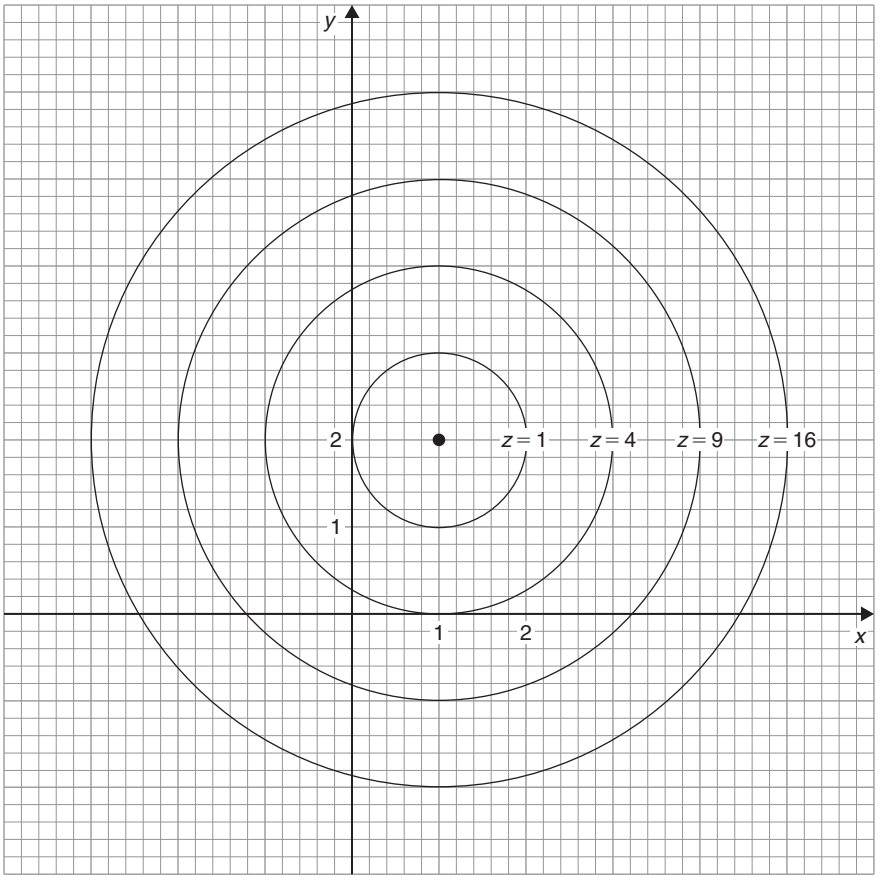


Figure 36.8

and substituting in equation (2) gives:

$$-6x+3\left(\frac{1}{2}x^2\right)^2=0$$

$$-6x+\frac{3}{4}x^4=0$$

$$3x\left(\frac{x^3}{4}-2\right)=0$$

from which, $x=0$ or $\frac{x^3}{4}-2=0$

i.e. $x^3=8$ and $x=2$

When $x=0$, $y=0$ and when $x=2$, $y=2$ from equations (1) and (2).

Thus stationary points occur at (0, 0) and (2, 2).

(iv) $\frac{\partial^2 z}{\partial x^2}=6x$, $\frac{\partial^2 z}{\partial y^2}=6y$ and $\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$

$$=\frac{\partial}{\partial x}(-6x+3y^2)=-6$$

(v) for (0, 0) $\frac{\partial^2 z}{\partial x^2}=0$, $\frac{\partial^2 z}{\partial y^2}=0$

and $\frac{\partial^2 z}{\partial x\partial y}=-6$

for (2, 2), $\frac{\partial^2 z}{\partial x^2}=12$, $\frac{\partial^2 z}{\partial y^2}=12$

and $\frac{\partial^2 z}{\partial x\partial y}=-6$

(vi) for (0, 0), $\left(\frac{\partial^2 z}{\partial x\partial y}\right)^2=(-6)^2=36$

for (2, 2), $\left(\frac{\partial^2 z}{\partial x\partial y}\right)^2=(-6)^2=36$

(vii) $\Delta_{(0,\,0)}=\left(\frac{\partial^2 z}{\partial x\partial y}\right)^2-\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right)$

$=36-(0)(0)=36$

$\Delta_{(2,\,2)}=36-(12)(12)=-108$

(viii) Since $\Delta_{(0,\,0)}>0$ then **(0, 0) is a saddle point.**

Since $\Delta_{(2,\,2)}<0$ and $\frac{\partial^2 z}{\partial x^2}>0$, then **(2, 2) is a minimum point.**

Now try the following exercise

Exercise 143 Further problems on maxima, minima and saddle points for functions of two variables

1. Find the stationary point of the surface $f(x, y)=x^2+y^2$ and determine its nature. Sketch the surface represented by z.
   [Minimum at (0, 0)]
2. Find the maxima, minima and saddle points for the following functions:
   (a) $f(x, y)=x^2+y^2-2x+4y+8$
   (b) $f(x, y)=x^2-y^2-2x+4y+8$
   (c) $f(x, y)=2x+2y-2xy-2x^2-y^2+4$.
   [(a) Minimum at (1, −2); (b) Saddle point at (1, 2); (c) Maximum at (0, 1)]
3. Determine the stationary values of the function $f(x, y)=x^3-6x^2-8y^2$ and distinguish between them. Sketch an approximate contour map to represent the surface $f(x, y)$.
   [Maximum point at (0, 0), saddle point at (4, 0)]
4. Locate the stationary point of the function $z=12x^2+6xy+15y^2$.
   [Minimum at (0, 0)]
5. Find the stationary points of the surface $z=x^3-xy+y^3$ and distinguish between them.
   [saddle point at (0, 0), minimum at $\left(\frac{1}{3}, \frac{1}{3}\right)$]

36.5 Further worked problems on maxima, minima and saddle points for functions of two variables

Problem 3. Find the co-ordinates of the stationary points on the surface

$$z=(x^2+y^2)^2-8(x^2-y^2)$$

and distinguish between them. Sketch the approximate contour map associated with z.

Following the procedure:

(i) $\dfrac{\partial z}{\partial x} = 2(x^2+y^2)2x - 16x$ and

$\dfrac{\partial z}{\partial y} = 2(x^2+y^2)2y + 16y$

(ii) for stationary points,

$2(x^2+y^2)2x - 16x = 0$

i.e. $4x^3 + 4xy^2 - 16x = 0$ (1)

and $2(x^2+y^2)2y + 16y = 0$

i.e. $4y(x^2+y^2+4) = 0$ (2)

(iii) From equation (1), $y^2 = \dfrac{16x - 4x^3}{4x} = 4 - x^2$

Substituting $y^2 = 4 - x^2$ in equation (2) gives

$4y(x^2 + 4 - x^2 + 4) = 0$

i.e. $32y = 0$ and $y = 0$

When $y = 0$ in equation (1), $4x^3 - 16x = 0$

i.e. $4x(x^2 - 4) = 0$

from which, $x = 0$ or $x = \pm 2$

The co-ordinates of the stationary points are (0, 0), (2, 0) and (−2, 0).

(iv) $\dfrac{\partial^2 z}{\partial x^2} = 12x^2 + 4y^2 - 16$,

$\dfrac{\partial^2 z}{\partial y^2} = 4x^2 + 12y^2 + 16$ and $\dfrac{\partial^2 z}{\partial x \partial y} = 8xy$

(v) For the point (0, 0),

$$\frac{\partial^2 z}{\partial x^2} = -16, \frac{\partial^2 z}{\partial y^2} = 16 \text{ and } \frac{\partial^2 z}{\partial x \partial y} = 0$$

For the point (2, 0),

$$\frac{\partial^2 z}{\partial x^2} = 32, \frac{\partial^2 z}{\partial y^2} = 32 \text{ and } \frac{\partial^2 z}{\partial x \partial y} = 0$$

For the point (−2, 0),

$$\frac{\partial^2 z}{\partial x^2} = 32, \frac{\partial^2 z}{\partial y^2} = 32 \text{ and } \frac{\partial^2 z}{\partial x \partial y} = 0$$

(vi) $\left(\dfrac{\partial^2 z}{\partial x \partial y}\right)^2 = 0$ for each stationary point

(vii) $\Delta_{(0,\,0)} = (0)^2 - (-16)(16) = 256$

$\Delta_{(2,\,0)} = (0)^2 - (32)(32) = -1024$

$\Delta_{(-2,\,0)} = (0)^2 - (32)(32) = -1024$

(viii) Since $\Delta_{(0,\,0)} > 0$, **the point (0, 0) is a saddle point**.

Since $\Delta_{(0,\,0)} < 0$ and $\left(\dfrac{\partial^2 z}{\partial x^2}\right)_{(2,\,0)} > 0$, **the point (2, 0) is a minimum point**.

Since $\Delta_{(-2,\,0)} < 0$ and $\left(\dfrac{\partial^2 z}{\partial x^2}\right)_{(-2,\,0)} > 0$, **the point (−2, 0) is a minimum point**.

Looking down towards the x-y plane from above, an approximate contour map can be constructed to represent the value of z. Such a map is shown in Fig. 36.9. To produce a contour map requires a large number of x-y co-ordinates to be chosen and the values of z at each co-ordinate calculated. Here are a few examples of points used to construct the contour map.

When $z = 0$, $0 = (x^2+y^2)^2 - 8(x^2 - y)^2$

In addition, when, say, $y = 0$ (i.e. on the x-axis)

$$0 = x^4 - 8x^2, \text{ i.e. } x^2(x^2 - 8) = 0$$

from which, $x = 0$ or $x = \pm\sqrt{8}$

Hence the contour $z = 0$ crosses the x-axis at 0 and $\pm\sqrt{8}$, i.e. at co-ordinates (0, 0), (2.83, 0) and (−2.83, 0) shown as points, S, a and b respectively.

When $z = 0$ and $x = 2$ then

$0 = (4 + y^2)^2 - 8(4 - y^2)$

i.e. $0 = 16 + 8y^2 + y^4 - 32 + 8y^2$

i.e. $0 = y^4 + 16y^2 - 16$

Let $y^2 = p$, then $p^2 + 16p - 16 = 0$ and

$$p = \frac{-16 \pm \sqrt{16^2 - 4(1)(-16)}}{2} = \frac{-16 \pm 17.89}{2}$$

$= 0.945$ or -16.945

Hence $y = \sqrt{p} = \sqrt{(0.945)}$ or $\sqrt{(-16.945)}$

$= \pm 0.97$ or complex roots.

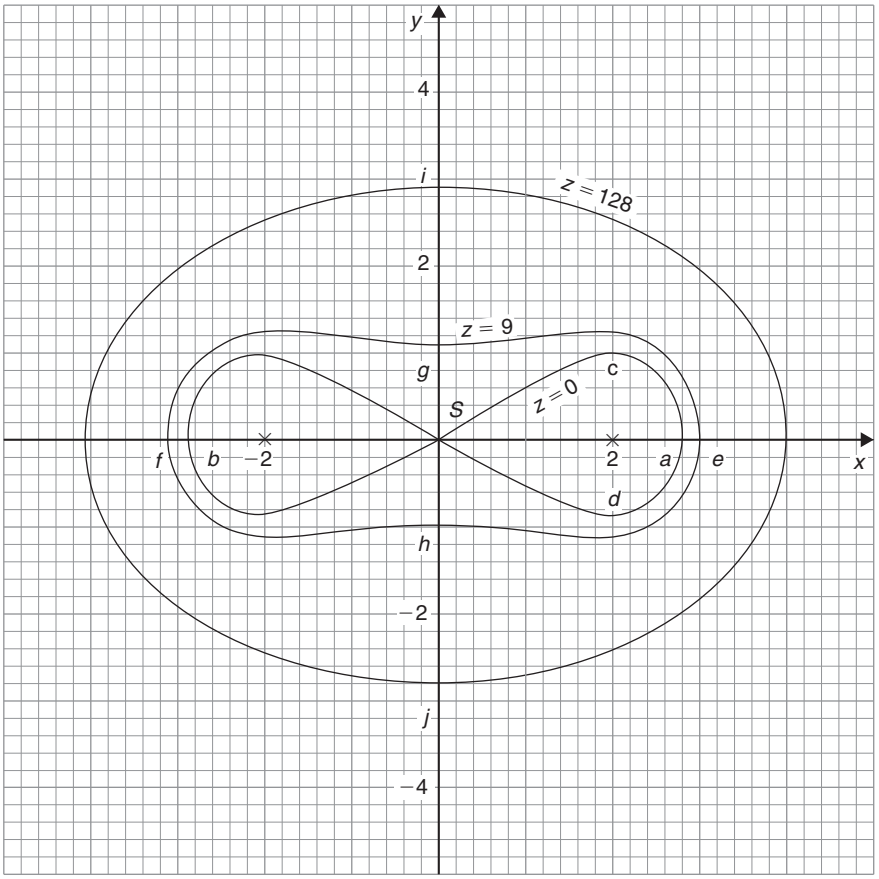


Figure 36.9

Hence the $z=0$ contour passes through the co-ordinates (2, 0.97) and (2, −0.97) shown as a c and d in Fig. 36.9.

Similarly, for the $\boldsymbol{z=9}$ contour, when $y=0$,

$$9=(x^2+0^2)^2-8(x^2-0^2)$$

i.e. $9=x^4-8x^2$

i.e. $x^4-8x^2-9=0$

Hence $(x^2-9)(x^2+1)=0$.

from which, $x=\pm 3$ or complex roots.

Thus the $z=9$ contour passes through (3, 0) and (−3, 0), shown as e and f in Fig. 36.9.

If $z=9$ and $x=0, 9 = y^4+8y^2$

i.e. $y^4+8y^2-9=0$

i.e. $(y^2+9)(y^2-1)=0$

from which, $y=\pm 1$ or complex roots.

Thus the $z=9$ contour also passes through (0, 1) and (0, −1), shown as g and h in Fig. 36.9.

When, say, $x=4$ and $y=0$,

$z=(4^2)^2-8(4^2)=128$.

when $z=128$ and $x=0$, $128=y^4+8y^2$

i.e. $y^4+8y^2-128=0$

i.e. $(y^2+16)(y^2-8)=0$

from which, $y=\pm\sqrt{8}$ or complex roots.

Thus the $z=128$ contour passes through (0, 2.83) and (0, −2.83), shown as i and j in Fig. 36.9.

In a similar manner many other points may be calculated with the resulting approximate contour map shown in Fig. 36.9. It is seen that two 'hollows' occur at the minimum points, and a 'cross-over' occurs at the saddle point S, which is typical of such contour maps.

Problem 4. Show that the function

$$f(x, y)=x^3-3x^2-4y^2+2$$

has one saddle point and one maximum point. Determine the maximum value.

Let $z = f(x, y) = x^3 - 3x^2 - 4y^2 + 2$.

Following the procedure:

(i) $\dfrac{\partial z}{\partial x} = 3x^2 - 6x$ and $\dfrac{\partial z}{\partial y} = -8y$

(ii) for stationary points, $3x^2 - 6x = 0$ (1)

and $-8y = 0$ (2)

(iii) From equation (1), $3x(x-2) = 0$ from which, $x = 0$ and $x = 2$.

From equation (2), $y = 0$.

Hence the stationary points are (0, 0) and (2, 0).

(iv) $\dfrac{\partial^2 z}{\partial x^2} = 6x - 6$, $\dfrac{\partial^2 z}{\partial y^2} = -8$ and $\dfrac{\partial^2 z}{\partial x \partial y} = 0$

(v) For the point (0, 0),

$$\frac{\partial^2 z}{\partial x^2} = -6, \frac{\partial^2 z}{\partial y^2} = -8 \text{ and } \frac{\partial^2 z}{\partial x \partial y} = 0$$

For the point (2, 0),

$$\frac{\partial^2 z}{\partial x^2} = 6, \frac{\partial^2 z}{\partial y^2} = -8 \text{ and } \frac{\partial^2 z}{\partial x \partial y} = 0$$

(vi) $\left(\dfrac{\partial^2 z}{\partial x \partial y}\right)^2 = (0)^2 = 0$

(vii) $\Delta_{(0,\,0)} = 0 - (-6)(-8) = -48$
$\Delta_{(2,\,0)} = 0 - (6)(-8) = 48$

(viii) Since $\Delta_{(0,\,0)} < 0$ and $\left(\dfrac{\partial^2 z}{\partial x^2}\right)_{(0,\,0)} < 0$, **the point (0, 0) is a maximum point** and hence **the maximum value is 0.**

Since $\Delta_{(2,\,0)} > 0$, **the point (2, 0) is a saddle point**.

The value of z at the saddle point is $2^3 - 3(2)^2 - 4(0)^2 + 2 = -2$.

An approximate contour map representing the surface $f(x, y)$ is shown in Fig. 36.10 where a 'hollow effect' is seen surrounding the maximum point and a 'cross-over' occurs at the saddle point S.

Problem 5. An open rectangular container is to have a volume of 62.5 m^3. Determine the least surface area of material required.

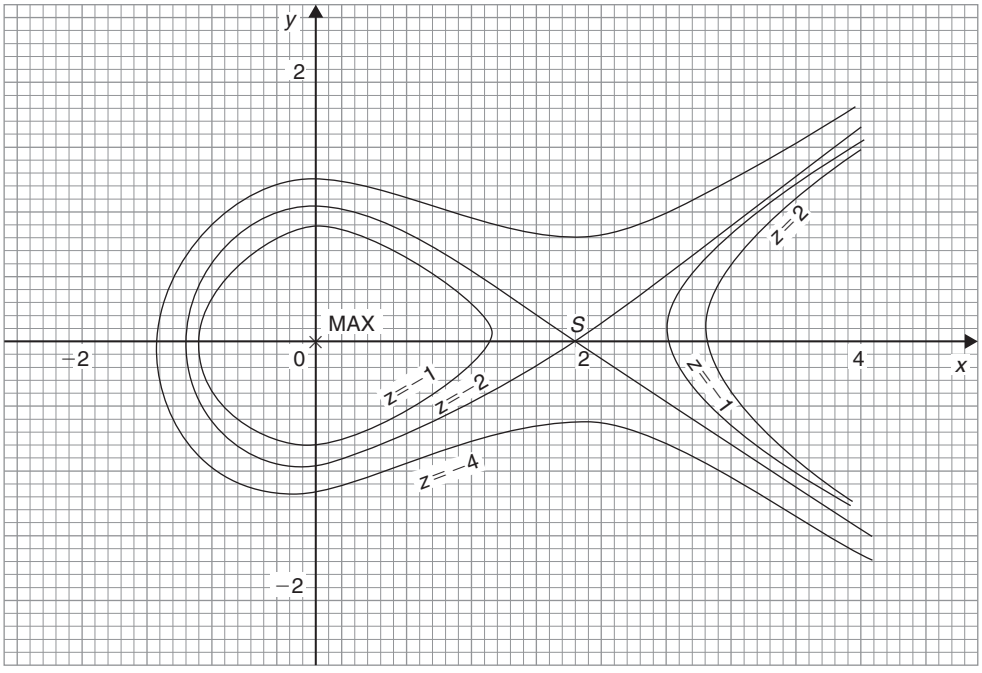


Figure 36.10

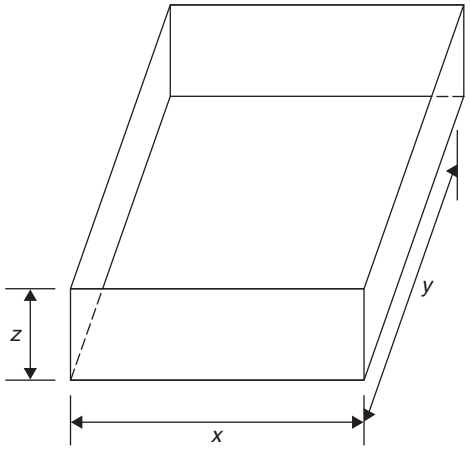


Figure 36.11

Let the dimensions of the container be x, y and z as shown in Fig. 36.11.

Volume $V = xyz = 62.5$ (1)

Surface area, $S = xy + 2yz + 2xz$ (2)

From equation (1), $z = \dfrac{62.5}{xy}$

Substituting in equation (2) gives:

$$S = xy + 2y\left(\frac{62.5}{xy}\right) + 2x\left(\frac{62.5}{xy}\right)$$

i.e. $$S = xy + \frac{125}{x} + \frac{125}{y}$$

which is a function of two variables

$$\frac{\partial s}{\partial x} = y - \frac{125}{x^2} = 0 \text{ for a stationary point,}$$

hence $x^2 y = 125$ (3)

$$\frac{\partial s}{\partial y} = x - \frac{125}{y^2} = 0 \text{ for a stationary point,}$$

hence $xy^2 = 125$ (4)

Dividing equation (3) by (4) gives:

$$\frac{x^2 y}{xy^2} = 1, \text{ i.e. } \frac{x}{y} = 1, \text{ i.e. } x = y$$

Substituting $y = x$ in equation (3) gives $x^3 = 125$, from which, $x = 5$ m.

Hence $y = 5$ m also

From equation (1), (5) (5) $z = 62.5$

from which, $$z = \frac{62.5}{25} = 2.5 \text{ m}$$

$$\frac{\partial^2 S}{\partial x^2} = \frac{250}{x^3}, \frac{\partial^2 S}{\partial y^2} = \frac{250}{y^3} \text{ and } \frac{\partial^2 S}{\partial x \partial y} = 1$$

When $x = y = 5$, $\dfrac{\partial^2 S}{\partial x^2} = 2$, $\dfrac{\partial^2 S}{\partial y^2} = 2$ and $\dfrac{\partial^2 S}{\partial x \partial y} = 1$

$$\Delta = (1)^2 - (2)(2) = -3$$

Since $\Delta < 0$ and $\dfrac{\partial^2 S}{\partial x^2} > 0$, then the surface area S is a **minimum**.

Hence the minimum dimensions of the container to have a volume of 62.5 m^3 are **5 m by 5 m by 2.5 m**.

From equation (2), **minimum surface area, *S***

$$= (5)(5) + 2(5)(2.5) + 2(5)(2.5)$$

$$= \mathbf{75\,m^2}$$

Now try the following exercise

Exercise 144 Further problems on maxima, minima and saddle points for functions of two variables

1. The function $z = x^2 + y^2 + xy + 4x - 4y + 3$ has one stationary value. Determine its co-ordinates and its nature.

   [Minimum at $(-4, 4)$]

2. An open rectangular container is to have a volume of 32 m^3. Determine the dimensions and the total surface area such that the total surface area is a minimum.

   [4 m by 4 m by 2 m, surface area = 48m^2]

3. Determine the stationary values of the function

   $$f(x, y) = x^4 + 4x^2 y^2 - 2x^2 + 2y^2 - 1$$

   and distinguish between them.

   [Minimum at $(1, 0)$, minimum at $(-1, 0)$, saddle point at $(0, 0)$]

4. Determine the stationary points of the surface $f(x, y) = x^3 - 6x^2 - y^2$.

[Maximum at (0, 0), saddle point at (4, 0)]

5. Locate the stationary points on the surface

$$f(x, y) = 2x^3 + 2y^3 - 6x - 24y + 16$$

and determine their nature.

[Minimum at (1, 2), maximum at (−1, −2), saddle points at (1, −2) and (−1, 2)]

6. A large marquee is to be made in the form of a rectangular box-like shape with canvas covering on the top, back and sides. Determine the minimum surface area of canvas necessary if the volume of the marquee is to the 250 m^3. [150 m^2]

Revision Test 10

This Revision Test covers the material contained in Chapters 32 to 36. *The marks for each question are shown in brackets at the end of each question.*

1. Differentiate the following functions with respect to x:

   (a) $5\ln(\text{sh}x)$ (b) $3\text{ch}^3 2x$

   (c) $e^{2x}\,\text{sech}\,2x$ (7)

2. Differentiate the following functions with respect to the variable:

   (a) $y = \frac{1}{5}\cos^{-1}\frac{x}{2}$

   (b) $y = 3e^{\sin^{-1} t}$

   (c) $y = \frac{2\sec^{-1} 5x}{x}$

   (d) $y = 3\sinh^{-1}\sqrt{(2x^2-1)}$ (14)

3. Evaluate the following, each correct to 3 decimal places:

   (a) $\sinh^{-1} 3$ (b) $\cosh^{-1} 2.5$ (c) $\tanh^{-1} 0.8$ (6)

4. If $z = f(x, y)$ and $z = x\cos(x + y)$ determine

   $$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x \partial y} \text{ and } \frac{\partial^2 z}{\partial y \partial x}. \quad (12)$$

5. The magnetic field vector H due to a steady current I flowing around a circular wire of radius r and at a distance x from its centre is given by

   $$H = \pm\frac{I}{2}\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{r^2 + x^2}}\right)$$

   Show that $H = \pm\dfrac{r^2 I}{2\sqrt{(r^2 + x^2)^3}}$ (7)

6. If $xyz = c$, where c is constant, show that

   $$dz = -z\left(\frac{dx}{x} + \frac{dy}{y}\right) \quad (6)$$

7. An engineering function $z = f(x, y)$ and $z = e^{\frac{y}{2}}\ln(2x + 3y)$. Determine the rate of increase of z, correct to 4 significant figures, when $x = 2$ cm, $y = 3$ cm, x is increasing at 5 cm/s and y is increasing at 4 cm/s. (8)

8. The volume V of a liquid of viscosity coefficient η delivered after time t when passed through a tube of length L and diameter d by a pressure p is given by $V = \dfrac{pd^4 t}{128\eta L}$. If the errors in V, p and L are 1%, 2% and 3% respectively, determine the error in η. (8)

9. Determine and distinugish between the stationary values of the function

   $$f(x, y) = x^3 - 6x^2 - 8y^2$$

   and sketch an approximate contour map to represent the surface $f(x, y)$. (20)

10. An open, rectangular fish tank is to have a volume of 13.5 m^3. Determine the least surface area of glass required. (12)

Chapter 37

Standard integration

37.1 The process of integration

The process of integration reverses the process of differentiation. In differentiation, if $f(x)=2x^2$ then $f'(x)=4x$. Thus the integral of $4x$ is $2x^2$, i.e. integration is the process of moving from $f'(x)$ to $f(x)$. By similar reasoning, the integral of $2t$ is t^2.

Integration is a process of summation or adding parts together and an elongated S, shown as $\int$, is used to replace the words 'the integral of'. Hence, from above, $\int 4x = 2x^2$ and $\int 2t$ is t^2.

In differentiation, the differential coefficient $\frac{dy}{dx}$ indicates that a function of x is being differentiated with respect to x, the dx indicating that it is 'with respect to x'. In integration the variable of integration is shown by adding d (the variable) after the function to be integrated.

Thus $\int 4x\,dx$ means 'the integral of $4x$ with respect to x',

and $\int 2t\,dt$ means 'the integral of $2t$ with respect to t'.

As stated above, the differential coefficient of $2x^2$ is $4x$, hence $\int 4x\,dx = 2x^2$. However, the differential coefficient of $2x^2+7$ is also $4x$. Hence $\int 4x\,dx$ is also equal to $2x^2+7$. To allow for the possible presence of a constant, whenever the process of integration is performed, a constant 'c' is added to the result.

Thus $\int 4x\,dx = 2x^2 + c$ and $\int 2t\,dt = t^2 + c$

'c' is called the **arbitrary constant of integration**.

37.2 The general solution of integrals of the form ax^n

The general solution of integrals of the form $\int ax^n dx$, where a and n are constants is given by:

$$\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c$$

This rule is true when n is fractional, zero, or a positive or negative integer, with the exception of $n=-1$.

Using this rule gives:

(i) $$\int 3x^4\,dx = \frac{3x^{4+1}}{4+1} + c = \frac{3}{5}x^5 + c$$

(ii) $$\int \frac{2}{x^2}\,dx = \int 2x^{-2}\,dx = \frac{2x^{-2+1}}{-2+1} + c$$
$$= \frac{2x^{-1}}{-1} + c = \frac{-2}{x} + c, \text{ and}$$

(iii) $$\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$$
$$= \frac{2}{3}\sqrt{x^3} + c$$

Each of these three results may be checked by differentiation.

(a) The integral of a constant k is $kx+c$. For example,

$$\int 8\,dx = 8x + c$$

(b) When a sum of several terms is integrated the result is the sum of the integrals of the separate terms.

For example,

$$\int (3x + 2x^2 - 5)\,dx$$
$$= \int 3x\,dx + \int 2x^2\,dx - \int 5\,dx$$
$$= \frac{3x^2}{2} + \frac{2x^3}{3} - 5x + c$$

37.3 Standard integrals

Since integration is the reverse process of differentiation the **standard integrals** listed in Table 37.1 may be deduced and readily checked by differentiation.

Table 37.1 Standard integrals

(i)	$\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c$ (except when $n = -1$)
(ii)	$\int \cos ax\,dx = \frac{1}{a}\sin ax + c$
(iii)	$\int \sin ax\,dx = -\frac{1}{a}\cos ax + c$
(iv)	$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + c$
(v)	$\int \text{cosec}^2 ax\,dx = -\frac{1}{a}\cot ax + c$
(vi)	$\int \text{cosec}\,ax\cot ax\,dx = -\frac{1}{a}\text{cosec}\,ax + c$
(vii)	$\int \sec ax\tan ax\,dx = \frac{1}{a}\sec ax + c$
(viii)	$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$
(ix)	$\int \frac{1}{x}\,dx = \ln x + c$

Problem 1. Determine (a) $\int 5x^2\,dx$ (b) $\int 2t^3\,dt$.

The standard integral, $\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c$

(a) When $a = 5$ and $n = 2$ then

$$\int 5x^2\,dx = \frac{5x^{2+1}}{2+1} + c = \frac{5x^3}{3} + c$$

(b) When $a = 2$ and $n = 3$ then

$$\int 2t^3\,dt = \frac{2t^{3+1}}{3+1} + c = \frac{2t^4}{4} + c = \frac{1}{2}t^4 + c$$

Each of these results may be checked by differentiating them.

Problem 2. Determine

$$\int \left(4 + \frac{3}{7}x - 6x^2\right) dx.$$

$\int (4 + \frac{3}{7}x - 6x^2)\,dx$ may be written as $\int 4\,dx + \int \frac{3}{7}x\,dx - \int 6x^2\,dx$, i.e. each term is integrated separately. (This splitting up of terms only applies, however, for addition and subtraction.)

Hence $\int \left(4 + \frac{3}{7}x - 6x^2\right) dx$

$$= 4x + \left(\frac{3}{7}\right)\frac{x^{1+1}}{1+1} - (6)\frac{x^{2+1}}{2+1} + c$$
$$= 4x + \left(\frac{3}{7}\right)\frac{x^2}{2} - (6)\frac{x^3}{3} + c$$
$$= 4x + \frac{3}{14}x^2 - 2x^3 + c$$

Note that when an integral contains more than one term there is no need to have an arbitrary constant for each; just a single constant at the end is sufficient.

Problem 3. Determine

(a) $\int \frac{2x^3 - 3x}{4x}\,dx$ (b) $\int (1-t)^2\,dt$.

(a) Rearranging into standard integral form gives:

$$\int \frac{2x^3 - 3x}{4x}\,dx$$
$$= \int \frac{2x^3}{4x} - \frac{3x}{4x}\,dx = \int \frac{x^2}{2} - \frac{3}{4}\,dx$$
$$= \left(\frac{1}{2}\right)\frac{x^{2+1}}{2+1} - \frac{3}{4}x + c$$
$$= \left(\frac{1}{2}\right)\frac{x^3}{3} - \frac{3}{4}x + c = \frac{1}{6}x^3 - \frac{3}{4}x + c$$

(b) Rearranging $\int (1-t)^2\,dt$ gives:

$$\int (1-2t+t^2)\,dt = t - \frac{2t^{1+1}}{1+1} + \frac{t^{2+1}}{2+1} + c$$

$$= t - \frac{2t^2}{2} + \frac{t^3}{3} + c$$

$$= \boldsymbol{t - t^2 + \frac{1}{3}t^3 + c}$$

This problem shows that functions often have to be rearranged into the standard form of $\int ax^n\,dx$ before it is possible to integrate them.

Problem 4. Determine $\int \frac{3}{x^2}\,dx$.

$\int \frac{3}{x^2}\,dx = \int 3x^{-2}\,dx$. Using the standard integral, $\int ax^n\,dx$ when $a=3$ and $n=-2$ gives:

$$\int 3x^{-2}\,dx = \frac{3x^{-2+1}}{-2+1} + c = \frac{3x^{-1}}{-1} + c$$

$$= -3x^{-1} + c = \boldsymbol{\frac{-3}{x} + c}$$

Problem 5. Determine $\int 3\sqrt{x}\,dx$.

For fractional powers it is necessary to appreciate $\sqrt[n]{a^m} = a^{\frac{m}{n}}$

$$\int 3\sqrt{x}\,dx = \int 3x^{\frac{1}{2}}\,dx = \frac{3x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$$

$$= \frac{3x^{\frac{3}{2}}}{\frac{3}{2}} + c = 2x^{\frac{3}{2}} + c = \boldsymbol{2\sqrt{x^3} + c}$$

Problem 6. Determine $\int \frac{-5}{9\sqrt[4]{t^3}}\,dt$.

$$\int \frac{-5}{9\sqrt[4]{t^3}}\,dt = \int \frac{-5}{9t^{\frac{3}{4}}}\,dt = \int \left(-\frac{5}{9}\right) t^{-\frac{3}{4}}\,dt$$

$$= \left(-\frac{5}{9}\right)\frac{t^{-\frac{3}{4}+1}}{-\frac{3}{4}+1} + c$$

$$= \left(-\frac{5}{9}\right)\frac{t^{\frac{1}{4}}}{\frac{1}{4}} + c = \left(-\frac{5}{9}\right)\left(\frac{4}{1}\right)t^{\frac{1}{4}} + c$$

$$= \boldsymbol{-\frac{20}{9}\sqrt[4]{t} + c}$$

Problem 7. Determine $\int \frac{(1+\theta)^2}{\sqrt{\theta}}\,d\theta$.

$$\int \frac{(1+\theta)^2}{\sqrt{\theta}}\,d\theta = \int \frac{(1+2\theta+\theta^2)}{\sqrt{\theta}}\,d\theta$$

$$= \int \left(\frac{1}{\theta^{\frac{1}{2}}} + \frac{2\theta}{\theta^{\frac{1}{2}}} + \frac{\theta^2}{\theta^{\frac{1}{2}}}\right)d\theta$$

$$= \int \left(\theta^{\frac{-1}{2}} + 2\theta^{1-\left(\frac{1}{2}\right)} + \theta^{2-\left(\frac{1}{2}\right)}\right)d\theta$$

$$= \int \left(\theta^{\frac{-1}{2}} + 2\theta^{\frac{1}{2}} + \theta^{\frac{3}{2}}\right)d\theta$$

$$= \frac{\theta^{\left(\frac{-1}{2}\right)+1}}{-\frac{1}{2}+1} + \frac{2\theta^{\left(\frac{1}{2}\right)+1}}{\frac{1}{2}+1} + \frac{\theta^{\left(\frac{3}{2}\right)+1}}{\frac{3}{2}+1} + c$$

$$= \frac{\theta^{\frac{1}{2}}}{\frac{1}{2}} + \frac{2\theta^{\frac{3}{2}}}{\frac{3}{2}} + \frac{\theta^{\frac{5}{2}}}{\frac{5}{2}} + c$$

$$= 2\theta^{\frac{1}{2}} + \frac{4}{3}\theta^{\frac{3}{2}} + \frac{2}{5}\theta^{\frac{5}{2}} + c$$

$$= \boldsymbol{2\sqrt{\theta} + \frac{4}{3}\sqrt{\theta^3} + \frac{2}{5}\sqrt{\theta^5} + c}$$

Problem 8. Determine
(a) $\int 4\cos 3x\,dx$ (b) $\int 5\sin 2\theta\,d\theta$.

(a) From Table 37.1(ii),

$$\int 4\cos 3x\,dx = (4)\left(\frac{1}{3}\right)\sin 3x + c$$

$$= \boldsymbol{\frac{4}{3}\sin 3x + c}$$

(b) From Table 37.1(iii),

$$\int 5\sin 2\theta\,d\theta = (5)\left(-\frac{1}{2}\right)\cos 2\theta + c$$

$$= \boldsymbol{-\frac{5}{2}\cos 2\theta + c}$$

Problem 9. Determine

(a) $\int 7\sec^2 4t\,\mathrm{d}t$ (b) $3\int \operatorname{cosec}^2 2\theta\,\mathrm{d}\theta$.

(a) From Table 37.1(iv),

$$\int 7\sec^2 4t\,\mathrm{d}t = (7)\left(\frac{1}{4}\right)\tan 4t + c$$

$$= \frac{7}{4}\tan 4t + c$$

(b) From Table 37.1(v),

$$3\int \operatorname{cosec}^2 2\theta\,\mathrm{d}\theta = (3)\left(-\frac{1}{2}\right)\cot 2\theta + c$$

$$= -\frac{3}{2}\cot 2\theta + c$$

Problem 10. Determine

(a) $\int 5\mathrm{e}^{3x}\,\mathrm{d}x$ (b) $\int \frac{2}{3\mathrm{e}^{4t}}\,\mathrm{d}t$.

(a) From Table 37.1(viii),

$$\int 5\mathrm{e}^{3x}\,\mathrm{d}x = (5)\left(\frac{1}{3}\right)\mathrm{e}^{3x} + c = \frac{5}{3}\mathrm{e}^{3x} + c$$

(b) $$\int \frac{2}{3\mathrm{e}^{4t}}\,\mathrm{d}t = \int \frac{2}{3}\mathrm{e}^{-4t}\,\mathrm{d}t = \left(\frac{2}{3}\right)\left(-\frac{1}{4}\right)\mathrm{e}^{-4t} + c$$

$$= -\frac{1}{6}\mathrm{e}^{-4t} + c = -\frac{1}{6\mathrm{e}^{4t}} + c$$

Problem 11. Determine

(a) $\int \frac{3}{5x}\,\mathrm{d}x$ (b) $\int \left(\frac{2m^2+1}{m}\right)\mathrm{d}m$.

(a) $$\int \frac{3}{5x}\,\mathrm{d}x = \int \left(\frac{3}{5}\right)\left(\frac{1}{x}\right)\mathrm{d}x = \frac{3}{5}\ln x + c$$

(from Table 37.1(ix))

(b) $$\int \left(\frac{2m^2+1}{m}\right)\mathrm{d}m = \int \left(\frac{2m^2}{m} + \frac{1}{m}\right)\mathrm{d}m$$

$$= \int \left(2m + \frac{1}{m}\right)\mathrm{d}m$$

$$= \frac{2m^2}{2} + \ln m + c$$

$$= m^2 + \ln m + c$$

Now try the following exercise

Exercise 145 Further problems on standard integrals

In Problems 1 to 12, determine the indefinite integrals.

1. (a) $\int 4\,\mathrm{d}x$ (b) $\int 7x\,\mathrm{d}x$

$\left[\text{(a) } 4x + c \quad \text{(b) } \frac{7x^2}{2} + c\right]$

2. (a) $\int \frac{2}{5}x^2\,\mathrm{d}x$ (b) $\int \frac{5}{6}x^3\,\mathrm{d}x$

$\left[\text{(a) } \frac{2}{15}x^3 + c \quad \text{(b) } \frac{5}{24}x^4 + c\right]$

3. (a) $\int \left(\frac{3x^2 - 5x}{x}\right)\mathrm{d}x$ (b) $\int (2+\theta)^2\,\mathrm{d}\theta$

$\left[\begin{array}{l}\text{(a) } \frac{3x^2}{2} - 5x + c \\ \text{(b) } 4\theta + 2\theta^2 + \frac{\theta^3}{3} + c\end{array}\right]$

4. (a) $\int \frac{4}{3x^2}\,\mathrm{d}x$ (b) $\int \frac{3}{4x^4}\,\mathrm{d}x$

$\left[\text{(a) } \frac{-4}{3x} + c \quad \text{(b) } \frac{-1}{4x^3} + c\right]$

5. (a) $2\int \sqrt{x^3}\,\mathrm{d}x$ (b) $\int \frac{1}{4}\sqrt[4]{x^5}\,\mathrm{d}x$

$\left[\text{(a) } \frac{4}{5}\sqrt{x^5} + c \quad \text{(b) } \frac{1}{9}\sqrt[4]{x^9} + c\right]$

6. (a) $\int \frac{-5}{\sqrt{t^3}}\,\mathrm{d}t$ (b) $\int \frac{3}{7\sqrt[5]{x^4}}\,\mathrm{d}x$

$\left[\text{(a) } \frac{10}{\sqrt{t}} + c \quad \text{(b) } \frac{15}{7}\sqrt[5]{x} + c\right]$

7. (a) $\int 3\cos 2x\,dx$ (b) $\int 7\sin 3\theta\,d\theta$

$$\left[\begin{array}{l}\text{(a) } \frac{3}{2}\sin 2x + c \\ \text{(b) } -\frac{7}{3}\cos 3\theta + c\end{array}\right]$$

8. (a) $\int \frac{3}{4}\sec^2 3x\,dx$ (b) $\int 2\,\text{cosec}^2 4\theta\,d\theta$

$$\left[\text{(a) } \frac{1}{4}\tan 3x + c \quad \text{(b) } -\frac{1}{2}\cot 4\theta + c\right]$$

9. (a) $5\int \cot 2t\,\text{cosec}\,2t\,dt$

(b) $\int \frac{4}{3}\sec 4t\tan 4t\,dt$

$$\left[\begin{array}{l}\text{(a) } -\frac{5}{2}\,\text{cosec}\,2t + c \\ \text{(b) } \frac{1}{3}\sec 4t + c\end{array}\right]$$

10. (a) $\int \frac{3}{4}e^{2x}\,dx$ (b) $\frac{2}{3}\int \frac{dx}{e^{5x}}$

$$\left[\text{(a) } \frac{3}{8}e^{2x} + c \quad \text{(b) } \frac{-2}{15e^{5x}} + c\right]$$

11. (a) $\int \frac{2}{3x}\,dx$ (b) $\int\left(\frac{u^2-1}{u}\right)du$

$$\left[\text{(a) } \frac{2}{3}\ln x + c \quad \text{(b) } \frac{u^2}{2} - \ln u + c\right]$$

12. (a) $\int \frac{(2+3x)^2}{\sqrt{x}}\,dx$ (b) $\int\left(\frac{1}{t} + 2t\right)^2 dt$

$$\left[\begin{array}{l}\text{(a) } 8\sqrt{x} + 8\sqrt{x^3} + \frac{18}{5}\sqrt{x^5} + c \\ \text{(b) } -\frac{1}{t} + 4t + \frac{4t^3}{3} + c\end{array}\right]$$

37.4 Definite integrals

Integrals containing an arbitrary constant c in their results are called **indefinite integrals** since their precise value cannot be determined without further information. **Definite integrals** are those in which limits are applied. If an expression is written as $[x]_a^b$, 'b' is called the **upper limit** and 'a' the **lower limit**. The operation of applying the limits is defined as $[x]_a^b = (b) - (a)$.

The increase in the value of the integral x^2 as x increases from 1 to 3 is written as $\int_1^3 x^2\,dx$.

Applying the limits gives:

$$\int_1^3 x^2\,dx = \left[\frac{x^3}{3} + c\right]_1^3 = \left(\frac{3^3}{3} + c\right) - \left(\frac{1^3}{3} + c\right)$$
$$= (9 + c) - \left(\frac{1}{3} + c\right) = \mathbf{8\frac{2}{3}}$$

Note that the 'c' term always cancels out when limits are applied and it need not be shown with definite integrals.

Problem 12. Evaluate

(a) $\int_1^2 3x\,dx$ (b) $\int_{-2}^3 (4 - x^2)\,dx$.

(a) $$\int_1^2 3x\,dx = \left[\frac{3x^2}{2}\right]_1^2 = \left\{\frac{3}{2}(2)^2\right\} - \left\{\frac{3}{2}(1)^2\right\}$$
$$= 6 - 1\frac{1}{2} = \mathbf{4\frac{1}{2}}$$

(b) $$\int_{-2}^3 (4 - x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_{-2}^3$$
$$= \left\{4(3) - \frac{(3)^3}{3}\right\} - \left\{4(-2) - \frac{(-2)^3}{3}\right\}$$
$$= \{12 - 9\} - \left\{-8 - \frac{-8}{3}\right\}$$
$$= \{3\} - \left\{-5\frac{1}{3}\right\} = \mathbf{8\frac{1}{3}}$$

Problem 13. Evaluate $\int_1^4 \left(\frac{\theta + 2}{\sqrt{\theta}}\right)d\theta$, taking positive square roots only.

$$\int_1^4 \left(\frac{\theta + 2}{\sqrt{\theta}}\right)d\theta = \int_1^4 \left(\frac{\theta}{\theta^{\frac{1}{2}}} + \frac{2}{\theta^{\frac{1}{2}}}\right)d\theta$$
$$= \int_1^4 \left(\theta^{\frac{1}{2}} + 2\theta^{\frac{-1}{2}}\right)d\theta$$
$$= \left[\frac{\theta^{\left(\frac{1}{2}\right)+1}}{\frac{1}{2}+1} + \frac{2\theta^{\left(\frac{-1}{2}\right)+1}}{-\frac{1}{2}+1}\right]_1^4$$

$$= \left[\frac{\theta^{\frac{3}{2}}}{\frac{3}{2}} + \frac{2\theta^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^4 = \left[\frac{2}{3}\sqrt{\theta^3} + 4\sqrt{\theta}\right]_1^4$$

$$= \left\{\frac{2}{3}\sqrt{(4)^3} + 4\sqrt{4}\right\} - \left\{\frac{2}{3}\sqrt{(1)^3} + 4\sqrt{(1)}\right\}$$

$$= \left\{\frac{16}{3} + 8\right\} - \left\{\frac{2}{3} + 4\right\}$$

$$= 5\frac{1}{3} + 8 - \frac{2}{3} - 4 = \mathbf{8\frac{2}{3}}$$

Problem 14. Evaluate $\int_0^{\frac{\pi}{2}} 3\sin 2x\,\mathrm{d}x$.

$$\int_0^{\frac{\pi}{2}} 3\sin 2x\,\mathrm{d}x$$

$$= \left[(3)\left(-\frac{1}{2}\right)\cos 2x\right]_0^{\frac{\pi}{2}} = \left[-\frac{3}{2}\cos 2x\right]_0^{\frac{\pi}{2}}$$

$$= \left\{-\frac{3}{2}\cos 2\left(\frac{\pi}{2}\right)\right\} - \left\{-\frac{3}{2}\cos 2(0)\right\}$$

$$= \left\{-\frac{3}{2}\cos\pi\right\} - \left\{-\frac{3}{2}\cos 0\right\}$$

$$= \left\{-\frac{3}{2}(-1)\right\} - \left\{-\frac{3}{2}(1)\right\} = \frac{3}{2} + \frac{3}{2} = \mathbf{3}$$

Problem 15. Evaluate $\int_1^2 4\cos 3t\,\mathrm{d}t$.

$$\int_1^2 4\cos 3t\,\mathrm{d}t = \left[(4)\left(\frac{1}{3}\right)\sin 3t\right]_1^2 = \left[\frac{4}{3}\sin 3t\right]_1^2$$

$$= \left\{\frac{4}{3}\sin 6\right\} - \left\{\frac{4}{3}\sin 3\right\}$$

Note that limits of trigonometric functions are always expressed in radians—thus, for example, $\sin 6$ means the sine of 6 radians $= -0.279415\ldots$

Hence $\int_1^2 4\cos 3t\,\mathrm{d}t$

$$= \left\{\frac{4}{3}(-0.279415\ldots)\right\} - \left\{\frac{4}{3}(0.141120\ldots)\right\}$$

$$= (-0.37255) - (0.18816) = \mathbf{-0.5607}$$

Problem 16. Evaluate

$$\text{(a)} \int_1^2 4\mathrm{e}^{2x}\,\mathrm{d}x \quad \text{(b)} \int_1^4 \frac{3}{4u}\,\mathrm{d}u,$$

each correct to 4 significant figures.

$$\text{(a)} \int_1^2 4\mathrm{e}^{2x}\,\mathrm{d}x = \left[\frac{4}{2}\mathrm{e}^{2x}\right]_1^2 = 2[\mathrm{e}^{2x}]_1^2 = 2[\mathrm{e}^4 - \mathrm{e}^2]$$

$$= 2[54.5982 - 7.3891] = \mathbf{94.42}$$

$$\text{(b)} \int_1^4 \frac{3}{4u}\,\mathrm{d}u = \left[\frac{3}{4}\ln u\right]_1^4 = \frac{3}{4}[\ln 4 - \ln 1]$$

$$= \frac{3}{4}[1.3863 - 0] = \mathbf{1.040}$$

Now try the following exercise

Exercise 146 Further problems on definite integrals

In problems 1 to 8, evaluate the definite integrals (where necessary, correct to 4 significant figures).

1. (a) $\int_1^4 5x^2\,\mathrm{d}x$ (b) $\int_{-1}^1 -\frac{3}{4}t^2\,\mathrm{d}t$

   $\left[\text{(a) } 105 \quad \text{(b) } -\frac{1}{2}\right]$

2. (a) $\int_{-1}^2 (3 - x^2)\,\mathrm{d}x$ (b) $\int_1^3 (x^2 - 4x + 3)\,\mathrm{d}x$

   $\left[\text{(a) } 6 \quad \text{(b) } -1\frac{1}{3}\right]$

3. (a) $\int_0^{\pi} \frac{3}{2}\cos\theta\,\mathrm{d}\theta$ (b) $\int_0^{\frac{\pi}{2}} 4\cos\theta\,\mathrm{d}\theta$

   [(a) 0 (b) 4]

4. (a) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 2\sin 2\theta\,\mathrm{d}\theta$ (b) $\int_0^2 3\sin t\,\mathrm{d}t$

   [(a) 1 (b) 4.248]

5. (a) $\int_0^1 5\cos 3x\,\mathrm{d}x$ (b) $\int_0^{\frac{\pi}{6}} 3\sec^2 2x\,\mathrm{d}x$

   [(a) 0.2352 (b) 2.598]

6. (a) $\int_1^2 \operatorname{cosec}^2 4t \, \mathrm{d}t$

(b) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (3\sin 2x - 2\cos 3x)\,\mathrm{d}x$

[(a) 0.2527 (b) 2.638]

7. (a) $\int_0^1 3\mathrm{e}^{3t}\,\mathrm{d}t$ (b) $\int_{-1}^2 \frac{2}{3\mathrm{e}^{2x}}\,\mathrm{d}x$

[(a) 19.09 (b) 2.457]

8. (a) $\int_2^3 \frac{2}{3x}\,\mathrm{d}x$ (b) $\int_1^3 \frac{2x^2+1}{x}\,\mathrm{d}x$

[(a) 0.2703 (b) 9.099]

9. The entropy change ΔS, for an ideal gas is given by:

$$\Delta S = \int_{T_1}^{T_2} C_v \frac{\mathrm{d}T}{T} - R\int_{V_1}^{V_2} \frac{\mathrm{d}V}{V}$$

where T is the thermodynamic temperature, V is the volume and $R = 8.314$. Determine the entropy change when a gas expands from 1 litre to 3 litres for a temperature rise from 100 K to 400 K given that:

$$C_v = 45 + 6\times 10^{-3}T + 8\times 10^{-6}T^2.$$

[55.65]

10. The p.d. between boundaries a and b of an electric field is given by: $V = \int_a^b \frac{Q}{2\pi r\varepsilon_0\varepsilon_r}\mathrm{d}r$

If $a = 10$, $b = 20$, $Q = 2\times 10^{-6}$ coulombs, $\varepsilon_0 = 8.85\times 10^{-12}$ and $\varepsilon_r = 2.77$, show that $V = 9$ kV.

11. The average value of a complex voltage waveform is given by:

$$V_{AV} = \frac{1}{\pi}\int_0^{\pi} (10\sin\omega t + 3\sin 3\omega t + 2\sin 5\omega t)\,\mathrm{d}(\omega t)$$

Evaluate V_{AV} correct to 2 decimal places.

[7.26]

Chapter 38

Some applications of integration

38.1 Introduction

There are a number of applications of integral calculus in engineering. The determination of areas, mean and r.m.s. values, volumes, centroids and second moments of area and radius of gyration are included in this chapter.

38.2 Areas under and between curves

In Fig. 38.1,

$$\textbf{total shaded area} = \int_a^b f(x)\mathrm{d}x - \int_b^c f(x)\mathrm{d}x + \int_c^d f(x)\mathrm{d}x$$

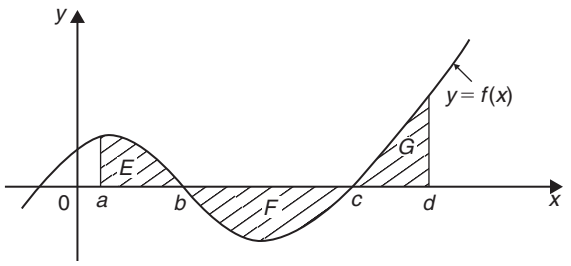


Figure 38.1

Problem 1. Determine the area between the curve $y = x^3 - 2x^2 - 8x$ and the x-axis.

$y = x^3 - 2x^2 - 8x = x(x^2 - 2x - 8) = x(x+2)(x-4)$

When $y = 0$, $x = 0$ or $(x+2) = 0$ or $(x-4) = 0$, i.e. when $y = 0$, $x = 0$ or -2 or 4, which means that the curve crosses the x-axis at 0, -2, and 4. Since the curve is a continuous function, only one other co-ordinate value needs to be calculated before a sketch of the curve can be produced. When $x = 1$, $y = -9$, showing that the part of the curve between $x = 0$ and $x = 4$ is negative. A sketch of $y = x^3 - 2x^2 - 8x$ is shown in Fig. 38.2. (Another method of sketching Fig. 38.2 would have been to draw up a table of values.)

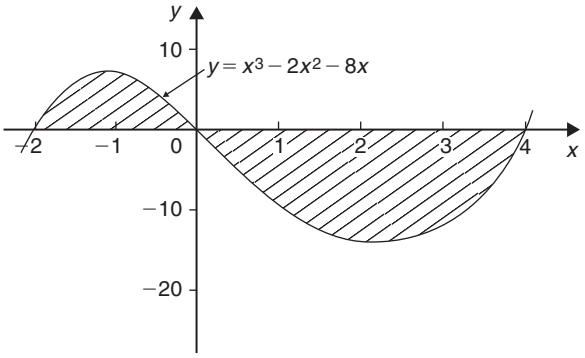


Figure 38.2

Shaded area

$$= \int_{-2}^{0} (x^3 - 2x^2 - 8x)\mathrm{d}x - \int_{0}^{4} (x^3 - 2x^2 - 8x)\mathrm{d}x$$

$$= \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{8x^2}{2}\right]_{-2}^{0} - \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{8x^2}{2}\right]_{0}^{4}$$

$$= \left(6\frac{2}{3}\right) - \left(-42\frac{2}{3}\right) = \mathbf{49\frac{1}{3}}\ \textbf{square units}$$

Problem 2. Determine the area enclosed between the curves $y = x^2 + 1$ and $y = 7 - x$.

At the points of intersection the curves are equal. Thus, equating the y values of each curve gives:

$$x^2 + 1 = 7 - x$$

from which, $x^2 + x - 6 = 0$

Factorizing gives $(x - 2)(x + 3) = 0$

from which $x = 2$ and $x = -3$

By firstly determining the points of intersection the range of x-values has been found. Tables of values are produced as shown below.

x	-3	-2	-1	0	1	2
$y = x^2 + 1$	10	5	2	1	2	5

x	-3	0	2
$y = 7 - x$	10	7	5

A sketch of the two curves is shown in Fig. 38.3.

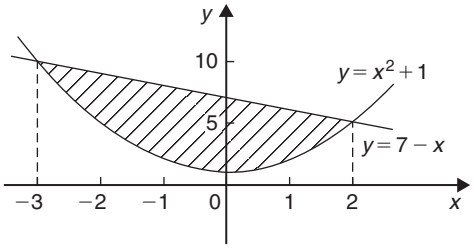


Figure 38.3

$$\text{Shaded area} = \int_{-3}^{2} (7 - x)\,dx - \int_{-3}^{2} (x^2 + 1)\,dx$$

$$= \int_{-3}^{2} [(7 - x) - (x^2 + 1)]\,dx$$

$$= \int_{-3}^{2} (6 - x - x^2)\,dx$$

$$= \left[6x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-3}^{2}$$

$$= \left(12 - 2 - \frac{8}{3}\right) - \left(-18 - \frac{9}{2} + 9\right)$$

$$= \left(7\frac{1}{3}\right) - \left(-13\frac{1}{2}\right)$$

$$= \mathbf{20\frac{5}{6}\ \text{square units}}$$

Problem 3. Determine by integration the area bounded by the three straight lines $y = 4 - x$, $y = 3x$ and $3y = x$.

Each of the straight lines are shown sketched in Fig. 38.4.

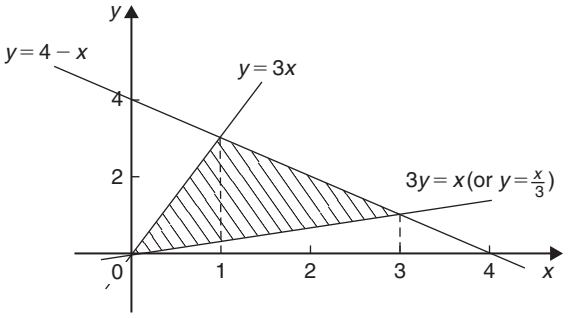


Figure 38.4

Shaded area

$$= \int_0^1 \left(3x - \frac{x}{3}\right) dx + \int_1^3 \left[(4 - x) - \frac{x}{3}\right] dx$$

$$= \left[\frac{3x^2}{2} - \frac{x^2}{6}\right]_0^1 + \left[4x - \frac{x^2}{2} - \frac{x^2}{6}\right]_1^3$$

$$= \left[\left(\frac{3}{2} - \frac{1}{6}\right) - (0)\right] + \left[\left(12 - \frac{9}{2} - \frac{9}{6}\right) - \left(4 - \frac{1}{2} - \frac{1}{6}\right)\right]$$

$$= \left(1\frac{1}{3}\right) + \left(6 - 3\frac{1}{3}\right) = \mathbf{4\ \text{square units}}$$

Now try the following exercise

Exercise 147 Further problems on areas under and between curves

1. Find the area enclosed by the curve $y = 4\cos 3x$, the x-axis and ordinates $x = 0$ and $x = \frac{\pi}{6}$ [$1\frac{1}{3}$ square units]

2. Sketch the curves $y = x^2 + 3$ and $y = 7 - 3x$ and determine the area enclosed by them. [$20\frac{5}{6}$ square units]

3. Determine the area enclosed by the three straight lines $y = 3x$, $2y = x$ and $y + 2x = 5$. [$2\frac{1}{2}$ square units]

38.3 Mean and r.m.s. values

With reference to Fig. 38.5,

$$\textbf{mean value, } \bar{y} = \frac{1}{b-a}\int_a^b y\,dx$$

and $$\textbf{r.m.s. value} = \sqrt{\left\{\frac{1}{b-a}\int_a^b y^2\,dx\right\}}$$

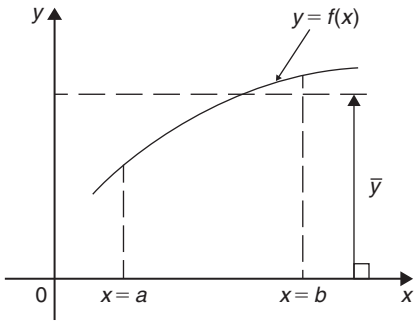


Figure 38.5

Problem 4. A sinusoidal voltage $v = 100\sin\omega t$ volts. Use integration to determine over half a cycle (a) the mean value, and (b) the r.m.s. value.

(a) Half a cycle means the limits are 0 to π radians.

$$\text{Mean value, } \bar{y} = \frac{1}{\pi - 0}\int_0^{\pi} v\,d(\omega t)$$

$$= \frac{1}{\pi}\int_0^{\pi} 100\sin\omega t\,d(\omega t)$$

$$= \frac{100}{\pi}[-\cos\omega t]_0^{\pi}$$

$$= \frac{100}{\pi}[(-\cos\pi) - (-\cos 0)]$$

$$= \frac{100}{\pi}[(+1) - (-1)] = \frac{200}{\pi}$$

$$= \textbf{63.66 volts}$$

[Note that for a sine wave,

$$\textbf{mean value} = \frac{\mathbf{2}}{\boldsymbol{\pi}} \times \textbf{maximum value}$$

In this case, mean value $= \frac{2}{\pi} \times 100 = 63.66$ V]

(b) r.m.s. value

$$= \sqrt{\left\{\frac{1}{\pi - 0}\int_0^{\pi} v^2\,d(\omega t)\right\}}$$

$$= \sqrt{\left\{\frac{1}{\pi}\int_0^{\pi} (100\sin\omega t)^2\,d(\omega t)\right\}}$$

$$= \sqrt{\left\{\frac{10000}{\pi}\int_0^{\pi} \sin^2\omega t\,d(\omega t)\right\}},$$

which is not a 'standard' integral.

It is shown in Chapter 17 that $\cos 2A = 1 - 2\sin^2 A$ and this formula is used whenever $\sin^2 A$ needs to be integrated.

Rearranging $\cos 2A = 1 - 2\sin^2 A$ gives

$$\sin^2 A = \frac{1}{2}(1 - \cos 2A)$$

Hence $$\sqrt{\left\{\frac{10000}{\pi}\int_0^{\pi} \sin^2\omega t\,d(\omega t)\right\}}$$

$$= \sqrt{\left\{\frac{10000}{\pi}\int_0^{\pi} \frac{1}{2}(1 - \cos 2\omega t)\,d(\omega t)\right\}}$$

$$= \sqrt{\left\{\frac{10000}{\pi}\frac{1}{2}\left[\omega t - \frac{\sin 2\omega t}{2}\right]_0^{\pi}\right\}}$$

$$= \sqrt{\left\{\frac{10000}{\pi}\frac{1}{2}\left[\left(\pi - \frac{\sin 2\pi}{2}\right) - \left(0 - \frac{\sin 0}{2}\right)\right]\right\}}$$

$$= \sqrt{\left\{\frac{10000}{\pi}\frac{1}{2}[\pi]\right\}}$$

$$= \sqrt{\left\{\frac{10000}{2}\right\}} = \frac{100}{\sqrt{2}} = \textbf{70.71 volts}$$

[Note that for a sine wave,

$$\textbf{r.m.s. value} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}} \times \textbf{maximum value}.$$

In this case,

$$\text{r.m.s. value} = \frac{1}{\sqrt{2}} \times 100 = 70.71\,\text{V}]$$

Now try the following exercise

Exercise 148 Further problems on mean and r.m.s. values

1. The vertical height h km of a missile varies with the horizontal distance d km, and is given by $h = 4d - d^2$. Determine the mean height of the missile from $d = 0$ to $d = 4$ km. [$2\frac{2}{3}$ km].
2. The distances of points y from the mean value of a frequency distribution are related to the variate x by the equation $y = x + \frac{1}{x}$. Determine the standard deviation (i.e. the r.m.s. value), correct to 4 significant figures for values of x from 1 to 2. [2.198]
3. A current $i = 25\sin 100\pi t$ mA flows in an electrical circuit. Determine, using integral calculus, its mean and r.m.s. values each correct to 2 decimal places over the range $t = 0$ to $t = 10$ ms. [15.92 mA, 17.68 mA]
4. A wave is defined by the equation:

$$v = E_1 \sin\omega t + E_3 \sin 3\omega t$$

where E_1, E_3 and ω are constants. Determine the r.m.s. value of v over the interval $0 \leq t \leq \frac{\pi}{\omega}$.

$$\left[\sqrt{\frac{E_1^2 + E_3^2}{2}}\right]$$

38.4 Volumes of solids of revolution

With reference to Fig. 38.6, the volume of revolution, V, obtained by rotating area A through one revolution about the x-axis is given by:

$$V = \int_a^b \pi y^2\,\mathrm{d}x$$

If a curve $x = f(y)$ is rotated 360° about the y-axis between the limits $y = c$ and $y = d$ then the volume

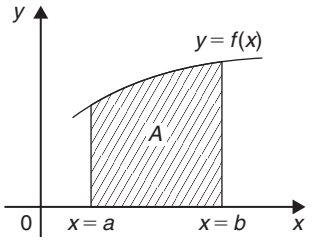


Figure 38.6

generated, V, is given by:

$$V = \int_c^d \pi x^2\,\mathrm{d}y$$

Problem 5. The curve $y = x^2 + 4$ is rotated one revolution about the x-axis between the limits $x = 1$ and $x = 4$. Determine the volume of solid of revolution produced.

Revolving the shaded area shown in Fig. 38.7, 360° about the x-axis produces a solid of revolution given by:

$$\begin{aligned}\text{Volume} &= \int_1^4 \pi y^2\,\mathrm{d}x = \int_1^4 \pi(x^2+4)^2\,\mathrm{d}x \\ &= \int_1^4 \pi(x^4 + 8x^2 + 16)\,\mathrm{d}x \\ &= \pi\left[\frac{x^5}{5} + \frac{8x^3}{3} + 16x\right]_1^4 \\ &= \pi[(204.8 + 170.67 + 64) \\ &\quad - (0.2 + 2.67 + 16)] \\ &= \mathbf{420.6\pi\ \text{cubic units}}\end{aligned}$$

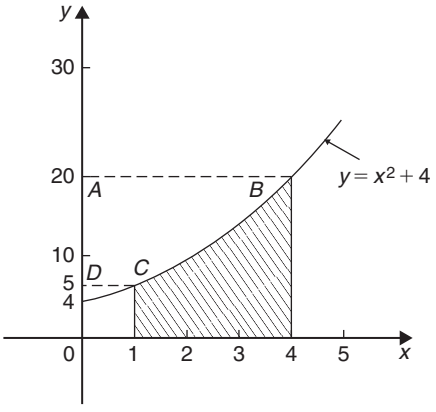


Figure 38.7

Problem 6. Determine the area enclosed by the two curves $y = x^2$ and $y^2 = 8x$. If this area is rotated 360° about the x-axis determine the volume of the solid of revolution produced.

At the points of intersection the co-ordinates of the curves are equal. Since $y = x^2$ then $y^2 = x^4$. Hence equating the y^2 values at the points of intersection:

$$x^4 = 8x$$

from which, $x^4 - 8x = 0$

and $x(x^3 - 8) = 0$

Hence, at the points of intersection, $x = 0$ and $x = 2$.

When $x = 0$, $y = 0$ and when $x = 2$, $y = 4$. The points of intersection of the curves $y = x^2$ and $y^2 = 8x$ are therefore at (0,0) and (2,4). A sketch is shown in Fig. 38.8. If $y^2 = 8x$ then $y = \sqrt{8x}$.

Shaded area

$$= \int_0^2 \left(\sqrt{8x} - x^2\right) dx = \int_0^2 \left(\sqrt{8}\right) x^{\frac{1}{2}} - x^2 \Big) dx$$

$$= \left[\left(\sqrt{8}\right) \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{3} \right]_0^2 = \left\{ \frac{\sqrt{8}\sqrt{8}}{\frac{3}{2}} - \frac{8}{3} \right\} - \{0\}$$

$$= \frac{16}{3} - \frac{8}{3} = \frac{8}{3} = \mathbf{2\frac{2}{3}}\ \textbf{square units}$$

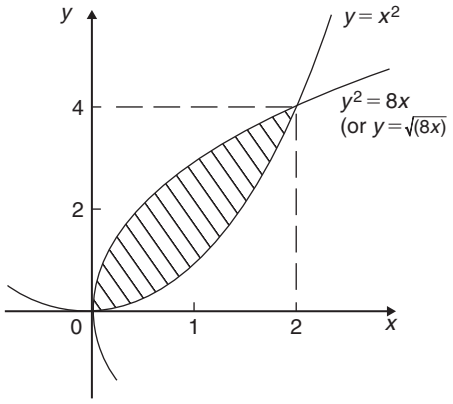


Figure 38.8

The volume produced by revolving the shaded area about the x-axis is given by:

{(volume produced by revolving $y^2 = 8x$) − (volume produced by revolving $y = x^2$)}

i.e. **volume** $= \int_0^2 \pi(8x)dx - \int_0^2 \pi(x^4)dx$

$$= \pi \int_0^2 (8x - x^4)dx = \pi \left[\frac{8x^2}{2} - \frac{x^5}{5} \right]_0^2$$

$$= \pi \left[\left(16 - \frac{32}{5}\right) - (0) \right]$$

$$= \mathbf{9.6\pi}\ \textbf{cubic units}$$

Now try the following exercise

Exercise 149 Further problems on volumes

1. The curve $xy = 3$ is revolved one revolution about the x-axis between the limits $x = 2$ and $x = 3$. Determine the volume of the solid produced. [1.5π cubic units]
2. The area between $\frac{y}{x^2} = 1$ and $y + x^2 = 8$ is rotated 360° about the x-axis. Find the volume produced. [$170\frac{2}{3}\pi$ cubic units]
3. The curve $y = 2x^2 + 3$ is rotated about (a) the x-axis between the limits $x = 0$ and $x = 3$, and (b) the y-axis, between the same limits. Determine the volume generated in each case. [(a) 329.4π (b) 81π]
4. The profile of a rotor blade is bounded by the lines $x = 0.2$, $y = 2x$, $y = e^{-x}$, $x = 1$ and the x-axis. The blade thickness t varies linearly with x and is given by: $t = (1.1 - x)$K, where K is a constant.
   (a) Sketch the rotor blade, labelling the limits.
   (b) Determine, using an iterative method, the value of x, correct to 3 decimal places, where $2x = e^{-x}$
   (c) Calculate the cross-sectional area of the blade, correct to 3 decimal places.
   (d) Calculate the volume of the blade in terms of K, correct to 3 decimal places.
   [(b) 0.352 (c) 0.419 square units (d) 0.222 K]

38.5 Centroids

A **lamina** is a thin flat sheet having uniform thickness. The **centre of gravity** of a lamina is the point where it balances perfectly, i.e. the lamina's **centre of mass**. When dealing with an area (i.e. a lamina of negligible thickness and mass) the term **centre of area** or **centroid** is used for the point where the centre of gravity of a lamina of that shape would lie.

If $\bar{x}$ and $\bar{y}$ denote the co-ordinates of the centroid C of area A of Fig. 38.9, then:

$$\bar{x} = \frac{\int_a^b xy\,dx}{\int_a^b y\,dx} \quad \text{and} \quad \bar{y} = \frac{\frac{1}{2}\int_a^b y^2\,dx}{\int_a^b y\,dx}$$

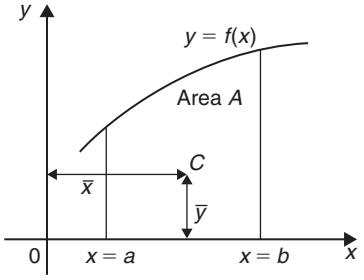


Figure 38.9

Problem 7. Find the position of the centroid of the area bounded by the curve $y = 3x^2$, the x-axis and the ordinates $x = 0$ and $x = 2$.

If $(\bar{x}, \bar{y})$ are co-ordinates of the centroid of the given area then:

$$\bar{x} = \frac{\int_0^2 xy\,dx}{\int_0^2 y\,dx} = \frac{\int_0^2 x(3x^2)\,dx}{\int_0^2 3x^2\,dx}$$

$$= \frac{\int_0^2 3x^3\,dx}{\int_0^2 3x^2\,dx} = \frac{\left[\frac{3x^4}{4}\right]_0^2}{[x^3]_0^2}$$

$$= \frac{12}{8} = \mathbf{1.5}$$

$$\bar{y} = \frac{\frac{1}{2}\int_0^2 y^2\,dx}{\int_0^2 y\,dx} = \frac{\frac{1}{2}\int_0^2 (3x^2)^2\,dx}{8}$$

$$= \frac{\frac{1}{2}\int_0^2 9x^4\,dx}{8} = \frac{\frac{9}{2}\left[\frac{x^5}{5}\right]_0^2}{8}$$

$$= \frac{\frac{9}{2}\left(\frac{32}{5}\right)}{8} = \frac{18}{5} = \mathbf{3.6}$$

Hence the centroid lies at (1.5, 3.6)

Problem 8. Determine the co-ordinates of the centroid of the area lying between the curve $y = 5x - x^2$ and the x-axis.

$y = 5x - x^2 = x(5 - x)$. When $y = 0$, $x = 0$ or $x = 5$. Hence the curve cuts the x-axis at 0 and 5 as shown in Fig. 38.10. Let the co-ordinates of the centroid be $(\bar{x}, \bar{y})$ then, by integration,

$$\bar{x} = \frac{\int_0^5 xy\,dx}{\int_0^5 y\,dx} = \frac{\int_0^5 x(5x - x^2)\,dx}{\int_0^5 (5x - x^2)\,dx}$$

$$= \frac{\int_0^5 (5x^2 - x^3)\,dx}{\int_0^5 (5x - x^2)\,dx} = \frac{\left[\frac{5x^3}{3} - \frac{x^4}{4}\right]_0^5}{\left[\frac{5x^2}{2} - \frac{x^3}{3}\right]_0^5}$$

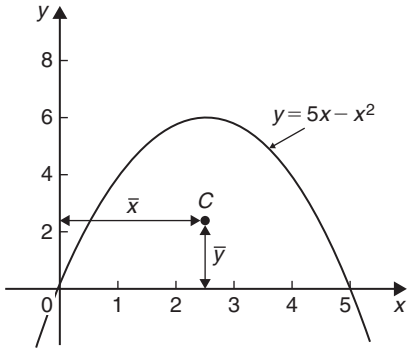


Figure 38.10

$$= \frac{\frac{625}{3} - \frac{625}{4}}{\frac{125}{2} - \frac{125}{3}} = \frac{\frac{625}{12}}{\frac{125}{6}}$$

$$= \left(\frac{625}{12}\right)\left(\frac{6}{125}\right) = \frac{5}{2} = \mathbf{2.5}$$

$$\bar{y} = \frac{\frac{1}{2}\int_0^5 y^2\,dx}{\int_0^5 y\,dx} = \frac{\frac{1}{2}\int_0^5 (5x - x^2)^2\,dx}{\int_0^5 (5x - x^2)\,dx}$$

$$= \frac{\frac{1}{2}\int_0^5 (25x^2 - 10x^3 + x^4)\,dx}{\frac{125}{6}}$$

$$= \frac{\frac{1}{2}\left[\frac{25x^3}{3} - \frac{10x^4}{4} + \frac{x^5}{5}\right]_0^5}{\frac{125}{6}}$$

$$= \frac{\frac{1}{2}\left(\frac{25(125)}{3} - \frac{6250}{4} + 625\right)}{\frac{125}{6}} = \mathbf{2.5}$$

Hence the centroid of the area lies at (2.5, 2.5).

(Note from Fig. 38.10 that the curve is symmetrical about $x = 2.5$ and thus $\bar{x}$ could have been determined 'on sight'.)

Now try the following exercise

Exercise 150 Further problems on centroids

In Problems 1 and 2, find the position of the centroids of the areas bounded by the given curves, the x-axis and the given ordinates.

1. $y = 3x + 2 \quad x = 0, x = 4$ [(2.5, 4.75)]
2. $y = 5x^2 \quad x = 1, x = 4$ [(3.036, 24.36)]
3. Determine the position of the centroid of a sheet of metal formed by the curve $y = 4x - x^2$ which lies above the x-axis. [(2, 1.6)]
4. Find the co-ordinates of the centroid of the area which lies between the curve $y/x = x - 2$ and the x-axis. [(1, −0.4)]
5. Sketch the curve $y^2 = 9x$ between the limits $x = 0$ and $x = 4$. Determine the position of the centroid of this area. [(2.4, 0)]

38.6 Theorem of Pappus

A theorem of Pappus states:

'If a plane area is rotated about an axis in its own plane but not intersecting it, the volume of the solid formed is given by the product of the area and the distance moved by the centroid of the area'.

With reference to Fig. 38.11, when the curve $y = f(x)$ is rotated one revolution about the x-axis between the limits $x = a$ and $x = b$, the volume V generated is given by:

volume $V = (A)(2\pi\bar{y})$, from which, $\bar{y} = \dfrac{V}{2\pi A}$

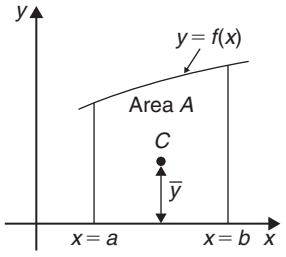


Figure 38.11

Problem 9. (a) Calculate the area bounded by the curve $y = 2x^2$, the x-axis and ordinates $x = 0$ and $x = 3$. (b) If this area is revolved (i) about the x-axis and (ii) about the y-axis, find the volumes of the solids produced. (c) Locate the position of the centroid using (i) integration, and (ii) the theorem of Pappus.

(a) The required area is shown shaded in Fig. 38.12.

$$\text{Area} = \int_0^3 y\,dx = \int_0^3 2x^2\,dx$$

$$= \left[\frac{2x^3}{3}\right]_0^3 = \mathbf{18 \text{ square units}}$$

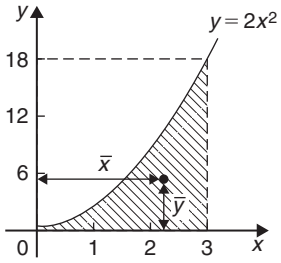


Figure 38.12

(b) (i) When the shaded area of Fig. 38.12 is revolved 360° about the x-axis, the volume generated

$$= \int_0^3 \pi y^2 \, dx = \int_0^3 \pi (2x^2)^2 \, dx$$

$$= \int_0^3 4\pi x^4 \, dx = 4\pi \left[\frac{x^5}{5} \right]_0^3$$

$$= 4\pi \left(\frac{243}{5} \right) = \mathbf{194.4\pi \text{ cubic units}}$$

(ii) When the shaded area of Fig. 38.12 is revolved 360° about the y-axis, the volume generated

$$= (\text{volume generated by } x = 3) - (\text{volume generated by } y = 2x^2)$$

$$= \int_0^{18} \pi (3)^2 \, dy - \int_0^{18} \pi \left(\frac{y}{2} \right) dy$$

$$= \pi \int_0^{18} \left(9 - \frac{y}{2} \right) dy = \pi \left[9y - \frac{y^2}{4} \right]_0^{18}$$

$$= \mathbf{81\pi \text{ cubic units}}$$

(c) If the co-ordinates of the centroid of the shaded area in Fig. 38.12 are $(\bar{x}, \bar{y})$ then:

(i) by integration,

$$\bar{x} = \frac{\int_0^3 xy \, dx}{\int_0^3 y \, dx} = \frac{\int_0^3 x(2x^2) \, dx}{18}$$

$$= \frac{\int_0^3 2x^3 \, dx}{18} = \frac{\left[\frac{2x^4}{4} \right]_0^3}{18}$$

$$= \frac{81}{36} = \mathbf{2.25}$$

$$\bar{y} = \frac{\frac{1}{2}\int_0^3 y^2 \, dx}{\int_0^3 y \, dx} = \frac{\frac{1}{2}\int_0^3 (2x^2)^2 \, dx}{18}$$

$$= \frac{\frac{1}{2}\int_0^3 4x^4 \, dx}{18} = \frac{\frac{1}{2}\left[\frac{4x^5}{5} \right]_0^3}{18} = \mathbf{5.4}$$

(ii) using the theorem of Pappus:

Volume generated when shaded area is revolved about OY = (area)$(2\pi\bar{x})$.

i.e. $81\pi = (18)(2\pi\bar{x})$,

from which, $\bar{x} = \dfrac{81\pi}{36\pi} = \mathbf{2.25}$

Volume generated when shaded area is revolved about OX = (area)$(2\pi\bar{y})$.

i.e. $194.4\pi = (18)(2\pi\bar{y})$,

from which, $\bar{y} = \dfrac{194.4\pi}{36\pi} = \mathbf{5.4}$

Hence the centroid of the shaded area in Fig. 38.12 is at (2.25, 5.4).

Problem 10. A metal disc has a radius of 5.0 cm and is of thickness 2.0 cm. A semicircular groove of diameter 2.0 cm is machined centrally around the rim to form a pulley. Determine, using Pappus' theorem, the volume and mass of metal removed and the volume and mass of the pulley if the density of the metal is 8000 kg m^{-3}.

A side view of the rim of the disc is shown in Fig. 38.13.

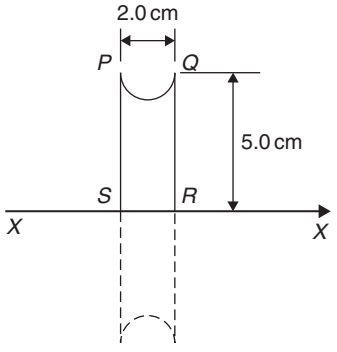


Figure 38.13

When area $PQRS$ is rotated about axis XX the volume generated is that of the pulley. The centroid of the semicircular area removed is at a distance of $\frac{4r}{3\pi}$ from its diameter (see '*Engineering Mathematics* 6th edition', Chapter 58), i.e. $\frac{4(1.0)}{3\pi}$, i.e. 0.424 cm from PQ. Thus the distance of the centroid from XX is $5.0 - 0.424$, i.e. 4.576 cm.

The distance moved through in one revolution by the centroid is $2\pi(4.576)$ cm.

Area of semicircle $= \frac{\pi r^2}{2} = \frac{\pi(1.0)^2}{2} = \frac{\pi}{2}\text{cm}^2$

By the theorem of Pappus,

volume generated = area × distance moved by centroid $= \left(\frac{\pi}{2}\right)(2\pi)(4.576)$.

i.e. **volume of metal removed = 45.16 cm^3**

Mass of metal removed = density × volume

$$= 8000\,\text{kg m}^{-3} \times \frac{45.16}{10^6}\text{m}^3$$

$$= 0.3613\,\text{kg or } 361.3\,\text{g}$$

volume of pulley = volume of cylindrical disc − volume of metal removed

$$= \pi(5.0)^2(2.0) - 45.16$$

$$= \mathbf{111.9\,cm^3}$$

Mass of pulley = density × volume

$$= 8000\,\text{kg m}^{-3} \times \frac{111.9}{10^6}\text{m}^3$$

$$= \mathbf{0.8952\,kg} \text{ or } \mathbf{895.2\,g}$$

Now try the following exercise

Exercise 151 Further problems on the theorem of Pappus

1. A right angled isosceles triangle having a hypotenuse of 8 cm is revolved one revolution about one of its equal sides as axis. Determine the volume of the solid generated using Pappus' theorem. [189.6 cm^3]
2. Using (a) the theorem of Pappus, and (b) integration, determine the position of the centroid of a metal template in the form of a quadrant of a circle of radius 4 cm. (The equation of a circle, centre 0, radius r is $x^2 + y^2 = r^2$).

   [On the centre line, distance 2.40 cm from the centre, i.e. at co-ordinates (1.70, 1.70)]
3. (a) Determine the area bounded by the curve $y = 5x^2$, the x-axis and the ordinates $x = 0$ and $x = 3$.

   (b) If this area is revolved 360° about (i) the x-axis, and (ii) the y-axis, find the volumes of the solids of revolution produced in each case.

   (c) Determine the co-ordinates of the centroid of the area using (i) integral calculus, and (ii) the theorem of Pappus.

   [(a) 45 square units
   (b) (i) 1215π cubic units
   (ii) 202.5π cubic units
   (c) (2.25, 13.5)]
4. A metal disc has a radius of 7.0 cm and is of thickness 2.5 cm. A semicircular groove of diameter 2.0 cm is machined centrally around the rim to form a pulley. Determine the volume of metal removed using Pappus' theorem and express this as a percentage of the original volume of the disc. Find also the mass of metal removed if the density of the metal is 7800 kg m^{-3}.

   [64.90 cm^3, 16.86%, 506.2 g]

For more on areas, mean and r.m.s. values, volumes and centroids, see '*Engineering Mathematics* 6th edition', Chapters 55 to 58.

38.7 Second moments of area of regular sections

The **first moment of area** about a fixed axis of a lamina of area A, perpendicular distance y from the centroid of the lamina is defined as Ay cubic units. The **second moment of area** of the same lamina as above is given by Ay^2, i.e. the perpendicular distance from the centroid of the area to the fixed axis is squared.

Second moments of areas are usually denoted by I and have units of mm^4, cm^4, and so on.

Radius of gyration

Several areas, $a_1, a_2, a_3, \ldots$ at distances $y_1, y_2, y_3, \ldots$ from a fixed axis, may be replaced by a single area A, where $A = a_1 + a_2 + a_3 + \cdots$ at distance k from the axis, such that $Ak^2 = \sum ay^2$.

k is called the **radius of gyration** of area A about the given axis. Since $Ak^2 = \sum ay^2 = I$ then the radius of gyration,

$$k = \sqrt{\frac{I}{A}}$$

The second moment of area is a quantity much used in the theory of bending of beams, in the torsion of shafts, and in calculations involving water planes and centres of pressure.

The procedure to determine the second moment of area of regular sections about a given axis is (i) to find the second moment of area of a typical element and (ii) to sum all such second moments of area by integrating between appropriate limits.

For example, the second moment of area of the rectangle shown in Fig. 38.14 about axis PP is found by initially considering an elemental strip of width δx, parallel to and distance x from axis PP. Area of shaded strip $= b\delta x$.

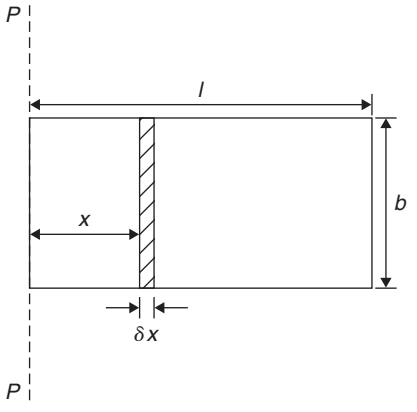


Figure 38.14

Second moment of area of the shaded strip about $PP = (x^2)(b\,\delta x)$.

The second moment of area of the whole rectangle about PP is obtained by summing all such strips between $x = 0$ and $x = l$, i.e. $\sum_{x=0}^{x=l} x^2\, b\delta x$.

It is a fundamental theorem of integration that

$$\lim_{\delta x \to 0} \sum_{x=0}^{x=l} x^2 b\,\delta x = \int_0^l x^2 b\,\mathrm{d}x$$

Thus the second moment of area of the rectangle about PP

$$= b\int_0^l x^2\,\mathrm{d}x = b\left[\frac{x^3}{3}\right]_0^l = \frac{\boldsymbol{bl^3}}{\boldsymbol{3}}$$

Since the total area of the rectangle, $A = lb$, then

$$I_{pp} = (lb)\left(\frac{l^2}{3}\right) = \frac{\boldsymbol{Al^2}}{\boldsymbol{3}}$$

$$I_{pp} = Ak_{pp}^2 \text{ thus } k_{pp}^2 = \frac{l^2}{3}$$

i.e. the radius of gyration about axes PP,

$$\boldsymbol{k_{pp}} = \sqrt{\frac{l^2}{3}} = \frac{\boldsymbol{l}}{\sqrt{\boldsymbol{3}}}$$

Parallel axis theorem

In Fig. 38.15, axis GG passes through the centroid C of area A. Axes DD and GG are in the same plane, are parallel to each other and distance d apart. The parallel axis theorem states:

$$\boldsymbol{I_{DD} = I_{GG} + Ad^2}$$

Using the parallel axis theorem the second moment of area of a rectangle about an axis through the centroid

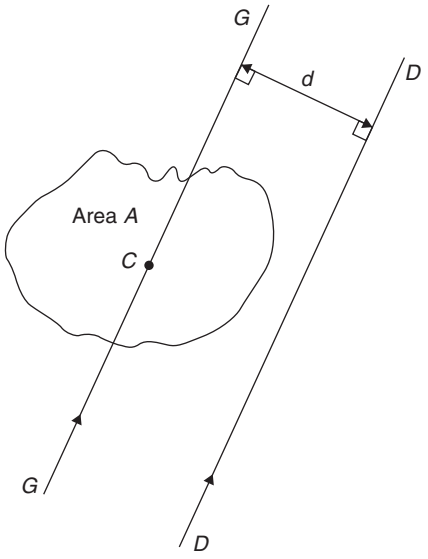


Figure 38.15

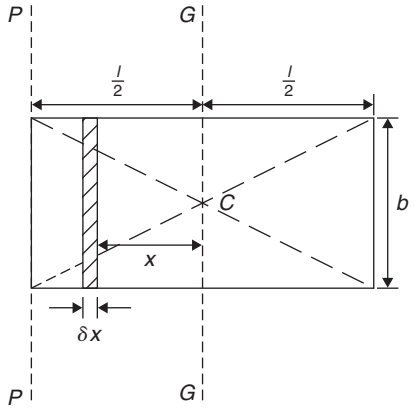


Figure 38.16

may be determined. In the rectangle shown in Fig. 38.16, $I_{pp} = \dfrac{bl^3}{3}$ (from above).

From the parallel axis theorem

$$I_{pp} = I_{GG} + (bl)\left(\frac{1}{2}\right)^2$$

i.e. $$\frac{bl^3}{3} = I_{GG} + \frac{bl^3}{4}$$

from which, $$\boldsymbol{I_{GG}} = \frac{bl^3}{3} - \frac{bl^3}{4} = \boldsymbol{\frac{bl^3}{12}}$$

Perpendicular axis theorem

In Fig. 38.17, axes OX, OY and OZ are mutually perpendicular. If OX and OY lie in the plane of area A then the perpendicular axis theorem states:

$$\boldsymbol{I_{OZ} = I_{OX} + I_{OY}}$$

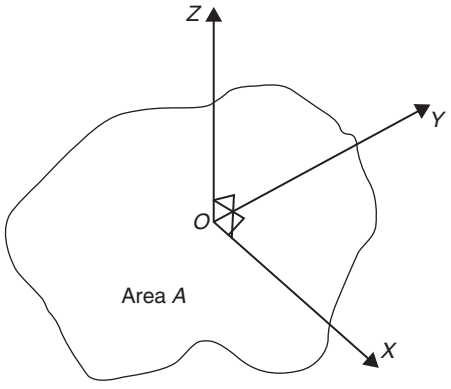


Figure 38.17

A summary of derived standard results for the second moment of area and radius of gyration of regular sections are listed in Table 38.1.

Problem 11. Determine the second moment of area and the radius of gyration about axes AA, BB and CC for the rectangle shown in Fig. 38.18.

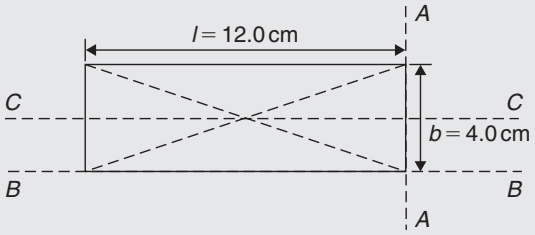


Figure 38.18

From Table 38.1, the second moment of area about axis AA,

$$\boldsymbol{I_{AA}} = \frac{bl^3}{3} = \frac{(4.0)(12.0)^3}{3} = \boldsymbol{2304\,cm^4}$$

Radius of gyration, $\boldsymbol{k_{AA}} = \dfrac{l}{\sqrt{3}} = \dfrac{12.0}{\sqrt{3}} = \boldsymbol{6.93\,cm}$

Similarly, $$\boldsymbol{I_{BB}} = \frac{lb^3}{3} = \frac{(12.0)(4.0)^3}{3} = \boldsymbol{256\,cm^4}$$

and $$\boldsymbol{k_{BB}} = \frac{b}{\sqrt{3}} = \frac{4.0}{\sqrt{3}} = \boldsymbol{2.31\,cm}$$

The second moment of area about the centroid of a rectangle is $\dfrac{bl^3}{12}$ when the axis through the centroid is parallel with the breadth b. In this case, the axis CC is parallel with the length l.

Hence $$\boldsymbol{I_{CC}} = \frac{lb^3}{12} = \frac{(12.0)(4.0)^3}{12} = \boldsymbol{64\,cm^4}$$

and $$\boldsymbol{k_{CC}} = \frac{b}{\sqrt{12}} = \frac{4.0}{\sqrt{12}} = \boldsymbol{1.15\,cm}$$

Table 38.1 Summary of standard results of the second moments of areas of regular sections

Shape	Position of axis	Second moment of area, I	Radius of gyration, k
Rectangle length l, breadth b	(1) Coinciding with b	$\frac{bl^3}{3}$	$\frac{l}{\sqrt{3}}$
	(2) Coinciding with l	$\frac{lb^3}{3}$	$\frac{b}{\sqrt{3}}$
	(3) Through centroid, parallel to b	$\frac{bl^3}{12}$	$\frac{l}{\sqrt{12}}$
	(4) Through centroid, parallel to l	$\frac{lb^3}{12}$	$\frac{b}{\sqrt{12}}$
Triangle Perpendicular height h, base b	(1) Coinciding with b	$\frac{bh^3}{12}$	$\frac{h}{\sqrt{6}}$
	(2) Through centroid, parallel to base	$\frac{bh^3}{36}$	$\frac{h}{\sqrt{18}}$
	(3) Through vertex, parallel to base	$\frac{bh^3}{4}$	$\frac{h}{\sqrt{2}}$
Circle radius r	(1) Through centre, perpendicular to plane (i.e. polar axis)	$\frac{\pi r^4}{2}$	$\frac{r}{\sqrt{2}}$
	(2) Coinciding with diameter	$\frac{\pi r^4}{4}$	$\frac{r}{2}$
	(3) About a tangent	$\frac{5\pi r^4}{4}$	$\frac{\sqrt{5}}{2}r$
Semicircle radius r	Coinciding with diameter	$\frac{\pi r^4}{8}$	$\frac{r}{2}$

Problem 12. Find the second moment of area and the radius of gyration about axis PP for the rectangle shown in Fig. 38.19.

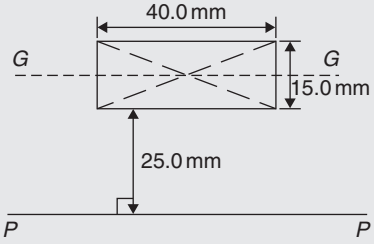


Figure 38.19

$I_{GG} = \dfrac{lb^3}{12}$ where $l = 40.0\,\text{mm}$ and $b = 15.0\,\text{mm}$

Hence $I_{GG} = \dfrac{(40.0)(15.0)^3}{12} = 11250\,\text{mm}^4$

From the parallel axis theorem, $I_{PP} = I_{GG} + Ad^2$, where $A = 40.0 \times 15.0 = 600\,\text{mm}^2$ and $d = 25.0 + 7.5 = 32.5\,\text{mm}$, the perpendicular distance between GG and PP. Hence,

$$I_{PP} = 11\,250 + (600)(32.5)^2$$

$$= \mathbf{645000\,mm^4}$$

$I_{PP} = Ak_{PP}^2$, from which,

$$k_{PP} = \sqrt{\frac{I_{PP}}{\text{area}}} = \sqrt{\left(\frac{645000}{600}\right)} = \mathbf{32.79\,mm}$$

Problem 13. Determine the second moment of area and radius of gyration about axis QQ of the triangle BCD shown in Fig. 38.20.

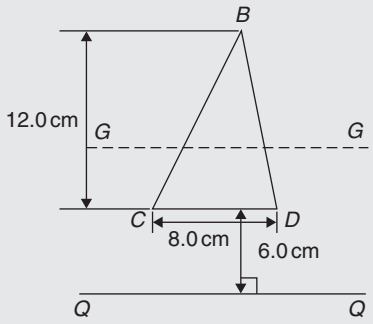


Figure 38.20

Using the parallel axis theorem: $I_{QQ} = I_{GG} + Ad^2$, where I_{GG} is the second moment of area about the centroid of the triangle,

i.e. $\dfrac{bh^3}{36} = \dfrac{(8.0)(12.0)^3}{36} = 384\,\text{cm}^4$,

A is the area of the triangle,

$$= \tfrac{1}{2}bh = \tfrac{1}{2}(8.0)(12.0) = 48\,\text{cm}^2$$

and d is the distance between axes GG and QQ,

$$= 6.0 + \tfrac{1}{3}(12.0) = 10\,\text{cm}.$$

Hence the second moment of area about axis QQ,

$$I_{QQ} = 384 + (48)(10)^2 = \mathbf{5184\,cm^4}$$

Radius of gyration,

$$k_{QQ} = \sqrt{\frac{I_{QQ}}{\text{area}}} = \sqrt{\left(\frac{5184}{48}\right)} = \mathbf{10.4\,cm}$$

Problem 14. Determine the second moment of area and radius of gyration of the circle shown in Fig. 38.21 about axis YY.

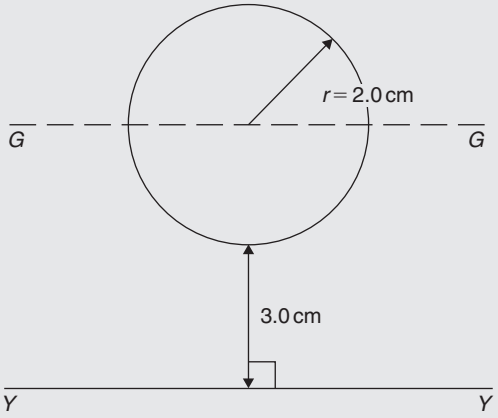


Figure 38.21

In Fig. 38.21, $I_{GG} = \dfrac{\pi r^4}{4} = \dfrac{\pi}{4}(2.0)^4 = 4\pi\ \text{cm}^4$.

Using the parallel axis theorem, $I_{YY} = I_{GG} + Ad^2$, where $d = 3.0 + 2.0 = 5.0\,\text{cm}$.

Hence $I_{YY} = 4\pi + [\pi(2.0)^2](5.0)^2$

$$= 4\pi + 100\pi = 104\pi = \mathbf{327\,cm^4}$$

Radius of gyration,

$$k_{YY} = \sqrt{\frac{I_{YY}}{\text{area}}} = \sqrt{\left(\frac{104\pi}{\pi(2.0)^2}\right)} = \sqrt{26} = \mathbf{5.10\,cm}$$

Problem 15. Determine the second moment of area and radius of gyration for the semicircle shown in Fig. 38.22 about axis XX.

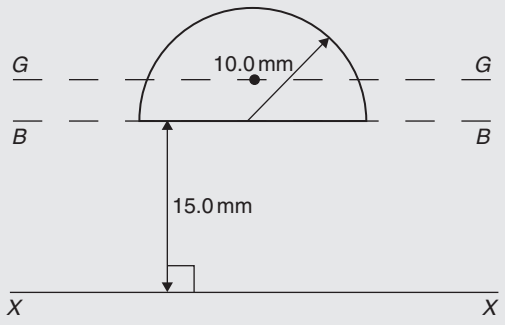


Figure 38.22

The centroid of a semicircle lies at $\frac{4r}{3\pi}$ from its diameter.

Using the parallel axis theorem:

$$I_{BB} = I_{GG} + Ad^2,$$

where $I_{BB} = \frac{\pi r^4}{8}$ (from Table 38.1)

$$= \frac{\pi(10.0)^4}{8} = 3927\,\text{mm}^4,$$

$$A = \frac{\pi r^2}{2} = \frac{\pi(10.0)^2}{2} = 157.1\,\text{mm}^2$$

and $d = \frac{4r}{3\pi} = \frac{4(10.0)}{3\pi} = 4.244\,\text{mm}$

Hence $3927 = I_{GG} + (157.1)(4.244)^2$

i.e. $3927 = I_{GG} + 2830,$

from which, $I_{GG} = 3927 - 2830 = 1097\,\text{mm}^4$

Using the parallel axis theorem again:

$$I_{XX} = I_{GG} + A(15.0 + 4.244)^2$$

i.e. $I_{XX} = 1097 + (157.1)(19.244)^2$

$$= 1097 + 58\,179$$

$$= 59276\,\text{mm}^4 \text{ or } \mathbf{59280\,mm^4},$$

correct to 4 significant figures.

Radius of gyration, $k_{XX} = \sqrt{\frac{I_{XX}}{\text{area}}} = \sqrt{\left(\frac{59\,276}{157.1}\right)}$

$$= \mathbf{19.42\,mm}$$

Problem 16. Determine the polar second moment of area of the propeller shaft cross-section shown in Fig. 38.23.

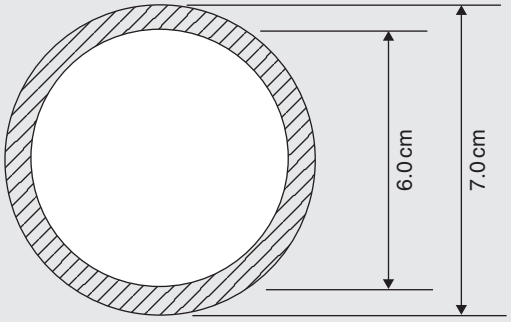


Figure 38.23

The polar second moment of area of a circle $= \frac{\pi r^4}{2}$

The polar second moment of area of the shaded area is given by the polar second moment of area of the 7.0 cm diameter circle minus the polar second moment of area of the 6.0 cm diameter circle.

Hence the polar second moment of area of the cross-section shown

$$= \frac{\pi}{2}\left(\frac{7.0}{2}\right)^4 - \frac{\pi}{2}\left(\frac{6.0}{2}\right)^4$$

$$= 235.7 - 127.2 = \mathbf{108.5\,cm^4}$$

Problem 17. Determine the second moment of area and radius of gyration of a rectangular lamina of length 40 mm and width 15 mm about an axis through one corner, perpendicular to the plane of the lamina.

The lamina is shown in Fig. 38.24.

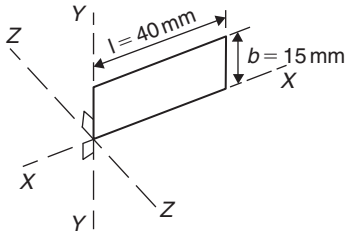


Figure 38.24

From the perpendicular axis theorem:

$$I_{ZZ} = I_{XX} + I_{YY}$$

$$I_{XX} = \frac{lb^3}{3} = \frac{(40)(15)^3}{3} = 45000\,\text{mm}^4$$

and $I_{YY} = \frac{bl^3}{3} = \frac{(15)(40)^3}{3} = 320000\,\text{mm}^4$

Hence $I_{ZZ} = 45\,000 + 320\,000$

$$= \mathbf{365000\,mm^4} \text{ or } \mathbf{36.5\,cm^4}$$

Radius of gyration,

$$k_{ZZ} = \sqrt{\frac{I_{ZZ}}{\text{area}}} = \sqrt{\left(\frac{365\,000}{(40)(15)}\right)}$$

$$= \mathbf{24.7\,mm} \text{ or } \mathbf{2.47\,cm}$$

Problem 18. Determine correct to 3 significant figures, the second moment of area about axis XX for the composite area shown in Fig. 38.25.

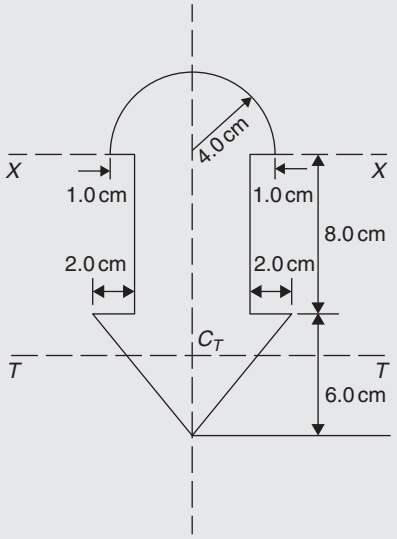


Figure 38.25

For the semicircle,

$$I_{XX} = \frac{\pi r^4}{8} = \frac{\pi(4.0)^4}{8} = 100.5\,\text{cm}^4$$

For the rectangle,

$$I_{XX} = \frac{bl^3}{3} = \frac{(6.0)(8.0)^3}{3} = 1024\,\text{cm}^4$$

For the triangle, about axis TT through centroid C_T,

$$I_{TT} = \frac{bh^3}{36} = \frac{(10)(6.0)^3}{36} = 60\,\text{cm}^4$$

By the parallel axis theorem, the second moment of area of the triangle about axis XX
$= 60 + \left[\frac{1}{2}(10)(6.0)\right]\left[8.0 + \frac{1}{3}(6.0)\right]^2 = 3060\,\text{cm}^4$.

Total second moment of area about XX

$= 100.5 + 1024 + 3060$

$= 4184.5$

$= \mathbf{4180\,cm^4}$, correct to 3 significant figures.

Problem 19. Determine the second moment of area and the radius of gyration about axis XX for the I-section shown in Fig. 38.26.

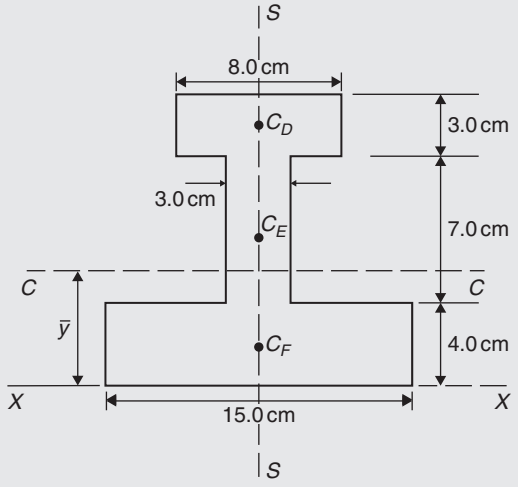


Figure 38.26

The I-section is divided into three rectangles, D, E and F and their centroids denoted by C_D, C_E and C_F respectively.

For rectangle D:
The second moment of area about C_D (an axis through C_D parallel to XX)

$$= \frac{bl^3}{12} = \frac{(8.0)(3.0)^3}{12} = 18\,\text{cm}^4$$

Using the parallel axis theorem:

$$I_{XX} = 18 + Ad^2$$

where $A = (8.0)(3.0) = 24\,\text{cm}^2$ and $d = 12.5\,\text{cm}$

Hence $I_{XX} = 18 + 24(12.5)^2 = 3768\,\text{cm}^4$.

For rectangle E:
The second moment of area about C_E (an axis through C_E parallel to XX)

$$= \frac{bl^3}{12} = \frac{(3.0)(7.0)^3}{12} = 85.75\,\text{cm}^4$$

Using the parallel axis theorem:

$$I_{XX} = 85.75 + (7.0)(3.0)(7.5)^2 = 1267\,\text{cm}^4.$$

For rectangle F:

$$I_{XX} = \frac{bl^3}{3} = \frac{(15.0)(4.0)^3}{3} = 320\,\text{cm}^4$$

Total second moment of area for the *I*-section about axis *XX*,

$$I_{XX} = 3768 + 1267 + 320 = \mathbf{5355\,cm^4}$$

Total area of *I*-section

$$= (8.0)(3.0) + (3.0)(7.0) + (15.0)(4.0)$$

$$= 105\,\text{cm}^2.$$

Radius of gyration,

$$k_{XX} = \sqrt{\frac{I_{XX}}{\text{area}}} = \sqrt{\left(\frac{5355}{105}\right)} = \mathbf{7.14\,cm}$$

Now try the following exercise

Exercise 152 Further problems on second moment of areas of regular sections

1. Determine the second moment of area and radius of gyration for the rectangle shown in Fig. 38.27 about (a) axis *AA* (b) axis *BB* and (c) axis *CC*.

[(a) 72 cm^4, 1.73 cm
(b) 128 cm^4, 2.31 cm
(c) 512 cm^4, 4.62 cm]

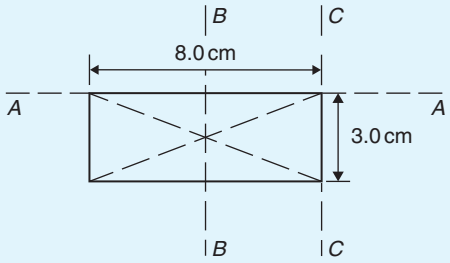


Figure 38.27

2. Determine the second moment of area and radius of gyration for the triangle shown in Fig. 38.28 about (a) axis *DD* (b) axis *EE* and (c) an axis through the centroid of the triangle parallel to axis *DD*.

[(a) 729 cm^4, 3.67 cm
(b) 2187 cm^4, 6.36 cm
(c) 243 cm^4, 2.12 cm]

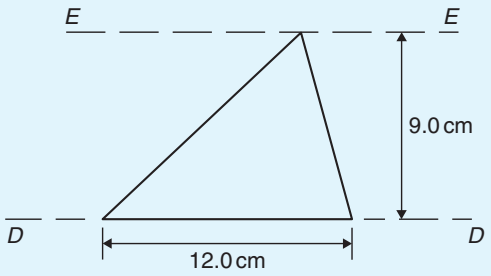


Figure 38.28

3. For the circle shown in Fig. 38.29, find the second moment of area and radius of gyration about (a) axis *FF* and (b) axis *HH*.

[(a) 201 cm^4, 2.0 cm
(b) 1005 cm^4, 4.47 cm]

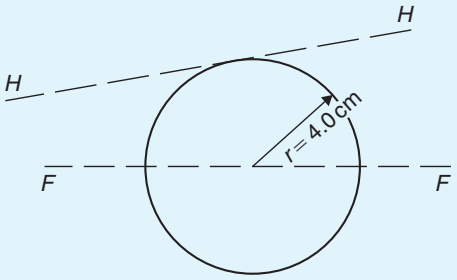


Figure 38.29

4. For the semicircle shown in Fig. 38.30, find the second moment of area and radius of gyration about axis *JJ*.

[3927 mm^4, 5.0 mm]

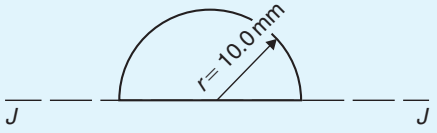


Figure 38.30

5. For each of the areas shown in Fig. 38.31 determine the second moment of area and radius of gyration about axis *LL*, by using the parallel axis theorem.

[(a) 335 cm^4, 4.73 cm
(b) 22030 cm^4, 14.3 cm
(c) 628 cm^4, 7.07 cm]

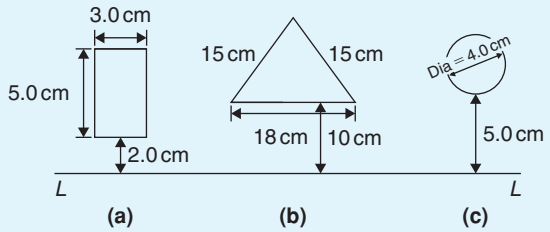


Figure 38.31

6. Calculate the radius of gyration of a rectangular door 2.0 m high by 1.5 m wide about a vertical axis through its hinge.

[0.866 m]

7. A circular door of a boiler is hinged so that it turns about a tangent. If its diameter is 1.0 m, determine its second moment of area and radius of gyration about the hinge.

[0.245 m^4, 0.559 m]

8. A circular cover, centre 0, has a radius of 12.0 cm. A hole of radius 4.0 cm and centre X, where $OX = 6.0$ cm, is cut in the cover. Determine the second moment of area and the radius of gyration of the remainder about a diameter through 0 perpendicular to OX.

[14280 cm^4, 5.96 cm]

9. For the sections shown in Fig. 38.32, find the second moment of area and the radius of gyration about axis XX.

[(a) 12190 mm^4, 10.9 mm
(b) 549.5 cm^4, 4.18 cm]

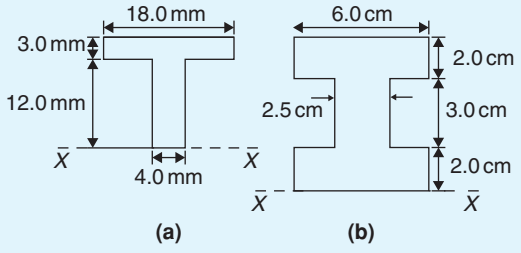


Figure 38.32

10. Determine the second moments of areas about the given axes for the shapes shown in Fig. 38.33. (In Fig. 38.33(b), the circular area is removed.)

[$I_{AA} = 4224\,cm^4$,
$I_{BB} = 6718\,cm^4$,
$I_{CC} = 37300\,cm^4$]

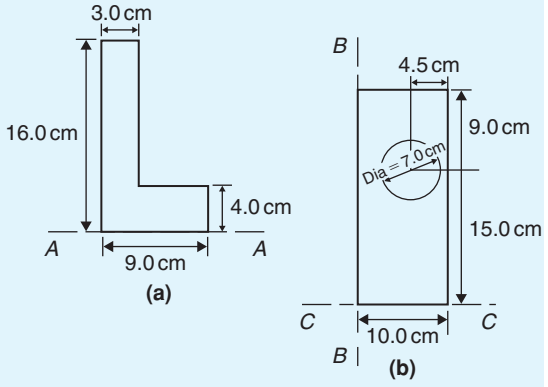


Figure 38.33

11. Find the second moment of area and radius of gyration about the axis XX for the beam section shown in Fig. 38.34.

[1350 cm^4,
5.67 cm]

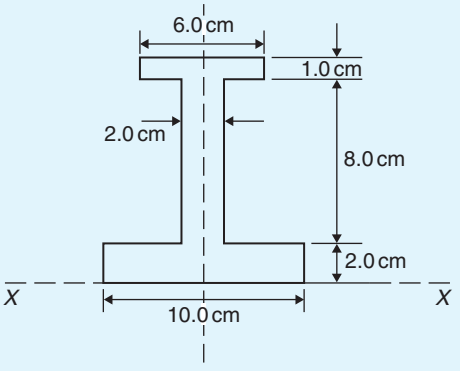


Figure 38.34

Chapter 39

Integration using algebraic substitutions

39.1 Introduction

Functions which require integrating are not always in the 'standard form' shown in Chapter 37. However, it is often possible to change a function into a form which can be integrated by using either:

(i) an algebraic substitution (see Section 39.2),

(ii) a trigonometric or hyperbolic substitution (see Chapter 40),

(iii) partial fractions (see Chapter 41),

(iv) the $t = \tan\theta/2$ substitution (see Chapter 42),

(v) integration by parts (see Chapter 43), or

(vi) reduction formulae (see Chapter 44).

39.2 Algebraic substitutions

With **algebraic substitutions**, the substitution usually made is to let u be equal to $f(x)$ such that $f(u)\,du$ is a standard integral. It is found that integrals of the forms,

$$k\int [f(x)]^n f'(x)\,dx \quad \text{and} \quad k\int \frac{f'(x)}{[f(x)]^n}\,dx$$

(where k and n are constants) can both be integrated by substituting u for $f(x)$.

39.3 Worked problems on integration using algebraic substitutions

Problem 1. Determine $\int \cos(3x+7)\,dx$.

$\int \cos(3x+7)\,dx$ is not a standard integral of the form shown in Table 37.1, page 369, thus an algebraic substitution is made.

Let $u = 3x+7$ then $\dfrac{du}{dx} = 3$ and rearranging gives $dx = \dfrac{du}{3}$. Hence,

$$\int \cos(3x+7)\,dx = \int (\cos u)\frac{du}{3} = \int \frac{1}{3}\cos u\,du,$$

which is a standard integral

$$= \frac{1}{3}\sin u + c$$

Rewriting u as $(3x+7)$ gives:

$$\int \cos(3x+7)\,dx = \frac{\mathbf{1}}{\mathbf{3}}\,\mathbf{\sin(3x+7)+c},$$

which may be checked by differentiating it.

Problem 2. Find $\int (2x-5)^7\,dx$.

$(2x-5)$ may be multiplied by itself 7 times and then each term of the result integrated. However, this would

be a lengthy process, and thus an algebraic substitution is made.

Let $u = (2x - 5)$ then $\frac{du}{dx} = 2$ and $dx = \frac{du}{2}$

Hence

$$\int (2x-5)^7\,dx = \int u^7 \frac{du}{2} = \frac{1}{2}\int u^7\,du$$

$$= \frac{1}{2}\left(\frac{u^8}{8}\right) + c = \frac{1}{16}u^8 + c$$

Rewriting u as $(2x - 5)$ gives:

$$\int (2x-5)^7\,dx = \frac{1}{16}(2x-5)^8 + c$$

Problem 3. Find $\int \frac{4}{(5x-3)}\,dx$.

Let $u = (5x - 3)$ then $\frac{du}{dx} = 5$ and $dx = \frac{du}{5}$

Hence

$$\int \frac{4}{(5x-3)}\,dx = \int \frac{4}{u}\frac{du}{5} = \frac{4}{5}\int \frac{1}{u}\,du$$

$$= \frac{4}{5}\ln u + c = \frac{4}{5}\ln(5x-3) + c$$

Problem 4. Evaluate $\int_0^1 2e^{6x-1}\,dx$, correct to 4 significant figures.

Let $u = 6x - 1$ then $\frac{du}{dx} = 6$ and $dx = \frac{du}{6}$

Hence

$$\int 2e^{6x-1}\,dx = \int 2e^u \frac{du}{6} = \frac{1}{3}\int e^u\,du$$

$$= \frac{1}{3}e^u + c = \frac{1}{3}e^{6x-1} + c$$

Thus

$$\int_0^1 2e^{6x-1}\,dx = \frac{1}{3}[e^{6x-1}]_0^1 = \frac{1}{3}[e^5 - e^{-1}] = \mathbf{49.35},$$

correct to 4 significant figures.

Problem 5. Determine $\int 3x(4x^2 + 3)^5\,dx$.

Let $u = (4x^2 + 3)$ then $\frac{du}{dx} = 8x$ and $dx = \frac{du}{8x}$

Hence

$$\int 3x(4x^2+3)^5\,dx = \int 3x(u)^5 \frac{du}{8x}$$

$$= \frac{3}{8}\int u^5\,du, \text{ by cancelling}$$

The original variable 'x' has been completely removed and the integral is now only in terms of u and is a standard integral.

Hence $\frac{3}{8}\int u^5\,du = \frac{3}{8}\left(\frac{u^6}{6}\right) + c$

$$= \frac{1}{16}u^6 + c = \frac{1}{16}(4x^2+3)^6 + c$$

Problem 6. Evaluate $\int_0^{\frac{\pi}{6}} 24\sin^5\theta\cos\theta\,d\theta$.

Let $u = \sin\theta$ then $\frac{du}{d\theta} = \cos\theta$ and $d\theta = \frac{du}{\cos\theta}$

Hence $\int 24\sin^5\theta\cos\theta\,d\theta = \int 24u^5\cos\theta\frac{du}{\cos\theta}$

$$= 24\int u^5\,du, \text{ by cancelling}$$

$$= 24\frac{u^6}{6} + c = 4u^6 + c = 4(\sin\theta)^6 + c$$

$$= 4\sin^6\theta + c$$

Thus $\int_0^{\frac{\pi}{6}} 24\sin^5\theta\cos\theta\,d\theta = [4\sin^6\theta]_0^{\frac{\pi}{6}}$

$$= 4\left[\left(\sin\frac{\pi}{6}\right)^6 - (\sin 0)^6\right]$$

$$= 4\left[\left(\frac{1}{2}\right)^6 - 0\right] = \frac{1}{16} \text{ or } \mathbf{0.0625}$$

Now try the following exercise

Exercise 153 Further problems on integration using algebraic substitutions

In Problems 1 to 6, integrate with respect to the variable.

1. $2\sin(4x+9)$ $\left[-\frac{1}{2}\cos(4x+9)+c\right]$
2. $3\cos(2\theta-5)$ $\left[\frac{3}{2}\sin(2\theta-5)+c\right]$
3. $4\sec^2(3t+1)$ $\left[\frac{4}{3}\tan(3t+1)+c\right]$
4. $\frac{1}{2}(5x-3)^6$ $\left[\frac{1}{70}(5x-3)^7+c\right]$
5. $\frac{-3}{(2x-1)}$ $\left[-\frac{3}{2}\ln(2x-1)+c\right]$
6. $3e^{3\theta+5}$ $[e^{3\theta+5}+c]$

In Problems 7 to 10, evaluate the definite integrals correct to 4 significant figures.

7. $\int_0^1 (3x+1)^5\,dx$ [227.5]
8. $\int_0^2 x\sqrt{(2x^2+1)}\,dx$ [4.333]
9. $\int_0^{\frac{\pi}{3}} 2\sin\left(3t+\frac{\pi}{4}\right)dt$ [0.9428]
10. $\int_0^1 3\cos(4x-3)\,dx$ [0.7369]

39.4 Further worked problems on integration using algebraic substitutions

Problem 7. Find $\int \frac{x}{2+3x^2}\,dx$.

Let $u=2+3x^2$ then $\frac{du}{dx}=6x$ and $dx=\frac{du}{6x}$

Hence

$$\int \frac{x}{2+3x^2}\,dx = \int \frac{x}{u}\frac{du}{6x} = \frac{1}{6}\int \frac{1}{u}\,du,$$

by cancelling

$$= \frac{1}{6}\ln u + c = \mathbf{\frac{1}{6}\ln(2+3x^2)+c}$$

Problem 8. Determine $\int \frac{2x}{\sqrt{(4x^2-1)}}\,dx$.

Let $u=4x^2-1$ then $\frac{du}{dx}=8x$ and $dx=\frac{du}{8x}$

Hence $\int \frac{2x}{\sqrt{(4x^2-1)}}\,dx = \int \frac{2x}{\sqrt{u}}\frac{du}{8x}$

$$= \frac{1}{4}\int \frac{1}{\sqrt{u}}\,du, \text{ by cancelling}$$

$$= \frac{1}{4}\int u^{\frac{-1}{2}}\,du = \frac{1}{4}\left[\frac{u^{\left(\frac{-1}{2}\right)+1}}{-\frac{1}{2}+1}\right]+c$$

$$= \frac{1}{4}\left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right]+c = \frac{1}{2}\sqrt{u}+c$$

$$= \mathbf{\frac{1}{2}\sqrt{(4x^2-1)}+c}$$

Problem 9. Show that

$$\int \tan\theta\,d\theta = \ln(\sec\theta)+c.$$

$\int \tan\theta\,d\theta = \int \frac{\sin\theta}{\cos\theta}\,d\theta$. Let $u=\cos\theta$

then $\frac{du}{d\theta} = -\sin\theta$ and $d\theta = \frac{-du}{\sin\theta}$

Hence

$$\int \frac{\sin\theta}{\cos\theta}\,d\theta = \int \frac{\sin\theta}{u}\left(\frac{-du}{\sin\theta}\right)$$

$$= -\int \frac{1}{u}\,du = -\ln u + c$$

$$= -\ln(\cos\theta)+c = \ln(\cos\theta)^{-1}+c,$$

by the laws of logarithms.

Hence $\int \tan\theta\,d\theta = \ln(\sec\theta) + c$,

since $(\cos\theta)^{-1} = \dfrac{1}{\cos\theta} = \sec\theta$

39.5 Change of limits

When evaluating definite integrals involving substitutions it is sometimes more convenient to **change the limits** of the integral as shown in Problems 10 and 11.

Problem 10. Evaluate $\int_1^3 5x\sqrt{(2x^2+7)}\,dx$, taking positive values of square roots only.

Let $u = 2x^2+7$, then $\dfrac{du}{dx} = 4x$ and $dx = \dfrac{du}{4x}$

It is possible in this case to change the limits of integration. Thus when $x=3$, $u = 2(3)^2+7 = 25$ and when $x=1$, $u = 2(1)^2+7 = 9$.

Hence

$$\int_{x=1}^{x=3} 5x\sqrt{(2x^2+7)}\,dx = \int_{u=9}^{u=25} 5x\sqrt{u}\frac{du}{4x}$$

$$= \frac{5}{4}\int_9^{25} \sqrt{u}\,du$$

$$= \frac{5}{4}\int_9^{25} u^{\frac{1}{2}}\,du$$

Thus the limits have been changed, and it is unnecessary to change the integral back in terms of x.

$$\text{Thus } \int_{x=1}^{x=3} 5x\sqrt{(2x^2+7)}\,dx = \frac{5}{4}\left[\frac{u^{\frac{3}{2}}}{3/2}\right]_9^{25}$$

$$= \frac{5}{6}\left[\sqrt{u^3}\right]_9^{25} = \frac{5}{6}\left[\sqrt{25^3} - \sqrt{9^3}\right]$$

$$= \frac{5}{6}(125-27) = \mathbf{81\frac{2}{3}}$$

Problem 11. Evaluate $\int_0^2 \dfrac{3x}{\sqrt{(2x^2+1)}}\,dx$, taking positive values of square roots only.

Let $u = 2x^2+1$ then $\dfrac{du}{dx} = 4x$ and $dx = \dfrac{du}{4x}$

$$\text{Hence } \int_0^2 \frac{3x}{\sqrt{(2x^2+1)}}\,dx = \int_{x=0}^{x=2} \frac{3x}{\sqrt{u}}\frac{du}{4x}$$

$$= \frac{3}{4}\int_{x=0}^{x=2} u^{\frac{-1}{2}}\,du$$

Since $u = 2x^2+1$, when $x=2, u=9$ and when $x=0, u=1$.

$$\text{Thus } \frac{3}{4}\int_{x=0}^{x=2} u^{\frac{-1}{2}}\,du = \frac{3}{4}\int_{u=1}^{u=9} u^{\frac{-1}{2}}\,du,$$

i.e. the limits have been changed

$$= \frac{3}{4}\left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^9 = \frac{3}{2}\left[\sqrt{9} - \sqrt{1}\right] = \mathbf{3},$$

taking positive values of square roots only.

Now try the following exercise

Exercise 154 Further problems on integration using algebraic substitutions

In Problems 1 to 7, integrate with respect to the variable.

1. $2x(2x^2-3)^5$ $\left[\dfrac{1}{12}(2x^2-3)^6 + c\right]$
2. $5\cos^5 t \sin t$ $\left[-\dfrac{5}{6}\cos^6 t + c\right]$
3. $3\sec^2 3x \tan 3x$ $\left[\dfrac{1}{2}\sec^2 3x + c \text{ or } \dfrac{1}{2}\tan^2 3x + c\right]$
4. $2t\sqrt{(3t^2-1)}$ $\left[\dfrac{2}{9}\sqrt{(3t^2-1)^3} + c\right]$
5. $\dfrac{\ln\theta}{\theta}$ $\left[\dfrac{1}{2}(\ln\theta)^2 + c\right]$

6. $3\tan 2t$ $\left[\frac{3}{2}\ln(\sec 2t)+c\right]$

7. $\dfrac{2e^t}{\sqrt{(e^t+4)}}$ $\left[4\sqrt{(e^t+4)}+c\right]$

In Problems 8 to 10, evaluate the definite integrals correct to 4 significant figures.

8. $\displaystyle\int_0^1 3x\,e^{(2x^2-1)}\,dx$ [1.763]

9. $\displaystyle\int_0^{\frac{\pi}{2}} 3\sin^4\theta\cos\theta\,d\theta$ [0.6000]

10. $\displaystyle\int_0^1 \frac{3x}{(4x^2-1)^5}\,dx$ [0.09259]

11. The electrostatic potential on all parts of a conducting circular disc of radius r is given by the equation:

$$V = 2\pi\sigma\int_0^9 \frac{R}{\sqrt{R^2+r^2}}\,dR$$

Solve the equation by determining the integral.

$$\left[V = 2\pi\sigma\left\{\sqrt{(9^2+r^2)}-r\right\}\right]$$

12. In the study of a rigid rotor the following integration occurs:

$$Z_r = \int_0^{\infty}(2J+1)e^{\frac{-J(J+1)h^2}{8\pi^2 IkT}}\,dJ$$

Determine Z_r for constant temperature T assuming h, I and k are constants.

$$\left[\frac{8\pi^2 IkT}{h^2}\right]$$

13. In electrostatics,

$$E = \int_0^{\pi}\left\{\frac{a^2\sigma\sin\theta}{2\varepsilon\sqrt{(a^2-x^2-2ax\cos\theta)}}d\theta\right\}$$

where a, σ and ε are constants, x is greater than a, and x is independent of θ. Show that $E = \dfrac{a^2\sigma}{\varepsilon x}$

Revision Test 11

This Revision Test covers the material contained in Chapters 37 to 39. *The marks for each question are shown in brackets at the end of each question.*

1. Determine: (a) $\int 3\sqrt{t^5}\,\mathrm{d}t$ (b) $\int \frac{2}{\sqrt[3]{x^2}}\,\mathrm{d}x$

   (c) $\int (2+\theta)^2\,\mathrm{d}\theta$ (9)

2. Evaluate the following integrals, each correct to 4 significant figures:

   (a) $\int_0^{\frac{\pi}{3}} 3\sin 2t\,\mathrm{d}t$ (b) $\int_1^2 \left(\frac{2}{x^2}+\frac{1}{x}+\frac{3}{4}\right)\mathrm{d}x$

   (c) $\int_0^1 \frac{3}{\mathrm{e}^{2t}}\,\mathrm{d}t$ (15)

3. Calculate the area between the curve $y=x^3-x^2-6x$ and the x-axis. (10)

4. A voltage $v=25\sin 50\pi t$ volts is applied across an electrical circuit. Determine, using integration, its mean and r.m.s. values over the range $t=0$ to $t=20\,\text{ms}$, each correct to 4 significant figures. (12)

5. Sketch on the same axes the curves $x^2=2y$ and $y^2=16x$ and determine the co-ordinates of the points of intersection. Determine (a) the area enclosed by the curves, and (b) the volume of the solid produced if the area is rotated one revolution about the x-axis. (13)

6. Calculate the position of the centroid of the sheet of metal formed by the x-axis and the part of the curve $y=5x-x^2$ which lies above the x-axis. (9)

7. A cylindrical pillar of diameter 400 mm has a groove cut around its circumference as shown in Fig. RT11.1. The section of the groove is a semi-circle of diameter 50 mm. Given that the centroid of a semicircle from its base is $\frac{4r}{3\pi}$, use the theorem of Pappus to determine the volume of material removed, in cm^3, correct to 3 significant figures. (8)

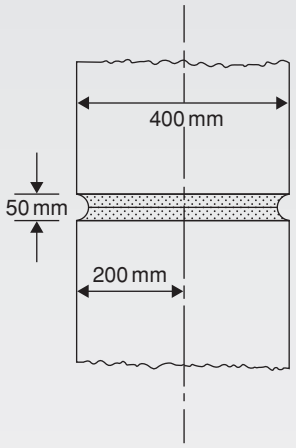


Figure RT11.1

8. A circular door is hinged so that it turns about a tangent. If its diameter is 1.0 m find its second moment of area and radius of gyration about the hinge. (5)

9. Determine the following integrals:

   (a) $\int 5(6t+5)^7\,\mathrm{d}t$ (b) $\int \frac{3\ln x}{x}\,\mathrm{d}x$

   (c) $\int \frac{2}{\sqrt{(2\theta-1)}}\,\mathrm{d}\theta$ (9)

10. Evaluate the following definite integrals:

    (a) $\int_0^{\frac{\pi}{2}} 2\sin\left(2t+\frac{\pi}{3}\right)\mathrm{d}t$ (b) $\int_0^1 3x\,\mathrm{e}^{4x^2-3}\,\mathrm{d}x$ (10)

Chapter 40

Integration using trigonometric and hyperbolic substitutions

40.1 Introduction

Table 40.1 gives a summary of the integrals that require the use of **trigonometric and hyperbolic substitutions** and their application is demonstrated in Problems 1 to 27.

40.2 Worked problems on integration of $\sin^2 x$, $\cos^2 x$, $\tan^2 x$ and $\cot^2 x$

Problem 1. Evaluate $\int_0^{\frac{\pi}{4}} 2\cos^2 4t \, dt$.

Since $\cos 2t = 2\cos^2 t - 1$ (from Chapter 17),

then $\cos^2 t = \frac{1}{2}(1 + \cos 2t)$ and

$\cos^2 4t = \frac{1}{2}(1 + \cos 8t)$

Hence $\int_0^{\frac{\pi}{4}} 2\cos^2 4t \, dt$

$= 2\int_0^{\frac{\pi}{4}} \frac{1}{2}(1 + \cos 8t)\, dt$

$= \left[t + \frac{\sin 8t}{8}\right]_0^{\frac{\pi}{4}}$

$= \left[\frac{\pi}{4} + \frac{\sin 8\left(\frac{\pi}{4}\right)}{8}\right] - \left[0 + \frac{\sin 0}{8}\right]$

$= \frac{\pi}{4}$ or **0.7854**

Problem 2. Determine $\int \sin^2 3x \, dx$.

Since $\cos 2x = 1 - 2\sin^2 x$ (from Chapter 17),

then $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and

$\sin^2 3x = \frac{1}{2}(1 - \cos 6x)$

Hence $\int \sin^2 3x \, dx = \int \frac{1}{2}(1 - \cos 6x)\, dx$

$= \frac{1}{2}\left(x - \frac{\sin 6x}{6}\right) + c$

Problem 3. Find $3\int \tan^2 4x \, dx$.

Since $1 + \tan^2 x = \sec^2 x$, then $\tan^2 x = \sec^2 x - 1$ and $\tan^2 4x = \sec^2 4x - 1$.

Hence $3\int \tan^2 4x \, dx = 3\int (\sec^2 4x - 1)\, dx$

$= 3\left(\frac{\tan 4x}{4} - x\right) + c$

Table 40.1 Integrals using trigonometric and hyperbolic substitutions

	$f(x)$	$\int f(x)\mathrm{d}x$	Method	See problem
1.	$\cos^2 x$	$\frac{1}{2}\left(x+\frac{\sin 2x}{2}\right)+c$	Use $\cos 2x = 2\cos^2 x - 1$	1
2.	$\sin^2 x$	$\frac{1}{2}\left(x-\frac{\sin 2x}{2}\right)+c$	Use $\cos 2x = 1 - 2\sin^2 x$	2
3.	$\tan^2 x$	$\tan x - x + c$	Use $1 + \tan^2 x = \sec^2 x$	3
4.	$\cot^2 x$	$-\cot x - x + c$	Use $\cot^2 x + 1 = \operatorname{cosec}^2 x$	4
5.	$\cos^m x \sin^n x$	(a) If either m or n is odd (but not both), use $\cos^2 x + \sin^2 x = 1$		5, 6
		(b) If both m and n are even, use either $\cos 2x = 2\cos^2 x - 1$ or $\cos 2x = 1 - 2\sin^2 x$		7, 8
6.	$\sin A \cos B$		Use $\frac{1}{2}[\sin(A+B) + \sin(A-B)]$	9
7.	$\cos A \sin B$		Use $\frac{1}{2}[\sin(A+B) - \sin(A-B)]$	10
8.	$\cos A \cos B$		Use $\frac{1}{2}[\cos(A+B) + \cos(A-B)]$	11
9.	$\sin A \sin B$		Use $-\frac{1}{2}[\cos(A+B) - \cos(A-B)]$	12
10.	$\frac{1}{\sqrt{(a^2-x^2)}}$	$\sin^{-1}\frac{x}{a}+c$	Use $x = a\sin\theta$ substitution	13, 14
11.	$\sqrt{(a^2-x^2)}$	$\frac{a^2}{2}\sin^{-1}\frac{x}{a}+\frac{x}{2}\sqrt{(a^2-x^2)}+c$	Use $x = a\sin\theta$ substitution	15, 16
12.	$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}+c$	Use $x = a\tan\theta$ substitution	17–19
13.	$\frac{1}{\sqrt{(x^2+a^2)}}$	$\sinh^{-1}\frac{x}{a}+c$ or $\ln\left\{\frac{x+\sqrt{(x^2+a^2)}}{a}\right\}+c$	Use $x = a\sinh\theta$ substitution	20–22
14.	$\sqrt{(x^2+a^2)}$	$\frac{a^2}{2}\sinh^{-1}\frac{x}{a}+\frac{x}{2}\sqrt{(x^2+a^2)}+c$	Use $x = a\sinh\theta$ substitution	23
15.	$\frac{1}{\sqrt{(x^2-a^2)}}$	$\cosh^{-1}\frac{x}{a}+c$ or $\ln\left\{\frac{x+\sqrt{(x^2-a^2)}}{a}\right\}+c$	Use $x = a\cosh\theta$ substitution	24, 25
16.	$\sqrt{(x^2-a^2)}$	$\frac{x}{2}\sqrt{(x^2-a^2)}-\frac{a^2}{2}\cosh^{-1}\frac{x}{a}+c$	Use $x = a\cosh\theta$ substitution	26, 27

Problem 4. Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2}\cot^2 2\theta \, d\theta$.

Since $\cot^2\theta + 1 = \operatorname{cosec}^2\theta$, then $\cot^2\theta = \operatorname{cosec}^2\theta - 1$ and $\cot^2 2\theta = \operatorname{cosec}^2 2\theta - 1$.

Hence $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2}\cot^2 2\theta \, d\theta$

$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (\operatorname{cosec}^2 2\theta - 1)\, d\theta = \frac{1}{2}\left[\frac{-\cot 2\theta}{2} - \theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{1}{2}\left[\left(\frac{-\cot 2\left(\frac{\pi}{3}\right)}{2} - \frac{\pi}{3}\right) - \left(\frac{-\cot 2\left(\frac{\pi}{6}\right)}{2} - \frac{\pi}{6}\right)\right]$$

$$= \frac{1}{2}[(0.2887 - 1.0472) - (-0.2887 - 0.5236)]$$

$$= \mathbf{0.0269}$$

Now try the following exercise

Exercise 155 Further problems on integration of $\sin^2 x$, $\cos^2 x$, $\tan^2 x$ and $\cot^2 x$

In Problems 1 to 4, integrate with respect to the variable.

1. $\sin^2 2x$ $\left[\frac{1}{2}\left(x - \frac{\sin 4x}{4}\right) + c\right]$
2. $3\cos^2 t$ $\left[\frac{3}{2}\left(t + \frac{\sin 2t}{2}\right) + c\right]$
3. $5\tan^2 3\theta$ $\left[5\left(\frac{1}{3}\tan 3\theta - \theta\right) + c\right]$
4. $2\cot^2 2t$ $[-(\cot 2t + 2t) + c]$

In Problems 5 to 8, evaluate the definite integrals, correct to 4 significant figures.

5. $\int_0^{\frac{\pi}{3}} 3\sin^2 3x \, dx$ $\left[\frac{\pi}{2}\text{ or }1.571\right]$
6. $\int_0^{\frac{\pi}{4}} \cos^2 4x \, dx$ $\left[\frac{\pi}{8}\text{ or }0.3927\right]$
7. $\int_0^1 2\tan^2 2t \, dt$ $[-4.185]$
8. $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cot^2\theta \, d\theta$ $[0.6311]$

40.3 Worked problems on powers of sines and cosines

Problem 5. Determine $\int \sin^5\theta \, d\theta$.

Since $\cos^2\theta + \sin^2\theta = 1$ then $\sin^2\theta = (1 - \cos^2\theta)$.

Hence $\int \sin^5\theta \, d\theta$

$$= \int \sin\theta(\sin^2\theta)^2 \, d\theta = \int \sin\theta(1 - \cos^2\theta)^2 \, d\theta$$

$$= \int \sin\theta(1 - 2\cos^2\theta + \cos^4\theta)\, d\theta$$

$$= \int (\sin\theta - 2\sin\theta\cos^2\theta + \sin\theta\cos^4\theta)\, d\theta$$

$$= \mathbf{-\cos\theta + \frac{2\cos^3\theta}{3} - \frac{\cos^5\theta}{5} + c}$$

Whenever a power of a cosine is multiplied by a sine of power 1, or vice-versa, the integral may be determined by inspection as shown.

In general, $\int \cos^n\theta \sin\theta \, d\theta = \frac{-\cos^{n+1}\theta}{(n+1)} + c$

and $\int \sin^n\theta \cos\theta \, d\theta = \frac{\sin^{n+1}\theta}{(n+1)} + c$

Problem 6. Evaluate $\int_0^{\frac{\pi}{2}} \sin^2 x \cos^3 x \, dx$.

$$\int_0^{\frac{\pi}{2}} \sin^2 x \cos^3 x \, dx = \int_0^{\frac{\pi}{2}} \sin^2 x \cos^2 x \cos x \, dx$$

$$= \int_0^{\frac{\pi}{2}} (\sin^2 x)(1 - \sin^2 x)(\cos x)\, dx$$

$$= \int_0^{\frac{\pi}{2}} (\sin^2 x \cos x - \sin^4 x \cos x)\, dx$$

$$= \left[\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5}\right]_0^{\frac{\pi}{2}}$$

$$= \left[\frac{\left(\sin\frac{\pi}{2}\right)^3}{3} - \frac{\left(\sin\frac{\pi}{2}\right)^5}{5}\right] - [0 - 0]$$

$$= \frac{1}{3} - \frac{1}{5} = \mathbf{\frac{2}{15}}\text{ or }\mathbf{0.1333}$$

Problem 7. Evaluate $\int_0^{\frac{\pi}{4}} 4\cos^4\theta \, d\theta$, correct to 4 significant figures.

$$\int_0^{\frac{\pi}{4}} 4\cos^4\theta\,d\theta = 4\int_0^{\frac{\pi}{4}} (\cos^2\theta)^2\,d\theta$$

$$= 4\int_0^{\frac{\pi}{4}} \left[\frac{1}{2}(1+\cos 2\theta)\right]^2 d\theta$$

$$= \int_0^{\frac{\pi}{4}} (1+2\cos 2\theta+\cos^2 2\theta)\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left[1+2\cos 2\theta+\frac{1}{2}(1+\cos 4\theta)\right] d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{3}{2}+2\cos 2\theta+\frac{1}{2}\cos 4\theta\right) d\theta$$

$$= \left[\frac{3\theta}{2}+\sin 2\theta+\frac{\sin 4\theta}{8}\right]_0^{\frac{\pi}{4}}$$

$$= \left[\frac{3}{2}\left(\frac{\pi}{4}\right)+\sin\frac{2\pi}{4}+\frac{\sin 4(\pi/4)}{8}\right]-[0]$$

$$= \frac{3\pi}{8}+1 = \mathbf{2.178},$$

correct to 4 significant figures.

Problem 8. Find $\int \sin^2 t\cos^4 t\,dt$.

$$\int \sin^2 t\cos^4 t\,dt = \int \sin^2 t(\cos^2 t)^2\,dt$$

$$= \int \left(\frac{1-\cos 2t}{2}\right)\left(\frac{1+\cos 2t}{2}\right)^2 dt$$

$$= \frac{1}{8}\int (1-\cos 2t)(1+2\cos 2t+\cos^2 2t)\,dt$$

$$= \frac{1}{8}\int (1+2\cos 2t+\cos^2 2t-\cos 2t$$
$$-2\cos^2 2t-\cos^3 2t)\,dt$$

$$= \frac{1}{8}\int (1+\cos 2t-\cos^2 2t-\cos^3 2t)\,dt$$

$$= \frac{1}{8}\int \left[1+\cos 2t-\left(\frac{1+\cos 4t}{2}\right)\right.$$
$$\left.-\cos 2t(1-\sin^2 2t)\right] dt$$

$$= \frac{1}{8}\int \left(\frac{1}{2}-\frac{\cos 4t}{2}+\cos 2t\sin^2 2t\right) dt$$

$$= \mathbf{\frac{1}{8}\left(\frac{t}{2}-\frac{\sin 4t}{8}+\frac{\sin^3 2t}{6}\right)+c}$$

Now try the following exercise

Exercise 156 Further problems on integration of powers of sines and cosines

In Problems 1 to 6, integrate with respect to the variable.

1. $\sin^3\theta$ $\left[(a)-\cos\theta+\frac{\cos^3\theta}{3}+c\right]$
2. $2\cos^3 2x$ $\left[\sin 2x-\frac{\sin^3 2x}{3}+c\right]$
3. $2\sin^3 t\cos^2 t$ $\left[\frac{-2}{3}\cos^3 t+\frac{2}{5}\cos^5 t+c\right]$
4. $\sin^3 x\cos^4 x$ $\left[\frac{-\cos^5 x}{5}+\frac{\cos^7 x}{7}+c\right]$
5. $2\sin^4 2\theta$ $\left[\frac{3\theta}{4}-\frac{1}{4}\sin 4\theta+\frac{1}{32}\sin 8\theta+c\right]$
6. $\sin^2 t\cos^2 t$ $\left[\frac{t}{8}-\frac{1}{32}\sin 4t+c\right]$

40.4 Worked problems on integration of products of sines and cosines

Problem 9. Determine $\int \sin 3t\cos 2t\,dt$.

$$\int \sin 3t\cos 2t\,dt$$

$$= \int \frac{1}{2}[\sin(3t+2t)+\sin(3t-2t)]\,dt,$$

from 6 of Table 40.1, which follows from Section 17.4, page 170,

$$= \frac{1}{2}\int (\sin 5t+\sin t)\,dt$$

$$= \mathbf{\frac{1}{2}\left(\frac{-\cos 5t}{5}-\cos t\right)+c}$$

Problem 10. Find $\int \frac{1}{3}\cos 5x\sin 2x\,dx$.

$$\int \frac{1}{3}\cos 5x \sin 2x \, dx$$
$$= \frac{1}{3}\int \frac{1}{2}[\sin(5x+2x) - \sin(5x-2x)]\,dx,$$
from 7 of Table 40.1
$$= \frac{1}{6}\int (\sin 7x - \sin 3x)\,dx$$
$$= \frac{1}{6}\left(\frac{-\cos 7x}{7} + \frac{\cos 3x}{3}\right) + c$$

Problem 11. Evaluate $\int_0^1 2\cos 6\theta \cos\theta \, d\theta$, correct to 4 decimal places.

$$\int_0^1 2\cos 6\theta \cos\theta \, d\theta$$
$$= 2\int_0^1 \frac{1}{2}[\cos(6\theta+\theta) + \cos(6\theta-\theta)]\,d\theta,$$
from 8 of Table 40.1
$$= \int_0^1 (\cos 7\theta + \cos 5\theta)\,d\theta = \left[\frac{\sin 7\theta}{7} + \frac{\sin 5\theta}{5}\right]_0^1$$
$$= \left(\frac{\sin 7}{7} + \frac{\sin 5}{5}\right) - \left(\frac{\sin 0}{7} + \frac{\sin 0}{5}\right)$$

'sin 7' means 'the sine of 7 radians' ($\equiv 401°4'$) and $\sin 5 \equiv 286°29'$.

Hence $\int_0^1 2\cos 6\theta \cos\theta \, d\theta$
$= (0.09386 + (-0.19178)) - (0)$
$= \mathbf{-0.0979}$, correct to 4 decimal places.

Problem 12. Find $3\int \sin 5x \sin 3x \, dx$.

$$3\int \sin 5x \sin 3x \, dx$$
$$= 3\int -\frac{1}{2}[\cos(5x+3x) - \cos(5x-3x)]\,dx,$$
from 9 of Table 40.1
$$= -\frac{3}{2}\int (\cos 8x - \cos 2x)\,dx$$
$$= -\frac{3}{2}\left(\frac{\sin 8}{8} - \frac{\sin 2x}{2}\right) + c \quad \text{or}$$
$$\frac{3}{16}(4\sin 2x - \sin 8x) + c$$

Now try the following exercise

Exercise 157 Further problems on integration of products of sines and cosines

In Problems 1 to 4, integrate with respect to the variable.

1. $\sin 5t \cos 2t$ $\left[-\frac{1}{2}\left(\frac{\cos 7t}{7} + \frac{\cos 3t}{3}\right) + c\right]$
2. $2\sin 3x \sin x$ $\left[\frac{\sin 2x}{2} - \frac{\sin 4x}{4} + c\right]$
3. $3\cos 6x \cos x$ $\left[\frac{3}{2}\left(\frac{\sin 7x}{7} + \frac{\sin 5x}{5}\right) + c\right]$
4. $\frac{1}{2}\cos 4\theta \sin 2\theta$ $\left[\frac{1}{4}\left(\frac{\cos 2\theta}{2} - \frac{\cos 6\theta}{6}\right) + c\right]$

In Problems 5 to 8, evaluate the definite integrals.

5. $\int_0^{\frac{\pi}{2}} \cos 4x \cos 3x \, dx$ [(a) $\frac{3}{7}$ or 0.4286]
6. $\int_0^1 2\sin 7t \cos 3t \, dt$ [0.5973]
7. $-4\int_0^{\frac{\pi}{3}} \sin 5\theta \sin 2\theta \, d\theta$ [0.2474]
8. $\int_1^2 3\cos 8t \sin 3t \, dt$ [−0.1999]

40.5 Worked problems on integration using the sin θ substitution

Problem 13. Determine $\int \frac{1}{\sqrt{(a^2 - x^2)}}\,dx$.

Let $x = a\sin\theta$, then $\frac{dx}{d\theta} = a\cos\theta$ and $dx = a\cos\theta\,d\theta$.

Hence $\int \frac{1}{\sqrt{(a^2 - x^2)}}\,dx$
$$= \int \frac{1}{\sqrt{(a^2 - a^2\sin^2\theta)}}\,a\cos\theta\,d\theta$$
$$= \int \frac{a\cos\theta\,d\theta}{\sqrt{[a^2(1 - \sin^2\theta)]}}$$

$$= \int \frac{a\cos\theta\,d\theta}{\sqrt{(a^2\cos^2\theta)}}, \quad \text{since } \sin^2\theta + \cos^2\theta = 1$$

$$= \int \frac{a\cos\theta\,d\theta}{a\cos\theta} = \int d\theta = \theta + c$$

Since $x = a\sin\theta$, then $\sin\theta = \frac{x}{a}$ and $\theta = \sin^{-1}\frac{x}{a}$.

Hence $\int \frac{1}{\sqrt{(a^2 - x^2)}}\,dx = \mathbf{\sin^{-1}\frac{x}{a} + c}$

Problem 14. Evaluate $\int_0^3 \frac{1}{\sqrt{(9 - x^2)}}\,dx$.

From Problem 13, $\int_0^3 \frac{1}{\sqrt{(9 - x^2)}}\,dx$

$$= \left[\sin^{-1}\frac{x}{3}\right]_0^3, \quad \text{since } a = 3$$

$$= (\sin^{-1} 1 - \sin^{-1} 0) = \mathbf{\frac{\pi}{2}} \text{ or } \mathbf{1.5708}$$

Problem 15. Find $\int \sqrt{(a^2 - x^2)}\,dx$.

Let $x = a\sin\theta$ then $\frac{dx}{d\theta} = a\cos\theta$ and $dx = a\cos\theta\,d\theta$.

Hence $\int \sqrt{(a^2 - x^2)}\,dx$

$$= \int \sqrt{(a^2 - a^2\sin^2\theta)}\,(a\cos\theta\,d\theta)$$

$$= \int \sqrt{[a^2(1 - \sin^2\theta)]}\,(a\cos\theta\,d\theta)$$

$$= \int \sqrt{(a^2\cos^2\theta)}\,(a\cos\theta\,d\theta)$$

$$= \int (a\cos\theta)(a\cos\theta\,d\theta)$$

$$= a^2 \int \cos^2\theta\,d\theta = a^2 \int \left(\frac{1 + \cos 2\theta}{2}\right) d\theta$$

(since $\cos 2\theta = 2\cos^2\theta - 1$)

$$= \frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + c$$

$$= \frac{a^2}{2}\left(\theta + \frac{2\sin\theta\cos\theta}{2}\right) + c$$

since from Chapter 17, $\sin 2\theta = 2\sin\theta\cos\theta$

$$= \frac{a^2}{2}[\theta + \sin\theta\cos\theta] + c$$

Since $x = a\sin\theta$, then $\sin\theta = \frac{x}{a}$ and $\theta = \sin^{-1}\frac{x}{a}$

Also, $\cos^2\theta + \sin^2\theta = 1$, from which,

$$\cos\theta = \sqrt{(1 - \sin^2\theta)} = \sqrt{\left[1 - \left(\frac{x}{a}\right)^2\right]}$$

$$= \sqrt{\left(\frac{a^2 - x^2}{a^2}\right)} = \frac{\sqrt{(a^2 - x^2)}}{a}$$

Thus $\int \sqrt{(a^2 - x^2)}\,dx = \frac{a^2}{2}[\theta + \sin\theta\cos\theta]$

$$= \frac{a^2}{2}\left[\sin^{-1}\frac{x}{a} + \left(\frac{x}{a}\right)\frac{\sqrt{(a^2 - x^2)}}{a}\right] + c$$

$$= \mathbf{\frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{(a^2 - x^2)} + c}$$

Problem 16. Evaluate $\int_0^4 \sqrt{(16 - x^2)}\,dx$.

From Problem 15, $\int_0^4 \sqrt{(16 - x^2)}\,dx$

$$= \left[\frac{16}{2}\sin^{-1}\frac{x}{4} + \frac{x}{2}\sqrt{(16 - x^2)}\right]_0^4$$

$$= \left[8\sin^{-1}1 + 2\sqrt{(0)}\right] - [8\sin^{-1}0 + 0]$$

$$= 8\sin^{-1}1 = 8\left(\frac{\pi}{2}\right) = \mathbf{4\pi} \text{ or } \mathbf{12.57}$$

Now try the following exercise

Exercise 158 Further problems on integration using the sine θ substitution

1. Determine $\int \frac{5}{\sqrt{(4 - t^2)}}\,dt$.

   $\left[5\sin^{-1}\frac{x}{2} + c\right]$

2. Determine $\int \frac{3}{\sqrt{(9 - x^2)}}\,dx$.

   $\left[3\sin^{-1}\frac{x}{3} + c\right]$

3. Determine $\int \sqrt{(4 - x^2)}\,dx$.

   $\left[2\sin^{-1}\frac{x}{2} + \frac{x}{2}\sqrt{(4 - x^2)} + c\right]$

4. Determine $\int \sqrt{(16-9t^2)}\,dt$. $\left[\frac{8}{3}\sin^{-1}\frac{3t}{4}+\frac{t}{2}\sqrt{(16-9t^2)}+c\right]$

5. Evaluate $\int_0^4 \frac{1}{\sqrt{(16-x^2)}}\,dx$. $\left[\frac{\pi}{2}\text{ or }1.571\right]$

6. Evaluate $\int_0^1 \sqrt{(9-4x^2)}\,dx$. [2.760]

40.6 Worked problems on integration using $\tan\theta$ substitution

Problem 17. Determine $\int \frac{1}{(a^2+x^2)}\,dx$.

Let $x = a\tan\theta$ then $\frac{dx}{d\theta} = a\sec^2\theta$ and $dx = a\sec^2\theta\,d\theta$.

Hence
$$\int \frac{1}{(a^2+x^2)}\,dx$$
$$= \int \frac{1}{(a^2+a^2\tan^2\theta)}(a\sec^2\theta\,d\theta)$$
$$= \int \frac{a\sec^2\theta\,d\theta}{a^2(1+\tan^2\theta)}$$
$$= \int \frac{a\sec^2\theta\,d\theta}{a^2\sec^2\theta}, \quad \text{since } 1+\tan^2\theta=\sec^2\theta$$
$$= \int \frac{1}{a}\,d\theta = \frac{1}{a}(\theta)+c$$

Since $x = a\tan\theta$, $\theta = \tan^{-1}\frac{x}{a}$

Hence $\int \frac{\mathbf{1}}{(\boldsymbol{a^2+x^2})}\,\mathbf{dx} = \frac{\mathbf{1}}{\boldsymbol{a}}\,\mathbf{tan^{-1}}\,\frac{\boldsymbol{x}}{\boldsymbol{a}}+\boldsymbol{c}$

Problem 18. Evaluate $\int_0^2 \frac{1}{(4+x^2)}\,dx$.

From Problem 17, $\int_0^2 \frac{1}{(4+x^2)}\,dx$
$$= \frac{1}{2}\left[\tan^{-1}\frac{x}{2}\right]_0^2 \quad \text{since } a=2$$
$$= \frac{1}{2}(\tan^{-1}1-\tan^{-1}0) = \frac{1}{2}\left(\frac{\pi}{4}-0\right)$$
$$= \frac{\pi}{8} \text{ or } \mathbf{0.3927}$$

Problem 19. Evaluate $\int_0^1 \frac{5}{(3+2x^2)}\,dx$, correct to 4 decimal places.

$$\int_0^1 \frac{5}{(3+2x^2)}\,dx = \int_0^1 \frac{5}{2[(3/2)+x^2]}\,dx$$
$$= \frac{5}{2}\int_0^1 \frac{1}{[\sqrt{(3/2)}]^2+x^2}\,dx$$
$$= \frac{5}{2}\left[\frac{1}{\sqrt{(3/2)}}\tan^{-1}\frac{x}{\sqrt{(3/2)}}\right]_0^1$$
$$= \frac{5}{2}\sqrt{\left(\frac{2}{3}\right)}\left[\tan^{-1}\sqrt{\left(\frac{2}{3}\right)}-\tan^{-1}0\right]$$
$$= (2.0412)[0.6847-0]$$
$$= \mathbf{1.3976}, \text{ correct to 4 decimal places.}$$

Now try the following exercise

Exercise 159 Further problems on integration using the $\tan\theta$ substitution

1. Determine $\int \frac{3}{4+t^2}\,dt$. $\left[\frac{3}{2}\tan^{-1}\frac{t}{2}+c\right]$

2. Determine $\int \frac{5}{16+9\theta^2}\,d\theta$. $\left[\frac{5}{12}\tan^{-1}\frac{3\theta}{4}+c\right]$

3. Evaluate $\int_0^1 \frac{3}{1+t^2}\,dt$. [2.356]

4. Evaluate $\int_0^3 \frac{5}{4+x^2}\,dx$. [2.457]

40.7 Worked problems on integration using the $\sinh\theta$ substitution

Problem 20. Determine $\int \frac{1}{\sqrt{(x^2+a^2)}}\,dx$.

Let $x = a\sinh\theta$, then $\frac{dx}{d\theta} = a\cosh\theta$ and $dx = a\cosh\theta\,d\theta$

Hence $\int \frac{1}{\sqrt{(x^2+a^2)}}\,dx$

$= \int \frac{1}{\sqrt{(a^2\sinh^2\theta + a^2)}}(a\cosh\theta\,d\theta)$

$= \int \frac{a\cosh\theta\,d\theta}{\sqrt{(a^2\cosh^2\theta)}}$,

since $\cosh^2\theta - \sinh^2\theta = 1$

$= \int \frac{a\cosh\theta}{a\cosh\theta}\,d\theta = \int d\theta = \theta + c$

$= \mathbf{sinh^{-1}}\,\boldsymbol{\frac{x}{a}} + \boldsymbol{c}$, since $x = a\sinh\theta$

It is shown on page 339 that

$$\sinh^{-1}\frac{x}{a} = \ln\left\{\frac{x+\sqrt{(x^2+a^2)}}{a}\right\},$$

which provides an alternative solution to

$$\int \frac{1}{\sqrt{(x^2+a^2)}}\,dx$$

Problem 21. Evaluate $\int_0^2 \frac{1}{\sqrt{(x^2+4)}}\,dx$, correct to 4 decimal places.

$$\int_0^2 \frac{1}{\sqrt{(x^2+4)}}\,dx = \left[\sinh^{-1}\frac{x}{2}\right]_0^2 \text{ or } \left[\ln\left\{\frac{x+\sqrt{(x^2+4)}}{2}\right\}\right]_0^2$$

from Problem 20, where $a = 2$

Using the logarithmic form,

$\int_0^2 \frac{1}{\sqrt{(x^2+4)}}\,dx$

$= \left[\ln\left(\frac{2+\sqrt{8}}{2}\right) - \ln\left(\frac{0+\sqrt{4}}{2}\right)\right]$

$= \ln 2.4142 - \ln 1 = \mathbf{0.8814}$,

correct to 4 decimal places.

Problem 22. Evaluate $\int_1^2 \frac{2}{x^2\sqrt{(1+x^2)}}\,dx$, correct to 3 significant figures.

Since the integral contains a term of the form $\sqrt{(a^2+x^2)}$, then let $x = \sinh\theta$, from which $\frac{dx}{d\theta} = \cosh\theta$ and $dx = \cosh\theta\,d\theta$

Hence $\int \frac{2}{x^2\sqrt{(1+x^2)}}\,dx$

$= \int \frac{2(\cosh\theta\,d\theta)}{\sinh^2\theta\sqrt{(1+\sinh^2\theta)}}$

$= 2\int \frac{\cosh\theta\,d\theta}{\sinh^2\theta\cosh\theta}$,

since $\cosh^2\theta - \sinh^2\theta = 1$

$= 2\int \frac{d\theta}{\sinh^2\theta} = 2\int \operatorname{cosech}^2\theta\,d\theta$

$= -2\coth\theta + c$

$\coth\theta = \frac{\cosh\theta}{\sinh\theta} = \frac{\sqrt{(1+\sinh^2\theta)}}{\sinh\theta} = \frac{\sqrt{(1+x^2)}}{x}$

Hence $\int_1^2 \frac{2}{x^2\sqrt{1+x^2)}}\,dx$

$= -[2\coth\theta]_1^2 = -2\left[\frac{\sqrt{(1+x^2)}}{x}\right]_1^2$

$= -2\left[\frac{\sqrt{5}}{2} - \frac{\sqrt{2}}{1}\right] = \mathbf{0.592}$,

correct to 3 significant figures

Problem 23. Find $\int \sqrt{(x^2+a^2)}\,dx$.

Let $x = a\sinh\theta$ then $\frac{dx}{d\theta} = a\cosh\theta$ and $dx = a\cosh\theta\,d\theta$

Hence $\int \sqrt{(x^2+a^2)}\,dx$

$= \int \sqrt{(a^2\sinh^2\theta + a^2)}(a\cosh\theta\,d\theta)$

$= \int \sqrt{[a^2(\sinh^2\theta + 1)]}(a\cosh\theta\,d\theta)$

$= \int \sqrt{(a^2\cosh^2\theta)}\,(a\cosh\theta\,d\theta)$,

since $\cosh^2\theta - \sinh^2\theta = 1$

$= \int (a\cosh\theta)(a\cosh\theta)\,d\theta = a^2\int \cosh^2\theta\,d\theta$

$= a^2\int \left(\frac{1+\cosh 2\theta}{2}\right)d\theta$

$$= \frac{a^2}{2}\left(\theta + \frac{\sinh 2\theta}{2}\right) + c$$

$$= \frac{a^2}{2}[\theta + \sinh\theta\cosh\theta] + c,$$

since $\sinh 2\theta = 2\sinh\theta\cosh\theta$

Since $x = a\sinh\theta$, then $\sinh\theta = \frac{x}{a}$ and $\theta = \sinh^{-1}\frac{x}{a}$

Also since $\cosh^2\theta - \sinh^2\theta = 1$

then $\cosh\theta = \sqrt{(1+\sinh^2\theta)}$

$$= \sqrt{\left[1 + \left(\frac{x}{a}\right)^2\right]} = \sqrt{\left(\frac{a^2+x^2}{a^2}\right)}$$

$$= \frac{\sqrt{(a^2+x^2)}}{a}$$

Hence $\int \sqrt{(x^2+a^2)}\,dx$

$$= \frac{a^2}{2}\left[\sinh^{-1}\frac{x}{a} + \left(\frac{x}{a}\right)\frac{\sqrt{(x^2+a^2)}}{a}\right] + c$$

$$= \mathbf{\frac{a^2}{2}\sinh^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{(x^2+a^2)} + c}$$

Now try the following exercise

Exercise 160 Further problems on integration using the $\sinh\theta$ substitution

1. Find $\int \frac{2}{\sqrt{(x^2+16)}}\,dx$. $\left[2\sinh^{-1}\frac{x}{4} + c\right]$

2. Find $\int \frac{3}{\sqrt{(9+5x^2)}}\,dx$. $\left[\frac{3}{\sqrt{5}}\sinh^{-1}\frac{\sqrt{5}}{3}x + c\right]$

3. Find $\int \sqrt{(x^2+9)}\,dx$. $\left[\frac{9}{2}\sinh^{-1}\frac{x}{3} + \frac{x}{2}\sqrt{(x^2+9)} + c\right]$

4. Find $\int \sqrt{(4t^2+25)}\,dt$. $\left[\frac{25}{4}\sinh^{-1}\frac{2t}{5} + \frac{t}{2}\sqrt{(4t^2+25)} + c\right]$

5. Evaluate $\int_0^3 \frac{4}{\sqrt{(t^2+9)}}\,dt$. [3.525]

6. Evaluate $\int_0^1 \sqrt{(16+9\theta^2)}\,d\theta$. [4.348]

40.8 Worked problems on integration using the $\cosh\theta$ substitution

Problem 24. Determine $\int \frac{1}{\sqrt{(x^2-a^2)}}\,dx$.

Let $x = a\cosh\theta$ then $\frac{dx}{d\theta} = a\sinh\theta$ and $dx = a\sinh\theta\,d\theta$

Hence $\int \frac{1}{\sqrt{(x^2-a^2)}}\,dx$

$$= \int \frac{1}{\sqrt{(a^2\cosh^2\theta - a^2)}}(a\sinh\theta\,d\theta)$$

$$= \int \frac{a\sinh\theta\,d\theta}{\sqrt{[a^2(\cosh^2\theta - 1)]}}$$

$$= \int \frac{a\sinh\theta\,d\theta}{\sqrt{(a^2\sinh^2\theta)}},$$

since $\cosh^2\theta - \sinh^2\theta = 1$

$$= \int \frac{a\sinh\theta\,d\theta}{a\sinh\theta} = \int d\theta = \theta + c$$

$$= \mathbf{\cosh^{-1}\frac{x}{a} + c}, \quad \text{since } x = a\cosh\theta$$

It is shown on page 339 that

$$\mathbf{\cosh^{-1}\frac{x}{a} = \ln\left\{\frac{x+\sqrt{(x^2-a^2)}}{a}\right\}}$$

which provides as alternative solution to

$$\int \frac{1}{\sqrt{(x^2-a^2)}}\,dx$$

Problem 25. Determine $\int \frac{2x-3}{\sqrt{(x^2-9)}}\,dx$.

$$\int \frac{2x-3}{\sqrt{(x^2-9)}}\,dx = \int \frac{2x}{\sqrt{(x^2-9)}}\,dx - \int \frac{3}{\sqrt{(x^2-9)}}\,dx$$

The first integral is determined using the algebraic substitution $u=(x^2-9)$, and the second integral is of the form $\int \frac{1}{\sqrt{(x^2-a^2)}}\,dx$ (see Problem 24)

Hence $\int \frac{2x}{\sqrt{(x^2-9)}}\,dx - \int \frac{3}{\sqrt{(x^2-9)}}\,dx$

$$= \mathbf{2\sqrt{(x^2-9)} - 3\cosh^{-1}\frac{x}{3} + c}$$

Problem 26. $\int \sqrt{(x^2-a^2)}\,dx$.

Let $x = a\cosh\theta$ then $\frac{dx}{d\theta} = a\sinh\theta$ and $dx = a\sinh\theta\,d\theta$

Hence $\int \sqrt{(x^2-a^2)}\,dx$

$$= \int \sqrt{(a^2\cosh^2\theta - a^2)}\,(a\sinh\theta\,d\theta)$$

$$= \int \sqrt{[a^2(\cosh^2\theta - 1)]}\,(a\sinh\theta\,d\theta)$$

$$= \int \sqrt{(a^2\sinh^2\theta)}\,(a\sinh\theta\,d\theta)$$

$$= a^2\int \sinh^2\theta\,d\theta = a^2\int\left(\frac{\cosh 2\theta - 1}{2}\right)d\theta$$

since $\cosh 2\theta = 1 + 2\sinh^2\theta$

from Table 5.1, page 45,

$$= \frac{a^2}{2}\left[\frac{\sinh 2\theta}{2} - \theta\right] + c$$

$$= \frac{a^2}{2}[\sinh\theta\cosh\theta - \theta] + c,$$

since $\sinh 2\theta = 2\sinh\theta\cosh\theta$

Since $x = a\cosh\theta$ then $\cosh\theta = \frac{x}{a}$ and $\theta = \cosh^{-1}\frac{x}{a}$

Also, since $\cosh^2\theta - \sinh^2\theta = 1$, then

$$\sinh\theta = \sqrt{(\cosh^2\theta - 1)}$$

$$= \sqrt{\left[\left(\frac{x}{a}\right)^2 - 1\right]} = \frac{\sqrt{(x^2-a^2)}}{a}$$

Hence $\int \sqrt{(x^2-a^2)}\,dx$

$$= \frac{a^2}{2}\left[\frac{\sqrt{(x^2-a^2)}}{a}\left(\frac{x}{a}\right) - \cosh^{-1}\frac{x}{a}\right] + c$$

$$= \mathbf{\frac{x}{2}\sqrt{(x^2-a^2)} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a} + c}$$

Problem 27. Evaluate $\int_2^3 \sqrt{(x^2-4)}\,dx$.

$$\int_2^3 \sqrt{(x^2-4)}\,dx = \left[\frac{x}{2}\sqrt{(x^2-4)} - \frac{4}{2}\cosh^{-1}\frac{x}{2}\right]_2^3$$

from Problem 26, when $a = 2$,

$$= \left(\frac{3}{5}\sqrt{5} - 2\cosh^{-1}\frac{3}{2}\right) - (0 - 2\cosh^{-1}1)$$

Since $\cosh^{-1}\frac{x}{a} = \ln\left\{\frac{x+\sqrt{(x^2-a^2)}}{a}\right\}$ then

$$\cosh^{-1}\frac{3}{2} = \ln\left\{\frac{3+\sqrt{(3^2-2^2)}}{2}\right\}$$

$$= \ln 2.6180 = 0.9624$$

Similarly, $\cosh^{-1}1 = 0$

Hence $\int_2^3 \sqrt{(x^2-4)}\,dx$

$$= \left[\frac{3}{2}\sqrt{5} - 2(0.9624)\right] - [0]$$

$$= \mathbf{1.429}, \text{ correct to 4 significant figures.}$$

Now try the following exercise

Exercise 161 Further problems on integration using the $\cosh\theta$ substitution

1. Find $\int \frac{1}{\sqrt{(t^2-16)}}\,dt$. $\left[\cosh^{-1}\frac{x}{4} + c\right]$

2. Find $\int \frac{3}{\sqrt{(4x^2-9)}}\,\mathrm{d}x$. $\left[\frac{3}{2}\cosh^{-1}\frac{2x}{3}+c\right]$

3. Find $\int \sqrt{(\theta^2-9)}\,\mathrm{d}\theta$.

$$\left[\frac{\theta}{2}\sqrt{(\theta^2-9)}-\frac{9}{2}\cosh^{-1}\frac{\theta}{3}+c\right]$$

4. Find $\int \sqrt{(4\theta^2-25)}\,\mathrm{d}\theta$.

$$\left[\theta\sqrt{\left(\theta^2-\frac{25}{4}\right)}-\frac{25}{4}\cosh^{-1}\frac{2\theta}{5}+c\right]$$

5. Evaluate $\int_1^2 \frac{2}{\sqrt{(x^2-1)}}\,\mathrm{d}x$. [2.634]

6. Evaluate $\int_2^3 \sqrt{(t^2-4)}\,\mathrm{d}t$. [1.429]

Chapter 41

Integration using partial fractions

41.1 Introduction

The process of expressing a fraction in terms of simpler fractions—called **partial fractions**—is discussed in Chapter 2, with the forms of partial fractions used being summarized in Table 2.1, page 13.

Certain functions have to be resolved into partial fractions before they can be integrated as demonstrated in the following worked problems.

41.2 Worked problems on integration using partial fractions with linear factors

Problem 1. Determine $\int \frac{11-3x}{x^2+2x-3}\,dx$.

As shown in Problem 1, page 13:

$$\frac{11-3x}{x^2+2x-3} \equiv \frac{2}{(x-1)} - \frac{5}{(x+3)}$$

Hence $\int \frac{11-3x}{x^2+2x-3}\,dx$

$$= \int \left\{ \frac{2}{(x-1)} - \frac{5}{(x+3)} \right\} dx$$

$$= \mathbf{2\ln(x-1) - 5\ln(x+3) + c}$$

(by algebraic substitutions — see Chapter 39)

or $\mathbf{\ln\left\{\frac{(x-1)^2}{(x+3)^5}\right\} + c}$ by the laws of logarithms

Problem 2. Find

$$\int \frac{2x^2-9x-35}{(x+1)(x-2)(x+3)}\,dx.$$

It was shown in Problem 2, page 14:

$$\frac{2x^2-9x-35}{(x+1)(x-2)(x+3)} \equiv \frac{4}{(x+1)} - \frac{3}{(x-2)} + \frac{1}{(x+3)}$$

Hence $\int \frac{2x^2-9x-35}{(x+1)(x-2)(x+3)}\,dx$

$$\equiv \int \left\{ \frac{4}{(x+1)} - \frac{3}{(x-2)} + \frac{1}{(x+3)} \right\} dx$$

$$= \mathbf{4\ln(x+1) - 3\ln(x-2) + \ln(x+3) + c}$$

or $\mathbf{\ln\left\{\frac{(x+1)^4(x+3)}{(x-2)^3}\right\} + c}$

Problem 3. Determine $\int \frac{x^2+1}{x^2-3x+2}\,dx$.

By dividing out (since the numerator and denominator are of the same degree) and resolving into partial fractions it was shown in Problem 3, page 14:

$$\frac{x^2+1}{x^2-3x+2} \equiv 1 - \frac{2}{(x-1)} + \frac{5}{(x-2)}$$

Hence $\int \frac{x^2+1}{x^2-3x+2}\,dx$

$$\equiv \int \left\{1 - \frac{2}{(x-1)} + \frac{5}{(x-2)}\right\} dx$$

$$= \mathbf{(x-2)\ln(x-1) + 5\ln(x-2) + c}$$

$$\text{or } \mathbf{x + \ln\left\{\frac{(x-2)^5}{(x-1)^2}\right\} + c}$$

Problem 4. Evaluate

$$\int_2^3 \frac{x^3-2x^2-4x-4}{x^2+x-2}\,dx,$$

correct to 4 significant figures.

By dividing out and resolving into partial fractions it was shown in Problem 4, page 15:

$$\frac{x^3-2x^2-4x-4}{x^2+x-2} \equiv x - 3 + \frac{4}{(x+2)} - \frac{3}{(x-1)}$$

Hence $\int_2^3 \frac{x^3-2x^2-4x-4}{x^2+x-2}\,dx$

$$\equiv \int_2^3 \left\{x - 3 + \frac{4}{(x+2)} - \frac{3}{(x-1)}\right\} dx$$

$$= \left[\frac{x^2}{2} - 3x + 4\ln(x+2) - 3\ln(x-1)\right]_2^3$$

$$= \left(\frac{9}{2} - 9 + 4\ln 5 - 3\ln 2\right) - (2 - 6 + 4\ln 4 - 3\ln 1)$$

$$= \mathbf{-1.687}, \text{ correct to 4 significant figures.}$$

Now try the following exercise

Exercise 162 Further problems on integration using partial fractions with linear factors

In Problems 1 to 5, integrate with respect to x.

1. $\int \frac{12}{(x^2-9)}\,dx$

$$\left[\begin{array}{l} 2\ln(x-3) - 2\ln(x+3) + c \\ \text{or } \ln\left\{\frac{x-3}{x+3}\right\}^2 + c \end{array}\right]$$

2. $\int \frac{4(x-4)}{(x^2-2x-3)}\,dx$

$$\left[\begin{array}{l} 5\ln(x+1) - \ln(x-3) + c \\ \text{or } \ln\left\{\frac{(x+1)^5}{(x-3)}\right\} + c \end{array}\right]$$

3. $\int \frac{3(2x^2-8x-1)}{(x+4)(x+1)(2x-1)}\,dx$

$$\left[\begin{array}{l} 7\ln(x+4) - 3\ln(x+1) \\ -\ln(2x-1) + c \quad \text{or} \\ \ln\left\{\frac{(x+4)^7}{(x+1)^3(2x-1)}\right\} + c \end{array}\right]$$

4. $\int \frac{x^2+9x+8}{x^2+x-6}\,dx$

$$\left[\begin{array}{l} x + 2\ln(x+3) + 6\ln(x-2) + c \\ \text{or } x + \ln\{(x+3)^2(x-2)^6\} + c \end{array}\right]$$

5. $\int \frac{3x^3-2x^2-16x+20}{(x-2)(x+2)}\,dx$

$$\left[\begin{array}{l} \frac{3x^2}{2} - 2x + \ln(x-2) \\ -5\ln(x+2) + c \end{array}\right]$$

In Problems 6 and 7, evaluate the definite integrals correct to 4 significant figures.

6. $\int_3^4 \frac{x^2-3x+6}{x(x-2)(x-1)}\,dx$ [0.6275]

7. $\int_4^6 \frac{x^2 - x - 14}{x^2 - 2x - 3}\,dx$ [0.8122]

8. Determine the value of k, given that:

$$\int_0^1 \frac{(x-k)}{(3x+1)(x+1)}\,dx = 0 \qquad \left[\frac{1}{3}\right]$$

9. The velocity constant k of a given chemical reaction is given by:

$$kt = \int \left(\frac{1}{(3-0.4x)(2-0.6x)}\right) dx$$

where $x = 0$ when $t = 0$. Show that:

$$kt = \ln\left\{\frac{2(3-0.4x)}{3(2-0.6x)}\right\}$$

41.3 Worked problems on integration using partial fractions with repeated linear factors

Problem 5. Determine $\int \frac{2x+3}{(x-2)^2}\,dx$.

It was shown in Problem 5, page 16:

$$\frac{2x+3}{(x-2)^2} \equiv \frac{2}{(x-2)} + \frac{7}{(x-2)^2}$$

Thus $\int \frac{2x+3}{(x-2)^2}\,dx \equiv \int \left\{\frac{2}{(x-2)} + \frac{7}{(x-2)^2}\right\} dx$

$$= \mathbf{2\ln(x-2) - \frac{7}{(x-2)} + c}$$

$\left[\int \frac{7}{(x-2)^2}\,dx$ is determined using the algebraic substitution $u = (x-2)$ — see Chapter 39.$\right]$

Problem 6. Find $\int \frac{5x^2 - 2x - 19}{(x+3)(x-1)^2}\,dx$.

It was shown in Problem 6, page 16:

$$\frac{5x^2 - 2x - 19}{(x+3)(x-1)^2} \equiv \frac{2}{(x+3)} + \frac{3}{(x-1)} - \frac{4}{(x-1)^2}$$

Hence $\int \frac{5x^2 - 2x - 19}{(x+3)(x-1)^2}\,dx$

$$\equiv \int \left\{\frac{2}{(x+3)} + \frac{3}{(x-1)} - \frac{4}{(x-1)^2}\right\} dx$$

$$= \mathbf{2\ln(x+3) + 3\ln(x-1) + \frac{4}{(x-1)} + c}$$

or $\mathbf{\ln\left\{(x+3)^2(x-1)^3\right\} + \frac{4}{(x-1)} + c}$

Problem 7. Evaluate

$$\int_{-2}^{1} \frac{3x^2 + 16x + 15}{(x+3)^3}\,dx,$$

correct to 4 significant figures.

It was shown in Problem 7, page 17:

$$\frac{3x^2 + 16x + 15}{(x+3)^3} \equiv \frac{3}{(x+3)} - \frac{2}{(x+3)^2} - \frac{6}{(x+3)^3}$$

Hence $\int \frac{3x^2 + 16x + 15}{(x+3)^3}\,dx$

$$\equiv \int_{-2}^{1} \left\{\frac{3}{(x+3)} - \frac{2}{(x+3)^2} - \frac{6}{(x+3)^3}\right\} dx$$

$$= \left[3\ln(x+3) + \frac{2}{(x+3)} + \frac{3}{(x+3)^2}\right]_{-2}^{1}$$

$$= \left(3\ln 4 + \frac{2}{4} + \frac{3}{16}\right) - \left(3\ln 1 + \frac{2}{1} + \frac{3}{1}\right)$$

$$= \mathbf{-0.1536}$$, correct to 4 significant figures.

Now try the following exercise

Exercise 163 Further problems on integration using partial fractions with repeated linear factors

In Problems 1 and 2, integrate with respect to x.

1. $\int \frac{4x-3}{(x+1)^2}\,dx$

$$\left[4\ln(x+1)+\frac{7}{(x+1)}+c\right]$$

2. $\int \frac{5x^2-30x+44}{(x-2)^3}\,dx$

$$\left[5\ln(x-2)+\frac{10}{(x-2)}-\frac{2}{(x-2)^2}+c\right]$$

In Problems 3 and 4, evaluate the definite integrals correct to 4 significant figures.

3. $\int_1^2 \frac{x^2+7x+3}{x^2(x+3)}$ [1.663]

4. $\int_6^7 \frac{18+21x-x^2}{(x-5)(x+2)^2}\,dx$ [1.089]

5. Show that $\int_0^1 \left(\frac{4t^2+9t+8}{(t+2)(t+1)^2}\right)dt = 2.546$, correct to 4 significant figures.

41.4 Worked problems on integration using partial fractions with quadratic factors

Problem 8. Find $\int \frac{3+6x+4x^2-2x^3}{x^2(x^2+3)}\,dx$.

It was shown in Problem 9, page 18:

$$\frac{3+6x+4x^2-2x^3}{x^2(x^2+3)} \equiv \frac{2}{x}+\frac{1}{x^2}+\frac{3-4x}{(x^2+3)}$$

Thus $\int \frac{3+6x+4x^2-2x^3}{x^2(x^2+3)}\,dx$

$$\equiv \int \left(\frac{2}{x}+\frac{1}{x^2}+\frac{(3-4x)}{(x^2+3)}\right)dx$$

$$= \int \left\{\frac{2}{x}+\frac{1}{x^2}+\frac{3}{(x^2+3)}-\frac{4x}{(x^2+3)}\right\}dx$$

$$\int \frac{3}{(x^2+3)}\,dx = 3\int \frac{1}{x^2+(\sqrt{3})^2}\,dx$$

$$= \frac{3}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}, \text{ from 12, Table 40.1, page 399.}$$

$\int \frac{4x}{x^2+3}\,dx$ is determined using the algebraic substitution $u=(x^2+3)$.

Hence $\int \left\{\frac{2}{x}+\frac{1}{x^2}+\frac{3}{(x^2+3)}-\frac{4x}{(x^2+3)}\right\}dx$

$$= 2\ln x - \frac{1}{x}+\frac{3}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} - 2\ln(x^2+3)+c$$

$$= \mathbf{\ln\left(\frac{x}{x^2+3}\right)^2 - \frac{1}{x}+\sqrt{3}\tan^{-1}\frac{x}{\sqrt{3}}+c}$$

Problem 9. Determine $\int \frac{1}{(x^2-a^2)}\,dx$.

Let $\frac{1}{(x^2-a^2)} \equiv \frac{A}{(x-a)}+\frac{B}{(x+a)}$

$$\equiv \frac{A(x+a)+B(x-a)}{(x+a)(x-a)}$$

Equating the numerators gives:

$$1 \equiv A(x+a)+B(x-a)$$

Let $x=a$, then $A=\frac{1}{2a}$, and let $x=-a$, then $B=-\frac{1}{2a}$

Hence $\int \frac{1}{(x^2-a^2)}\,dx$

$$\equiv \int \frac{1}{2a}\left[\frac{1}{(x-a)}-\frac{1}{(x+a)}\right]dx$$

$$= \frac{1}{2a}[\ln(x-a) - \ln(x+a)] + c$$

$$= \mathbf{\frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right) + c}$$

Problem 10. Evaluate

$$\int_3^4 \frac{3}{(x^2-4)}\,dx,$$

correct to 3 significant figures.

From Problem 9,

$$\int_3^4 \frac{3}{(x^2-4)}\,dx = 3\left[\frac{1}{2(2)}\ln\left(\frac{x-2}{x+2}\right)\right]_3^4$$

$$= \frac{3}{4}\left[\ln\frac{2}{6} - \ln\frac{1}{5}\right]$$

$$= \frac{3}{4}\ln\frac{5}{3} = \mathbf{0.383},$$ correct to 3 significant figures.

Problem 11. Determine $\int \frac{1}{(a^2-x^2)}\,dx$.

Using partial fractions, let

$$\frac{1}{(a^2-x^2)} \equiv \frac{1}{(a-x)(a+x)} \equiv \frac{A}{(a-x)} + \frac{B}{(a+x)}$$

$$\equiv \frac{A(a+x)+B(a-x)}{(a-x)(a+x)}$$

Then $1 \equiv A(a+x)+B(a-x)$

Let $x=a$ then $A=\frac{1}{2a}$. Let $x=-a$ then $B=\frac{1}{2a}$

Hence $\int \frac{1}{(a^2-x^2)}\,dx$

$$= \int \frac{1}{2a}\left[\frac{1}{(a-x)} + \frac{1}{(a+x)}\right]dx$$

$$= \frac{1}{2a}[-\ln(a-x) + \ln(a+x)] + c$$

$$= \mathbf{\frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right) + c}$$

Problem 12. Evaluate

$$\int_0^2 \frac{5}{(9-x^2)}\,dx,$$

correct to 4 decimal places.

From Problem 11,

$$\int_0^2 \frac{5}{(9-x^2)}\,dx = 5\left[\frac{1}{2(3)}\ln\left(\frac{3+x}{3-x}\right)\right]_0^2$$

$$= \frac{5}{6}\left[\ln\frac{5}{1} - \ln 1\right]$$

$$= \mathbf{1.3412},$$ correct to 4 decimal places.

Now try the following exercise

Exercise 164 Further problems on integration using partial fractions with quadratic factors

1. Determine $\int \frac{x^2-x-13}{(x^2+7)(x-2)}\,dx$.

$$\left[\begin{array}{l}\ln(x^2+7) + \frac{3}{\sqrt{7}}\tan^{-1}\frac{x}{\sqrt{7}} \\ -\ln(x-2) + c\end{array}\right]$$

In Problems 2 to 4, evaluate the definite integrals correct to 4 significant figures.

2. $\int_5^6 \frac{6x-5}{(x-4)(x^2+3)}\,dx$ [0.5880]

3. $\int_1^2 \frac{4}{(16-x^2)}\,dx$ [0.2939]

4. $\int_4^5 \frac{2}{(x^2-9)}\,dx$ [0.1865]

5. Show that $\int_1^2 \left(\frac{2+\theta+6\theta^2-2\theta^3}{\theta^2(\theta^2+1)}\right)d\theta$ = 1.606, correct to 4 significant figures.

Chapter 42

The $t = \tan\frac{\theta}{2}$ substitution

42.1 Introduction

Integrals of the form $\int \frac{1}{a\cos\theta + b\sin\theta + c}\,d\theta$, where a, b and c are constants, may be determined by using the substitution $t = \tan\frac{\theta}{2}$. The reason is explained below.

If angle A in the right-angled triangle ABC shown in Fig. 42.1 is made equal to $\frac{\theta}{2}$ then, since tangent $= \frac{\text{opposite}}{\text{adjacent}}$, if $BC = t$ and $AB = 1$, then $\tan\frac{\theta}{2} = t$.

By Pythagoras' theorem, $AC = \sqrt{1+t^2}$

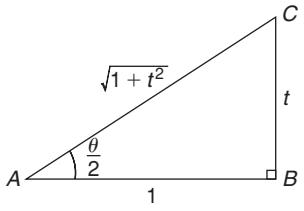


Figure 42.1

Therefore $\sin\frac{\theta}{2} = \frac{t}{\sqrt{1+t^2}}$ and $\cos\frac{\theta}{2} = \frac{1}{\sqrt{1+t^2}}$ Since $\sin 2x = 2\sin x\cos x$ (from double angle formulae, Chapter 17), then

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= 2\left(\frac{t}{\sqrt{1+t^2}}\right)\left(\frac{t}{\sqrt{1+t^2}}\right)$$

i.e. $$\sin\theta = \frac{2t}{(1+t^2)} \qquad (1)$$

Since $\cos 2x = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$

$$= \left(\frac{1}{\sqrt{1+t^2}}\right)^2 - \left(\frac{t}{\sqrt{1+t^2}}\right)^2$$

i.e. $$\cos\theta = \frac{1-t^2}{1+t^2} \qquad (2)$$

Also, since $t = \tan\frac{\theta}{2}$,

$\frac{dt}{d\theta} = \frac{1}{2}\sec^2\frac{\theta}{2} = \frac{1}{2}\left(1+\tan^2\frac{\theta}{2}\right)$ from trigonometric identities,

i.e. $$\frac{dt}{d\theta} = \frac{1}{2}(1+t^2)$$

from which, $$d\theta = \frac{2\,dt}{1+t^2} \qquad (3)$$

Equations (1), (2) and (3) are used to determine integrals of the form $\int \frac{1}{a\cos\theta + b\sin\theta + c}\,d\theta$ where a, b or c may be zero.

42.2 Worked problems on the $t = \tan\frac{\theta}{2}$ substitution

Problem 1. Determine $\int \frac{d\theta}{\sin\theta}$

If $t = \tan\frac{\theta}{2}$ then $\sin\theta = \frac{2t}{1+t^2}$ and $d\theta = \frac{2\,dt}{1+t^2}$ from equations (1) and (3).

Thus
$$\int \frac{d\theta}{\sin\theta} = \int \frac{1}{\sin\theta}\,d\theta$$
$$= \int \frac{1}{\frac{2t}{1+t^2}}\left(\frac{2\,dt}{1+t^2}\right)$$
$$= \int \frac{1}{t}\,dt = \ln t + c$$

Hence $$\int \frac{d\theta}{\sin\theta} = \ln\left(\tan\frac{\theta}{2}\right) + c$$

Problem 2. Determine $\int \frac{dx}{\cos x}$

If $\tan\frac{x}{2}$ then $\cos x = \frac{1-t^2}{1+t^2}$ and $dx = \frac{2\,dt}{1+t^2}$ from equations (2) and (3).

Thus
$$\int \frac{dx}{\cos x} = \int \frac{1}{\frac{1-t^2}{1+t^2}}\left(\frac{2\,dt}{1+t^2}\right)$$
$$= \int \frac{2}{1-t^2}\,dt$$

$\frac{2}{1-t^2}$ may be resolved into partial fractions (see Chapter 2).

Let
$$\frac{2}{1-t^2} = \frac{2}{(1-t)(1+t)}$$
$$= \frac{A}{(1-t)} + \frac{B}{(1+t)}$$
$$= \frac{A(1+t)+B(1-t)}{(1-t)(1+t)}$$

Hence $2 = A(1+t) + B(1-t)$

When $t = 1, 2 = 2A$, from which, $A = 1$

When $t = -1, 2 = 2B$, from which, $B = 1$

Hence
$$\int \frac{2\,dt}{1-t^2} = \int \frac{1}{(1-t)} + \frac{1}{(1+t)}\,dt$$
$$= -\ln(1-t) + \ln(1+t) + c$$
$$= \ln\left\{\frac{(1+t)}{(1-t)}\right\} + c$$

Thus
$$\int \frac{dx}{\cos x} = \ln\left\{\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right\} + c$$

Note that since $\tan\frac{\pi}{4} = 1$, the above result may be written as:

$$\int \frac{dx}{\cos x} = \ln\left\{\frac{\tan\frac{\pi}{4} + \tan\frac{x}{2}}{1 - \tan\frac{\pi}{4}\tan\frac{x}{2}}\right\} + c$$
$$= \ln\left\{\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right\} + c$$

from compound angles, Chapter 17.

Problem 3. Determine $\int \frac{dx}{1+\cos x}$

If $\tan\frac{x}{2}$ then $\cos x = \frac{1-t^2}{1+t^2}$ and $dx = \frac{2\,dt}{1+t^2}$ from equations (2) and (3).

Thus
$$\int \frac{dx}{1+\cos x} = \int \frac{1}{1+\cos x}\,dx$$
$$= \int \frac{1}{1+\frac{1-t^2}{1+t^2}}\left(\frac{2\,dt}{1+t^2}\right)$$
$$= \int \frac{1}{\frac{(1+t^2)+(1-t^2)}{1+t^2}}\left(\frac{2\,dt}{1+t^2}\right)$$
$$= \int dt$$

Hence $$\int \frac{dx}{1+\cos x} = t + c = \tan\frac{x}{2} + c$$

Problem 4. Determine $\int \frac{d\theta}{5+4\cos\theta}$

If $t=\tan\frac{\theta}{2}$ then $\cos\theta=\frac{1-t^2}{1+t^2}$ and $\mathrm{d}x=\frac{2\,\mathrm{d}t}{1+t^2}$ from equations (2) and (3).

Thus $\displaystyle\int\frac{\mathrm{d}\theta}{5+4\cos\theta}=\int\frac{\left(\frac{2\,\mathrm{d}t}{1+t^2}\right)}{5+4\left(\frac{1-t^2}{1+t^2}\right)}$

$$=\int\frac{\left(\frac{2\,\mathrm{d}t}{1+t^2}\right)}{\frac{5(1+t^2)+4(1-t^2)}{(1+t^2)}}$$

$$=2\int\frac{\mathrm{d}t}{t^2+9}=2\int\frac{\mathrm{d}t}{t^2+3^2}$$

$$=2\left(\frac{1}{3}\tan^{-1}\frac{t}{3}\right)+c,$$

from 12 of Table 40.1, page 399. Hence

$$\int\frac{\mathrm{d}\theta}{5+4\cos\theta}=\frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{\theta}{2}\right)+c$$

Now try the following exercise

Exercise 165 Further problems on the $t=\tan\frac{\theta}{2}$ substitution

Integrate the following with respect to the variable:

1. $\displaystyle\int\frac{\mathrm{d}\theta}{1+\sin\theta}$ $\left[\frac{-2}{1+\tan\frac{\theta}{2}}+c\right]$

2. $\displaystyle\int\frac{\mathrm{d}x}{1-\cos x+\sin x}$ $\left[\ln\left\{\frac{\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right\}+c\right]$

3. $\displaystyle\int\frac{\mathrm{d}\alpha}{3+2\cos\alpha}$ $\left[\frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{1}{\sqrt{5}}\tan\frac{\alpha}{2}\right)+c\right]$

4. $\displaystyle\int\frac{\mathrm{d}x}{3\sin x-4\cos x}$ $\left[\frac{1}{5}\ln\left\{\frac{2\tan\frac{x}{2}-1}{\tan\frac{x}{2}+2}\right\}+c\right]$

42.3 Further worked problems on the $t=\tan\frac{\theta}{2}$ substitution

Problem 5. Determine $\displaystyle\int\frac{\mathrm{d}x}{\sin x+\cos x}$

If $\tan\frac{x}{2}$ then $\sin x=\frac{2t}{1+t^2}$, $\cos x=\frac{1-t^2}{1+t^2}$ and $\mathrm{d}x=\frac{2\,\mathrm{d}t}{1+t^2}$ from equations (1), (2) and (3).

Thus

$$\int\frac{\mathrm{d}x}{\sin x+\cos x}=\int\frac{\frac{2\,\mathrm{d}t}{1+t^2}}{\left(\frac{2t}{1+t^2}\right)+\left(\frac{1-t^2}{1+t^2}\right)}$$

$$=\int\frac{\frac{2\,\mathrm{d}t}{1+t^2}}{\frac{2t+1-t^2}{1+t^2}}=\int\frac{2\,\mathrm{d}t}{1+2t-t^2}$$

$$=\int\frac{-2\,\mathrm{d}t}{t^2-2t-1}=\int\frac{-2\,\mathrm{d}t}{(t-1)^2-2}$$

$$=\int\frac{2\,\mathrm{d}t}{(\sqrt{2})^2-(t-1)^2}$$

$$=2\left[\frac{1}{2\sqrt{2}}\ln\left\{\frac{\sqrt{2}+(t-1)}{\sqrt{2}-(t-1)}\right\}\right]+c$$

(see Problem 11, Chapter 41, page 413),

i.e. $\displaystyle\int\frac{\mathrm{d}x}{\sin x+\cos x}$

$$=\frac{1}{\sqrt{2}}\ln\left\{\frac{\sqrt{2}-1+\tan\frac{x}{2}}{\sqrt{2}+1-\tan\frac{x}{2}}\right\}+c$$

Problem 6. Determine $\displaystyle\int\frac{\mathrm{d}x}{7-3\sin x+6\cos x}$

From equations (1) and (3),

$$\int\frac{\mathrm{d}x}{7-3\sin x+6\cos x}$$

$$=\int\frac{\frac{2\,\mathrm{d}t}{1+t^2}}{7-3\left(\frac{2t}{1+t^2}\right)+6\left(\frac{1-t^2}{1+t^2}\right)}$$

$$= \int \frac{\dfrac{2\,dt}{1+t^2}}{\dfrac{7(1+t^2)-3(2t)+6(1-t^2)}{1+t^2}}$$

$$= \int \frac{2\,dt}{7+7t^2-6t+6-6t^2}$$

$$= \int \frac{2\,dt}{t^2-6t+13} = \int \frac{2\,dt}{(t-3)^2+2^2}$$

$$= 2\left[\frac{1}{2}\tan^{-1}\left(\frac{t-3}{2}\right)\right] + c$$

from 12, Table 40.1, page 399. Hence

$$\int \frac{dx}{7-3\sin x+6\cos x} = \tan^{-1}\left(\frac{\tan\frac{x}{2}-3}{2}\right)+c$$

Problem 7. Determine $\int \dfrac{d\theta}{4\cos\theta+3\sin\theta}$

From equations (1) to (3),

$$\int \frac{d\theta}{4\cos\theta+3\sin\theta}$$

$$= \int \frac{\dfrac{2\,dt}{1+t^2}}{4\left(\dfrac{1-t^2}{1+t^2}\right)+3\left(\dfrac{2t}{1+t^2}\right)}$$

$$= \int \frac{2\,dt}{4-4t^2+6t} = \int \frac{dt}{2+3t-2t^2}$$

$$= -\frac{1}{2}\int \frac{dt}{t^2-\frac{3}{2}t-1}$$

$$= -\frac{1}{2}\int \frac{dt}{\left(t-\frac{3}{4}\right)^2-\frac{25}{16}}$$

$$= \frac{1}{2}\int \frac{dt}{\left(\frac{5}{4}\right)^2-\left(t-\frac{3}{4}\right)^2}$$

$$= \frac{1}{2}\left[\frac{1}{2\left(\frac{5}{4}\right)}\ln\left\{\frac{\frac{5}{4}+\left(t-\frac{3}{4}\right)}{\frac{5}{4}-\left(t-\frac{3}{4}\right)}\right\}\right]+c$$

from Problem 11, Chapter 41, page 413

$$= \frac{1}{5}\ln\left\{\frac{\frac{1}{2}+t}{2-t}\right\}+c$$

Hence $\displaystyle\int \frac{d\theta}{4\cos\theta+3\sin\theta}$

$$= \frac{1}{5}\ln\left\{\frac{\frac{1}{2}+\tan\frac{\theta}{2}}{2-\tan\frac{\theta}{2}}\right\}+c$$

or $$\frac{1}{5}\ln\left\{\frac{1+2\tan\frac{\theta}{2}}{4-2\tan\frac{\theta}{2}}\right\}+c$$

Now try the following exercise

Exercise 166 Further problems on the $t = \tan\frac{\theta}{2}$ substitution

In Problems 1 to 4, integrate with respect to the variable.

1. $\displaystyle\int \frac{d\theta}{5+4\sin\theta}$

$$\left[\frac{2}{3}\tan^{-1}\left(\frac{5\tan\frac{\theta}{2}+4}{3}\right)+c\right]$$

2. $\displaystyle\int \frac{dx}{1+2\sin x}$

$$\left[\frac{1}{\sqrt{3}}\ln\left\{\frac{\tan\frac{x}{2}+2-\sqrt{3}}{\tan\frac{x}{2}+2+\sqrt{3}}\right\}+c\right]$$

3. $\displaystyle\int \frac{dp}{3-4\sin p+2\cos p}$

$$\left[\frac{1}{\sqrt{11}}\ln\left\{\frac{\tan\frac{p}{2}-4-\sqrt{11}}{\tan\frac{p}{2}-4+\sqrt{11}}\right\}+c\right]$$

4. $\displaystyle\int \frac{\mathrm{d}\theta}{3-4\sin\theta}$

$$\left[\frac{1}{\sqrt{7}}\ln\left\{\frac{3\tan\frac{\theta}{2}-4-\sqrt{7}}{3\tan\frac{\theta}{2}-4+\sqrt{7}}\right\}+c\right]$$

5. Show that

$$\int \frac{\mathrm{d}t}{1+3\cos t}=\frac{1}{2\sqrt{2}}\ln\left\{\frac{\sqrt{2}+\tan\frac{t}{2}}{\sqrt{2}-\tan\frac{t}{2}}\right\}+c.$$

6. Show that $\displaystyle\int_0^{\pi/3}\frac{3\,\mathrm{d}\theta}{\cos\theta}=3.95$, correct to 3 significant figures.

7. Show that

$$\int_0^{\pi/2}\frac{\mathrm{d}\theta}{2+\cos\theta}=\frac{\pi}{3\sqrt{3}}.$$

Revision Test 12

This Revision Test covers the material contained in Chapters 40 to 42. *The marks for each question are shown in brackets at the end of each question.*

1. Determine the following integrals:

   (a) $\int \cos^3 x \sin^2 x \, dx$ (b) $\int \frac{2}{\sqrt{(9-4x^2)}} dx$

   (c) $\int \frac{2}{\sqrt{(4x^2-9)}} dx$ (14)

2. Evaluate the following definite integrals, correct to 4 significant figures:

   (a) $\int_0^{\frac{\pi}{2}} 3\sin^2 t \, dt$ (b) $\int_0^{\frac{\pi}{3}} 3\cos 5\theta \sin 3\theta \, d\theta$

   (c) $\int_0^2 \frac{5}{4+x^2} dx$ (15)

3. Determine:

   (a) $\int \frac{x-11}{x^2-x-2} dx$

   (b) $\int \frac{3-x}{(x^2+3)(x+3)} dx$ (21)

4. Evaluate $\int_1^2 \frac{3}{x^2(x+2)} dx$ correct to 4 significant figures. (12)

5. Determine: $\int \frac{dx}{2\sin x + \cos x}$ (8)

6. Evaluate $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{dx}{3-2\sin x}$ correct to 3 decimal places. (10)

Chapter 43

Integration by parts

43.1 Introduction

From the product rule of differentiation:

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx},$$

where u and v are both functions of x.

Rearranging gives: $u\dfrac{dv}{dx} = \dfrac{d}{dx}(uv) - v\dfrac{du}{dx}$

Integrating both sides with respect to x gives:

$$\int u\frac{dv}{dx}\,dx = \int \frac{d}{dx}(uv)\,dx - \int v\frac{du}{dx}\,dx$$

i.e. $$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

or $$\int u\,dv = uv - \int v\,du$$

This is known as the **integration by parts formula** and provides a method of integrating such products of simple functions as $\int xe^x\,dx$, $\int t\sin t\,dt$, $\int e^\theta\cos\theta\,d\theta$ and $\int x\ln x\,dx$.

Given a product of two terms to integrate the initial choice is: 'which part to make equal to u' and 'which part to make equal to v'. The choice must be such that the 'u part' becomes a constant after successive differentiation and the 'dv part' can be integrated from standard integrals. Invariably, the following rule holds: If a product to be integrated contains an algebraic term (such as x, t^2 or 3θ) then this term is chosen as the u part. The one exception to this rule is when a '$\ln x$' term is involved; in this case $\ln x$ is chosen as the 'u part'.

43.2 Worked problems on integration by parts

Problem 1. Determine $\int x\cos x\,dx$.

From the integration by parts formula,

$$\int u\,dv = uv - \int v\,du$$

Let $u = x$, from which $\dfrac{du}{dx} = 1$, i.e. $du = dx$ and let $dv = \cos x\,dx$, from which $v = \int\cos x\,dx = \sin x$.

Expressions for u, du and v are now substituted into the 'by parts' formula as shown below.

$$\int \boxed{u}\;\boxed{dv} = \boxed{u}\;\boxed{v} - \int \boxed{v}\;\boxed{du}$$

$$\int \boxed{x}\;\boxed{\cos x\,dx} = \boxed{(x)}\;\boxed{(\sin x)} - \int \boxed{(\sin x)}\;\boxed{(dx)}$$

i.e. $$\int x\cos x\,dx = x\sin x - (-\cos x) + c$$

$$= x\sin x + \cos x + c$$

[This result may be checked by differentiating the right hand side,

i.e. $\dfrac{d}{dx}(x\sin x + \cos x + c)$

$= [(x)(\cos x) + (\sin x)(1)] - \sin x + 0$

using the product rule

$= x\cos x$, which is the function

being integrated]

Problem 2. Find $\int 3te^{2t}\,dt$.

Let $u=3t$, from which, $\frac{du}{dt}=3$, i.e. $du=3\,dt$ and let $dv=e^{2t}\,dt$, from which, $v=\int e^{2t}\,dt=\frac{1}{2}e^{2t}$

Substituting into $\int u\,dv=uv-\int v\,du$ gives:

$$\int 3te^{2t}\,dt=(3t)\left(\frac{1}{2}e^{2t}\right)-\int\left(\frac{1}{2}e^{2t}\right)(3\,dt)$$

$$=\frac{3}{2}te^{2t}-\frac{3}{2}\int e^{2t}\,dt$$

$$=\frac{3}{2}te^{2t}-\frac{3}{2}\left(\frac{e^{2t}}{2}\right)+c$$

Hence

$$\int \mathbf{3}te^{\mathbf{2}t}\,\mathbf{d}t=\tfrac{\mathbf{3}}{\mathbf{2}}e^{\mathbf{2}t}\left(\mathbf{t}-\tfrac{\mathbf{1}}{\mathbf{2}}\right)+\boldsymbol{c},$$

which may be checked by differentiating.

Problem 3. Evaluate $\int_0^{\frac{\pi}{2}} 2\theta\sin\theta\,d\theta$.

Let $u=2\theta$, from which, $\frac{du}{d\theta}=2$, i.e. $du=2\,d\theta$ and let $dv=\sin\theta\,d\theta$, from which,

$$v=\int\sin\theta\,d\theta=-\cos\theta$$

Substituting into $\int u\,dv=uv-\int v\,du$ gives:

$$\int 2\theta\sin\theta\,d\theta=(2\theta)(-\cos\theta)-\int(-\cos\theta)(2\,d\theta)$$

$$=-2\theta\cos\theta+2\int\cos\theta\,d\theta$$

$$=-2\theta\cos\theta+2\sin\theta+c$$

Hence $\int_0^{\frac{\pi}{2}} 2\theta\sin\theta\,d\theta$

$$=[-2\theta\cos\theta+2\sin\theta]_0^{\frac{\pi}{2}}$$

$$=\left[-2\left(\frac{\pi}{2}\right)\cos\frac{\pi}{2}+2\sin\frac{\pi}{2}\right]-[0+2\sin 0]$$

$$=(-0+2)-(0+0)=\mathbf{2}$$

since $\cos\frac{\pi}{2}=0$ and $\sin\frac{\pi}{2}=1$

Problem 4. Evaluate $\int_0^1 5xe^{4x}\,dx$, correct to 3 significant figures.

Let $u=5x$, from which $\frac{du}{dx}=5$, i.e. $du=5\,dx$ and let $dv=e^{4x}\,dx$, from which, $v=\int e^{4x}\,dx=\frac{1}{4}e^{4x}$.

Substituting into $\int u\,dv=uv-\int v\,du$ gives:

$$\int 5xe^{4x}\,dx=(5x)\left(\frac{e^{4x}}{4}\right)-\int\left(\frac{e^{4x}}{4}\right)(5\,dx)$$

$$=\frac{5}{4}xe^{4x}-\frac{5}{4}\int e^{4x}\,dx$$

$$=\frac{5}{4}xe^{4x}-\frac{5}{4}\left(\frac{e^{4x}}{4}\right)+c$$

$$=\frac{5}{4}e^{4x}\left(x-\frac{1}{4}\right)+c$$

Hence $\int_0^1 5xe^{4x}\,dx$

$$=\left[\frac{5}{4}e^{4x}\left(x-\frac{1}{4}\right)\right]_0^1$$

$$=\left[\frac{5}{4}e^{4}\left(1-\frac{1}{4}\right)\right]-\left[\frac{5}{4}e^{0}\left(0-\frac{1}{4}\right)\right]$$

$$=\left(\frac{15}{16}e^{4}\right)-\left(-\frac{5}{16}\right)$$

$$=51.186+0.313=51.499=\mathbf{51.5},$$

correct to 3 significant figures

Problem 5. Determine $\int x^2\sin x\,dx$.

Let $u=x^2$, from which, $\frac{du}{dx}=2x$, i.e. $du=2x\,dx$, and let $dv=\sin x\,dx$, from which,

$$v=\int\sin x\,dx=-\cos x$$

Substituting into $\int u\,dv=uv-\int v\,du$ gives:

$$\int x^2\sin x\,dx=(x^2)(-\cos x)-\int(-\cos x)(2x\,dx)$$

$$=-x^2\cos x+2\left[\int x\cos x\,dx\right]$$

The integral, $\int x\cos x\,dx$, is not a 'standard integral' and it can only be determined by using the integration by parts formula again.
From Problem 1, $\int x\cos x\,dx = x\sin x + \cos x$

Hence $\int x^2\sin x\,dx$

$$= -x^2\cos x + 2\{x\sin x + \cos x\} + c$$

$$= -x^2\cos x + 2x\sin x + 2\cos x + c$$

$$= \mathbf{(2-x^2)\cos x + 2x\sin x + c}$$

In general, if the algebraic term of a product is of power n, then the integration by parts formula is applied n times.

Now try the following exercise

Exercise 167 Further problems on integration by parts

Determine the integrals in Problems 1 to 5 using integration by parts.

1. $\int xe^{2x}\,dx$ $\left[\left[\frac{e^{2x}}{2}\left(x-\frac{1}{2}\right)\right]+c\right]$

2. $\int \frac{4x}{e^{3x}}\,dx$ $\left[-\frac{4}{3}e^{-3x}\left(x+\frac{1}{3}\right)+c\right]$

3. $\int x\sin x\,dx$ $[-x\cos x+\sin x+c]$

4. $\int 5\theta\cos 2\theta\,d\theta$ $\left[\frac{5}{2}\left(\theta\sin 2\theta+\frac{1}{2}\cos 2\theta\right)+c\right]$

5. $\int 3t^2e^{2t}\,dt$ $\left[\frac{3}{2}e^{2t}\left(t^2-t+\frac{1}{2}\right)+c\right]$

Evaluate the integrals in Problems 6 to 9, correct to 4 significant figures.

6. $\int_0^2 2xe^x\,dx$ [16.78]

7. $\int_0^{\frac{\pi}{4}} x\sin 2x\,dx$ [0.2500]

8. $\int_0^{\frac{\pi}{2}} t^2\cos t\,dt$ [0.4674]

9. $\int_1^2 3x^2e^{\frac{x}{2}}\,dx$ [15.78]

43.3 Further worked problems on integration by parts

Problem 6. Find $\int x\ln x\,dx$.

The logarithmic function is chosen as the 'u part'.
Thus when $u=\ln x$, then $\frac{du}{dx}=\frac{1}{x}$, i.e. $du=\frac{dx}{x}$

Letting $dv=x\,dx$ gives $v=\int x\,dx=\frac{x^2}{2}$
Substituting into $\int u\,dv = uv-\int v\,du$ gives:

$$\int x\ln x\,dx = (\ln x)\left(\frac{x^2}{2}\right)-\int\left(\frac{x^2}{2}\right)\frac{dx}{x}$$

$$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx$$

$$= \frac{x^2}{2}\ln x - \frac{1}{2}\left(\frac{x^2}{2}\right)+c$$

Hence $\int x\ln x\,dx = \mathbf{\frac{x^2}{2}\left(\ln x-\frac{1}{2}\right)+c}$ **or**

$$\mathbf{\frac{x^2}{4}(2\ln x-1)+c}$$

Problem 7. Determine $\int \ln x\,dx$.

$\int \ln x\,dx$ is the same as $\int (1)\ln x\,dx$
Let $u=\ln x$, from which, $\frac{du}{dx}=\frac{1}{x}$, i.e. $du=\frac{dx}{x}$
and let $dv=1dx$, from which, $v=\int 1\,dx = x$
Substituting into $\int u\,dv = uv-\int v\,du$ gives:

$$\int \ln x\,dx = (\ln x)(x)-\int x\frac{dx}{x}$$

$$= x\ln x - \int dx = x\ln x - x + c$$

Hence $\int \mathbf{\ln x\,dx = x(\ln x-1)+c}$

Problem 8. Evaluate $\int_1^9 \sqrt{x}\ln x\,\mathrm{d}x$, correct to 3 significant figures.

Let $u=\ln x$, from which $\mathrm{d}u=\dfrac{\mathrm{d}x}{x}$

and let $\mathrm{d}v=\sqrt{x}\,\mathrm{d}x=x^{\frac{1}{2}}\,\mathrm{d}x$, from which,

$$v=\int x^{\frac{1}{2}}\,\mathrm{d}x=\frac{2}{3}x^{\frac{3}{2}}$$

Substituting into $\int u\,\mathrm{d}v=uv-\int v\,\mathrm{d}u$ gives:

$$\begin{aligned}\int\sqrt{x}\ln x\,\mathrm{d}x&=(\ln x)\left(\frac{2}{3}x^{\frac{3}{2}}\right)-\int\left(\frac{2}{3}x^{\frac{3}{2}}\right)\left(\frac{\mathrm{d}x}{x}\right)\\&=\frac{2}{3}\sqrt{x^3}\ln x-\frac{2}{3}\int x^{\frac{1}{2}}\,\mathrm{d}x\\&=\frac{2}{3}\sqrt{x^3}\ln x-\frac{2}{3}\left(\frac{2}{3}x^{\frac{3}{2}}\right)+c\\&=\frac{2}{3}\sqrt{x^3}\left[\ln x-\frac{2}{3}\right]+c\end{aligned}$$

Hence $\int_1^9\sqrt{x}\ln x\,\mathrm{d}x$

$$\begin{aligned}&=\left[\frac{2}{3}\sqrt{x^3}\left(\ln x-\frac{2}{3}\right)\right]_1^9\\&=\left[\frac{2}{3}\sqrt{9^3}\left(\ln 9-\frac{2}{3}\right)\right]-\left[\frac{2}{3}\sqrt{1^3}\left(\ln 1-\frac{2}{3}\right)\right]\\&=\left[18\left(\ln 9-\frac{2}{3}\right)\right]-\left[\frac{2}{3}\left(0-\frac{2}{3}\right)\right]\\&=27.550+0.444=27.994=\mathbf{28.0},\end{aligned}$$

correct to 3 significant figures.

Problem 9. Find $\int \mathrm{e}^{ax}\cos bx\,\mathrm{d}x$.

When integrating a product of an exponential and a sine or cosine function it is immaterial which part is made equal to 'u'.

Let $u=\mathrm{e}^{ax}$, from which $\dfrac{\mathrm{d}u}{\mathrm{d}x}=a\mathrm{e}^{ax}$,

i.e. $\mathrm{d}u=a\mathrm{e}^{ax}\,\mathrm{d}x$ and let $\mathrm{d}v=\cos bx\,\mathrm{d}x$, from which,

$$v=\int\cos bx\,\mathrm{d}x=\frac{1}{b}\sin bx$$

Substituting into $\int u\,\mathrm{d}v=uv-\int v\,\mathrm{d}u$ gives:

$$\begin{aligned}&\int \mathrm{e}^{ax}\cos bx\,\mathrm{d}x\\&\quad=(\mathrm{e}^{ax})\left(\frac{1}{b}\sin bx\right)-\int\left(\frac{1}{b}\sin bx\right)(a\mathrm{e}^{ax}\,\mathrm{d}x)\\&\quad=\frac{1}{b}\mathrm{e}^{ax}\sin bx-\frac{a}{b}\left[\int\mathrm{e}^{ax}\sin bx\,\mathrm{d}x\right]\end{aligned}\qquad(1)$$

$\int \mathrm{e}^{ax}\sin bx\,\mathrm{d}x$ is now determined separately using integration by parts again:

Let $u=\mathrm{e}^{ax}$ then $\mathrm{d}u=a\mathrm{e}^{ax}\,\mathrm{d}x$, and let $\mathrm{d}v=\sin bx\,\mathrm{d}x$, from which

$$v=\int\sin bx\,\mathrm{d}x=-\frac{1}{b}\cos bx$$

Substituting into the integration by parts formula gives:

$$\begin{aligned}\int\mathrm{e}^{ax}\sin bx\,\mathrm{d}x&=(\mathrm{e}^{ax})\left(-\frac{1}{b}\cos bx\right)\\&\quad-\int\left(-\frac{1}{b}\cos bx\right)(a\mathrm{e}^{ax}\,\mathrm{d}x)\\&=-\frac{1}{b}\mathrm{e}^{ax}\cos bx+\frac{a}{b}\int\mathrm{e}^{ax}\cos bx\,\mathrm{d}x\end{aligned}$$

Substituting this result into equation (1) gives:

$$\begin{aligned}\int\mathrm{e}^{ax}\cos bx\,\mathrm{d}x&=\frac{1}{b}\mathrm{e}^{ax}\sin bx-\frac{a}{b}\left[-\frac{1}{b}\mathrm{e}^{ax}\cos bx\right.\\&\quad\left.+\frac{a}{b}\int\mathrm{e}^{ax}\cos bx\,\mathrm{d}x\right]\\&=\frac{1}{b}\mathrm{e}^{ax}\sin bx+\frac{a}{b^2}\mathrm{e}^{ax}\cos bx\\&\quad-\frac{a^2}{b^2}\int\mathrm{e}^{ax}\cos bx\,\mathrm{d}x\end{aligned}$$

The integral on the far right of this equation is the same as the integral on the left hand side and thus they may be combined.

$$\begin{aligned}&\int\mathrm{e}^{ax}\cos bx\,\mathrm{d}x+\frac{a^2}{b^2}\int\mathrm{e}^{ax}\cos bx\,\mathrm{d}x\\&\qquad=\frac{1}{b}\mathrm{e}^{ax}\sin bx+\frac{a}{b^2}\mathrm{e}^{ax}\cos bx\end{aligned}$$

i.e. $\left(1+\frac{a^2}{b^2}\right)\int e^{ax}\cos bx\,dx$

$$=\frac{1}{b}e^{ax}\sin bx+\frac{a}{b^2}e^{ax}\cos bx$$

i.e. $\left(\frac{b^2+a^2}{b^2}\right)\int e^{ax}\cos bx\,dx$

$$=\frac{e^{ax}}{b^2}(b\sin bx+a\cos bx)$$

Hence $\int e^{ax}\cos bx\,dx$

$$=\left(\frac{b^2}{b^2+a^2}\right)\left(\frac{e^{ax}}{b^2}\right)(b\sin bx+a\cos bx)$$

$$=\frac{\mathbf{e^{ax}}}{\mathbf{a^2+b^2}}\mathbf{(b\sin bx+a\cos bx)+c}$$

Using a similar method to above, that is, integrating by parts twice, the following result may be proved:

$$\int \mathbf{e^{ax}\sin bx\,dx}$$

$$\mathbf{=\frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)+c} \qquad (2)$$

Problem 10. Evaluate $\int_0^{\frac{\pi}{4}} e^t\sin 2t\,dt$, correct to 4 decimal places.

Comparing $\int e^t\sin 2t\,dt$ with $\int e^{ax}\sin bx\,dx$ shows that $x=t$, $a=1$ and $b=2$.

Hence, substituting into equation (2) gives:

$$\int_0^{\frac{\pi}{4}} e^t\sin 2t\,dt$$

$$=\left[\frac{e^t}{1^2+2^2}(1\sin 2t-2\cos 2t)\right]_0^{\frac{\pi}{4}}$$

$$=\left[\frac{e^{\frac{\pi}{4}}}{5}\left(\sin 2\left(\frac{\pi}{4}\right)-2\cos 2\left(\frac{\pi}{4}\right)\right)\right]$$

$$-\left[\frac{e^0}{5}(\sin 0-2\cos 0)\right]$$

$$=\left[\frac{e^{\frac{\pi}{4}}}{5}(1-0)\right]-\left[\frac{1}{5}(0-2)\right]=\frac{e^{\frac{\pi}{4}}}{5}+\frac{2}{5}$$

$$=\mathbf{0.8387},\ \text{correct to 4 decimal places.}$$

Now try the following exercise

Exercise 168 Further problems on integration by parts

Determine the integrals in Problems 1 to 5 using integration by parts.

1. $\int 2x^2\ln x\,dx$ $\left[\frac{2}{3}x^3\left(\ln x-\frac{1}{3}\right)+c\right]$
2. $\int 2\ln 3x\,dx$ $[2x(\ln 3x-1)+c]$
3. $\int x^2\sin 3x\,dx$ $\left[\frac{\cos 3x}{27}(2-9x^2)+\frac{2}{9}x\sin 3x+c\right]$
4. $\int 2e^{5x}\cos 2x\,dx$ $\left[\frac{2}{29}e^{5x}(2\sin 2x+5\cos 2x)+c\right]$
5. $\int 2\theta\sec^2\theta\,d\theta$ $[2[\theta\tan\theta-\ln(\sec\theta)]+c]$

Evaluate the integrals in Problems 6 to 9, correct to 4 significant figures.

6. $\int_1^2 x\ln x\,dx$ [0.6363]
7. $\int_0^1 2e^{3x}\sin 2x\,dx$ [11.31]
8. $\int_0^{\frac{\pi}{2}} e^t\cos 3t\,dt$ [−1.543]
9. $\int_1^4 \sqrt{x^3}\ln x\,dx$ [12.78]

10. In determining a Fourier series to represent $f(x)=x$ in the range $-\pi$ to π, Fourier coefficients are given by:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,\mathrm{d}x$$

and $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,\mathrm{d}x$$

where n is a positive integer. Show by using integration by parts that $a_n=0$ and $b_n=-\frac{2}{n}\cos n\pi$.

11. The equation $C=\int_0^1 \mathrm{e}^{-0.4\theta}\cos 1.2\theta\,\mathrm{d}\theta$

and $S=\int_0^1 \mathrm{e}^{-0.4\theta}\sin 1.2\theta\,\mathrm{d}\theta$

are involved in the study of damped oscillations. Determine the values of C and S.

[$C=0.66$, $S=0.41$]

Chapter 44

Reduction formulae

44.1 Introduction

When using integration by parts in Chapter 43, an integral such as $\int x^2 e^x \, dx$ requires integration by parts twice. Similarly, $\int x^3 e^x \, dx$ requires integration by parts three times. Thus, integrals such as $\int x^5 e^x \, dx$, $\int x^6 \cos x \, dx$ and $\int x^8 \sin 2x \, dx$ for example, would take a long time to determine using integration by parts. **Reduction formulae** provide a quicker method for determining such integrals and the method is demonstrated in the following sections.

44.2 Using reduction formulae for integrals of the form $\int x^n e^x dx$

To determine $\int x^n e^x \, dx$ using integration by parts,

let $u = x^n$ from which,

$$\frac{du}{dx} = nx^{n-1} \text{ and } du = nx^{n-1} \, dx$$

and $dv = e^x \, dx$ from which,

$$v = \int e^x \, dx = e^x$$

Thus, $\int x^n e^x \, dx = x^n e^x - \int e^x n x^{n-1} \, dx$

using the integration by parts formula,

$$= x^n e^x - n \int x^{n-1} e^x \, dx$$

The integral on the far right is seen to be of the same form as the integral on the left-hand side, except that n has been replaced by $n-1$.

Thus, if we let,

$$\int x^n e^x \, dx = I_n,$$

then $\int x^{n-1} e^x \, dx = I_{n-1}$

Hence $\int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx$

can be written as:

$$I_n = x^n e^x - n I_{n-1} \qquad (1)$$

Equation (1) is an example of a reduction formula since it expresses an integral in n in terms of the same integral in $n-1$.

Problem 1. Determine $\int x^2 e^x \, dx$ using a reduction formula.

Using equation (1) with $n = 2$ gives:

$$\int x^2 e^x \, dx = I_2 = x^2 e^x - 2I_1$$

and $I_1 = x^1 e^x - 1 I_0$

$$I_0 = \int x^0 e^x \, dx = \int e^x \, dx = e^x + c_1$$

Hence $I_2 = x^2 e^x - 2[x e^x - 1 I_0]$

$$= x^2 e^x - 2[x e^x - 1(e^x + c_1)]$$

i.e. $\int \mathbf{x^2 e^x \, dx} = x^2 e^x - 2x e^x + 2e^x + 2c_1$

$$= \mathbf{e^x(x^2 - 2x + 2) + c}$$

(where $c = 2c_1$)

As with integration by parts, in the following examples the constant of integration will be added at the last step with indefinite integrals.

Problem 2. Use a reduction formula to determine $\int x^3 e^x \, dx$.

From equation (1), $I_n = x^n e^x - n I_{n-1}$

Hence $\int x^3 e^x \, dx = I_3 = x^3 e^x - 3I_2$

$I_2 = x^2 e^x - 2I_1$

$I_1 = x^1 e^x - 1I_0$

and $I_0 = \int x^0 e^x \, dx = \int e^x \, dx = e^x$

Thus $\int x^3 e^x \, dx = x^3 e^x - 3[x^2 e^x - 2I_1]$

$= x^3 e^x - 3[x^2 e^x - 2(x e^x - I_0)]$

$= x^3 e^x - 3[x^2 e^x - 2(x e^x - e^x)]$

$= x^3 e^x - 3x^2 e^x + 6(x e^x - e^x)$

$= x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x$

i.e. $\int \boldsymbol{x^3 e^x \, dx = e^x(x^3 - 3x^2 + 6x - 6) + c}$

Now try the following exercise

Exercise 169 Further problems on using reduction formulae for integrals of the form $\int x^n e^x \, dx$

1. Use a reduction formula to determine $\int x^4 e^x \, dx$.
   $[e^x(x^4 - 4x^3 + 12x^2 - 24x + 24) + c]$
2. Determine $\int t^3 e^{2t} dt$ using a reduction formula.
   $\left[e^{2t}\left(\frac{1}{2}t^3 - \frac{3}{4}t^2 + \frac{3}{4}t - \frac{3}{8}\right) + c\right]$
3. Use the result of Problem 2 to evaluate $\int_0^1 5t^3 e^{2t} dt$, correct to 3 decimal places.
   [6.493]

44.3 Using reduction formulae for integrals of the form $\int x^n \cos x \, dx$ and $\int x^n \sin x \, dx$

(a) $\int x^n \cos x \, dx$

Let $I_n = \int x^n \cos x \, dx$ then, using integration by parts:

if $u = x^n$ then $\frac{du}{dx} = nx^{n-1}$

and if $dv = \cos x \, dx$ then

$v = \int \cos x \, dx = \sin x$

Hence $I_n = x^n \sin x - \int (\sin x) n x^{n-1} \, dx$

$= x^n \sin x - n \int x^{n-1} \sin x \, dx$

Using integration by parts again, this time with $u = x^{n-1}$:

$\frac{du}{dx} = (n-1)x^{n-2}$, and $dv = \sin x \, dx$,

from which,

$v = \int \sin x \, dx = -\cos x$

Hence $I_n = x^n \sin x - n\left[x^{n-1}(-\cos x) - \int (-\cos x)(n-1)x^{n-2} \, dx\right]$

$= x^n \sin x + nx^{n-1} \cos x - n(n-1) \int x^{n-2} \cos x \, dx$

i.e. $\boldsymbol{I_n = x^n \sin x + nx^{n-1} \cos x - n(n-1) I_{n-2}}$ (2)

Problem 3. Use a reduction formula to determine $\int x^2 \cos x \, dx$.

Using the reduction formula of equation (2):

$\int x^2 \cos x \, dx = I_2$

$= x^2 \sin x + 2x^1 \cos x - 2(1) I_0$

and $I_0 = \int x^0 \cos x \, dx$

$= \int \cos x \, dx = \sin x$

Hence

$$\int \boldsymbol{x^2 \cos x \, dx = x^2 \sin x + 2x \cos x - 2 \sin x + c}$$

Problem 4. Evaluate $\int_1^2 4t^3 \cos t \, dt$, correct to 4 significant figures.

Let us firstly find a reduction formula for $\int t^3 \cos t \, dt$.

From equation (2),

$$\int t^3 \cos t \, dt = I_3 = t^3 \sin t + 3t^2 \cos t - 3(2)I_1$$

and

$$I_1 = t^1 \sin t + 1t^0 \cos t - 1(0)I_{n-2}$$
$$= t \sin t + \cos t$$

Hence

$$\int t^3 \cos t \, dt = t^3 \sin t + 3t^2 \cos t - 3(2)[t \sin t + \cos t]$$
$$= t^3 \sin t + 3t^2 \cos t - 6t \sin t - 6 \cos t$$

Thus

$$\int_1^2 4t^3 \cos t \, dt$$
$$= [4(t^3 \sin t + 3t^2 \cos t - 6t \sin t - 6 \cos t)]_1^2$$
$$= [4(8 \sin 2 + 12 \cos 2 - 12 \sin 2 - 6 \cos 2)]$$
$$- [4(\sin 1 + 3 \cos 1 - 6 \sin 1 - 6 \cos 1)]$$
$$= (-24.53628) - (-23.31305)$$
$$= \mathbf{-1.223}$$

Problem 5. Determine a reduction formula for $\int_0^{\pi} x^n \cos x \, dx$ and hence evaluate $\int_0^{\pi} x^4 \cos x \, dx$, correct to 2 decimal places.

From equation (2),

$$I_n = x^n \sin x + nx^{n-1} \cos x - n(n-1)I_{n-2}.$$

hence $\int_0^{\pi} x^n \cos x \, dx = [x^n \sin x + nx^{n-1} \cos x]_0^{\pi}$
$$- n(n-1)I_{n-2}$$
$$= [(\pi^n \sin \pi + n\pi^{n-1} \cos \pi)$$
$$- (0 + 0)] - n(n-1)I_{n-2}$$
$$= \mathbf{-n\pi^{n-1} - n(n-1)I_{n-2}}$$

Hence

$$\int_0^{\pi} x^4 \cos x \, dx = I_4$$
$$= -4\pi^3 - 4(3)I_2 \text{ since } n = 4$$

When $n = 2$,

$$\int_0^{\pi} x^2 \cos x \, dx = I_2 = -2\pi^1 - 2(1)I_0$$

and

$$I_0 = \int_0^{\pi} x^0 \cos x \, dx$$
$$= \int_0^{\pi} \cos x \, dx$$
$$= [\sin x]_0^{\pi} = 0$$

Hence

$$\int_0^{\pi} x^4 \cos x \, dx = -4\pi^3 - 4(3)[-2\pi - 2(1)(0)]$$
$$= \mathbf{-4\pi^3 + 24\pi} \text{ or } \mathbf{-48.63},$$

correct to 2 decimal places.

(b) $\int x^n \sin x \, dx$

Let $I_n = \int x^n \sin x \, dx$
Using integration by parts, if $u = x^n$ then $\frac{du}{dx} = nx^{n-1}$ and if $dv = \sin x \, dx$ then $v = \int \sin x \, dx = -\cos x$. Hence

$$\int x^n \sin x \, dx$$
$$= I_n = x^n(-\cos x) - \int (-\cos x) nx^{n-1} \, dx$$
$$= -x^n \cos x + n \int x^{n-1} \cos x \, dx$$

Using integration by parts again, with $u = x^{n-1}$, from which, $\frac{du}{dx} = (n-1)x^{n-2}$ and $dv = \cos x$, from which, $v = \int \cos x \, dx = \sin x$. Hence

$$I_n = -x^n \cos x + n \left[x^{n-1}(\sin x) - \int (\sin x)(n-1)x^{n-2} \, dx \right]$$
$$= -x^n \cos x + nx^{n-1}(\sin x) - n(n-1) \int x^{n-2} \sin x \, dx$$

i.e. $$I_n = -x^n \cos x + nx^{n-1} \sin x - n(n-1)I_{n-2} \quad (3)$$

Problem 6. Use a reduction formula to determine $\int x^3 \sin x \, dx$.

Using equation (3),

$$\int x^3 \sin x \, dx = I_3$$
$$= -x^3 \cos x + 3x^2 \sin x - 3(2)I_1$$

and $$I_1 = -x^1 \cos x + 1x^0 \sin x$$
$$= -x \cos x + \sin x$$

Hence

$$\int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x - 6[-x \cos x + \sin x]$$
$$= \mathbf{-x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + c}$$

Problem 7. Evaluate $\int_0^{\frac{\pi}{2}} 3\theta^4 \sin\theta \, d\theta$, correct to 2 decimal places.

From equation (3),

$$I_n = [-x^n \cos x + nx^{n-1}(\sin x)]_0^{\frac{\pi}{2}} - n(n-1)I_{n-2}$$
$$= \left[\left(-\left(\frac{\pi}{2}\right)^n \cos\frac{\pi}{2} + n\left(\frac{\pi}{2}\right)^{n-1} \sin\frac{\pi}{2}\right) - (0)\right] - n(n-1)I_{n-2}$$
$$= \mathbf{n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}}$$

Hence

$$\int_0^{\frac{\pi}{2}} 3\theta^4 \sin\theta \, d\theta = 3\int_0^{\frac{\pi}{2}} \theta^4 \sin\theta \, d\theta$$
$$= 3I_4$$
$$= 3\left[4\left(\frac{\pi}{2}\right)^3 - 4(3)I_2\right]$$

$$I_2 = 2\left(\frac{\pi}{2}\right)^1 - 2(1)I_0 \text{ and}$$

$$I_0 = \int_0^{\frac{\pi}{2}} \theta^0 \sin\theta \, d\theta = [-\cos x]_0^{\frac{\pi}{2}}$$
$$= [-0 - (-1)] = 1$$

Hence

$$3\int_0^{\frac{\pi}{2}} \theta^4 \sin\theta \, d\theta$$
$$= 3I_4$$
$$= 3\left[4\left(\frac{\pi}{2}\right)^3 - 4(3)\left\{2\left(\frac{\pi}{2}\right)^1 - 2(1)I_0\right\}\right]$$
$$= 3\left[4\left(\frac{\pi}{2}\right)^3 - 4(3)\left\{2\left(\frac{\pi}{2}\right)^1 - 2(1)(1)\right\}\right]$$
$$= 3\left[4\left(\frac{\pi}{2}\right)^3 - 24\left(\frac{\pi}{2}\right)^1 + 24\right]$$
$$= 3(15.503 - 37.699 + 24)$$
$$= 3(1.8039) = \mathbf{5.41}$$

Now try the following exercise

Exercise 170 Further problems on reduction formulae for integrals of the form $\int x^n \cos x \, dx$ and $\int x^n \sin x \, dx$

1. Use a reduction formula to determine $\int x^5 \cos x \, dx$. $\left[\begin{array}{l} x^5 \sin x + 5x^4 \cos x - 20x^3 \sin x \\ -60x^2 \cos x + 120x \sin x \\ +120 \cos x + c \end{array}\right]$
2. Evaluate $\int_0^{\pi} x^5 \cos x \, dx$, correct to 2 decimal places. [−134.87]
3. Use a reduction formula to determine $\int x^5 \sin x \, dx$. $\left[\begin{array}{l} -x^5 \cos x + 5x^4 \sin x + 20x^3 \cos x \\ -60x^2 \sin x - 120x \cos x \\ +120 \sin x + c \end{array}\right]$
4. Evaluate $\int_0^{\pi} x^5 \sin x \, dx$, correct to 2 decimal places. [62.89]

44.4 Using reduction formulae for integrals of the form $\int \sin^n x \, dx$ and $\int \cos^n x \, dx$

(a) $\int \sin^n x \, dx$

Let $I_n = \int \sin^n x \, dx \equiv \int \sin^{n-1} x \sin x \, dx$ from laws of indices.

Using integration by parts, let $u = \sin^{n-1} x$, from which,

$$\frac{du}{dx} = (n-1)\sin^{n-2}x\cos x \quad \text{and}$$

$$du = (n-1)\sin^{n-2}x\cos x\,dx$$

and let $dv = \sin x\,dx$, from which, $v = \int \sin x\,dx = -\cos x$. Hence,

$$I_n = \int \sin^{n-1}x\sin x\,dx$$

$$= (\sin^{n-1}x)(-\cos x) - \int (-\cos x)(n-1)\sin^{n-2}x\cos x\,dx$$

$$= -\sin^{n-1}x\cos x + (n-1)\int \cos^2 x\sin^{n-2}x\,dx$$

$$= -\sin^{n-1}x\cos x + (n-1)\int (1-\sin^2 x)\sin^{n-2}x\,dx$$

$$= -\sin^{n-1}x\cos x + (n-1)\left\{\int \sin^{n-2}x\,dx - \int \sin^n x\,dx\right\}$$

i.e. $$I_n = -\sin^{n-1}x\cos x + (n-1)I_{n-2} - (n-1)I_n$$

i.e. $$I_n + (n-1)I_n = -\sin^{n-1}x\cos x + (n-1)I_{n-2}$$

and $$nI_n = -\sin^{n-1}x\cos x + (n-1)I_{n-2}$$

from which,

$$\int \sin^n x\,dx =$$

$$\boldsymbol{I_n = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}I_{n-2}} \qquad (4)$$

Problem 8. Use a reduction formula to determine $\int \sin^4 x\,dx$.

Using equation (4),

$$\int \sin^4 x\,dx = I_4 = -\frac{1}{4}\sin^3 x\cos x + \frac{3}{4}I_2$$

$$I_2 = -\frac{1}{2}\sin^1 x\cos x + \frac{1}{2}I_0$$

and $$I_0 = \int \sin^0 x\,dx = \int 1\,dx = x$$

Hence

$$\int \sin^4 x\,dx = I_4 = -\frac{1}{4}\sin^3 x\cos x + \frac{3}{4}\left[-\frac{1}{2}\sin x\cos x + \frac{1}{2}(x)\right]$$

$$= \boldsymbol{-\frac{1}{4}\sin^3 x\cos x - \frac{3}{8}\sin x\cos x + \frac{3}{8}x + c}$$

Problem 9. Evaluate $\int_0^1 4\sin^5 t\,dt$, correct to 3 significant figures.

Using equation (4),

$$\int \sin^5 t\,dt = I_5 = -\frac{1}{5}\sin^4 t\cos t + \frac{4}{5}I_3$$

$$I_3 = -\frac{1}{3}\sin^2 t\cos t + \frac{2}{3}I_1$$

and $$I_1 = -\frac{1}{1}\sin^0 t\cos t + 0 = -\cos t$$

Hence

$$\int \sin^5 t\,dt = -\frac{1}{5}\sin^4 t\cos t + \frac{4}{5}\left[-\frac{1}{3}\sin^2 t\cos t + \frac{2}{3}(-\cos t)\right]$$

$$= -\frac{1}{5}\sin^4 t\cos t - \frac{4}{15}\sin^2 t\cos t - \frac{8}{15}\cos t + c$$

and $$\int_0^t 4\sin^5 t\,dt$$

$$= 4\left[-\frac{1}{5}\sin^4 t\cos t - \frac{4}{15}\sin^2 t\cos t - \frac{8}{15}\cos t\right]_0^1$$

$$= 4\left[\left(-\frac{1}{5}\sin^4 1\cos 1 - \frac{4}{15}\sin^2 1\cos 1 - \frac{8}{15}\cos 1\right) - \left(-0-0-\frac{8}{15}\right)\right]$$

$$= 4[(-0.054178 - 0.1020196 - 0.2881612) - (-0.533333)]$$

$$= 4(0.0889745) = \mathbf{0.356}$$

Problem 10. Determine a reduction formula for $\int_0^{\frac{\pi}{2}} \sin^n x\,dx$ and hence evaluate $\int_0^{\frac{\pi}{2}} \sin^6 x\,dx$

From equation (4),

$$\int \sin^n x\,dx = I_n = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n} I_{n-2}$$

hence

$$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \left[-\frac{1}{n}\sin^{n-1} x\cos x\right]_0^{\frac{\pi}{2}} + \frac{n-1}{n} I_{n-2}$$

$$= [0-0] + \frac{n-1}{n} I_{n-2}$$

i.e. $$\boldsymbol{I_n = \frac{n-1}{n} I_{n-2}}$$

Hence

$$\int_0^{\frac{\pi}{2}} \sin^6 x\,dx = I_6 = \frac{5}{6} I_4$$

$$I_4 = \frac{3}{4} I_2, \quad I_2 = \frac{1}{2} I_0$$

and $$I_0 = \int_0^{\frac{\pi}{2}} \sin^0 x\,dx = \int_0^{\frac{\pi}{2}} 1\,dx = \frac{\pi}{2}$$

Thus

$$\int_0^{\frac{\pi}{2}} \sin^6 x\,dx = I_6 = \frac{5}{6} I_4 = \frac{5}{6}\left[\frac{3}{4} I_2\right]$$

$$= \frac{5}{6}\left[\frac{3}{4}\left\{\frac{1}{2} I_0\right\}\right]$$

$$= \frac{5}{6}\left[\frac{3}{4}\left\{\frac{1}{2}\left[\frac{\pi}{2}\right]\right\}\right] = \mathbf{\frac{15}{96}\pi}$$

(b) $\int \cos^n x\,dx$

Let $I_n = \int \cos^n x\,dx \equiv \int \cos^{n-1} x\cos x\,dx$ from laws of indices.

Using integration by parts, let $u = \cos^{n-1} x$ from which,

$$\frac{du}{dx} = (n-1)\cos^{n-2} x(-\sin x)$$

and $$du = (n-1)\cos^{n-2} x(-\sin x)\,dx$$

and let $$dv = \cos x\,dx$$

from which, $v = \int \cos x\,dx = \sin x$

Then

$$I_n = (\cos^{n-1} x)(\sin x) - \int (\sin x)(n-1)\cos^{n-2} x(-\sin x)\,dx$$

$$= (\cos^{n-1} x)(\sin x) + (n-1)\int \sin^2 x\cos^{n-2} x\,dx$$

$$= (\cos^{n-1} x)(\sin x) + (n-1)\int (1-\cos^2 x)\cos^{n-2} x\,dx$$

$$= (\cos^{n-1} x)(\sin x) + (n-1)\left\{\int \cos^{n-2} x\,dx - \int \cos^n x\,dx\right\}$$

i.e. $I_n = (\cos^{n-1} x)(\sin x) + (n-1)I_{n-2} - (n-1)I_n$

i.e. $I_n + (n-1)I_n = (\cos^{n-1} x)(\sin x) + (n-1)I_{n-2}$

i.e. $nI_n = (\cos^{n-1} x)(\sin x) + (n-1)I_{n-2}$

Thus $$\boldsymbol{I_n = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n} I_{n-2}} \qquad (5)$$

Problem 11. Use a reduction formula to determine $\int \cos^4 x\,dx$.

Using equation (5),

$$\int \cos^4 x\,dx = I_4 = \frac{1}{4}\cos^3 x\sin x + \frac{3}{4} I_2$$

and $$I_2 = \frac{1}{2}\cos x\sin x + \frac{1}{2} I_0$$

and $$I_0 = \int \cos^0 x\,dx = \int 1\,dx = x$$

Hence $\int \cos^4 x\,dx$

$$= \frac{1}{4}\cos^3 x \sin x + \frac{3}{4}\left(\frac{1}{2}\cos x \sin x + \frac{1}{2}x\right)$$

$$= \mathbf{\frac{1}{4}\cos^3 x \sin x + \frac{3}{8}\cos x \sin x + \frac{3}{8}x + c}$$

Problem 12. Determine a reduction formula for $\int_0^{\frac{\pi}{2}} \cos^n x\,dx$ and hence evaluate $\int_0^{\frac{\pi}{2}} \cos^5 x\,dx$

From equation (5),

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n} I_{n-2}$$

and hence

$$\int_0^{\frac{\pi}{2}} \cos^n x\,dx = \left[\frac{1}{n}\cos^{n-1} x \sin x\right]_0^{\frac{\pi}{2}} + \frac{n-1}{n} I_{n-2}$$

$$= [0-0] + \frac{n-1}{n} I_{n-2}$$

i.e. $$\mathbf{\int_0^{\frac{\pi}{2}} \cos^n x\,dx = I_n = \frac{n-1}{n} I_{n-2}} \qquad (6)$$

(Note that this is the same reduction formula as for $\int_0^{\frac{\pi}{2}} \sin^n x\,dx$ (in Problem 10) and the result is usually known as **Wallis's formula**).

Thus, from equation (6),

$$\int_0^{\frac{\pi}{2}} \cos^5 x\,dx = \frac{4}{5} I_3, \qquad I_3 = \frac{2}{3} I_1$$

and $$I_1 = \int_0^{\frac{\pi}{2}} \cos^1 x\,dx$$

$$= [\sin x]_0^{\frac{\pi}{2}} = (1-0) = 1$$

Hence $$\int_0^{\frac{\pi}{2}} \cos^5 x\,dx = \frac{4}{5} I_3 = \frac{4}{5}\left[\frac{2}{3} I_1\right]$$

$$= \frac{4}{5}\left[\frac{2}{3}(1)\right] = \mathbf{\frac{8}{15}}$$

Now try the following exercise

Exercise 171 Further problems on reduction formulae for integrals of the form $\int \sin^n x\,dx$ and $\int \cos^n x\,dx$

1. Use a reduction formula to determine $\int \sin^7 x\,dx$.

$$\left[-\frac{1}{7}\sin^6 x \cos x - \frac{6}{35}\sin^4 x \cos x - \frac{8}{35}\sin^2 x \cos x - \frac{16}{35}\cos x + c\right]$$

2. Evaluate $\int_0^{\pi} 3\sin^3 x\,dx$ using a reduction formula. [4]

3. Evaluate $\int_0^{\frac{\pi}{2}} \sin^5 x\,dx$ using a reduction formula. $\left[\frac{8}{15}\right]$

4. Determine, using a reduction formula, $\int \cos^6 x\,dx$.

$$\left[\frac{1}{6}\cos^5 x \sin x + \frac{5}{24}\cos^3 x \sin x + \frac{5}{16}\cos x \sin x + \frac{5}{16}x + c\right]$$

5. Evaluate $\int_0^{\frac{\pi}{2}} \cos^7 x\,dx$. $\left[\frac{16}{35}\right]$

44.5 Further reduction formulae

The following worked problems demonstrate further examples where integrals can be determined using reduction formulae.

Problem 13. Determine a reduction formula for $\int \tan^n x\,dx$ and hence find $\int \tan^7 x\,dx$.

Let $I_n = \int \tan^n x\,dx \equiv \int \tan^{n-2} x \tan^2 x\,dx$

by the laws of indices

$$= \int \tan^{n-2} x(\sec^2 x - 1)\,dx$$

since $1 + \tan^2 x = \sec^2 x$

$$= \int \tan^{n-2} x \sec^2 x\,dx - \int \tan^{n-2} x\,dx$$

$$= \int \tan^{n-2} x \sec^2 x \, dx - I_{n-2}$$

i.e. $$\boldsymbol{I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}}$$

When $n = 7$,

$$I_7 = \int \tan^7 x \, dx = \frac{\tan^6 x}{6} - I_5$$

$$I_5 = \frac{\tan^4 x}{4} - I_3 \quad \text{and} \quad I_3 = \frac{\tan^2 x}{2} - I_1$$

$$I_1 = \int \tan x \, dx = \ln(\sec x)$$

from Problem 9, Chapter 39, page 394

Thus

$$\int \tan^7 x \, dx = \frac{\tan^6 x}{6} - \left[\frac{\tan^4 x}{4} - \left(\frac{\tan^2 x}{2} - \ln(\sec x) \right) \right]$$

Hence $$\boldsymbol{\int \tan^7 x \, dx}$$

$$\boldsymbol{= \frac{1}{6}\tan^6 x - \frac{1}{4}\tan^4 x + \frac{1}{2}\tan^2 x - \ln(\sec x) + c}$$

Problem 14. Evaluate, using a reduction formula, $\int_0^{\frac{\pi}{2}} \sin^2 t \cos^6 t \, dt$.

$$\int_0^{\frac{\pi}{2}} \sin^2 t \cos^6 t \, dt = \int_0^{\frac{\pi}{2}} (1 - \cos^2 t) \cos^6 t \, dt$$

$$= \int_0^{\frac{\pi}{2}} \cos^6 t \, dt - \int_0^{\frac{\pi}{2}} \cos^8 t \, dt$$

If $$I_n = \int_0^{\frac{\pi}{2}} \cos^n t \, dt$$

then

$$\int_0^{\frac{\pi}{2}} \sin^2 t \cos^6 t \, dt = I_6 - I_8$$

and from equation (6),

$$I_6 = \frac{5}{6} I_4 = \frac{5}{6}\left[\frac{3}{4} I_2\right]$$

$$= \frac{5}{6}\left[\frac{3}{4}\left(\frac{1}{2} I_0\right)\right]$$

and $$I_0 = \int_0^{\frac{\pi}{2}} \cos^0 t \, dt$$

$$= \int_0^{\frac{\pi}{2}} 1 \, dt = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

Hence $$I_6 = \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{15\pi}{96} \text{ or } \frac{5\pi}{32}$$

Similarly, $I_8 = \frac{7}{8} I_6 = \frac{7}{8} \cdot \frac{5\pi}{32}$

Thus

$$\int_0^{\frac{\pi}{2}} \sin^2 t \cos^6 t \, dt = I_6 - I_8$$

$$= \frac{5\pi}{32} - \frac{7}{8} \cdot \frac{5\pi}{32}$$

$$= \frac{1}{8} \cdot \frac{5\pi}{32} = \boldsymbol{\frac{5\pi}{256}}$$

Problem 15. Use integration by parts to determine a reduction formula for $\int (\ln x)^n \, dx$. Hence determine $\int (\ln x)^3 \, dx$.

Let $I_n = \int (\ln x)^n \, dx$.
Using integration by parts, let $u = (\ln x)^n$, from which,

$$\frac{du}{dx} = n(\ln x)^{n-1}\left(\frac{1}{x}\right)$$

and $$du = n(\ln x)^{n-1}\left(\frac{1}{x}\right) dx$$

and let $dv = dx$, from which, $v = \int dx = x$

Then $$I_n = \int (\ln x)^n \, dx$$

$$= (\ln x)^n (x) - \int (x) n (\ln x)^{n-1}\left(\frac{1}{x}\right) dx$$

$$= x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$$

i.e. $\boldsymbol{I_n = x(\ln x)^n - nI_{n-1}}$

When $n = 3$,

$$\int (\ln x)^3\,dx = I_3 = x(\ln x)^3 - 3I_2$$

$I_2 = x(\ln x)^2 - 2I_1$ and $I_1 = \int \ln x\,dx = x(\ln x - 1)$ from Problem 7, page 422.

Hence

$$\begin{aligned}\int (\ln x)^3\,dx &= x(\ln x)^3 - 3[x(\ln x)^2 - 2I_1] + c\\ &= x(\ln x)^3 - 3[x(\ln x)^2 \\ &\qquad - 2[x(\ln x - 1)]] + c\\ &= x(\ln x)^3 - 3[x(\ln x)^2 \\ &\qquad - 2x\ln x + 2x] + c\\ &= x(\ln x)^3 - 3x(\ln x)^2 \\ &\qquad + 6x\ln x - 6x + c\\ &= \boldsymbol{x[(\ln x)^3 - 3(\ln x)^2} \\ &\qquad \boldsymbol{+ 6\ln x - 6] + c}\end{aligned}$$

Now try the following exercise

Exercise 172 Further problems on reduction formulae

1. Evaluate $\int_0^{\frac{\pi}{2}} \cos^2 x \sin^5 x\,dx$. $\left[\frac{8}{105}\right]$
2. Determine $\int \tan^6 x\,dx$ by using reduction formulae and hence evaluate $\int_0^{\frac{\pi}{4}} \tan^6 x\,dx$. $\left[\frac{13}{15} - \frac{\pi}{4}\right]$
3. Evaluate $\int_0^{\frac{\pi}{2}} \cos^5 x \sin^4 x\,dx$. $\left[\frac{8}{315}\right]$
4. Use a reduction formula to determine $\int (\ln x)^4\,dx$. $\left[x(\ln x)^4 - 4x(\ln x)^3 + 12x(\ln x)^2 - 24x\ln x + 24x + c\right]$
5. Show that $\int_0^{\frac{\pi}{2}} \sin^3\theta\cos^4\theta\,d\theta = \frac{2}{35}$

Chapter 45

Numerical integration

45.1 Introduction

Even with advanced methods of integration there are many mathematical functions which cannot be integrated by analytical methods and thus approximate methods have then to be used. Approximate methods of definite integrals may be determined by what is termed **numerical integration**.

It may be shown that determining the value of a definite integral is, in fact, finding the area between a curve, the horizontal axis and the specified ordinates. Three methods of finding approximate areas under curves are the trapezoidal rule, the mid-ordinate rule and Simpson's rule, and these rules are used as a basis for numerical integration.

45.2 The trapezoidal rule

Let a required definite integral be denoted by $\int_a^b y\,dx$ and be represented by the area under the graph of $y = f(x)$ between the limits $x = a$ and $x = b$ as shown in Fig. 45.1. Let the range of integration be divided into n equal intervals each of width d, such that $nd = b - a$, i.e. $d = \dfrac{b-a}{n}$

The ordinates are labelled $y_1, y_2, y_3, \ldots, y_{n+1}$ as shown.

An approximation to the area under the curve may be determined by joining the tops of the ordinates by straight lines. Each interval is thus a trapezium, and since the area of a trapezium is given by:

$$\text{area} = \frac{1}{2}(\text{sum of parallel sides})(\text{perpendicular distance between them})$$ then

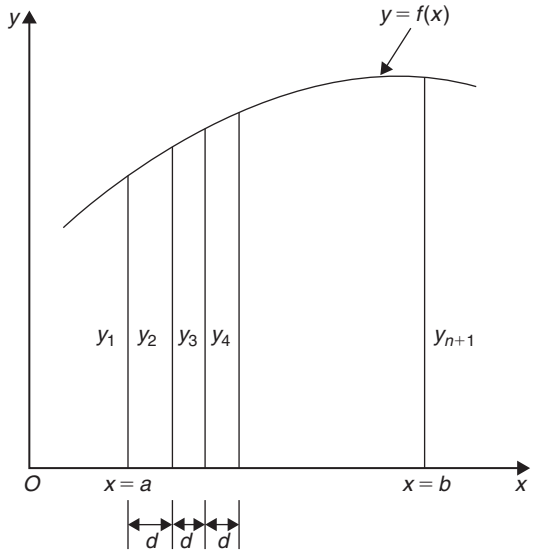


Figure 45.1

$$\int_a^b y\,dx \approx \frac{1}{2}(y_1 + y_2)d + \frac{1}{2}(y_2 + y_3)d + \frac{1}{2}(y_3 + y_4)d + \cdots \frac{1}{2}(y_n + y_{n+1})d$$

$$\approx d\left[\frac{1}{2}y_1 + y_2 + y_3 + y_4 + \cdots + y_n + \frac{1}{2}y_{n+1}\right]$$

i.e. **the trapezoidal rule states:**

$$\int_a^b y\,dx \approx \left(\begin{matrix}\textbf{width of}\\ \textbf{interval}\end{matrix}\right)\left\{\frac{1}{2}\left(\begin{matrix}\textbf{first + last}\\ \textbf{ordinate}\end{matrix}\right) + \left(\begin{matrix}\textbf{sum of remaining}\\ \textbf{ordinates}\end{matrix}\right)\right\} \quad (1)$$

Problem 1. (a) Use integration to evaluate, correct to 3 decimal places, $\int_1^3 \frac{2}{\sqrt{x}}\,dx$ (b) Use the trapezoidal rule with 4 intervals to evaluate the integral in part (a), correct to 3 decimal places.

(a) $$\int_1^3 \frac{2}{\sqrt{x}}\,dx = \int_1^3 2x^{-\frac{1}{2}}\,dx$$

$$= \left[\frac{2x^{\left(\frac{-1}{2}\right)+1}}{-\frac{1}{2}+1}\right]_1^3 = \left[4x^{\frac{1}{2}}\right]_1^3$$

$$= 4\left[\sqrt{x}\right]_1^3 = 4\left[\sqrt{3}-\sqrt{1}\right]$$

$$= \mathbf{2.928}, \text{ correct to 3 decimal places}$$

(b) The range of integration is the difference between the upper and lower limits, i.e. $3-1=2$. Using the trapezoidal rule with 4 intervals gives an interval width $d=\frac{3-1}{4}=0.5$ and ordinates situated at 1.0, 1.5, 2.0, 2.5 and 3.0. Corresponding values of $\frac{2}{\sqrt{x}}$ are shown in the table below, each correct to 4 decimal places (which is one more decimal place than required in the problem).

x	$\frac{2}{\sqrt{x}}$
1.0	2.0000
1.5	1.6330
2.0	1.4142
2.5	1.2649
3.0	1.1547

From equation (1):

$$\int_1^3 \frac{2}{\sqrt{x}}\,dx \approx (0.5)\left\{\frac{1}{2}(2.0000+1.1547) + 1.6330 + 1.4142 + 1.2649\right\}$$

$$= \mathbf{2.945}, \text{ correct to 3 decimal places}$$

This problem demonstrates that even with just 4 intervals a close approximation to the true value of 2.928 (correct to 3 decimal places) is obtained using the trapezoidal rule.

Problem 2. Use the trapezoidal rule with 8 intervals to evaluate, $\int_1^3 \frac{2}{\sqrt{x}}\,dx$ correct to 3 decimal places.

With 8 intervals, the width of each is $\frac{3-1}{8}$ i.e. 0.25 giving ordinates at 1.00, 1.25, 1.50, 1.75, 2.00, 2.25, 2.50, 2.75 and 3.00. Corresponding values of $\frac{2}{\sqrt{x}}$ are shown in the table below.

x	$\frac{2}{\sqrt{x}}$
1.00	2.0000
1.25	1.7889
1.50	1.6330
1.75	1.5119
2.00	1.4142
2.25	1.3333
2.50	1.2649
2.75	1.2060
3.00	1.1547

From equation (1):

$$\int_1^3 \frac{2}{\sqrt{x}}\,dx \approx (0.25)\left\{\frac{1}{2}(2.000+1.1547) + 1.7889 + 1.6330 + 1.5119 + 1.4142 + 1.3333 + 1.2649 + 1.2060\right\}$$

$$= \mathbf{2.932}, \text{ correct to 3 decimal places.}$$

This problem demonstrates that the greater the number of intervals chosen (i.e. the smaller the interval width) the more accurate will be the value of the definite integral. The exact value is found when the number of intervals is infinite, which is, of course, what the process of integration is based upon.

Problem 3. Use the trapezoidal rule to evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\,dx$ using 6 intervals. Give the answer correct to 4 significant figures.

With 6 intervals, each will have a width of $\frac{\frac{\pi}{2}-0}{6}$ i.e. $\frac{\pi}{12}$ rad (or 15°) and the ordinates occur at $0, \frac{\pi}{12}, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{5\pi}{12}$ and $\frac{\pi}{2}$

Corresponding values of $\frac{1}{1+\sin x}$ are shown in the table below.

x	$\frac{1}{1+\sin x}$
0	1.0000
$\frac{\pi}{12}$ (or 15°)	0.79440
$\frac{\pi}{6}$ (or 30°)	0.66667
$\frac{\pi}{4}$ (or 45°)	0.58579
$\frac{\pi}{3}$ (or 60°)	0.53590
$\frac{5\pi}{12}$ (or 75°)	0.50867
$\frac{\pi}{2}$ (or 90°)	0.50000

From equation (1):

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\,dx \approx \left(\frac{\pi}{12}\right)\left\{\frac{1}{2}(1.00000+0.50000) + 0.79440 + 0.66667 + 0.58579 + 0.53590 + 0.50867\right\}$$

$$= \mathbf{1.006}, \text{ correct to 4 significant figures.}$$

Now try the following exercise

Exercise 173 Further problems on the trapezoidal rule

In Problems 1 to 4, evaluate the definite integrals using the **trapezoidal rule**, giving the answers correct to 3 decimal places.

1. $\int_0^1 \frac{2}{1+x^2}\,dx$ (Use 8 intervals) [1.569]
2. $\int_1^3 2\ln 3x\,dx$ (Use 8 intervals) [6.979]
3. $\int_0^{\frac{\pi}{3}} \sqrt{(\sin\theta)}\,d\theta$ (Use 6 intervals) [0.672]
4. $\int_0^{1.4} e^{-x^2}\,dx$ (Use 7 intervals) [0.843]

45.3 The mid-ordinate rule

Let a required definite integral be denoted again by $\int_a^b y\,dx$ and represented by the area under the graph of $y = f(x)$ between the limits $x = a$ and $x = b$, as shown in Fig. 45.2.

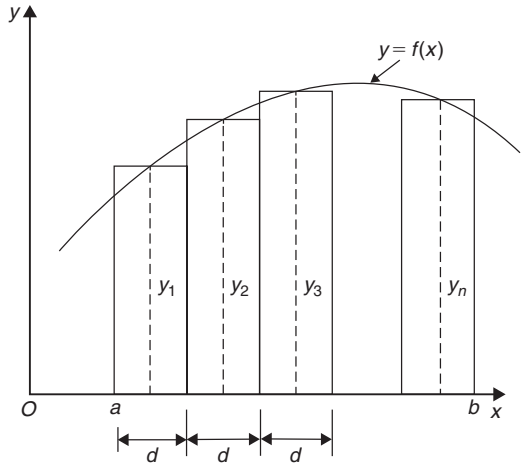


Figure 45.2

With the mid-ordinate rule each interval of width d is assumed to be replaced by a rectangle of height equal to the ordinate at the middle point of each interval, shown as $y_1, y_2, y_3, \ldots, y_n$ in Fig. 45.2.

Thus $\int_a^b y\,dx \approx dy_1 + dy_2 + dy_3 + \cdots + dy_n$

$$\approx d(y_1 + y_2 + y_3 + \cdots + y_n)$$

i.e. **the mid-ordinate rule states:**

$$\int_a^b y\,dx \approx \textbf{(width of interval) (sum of mid-ordinates)} \qquad (2)$$

Problem 4. Use the mid-ordinate rule with (a) 4 intervals, (b) 8 intervals, to evaluate $\int_1^3 \frac{2}{\sqrt{x}}\,dx$, correct to 3 decimal places.

(a) With 4 intervals, each will have a width of $\frac{3-1}{4}$, i.e. 0.5 and the ordinates will occur at 1.0, 1.5, 2.0, 2.5 and 3.0. Hence the mid-ordinates y_1, y_2, y_3 and y_4 occur at 1.25, 1.75, 2.25 and 2.75. Corresponding values of $\frac{2}{\sqrt{x}}$ are shown in the following table.

x	$\frac{2}{\sqrt{x}}$
1.25	1.7889
1.75	1.5119
2.25	1.3333
2.75	1.2060

From equation (2):

$$\int_1^3 \frac{2}{\sqrt{x}}\,dx \approx (0.5)[1.7889 + 1.5119 + 1.3333 + 1.2060]$$
$$= \mathbf{2.920}, \text{ correct to 3 decimal places.}$$

(b) With 8 intervals, each will have a width of 0.25 and the ordinates will occur at 1.00, 1.25, 1.50, 1.75,... and thus mid-ordinates at 1.125, 1.375, 1.625, 1.875...

Corresponding values of $\frac{2}{\sqrt{x}}$ are shown in the following table.

x	$\frac{2}{\sqrt{x}}$
1.125	1.8856
1.375	1.7056
1.625	1.5689
1.875	1.4606
2.125	1.3720
2.375	1.2978
2.625	1.2344
2.875	1.1795

From equation (2):

$$\int_1^3 \frac{2}{\sqrt{x}}\,dx \approx (0.25)[1.8856 + 1.7056 + 1.5689 + 1.4606 + 1.3720 + 1.2978 + 1.2344 + 1.1795]$$
$$= \mathbf{2.926}, \text{ correct to 3 decimal places.}$$

As previously, the greater the number of intervals the nearer the result is to the true value (of 2.928, correct to 3 decimal places).

Problem 5. Evaluate $\int_0^{2.4} e^{\frac{-x^2}{3}}\,dx$, correct to 4 significant figures, using the mid-ordinate rule with 6 intervals.

With 6 intervals each will have a width of $\frac{2.4-0}{6}$, i.e. 0.40 and the ordinates will occur at 0, 0.40, 0.80, 1.20, 1.60, 2.00 and 2.40 and thus mid-ordinates at 0.20, 0.60, 1.00, 1.40, 1.80 and 2.20. Corresponding values of $e^{\frac{-x^2}{3}}$ are shown in the following table.

x	$e^{\frac{-x^2}{3}}$
0.20	0.98676
0.60	0.88692
1.00	0.71653
1.40	0.52031
1.80	0.33960
2.20	0.19922

From equation (2):

$$\int_0^{2.4} e^{\frac{-x^2}{3}}\,dx \approx (0.40)[0.98676 + 0.88692 + 0.71653 + 0.52031 + 0.33960 + 0.19922]$$
$$= \mathbf{1.460}, \text{ correct to 4 significant figures.}$$

Now try the following exercise

Exercise 174 Further problems on the mid-ordinate rule

In Problems 1 to 4, evaluate the definite integrals using the **mid-ordinate rule**, giving the answers correct to 3 decimal places.

1. $\int_0^2 \frac{3}{1+t^2}\,dt$ (Use 8 intervals) [3.323]
2. $\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin\theta}\,d\theta$ (Use 6 intervals) [0.997]
3. $\int_1^3 \frac{\ln x}{x}\,dx$ (Use 10 intervals) [0.605]
4. $\int_0^{\frac{\pi}{3}} \sqrt{(\cos^3 x)}\,dx$ (Use 6 intervals) [0.799]

45.4 Simpson's rule

The approximation made with the trapezoidal rule is to join the top of two successive ordinates by a straight line, i.e. by using a linear approximation of the form $a+bx$. With Simpson's rule, the approximation made is to join the tops of three successive ordinates by a parabola, i.e. by using a quadratic approximation of the form $a+bx+cx^2$.

Figure 45.3 shows a parabola $y=a+bx+cx^2$ with ordinates y_1, y_2 and y_3 at $x=-d, x=0$ and $x=d$ respectively.

Thus the width of each of the two intervals is d. The area enclosed by the parabola, the x-axis and ordinates $x=-d$ and $x=d$ is given by:

$$\int_{-d}^{d}(a+bx+cx^2)dx = \left[ax+\frac{bx^2}{2}+\frac{cx^3}{3}\right]_{-d}^{d}$$

$$= \left(ad+\frac{bd^2}{2}+\frac{cd^3}{3}\right) - \left(-ad+\frac{bd^2}{2}-\frac{cd^3}{3}\right)$$

$$= 2ad+\frac{2}{3}cd^3 \text{ or}$$

$$\frac{1}{3}d(6a+2cd^2) \qquad (3)$$

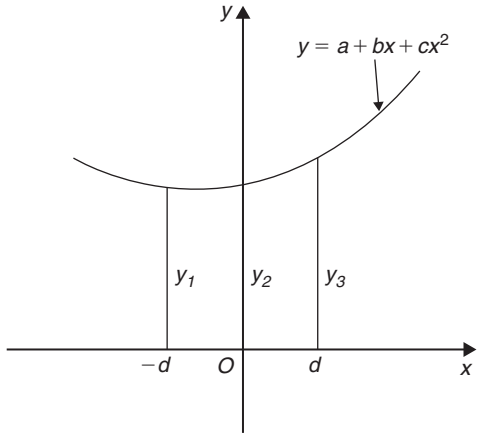


Figure 45.3

Since $y=a+bx+cx^2$,

at $x=-d$, $y_1 = a-bd+cd^2$

at $x=0$, $y_2 = a$

and at $x=d$, $y_3 = a+bd+cd^2$

Hence $y_1+y_3 = 2a+2cd^2$

And $y_1+4y_2+y_3 = 6a+2cd^2$ (4)

Thus the area under the parabola between $x=-d$ and $x=d$ in Fig. 45.3 may be expressed as $\frac{1}{3}d(y_1+4y_2+y_3)$, from equations (3) and (4), and the result is seen to be independent of the position of the origin.

Let a definite integral be denoted by $\int_a^b y\,dx$ and represented by the area under the graph of $y=f(x)$ between the limits $x=a$ and $x=b$, as shown in Fig. 45.4. The range of integration, $b-a$, is divided into an **even** number of intervals, say $2n$, each of width d.

Since an even number of intervals is specified, an odd number of ordinates, $2n+1$, exists. Let an approximation to the curve over the first two intervals be a parabola of the form $y=a+bx+cx^2$ which passes through the tops of the three ordinates y_1, y_2 and y_3. Similarly, let an approximation to the curve over the next two intervals be the parabola which passes through the tops of the ordinates y_3, y_4 and y_5, and so on.

Then $\int_a^b y\,dx$

$$\approx \frac{1}{3}d(y_1+4y_2+y_3)+\frac{1}{3}d(y_3+4y_4+y_5)$$

$$+\frac{1}{3}d(y_{2n-1}+4y_{2n}+y_{2n+1})$$

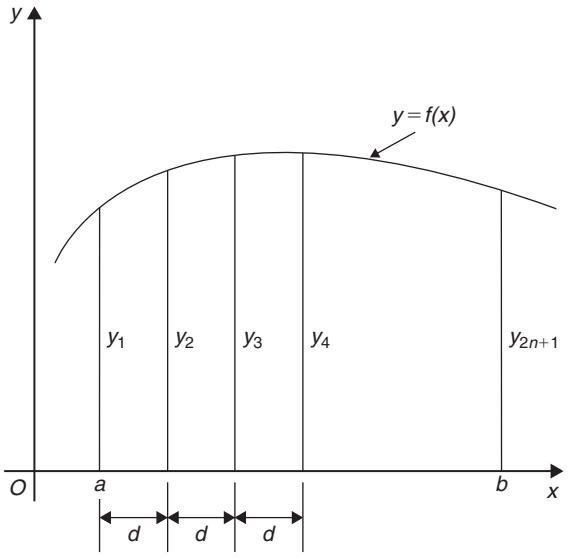


Figure 45.4

$$\approx \frac{1}{3}d[(y_1 + y_{2n+1}) + 4(y_2 + y_4 + \cdots + y_{2n}) + 2(y_3 + y_5 + \cdots + y_{2n-1})]$$

i.e. **Simpson's rule states:**

$$\int_a^b y\,dx \approx \frac{1}{3}\begin{pmatrix}\textbf{width of}\\ \textbf{interval}\end{pmatrix}\left\{\begin{pmatrix}\textbf{first + last}\\ \textbf{ordinate}\end{pmatrix} + 4\begin{pmatrix}\textbf{sum of even}\\ \textbf{ordinates}\end{pmatrix} + 2\begin{pmatrix}\textbf{sum of remaining}\\ \textbf{odd ordinates}\end{pmatrix}\right\} \quad (5)$$

Note that Simpson's rule can only be applied when an even number of intervals is chosen, i.e. an odd number of ordinates.

Problem 6. Use Simpson's rule with (a) 4 intervals, (b) 8 intervals, to evaluate $\int_1^3 \frac{2}{\sqrt{x}}\,dx$, correct to 3 decimal places.

(a) With 4 intervals, each will have a width of $\frac{3-1}{4}$, i.e. 0.5 and the ordinates will occur at 1.0, 1.5, 2.0, 2.5 and 3.0. The values of the ordinates are as shown in the table of Problem 1(b), page 436.

Thus, from equation (5):

$$\int_1^3 \frac{2}{\sqrt{x}}\,dx \approx \frac{1}{3}(0.5)\,[(2.0000 + 1.1547) + 4(1.6330 + 1.2649) + 2(1.4142)]$$

$$= \frac{1}{3}(0.5)[3.1547 + 11.5916 + 2.8284]$$

$$= \mathbf{2.929}, \text{ correct to 3 decimal places.}$$

(b) With 8 intervals, each will have a width of $\frac{3-1}{8}$, i.e. 0.25 and the ordinates occur at 1.00, 1.25, 1.50, 1.75, …, 3.0. The values of the ordinates are as shown in the table in Problem 2, page 436.
Thus, from equation (5):

$$\int_1^3 \frac{2}{\sqrt{x}}\,dx \approx \frac{1}{3}(0.25)\,[(2.0000 + 1.1547) + 4(1.7889 + 1.5119 + 1.3333 + 1.2060) + 2(1.6330 + 1.4142 + 1.2649)]$$

$$= \frac{1}{3}(0.25)[3.1547 + 23.3604 + 8.6242]$$

$$= \mathbf{2.928}, \text{ correct to 3 decimal places.}$$

It is noted that the latter answer is exactly the same as that obtained by integration. In general, Simpson's rule is regarded as the most accurate of the three approximate methods used in numerical integration.

Problem 7. Evaluate

$$\int_0^{\frac{\pi}{3}} \sqrt{\left(1 - \frac{1}{3}\sin^2\theta\right)}\,d\theta,$$

correct to 3 decimal places, using Simpson's rule with 6 intervals.

With 6 intervals, each will have a width of $\frac{\frac{\pi}{3}-0}{6}$ i.e. $\frac{\pi}{18}$ rad (or 10°), and the ordinates will occur at 0, $\frac{\pi}{18}$, $\frac{\pi}{9}$, $\frac{\pi}{6}$, $\frac{2\pi}{9}$, $\frac{5\pi}{18}$ and $\frac{\pi}{3}$

Corresponding values of $\sqrt{\left(1-\frac{1}{3}\sin^2\theta\right)}$ are shown in the table below.

θ	0	$\frac{\pi}{18}$ (or 10°)	$\frac{\pi}{9}$ (or 20°)	$\frac{\pi}{6}$ (or 30°)
$\sqrt{\left(1-\frac{1}{3}\sin^2\theta\right)}$	1.0000	0.9950	0.9803	0.9574

θ	$\frac{2\pi}{9}$ (or 40°)	$\frac{5\pi}{18}$ (or 50°)	$\frac{\pi}{3}$ (or 60°)
$\sqrt{\left(1-\frac{1}{3}\sin^2\theta\right)}$	0.9286	0.8969	0.8660

From Equation (5)

$$\int_0^{\frac{\pi}{3}} \sqrt{\left(1-\frac{1}{3}\sin^2\theta\right)}\,d\theta$$

$$\approx \frac{1}{3}\left(\frac{\pi}{18}\right)[(1.0000+0.8660)+4(0.9950+0.9574+0.8969)+2(0.9803+0.9286)]$$

$$=\frac{1}{3}\left(\frac{\pi}{18}\right)[1.8660+11.3972+3.8178]$$

$$=\mathbf{0.994}, \text{ correct to 3 decimal places.}$$

Problem 8. An alternating current i has the following values at equal intervals of 2.0 milliseconds:

Time (ms)	Current i (A)
0	0
2.0	3.5
4.0	8.2
6.0	10.0
8.0	7.3
10.0	2.0
12.0	0

Charge, q, in millicoulombs, is given by $q=\int_0^{12.0} i\,dt$.

Use Simpson's rule to determine the approximate charge in the 12 millisecond period.

From equation (5):

$$\text{Charge, } q=\int_0^{12.0} i\,dt \approx \frac{1}{3}(2.0)[(0+0)+4(3.5+10.0+2.0)+2(8.2+7.3)]$$

$$=\mathbf{62\,mC}$$

Now try the following exercise

Exercise 175 Further problems on Simpson's rule

In Problems 1 to 5, evaluate the definite integrals using **Simpson's rule**, giving the answers correct to 3 decimal places.

1. $\int_0^{\frac{\pi}{2}} \sqrt{(\sin x)}\,dx$ (Use 6 intervals) [1.187]
2. $\int_0^{1.6} \frac{1}{1+\theta^4}\,d\theta$ (Use 8 intervals) [1.034]
3. $\int_{0.2}^{1.0} \frac{\sin\theta}{\theta}\,d\theta$ (Use 8 intervals) [0.747]
4. $\int_0^{\frac{\pi}{2}} x\cos x\,dx$ (Use 6 intervals) [0.571]
5. $\int_0^{\frac{\pi}{3}} e^{x^2}\sin 2x\,dx$ (Use 10 intervals) [1.260]

In Problems 6 and 7 evaluate the definite integrals using (a) integration, (b) the trapezoidal rule,

(c) the mid-ordinate rule, (d) Simpson's rule. Give answers correct to 3 decimal places.

6. $\int_1^4 \frac{4}{x^3}\,dx$ (Use 6 intervals)

[(a) 1.875 (b) 2.107
(c) 1.765 (d) 1.916]

7. $\int_2^6 \frac{1}{\sqrt{(2x-1)}}dx$ (Use 8 intervals)

[(a) 1.585 (b) 1.588
(c) 1.583 (d) 1.585]

In Problems 8 and 9 evaluate the definite integrals using (a) the trapezoidal rule, (b) the mid-ordinate rule, (c) Simpson's rule. Use 6 intervals in each case and give answers correct to 3 decimal places.

8. $\int_0^3 \sqrt{(1+x^4)}\,dx$

[(a) 10.194 (b) 10.007
(c) 10.070]

9. $\int_{0.1}^{0.7} \frac{1}{\sqrt{(1-y^2)}}\,dy$

[(a) 0.677 (b) 0.674
(c) 0.675]

10. A vehicle starts from rest and its velocity is measured every second for 8 s, with values as follows:

time t (s)	velocity v (ms^{-1})
0	0
1.0	0.4
2.0	1.0
3.0	1.7
4.0	2.9
5.0	4.1
6.0	6.2
7.0	8.0
8.0	9.4

The distance travelled in 8.0 s is given by $\int_0^{8.0} v\,dt$

Estimate this distance using Simpson's rule, giving the answer correct to 3 significant figures. [28.8 m]

11. A pin moves along a straight guide so that its velocity v (m/s) when it is a distance $x(m)$ from the beginning of the guide at time t(s) is given in the table below.

t (s)	v (m/s)
0	0
0.5	0.052
1.0	0.082
1.5	0.125
2.0	0.162
2.5	0.175
3.0	0.186
3.5	0.160
4.0	0

Use Simpson's rule with 8 intervals to determine the approximate total distance travelled by the pin in the 4.0 s period. [0.485 m]

Revision Test 13

This Revision Test covers the material contained in Chapters 43 to 45. *The marks for each question are shown in brackets at the end of each question.*

1. Determine the following integrals:

   (a) $\int 5x\,e^{2x}\,dx$ (b) $\int t^2 \sin 2t\,dt$ (13)

2. Evaluate correct to 3 decimal places:

   $$\int_1^4 \sqrt{x}\ln x\,dx$$ (10)

3. Use reduction formulae to determine:

   (a) $\int x^3 e^{3x}\,dx$ (b) $\int t^4 \sin t\,dt$ (13)

4. Evaluate $\int_0^{\frac{\pi}{2}} \cos^6 x\,dx$ using a reduction formula. (6)

5. Evaluate $\int_1^3 \frac{5}{x^2}\,dx$ using (a) integration (b) the trapezoidal rule (c) the mid-ordinate rule (d) Simpson's rule. In each of the approximate methods use 8 intervals and give the answers correct to 3 decimal places. (19)

6. An alternating current i has the following values at equal intervals of 5 ms:

Time t (ms)	0	5	10	15	20	25	30
Current i (A)	0	4.8	9.1	12.7	8.8	3.5	0

   Charge q, in coulombs, is given by

   $$q = \int_0^{30\times10^{-3}} i\,dt.$$

   Use Simpson's rule to determine the approximate charge in the 30 ms period. (4)

Chapter 46

Solution of first order differential equations by separation of variables

46.1 Family of curves

Integrating both sides of the derivative $\frac{dy}{dx} = 3$ with respect to x gives $y = \int 3\,dx$, i.e., $y = 3x + c$, where c is an arbitrary constant.

$y = 3x + c$ represents a **family of curves**, each of the curves in the family depending on the value of c. Examples include $y = 3x + 8$, $y = 3x + 3$, $y = 3x$ and $y = 3x - 10$ and these are shown in Fig. 46.1.

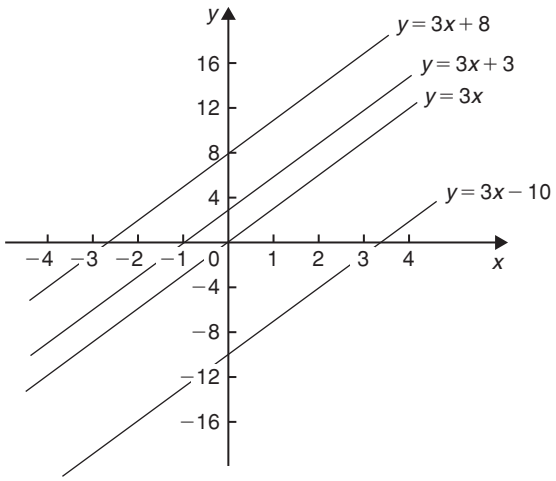


Figure 46.1

Each are straight lines of gradient 3. A particular curve of a family may be determined when a point on the curve is specified. Thus, if $y = 3x + c$ passes through the point (1, 2) then $2 = 3(1) + c$, from which, $c = -1$. The equation of the curve passing through (1, 2) is therefore $y = 3x - 1$.

Problem 1. Sketch the family of curves given by the equation $\frac{dy}{dx} = 4x$ and determine the equation of one of these curves which passes through the point (2, 3).

Integrating both sides of $\frac{dy}{dx} = 4x$ with respect to x gives:

$$\int \frac{dy}{dx}\,dx = \int 4x\,dx, \quad \text{i.e., } y = 2x^2 + c$$

Some members of the family of curves having an equation $y = 2x^2 + c$ include $y = 2x^2 + 15$, $y = 2x^2 + 8$, $y = 2x^2$ and $y = 2x^2 - 6$, and these are shown in Fig. 46.2. To determine the equation of the curve passing through the point (2, 3), $x = 2$ and $y = 3$ are substituted into the equation $y = 2x^2 + c$.

Thus $3 = 2(2)^2 + c$, from which $c = 3 - 8 = -5$.

Hence the equation of the curve passing through the point (2, 3) is $y = 2x^2 - 5$.

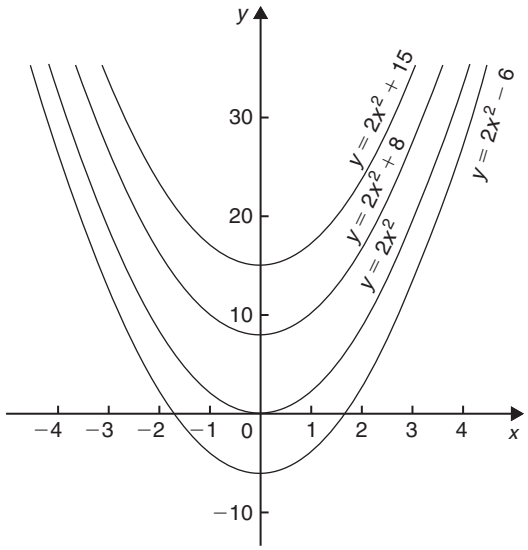


Figure 46.2

Now try the following exercise

Exercise 176 Further problems on families of curves

1. Sketch a family of curves represented by each of the following differential equations:
   (a) $\frac{dy}{dx} = 6$ (b) $\frac{dy}{dx} = 3x$ (c) $\frac{dy}{dx} = x + 2$
2. Sketch the family of curves given by the equation $\frac{dy}{dx} = 2x + 3$ and determine the equation of one of these curves which passes through the point (1, 3). [$y = x^2 + 3x - 1$]

46.2 Differential equations

A **differential equation** is one that contains differential coefficients.
Examples include

$$\text{(i) } \frac{dy}{dx} = 7x \quad \text{and} \quad \text{(ii) } \frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 2y = 0$$

Differential equations are classified according to the highest derivative which occurs in them. Thus example (i) above is a **first order differential equation**, and example (ii) is a **second order differential equation**.

The **degree** of a differential equation is that of the highest power of the highest differential which the equation contains after simplification.

Thus $\left(\frac{d^2x}{dt^2}\right)^3 + 2\left(\frac{dx}{dt}\right)^5 = 7$ is a second order differential equation of degree three.

Starting with a differential equation it is possible, by integration and by being given sufficient data to determine unknown constants, to obtain the original function. This process is called **'solving the differential equation'**. A solution to a differential equation which contains one or more arbitrary constants of integration is called the **general solution** of the differential equation.

When additional information is given so that constants may be calculated the **particular solution** of the differential equation is obtained. The additional information is called **boundary conditions**. It was shown in Section 46.1 that $y = 3x + c$ is the general solution of the differential equation $\frac{dy}{dx} = 3$.

Given the boundary conditions $x = 1$ and $y = 2$, produces the particular solution of $y = 3x - 1$.

Equations which can be written in the form

$$\frac{dy}{dx} = f(x), \frac{dy}{dx} = f(y) \text{ and } \frac{dy}{dx} = f(x) \cdot f(y)$$

can all be solved by integration. In each case it is possible to separate the y's to one side of the equation and the x's to the other. Solving such equations is therefore known as solution by **separation of variables**.

46.3 The solution of equations of the form $\frac{dy}{dx} = f(x)$

A differential equation of the form $\frac{dy}{dx} = f(x)$ is solved by direct integration,

i.e. $$y = \int f(x)\,dx$$

Problem 2. Determine the general solution of $x\frac{dy}{dx} = 2 - 4x^3$

Rearranging $x\frac{dy}{dx} = 2 - 4x^3$ gives:

$$\frac{dy}{dx} = \frac{2 - 4x^3}{x} = \frac{2}{x} - \frac{4x^3}{x} = \frac{2}{x} - 4x^2$$

Integrating both sides gives:

$$y = \int \left(\frac{2}{x} - 4x^2\right) dx$$

i.e. $\boldsymbol{y = 2\ln x - \frac{4}{3}x^3 + c}$,

which is the general solution.

Problem 3. Find the particular solution of the differential equation $5\frac{dy}{dx} + 2x = 3$, given the boundary conditions $y = 1\frac{2}{5}$ when $x = 2$.

Since $5\frac{dy}{dx} + 2x = 3$ then $\frac{dy}{dx} = \frac{3 - 2x}{5} = \frac{3}{5} - \frac{2x}{5}$

Hence $y = \int \left(\frac{3}{5} - \frac{2x}{5}\right) dx$

i.e. $y = \frac{3x}{5} - \frac{x^2}{5} + c$,

which is the general solution.

Substituting the boundary conditions $y = 1\frac{2}{5}$ and $x = 2$ to evaluate c gives:

$1\frac{2}{5} = \frac{6}{5} - \frac{4}{5} + c$, from which, $c = 1$

Hence the particular solution is $\boldsymbol{y = \frac{3x}{5} - \frac{x^2}{5} + 1}$.

Problem 4. Solve the equation $2t\left(t - \frac{d\theta}{dt}\right) = 5$, given $\theta = 2$ when $t = 1$.

Rearranging gives:

$$t - \frac{d\theta}{dt} = \frac{5}{2t} \quad \text{and} \quad \frac{d\theta}{dt} = t - \frac{5}{2t}$$

Integrating gives:

$$\theta = \int \left(t - \frac{5}{2t}\right) dt$$

i.e. $\theta = \frac{t^2}{2} - \frac{5}{2}\ln t + c$,

which is the general solution.

When $\theta = 2$, $t = 1$, thus $2 = \frac{1}{2} - \frac{5}{2}\ln 1 + c$ from which, $c = \frac{3}{2}$.

Hence the particular solution is:

$$\theta = \frac{t^2}{2} - \frac{5}{2}\ln t + \frac{3}{2}$$

i.e. $\boldsymbol{\theta = \frac{1}{2}(t^2 - 5\ln t + 3)}$

Problem 5. The bending moment M of the beam is given by $\frac{dM}{dx} = -w(l - x)$, where w and x are constants. Determine M in terms of x given: $M = \frac{1}{2}wl^2$ when $x = 0$.

$$\frac{dM}{dx} = -w(l - x) = -wl + wx$$

Integrating with respect to x gives:

$$M = -wlx + \frac{wx^2}{2} + c$$

which is the general solution.

When $M = \frac{1}{2}wl^2$, $x = 0$.

Thus $\frac{1}{2}wl^2 = -wl(0) + \frac{w(0)^2}{2} + c$

from which, $c = \frac{1}{2}wl^2$.

Hence the particular solution is:

$$M = -wlx + \frac{w(x)^2}{2} + \frac{1}{2}wl^2$$

i.e. $\boldsymbol{M = \frac{1}{2}w(l^2 - 2lx + x^2)}$

or $\boldsymbol{M = \frac{1}{2}w(l - x)^2}$

Now try the following exercise

Exercise 177 Further problems on equations of the form $\frac{dy}{dx} = f(x)$.

In Problems 1 to 5, solve the differential equations.

1. $\frac{dy}{dx} = \cos 4x - 2x$ $\left[y = \frac{\sin 4x}{4} - x^2 + c\right]$
2. $2x\frac{dy}{dx} = 3 - x^3$ $\left[y = \frac{3}{2}\ln x - \frac{x^3}{6} + c\right]$
3. $\frac{dy}{dx} + x = 3$, given $y = 2$ when $x = 1$. $\left[y = 3x - \frac{x^2}{2} - \frac{1}{2}\right]$
4. $3\frac{dy}{d\theta} + \sin\theta = 0$, given $y = \frac{2}{3}$ when $\theta = \frac{\pi}{3}$ $\left[y = \frac{1}{3}\cos\theta + \frac{1}{2}\right]$

5. $\frac{1}{e^x}+2=x-3\frac{dy}{dx}$, given $y=1$ when $x=0$.
$\left[y=\frac{1}{6}\left(x^2-4x+\frac{2}{e^x}+4\right)\right]$

6. The gradient of a curve is given by:
$\frac{dy}{dx}+\frac{x^2}{2}=3x$
Find the equation of the curve if it passes through the point $\left(1,\frac{1}{3}\right)$.
$\left[y=\frac{3}{2}x^2-\frac{x^3}{6}-1\right]$

7. The acceleration, a, of a body is equal to its rate of change of velocity, $\frac{dv}{dt}$. Find an equation for v in terms of t, given that when $t=0$, velocity $v=u$. [$v=u+at$]

8. An object is thrown vertically upwards with an initial velocity, u, of 20 m/s. The motion of the object follows the differential equation $\frac{ds}{dt}=u-gt$, where s is the height of the object in metres at time t seconds and $g=9.8\,m/s^2$. Determine the height of the object after 3 seconds if $s=0$ when $t=0$. [15.9 m]

46.4 The solution of equations of the form $\frac{dy}{dx}=f(y)$

A differential equation of the form $\frac{dy}{dx}=f(y)$ is initially rearranged to give $dx=\frac{dy}{f(y)}$ and then the solution is obtained by direct integration,

i.e. $$\int dx=\int \frac{dy}{f(y)}$$

Problem 6. Find the general solution of $\frac{dy}{dx}=3+2y$.

Rearranging $\frac{dy}{dx}=3+2y$ gives:

$$dx=\frac{dy}{3+2y}$$

Integrating both sides gives:

$$\int dx=\int \frac{dy}{3+2y}$$

Thus, by using the substitution $u=(3+2y)$ — see Chapter 39,

$$x=\tfrac{1}{2}\ln(3+2y)+c \qquad (1)$$

It is possible to give the general solution of a differential equation in a different form. For example, if $c=\ln k$, where k is a constant, then:

$$x=\tfrac{1}{2}\ln(3+2y)+\ln k,$$

i.e. $$x=\ln(3+2y)^{\frac{1}{2}}+\ln k$$

or $$x=\ln[k\sqrt{(3+2y)}] \qquad (2)$$

by the laws of logarithms, from which,

$$e^x=k\sqrt{(3+2y)} \qquad (3)$$

Equations (1), (2) and (3) are all acceptable general solutions of the differential equation

$$\frac{dy}{dx}=3+2y$$

Problem 7. Determine the particular solution of $(y^2-1)\frac{dy}{dx}=3y$ given that $y=1$ when $x=2\frac{1}{6}$

Rearranging gives:

$$dx=\left(\frac{y^2-1}{3y}\right)dy=\left(\frac{y}{3}-\frac{1}{3y}\right)dy$$

Integrating gives:

$$\int dx=\int\left(\frac{y}{3}-\frac{1}{3y}\right)dy$$

i.e. $$x=\frac{y^2}{6}-\frac{1}{3}\ln y+c,$$

which is the general solution.

When $y=1$, $x=2\frac{1}{6}$, thus $2\frac{1}{6}=\frac{1}{6}-\frac{1}{3}\ln 1+c$, from which, $c=2$.

Hence the particular solution is:

$$x=\frac{y^2}{6}-\frac{1}{3}\ln y+2$$

Problem 8. (a) The variation of resistance, R ohms, of an aluminium conductor with temperature $\theta°C$ is given by $\frac{dR}{d\theta} = \alpha R$, where α is the temperature coefficient of resistance of aluminium. If $R = R_0$ when $\theta = 0°C$, solve the equation for R. (b) If $\alpha = 38 \times 10^{-4}/°C$, determine the resistance of an aluminium conductor at 50°C, correct to 3 significant figures, when its resistance at 0°C is 24.0 Ω.

(a) $\frac{dR}{d\theta} = \alpha R$ is of the form $\frac{dy}{dx} = f(y)$

Rearranging gives: $d\theta = \frac{dR}{\alpha R}$

Integrating both sides gives:

$$\int d\theta = \int \frac{dR}{\alpha R}$$

i.e. $$\theta = \frac{1}{\alpha}\ln R + c,$$

which is the general solution.

Substituting the boundary conditions $R = R_0$ when $\theta = 0$ gives:

$$0 = \frac{1}{\alpha}\ln R_0 + c$$

from which $$c = -\frac{1}{\alpha}\ln R_0$$

Hence the particular solution is

$$\theta = \frac{1}{\alpha}\ln R - \frac{1}{\alpha}\ln R_0 = \frac{1}{\alpha}(\ln R - \ln R_0)$$

i.e. $$\theta = \frac{1}{\alpha}\ln\left(\frac{R}{R_0}\right) \text{ or } \alpha\theta = \ln\left(\frac{R}{R_0}\right)$$

Hence $e^{\alpha\theta} = \frac{R}{R_0}$ from which, $\boldsymbol{R = R_0 e^{\alpha\theta}}$

(b) Substituting $\alpha = 38 \times 10^{-4}$, $R_0 = 24.0$ and $\theta = 50$ into $R = R_0 e^{\alpha\theta}$ gives the resistance at 50°C, i.e. $\boldsymbol{R_{50} = 24.0\,e^{(38\times10^{-4}\times50)} = 24.0\,\text{ohms}}$

Now try the following exercise

Exercise 178 Further problems on equations of the form $\frac{dy}{dx} = f(y)$

In Problems 1 to 3, solve the differential equations.

1. $\frac{dy}{dx} = 2 + 3y$ $\left[x = \frac{1}{3}\ln(2+3y) + c\right]$
2. $\frac{dy}{dx} = 2\cos^2 y$ $[\tan y = 2x + c]$
3. $(y^2 + 2)\frac{dy}{dx} = 5y$, given $y = 1$ when $x = \frac{1}{2}$ $\left[\frac{y^2}{2} + 2\ln y = 5x - 2\right]$
4. The current in an electric circuit is given by the equation
$$Ri + L\frac{di}{dt} = 0,$$
where L and R are constants. Show that $i = Ie^{\frac{-Rt}{L}}$, given that $i = I$ when $t = 0$.
5. The velocity of a chemical reaction is given by $\frac{dx}{dt} = k(a - x)$, where x is the amount transferred in time t, k is a constant and a is the concentration at time $t = 0$ when $x = 0$. Solve the equation and determine x in terms of t. $[x = a(1 - e^{-kt})]$
6. (a) Charge Q coulombs at time t seconds is given by the differential equation $R\frac{dQ}{dt} + \frac{Q}{C} = 0$, where C is the capacitance in farads and R the resistance in ohms. Solve the equation for Q given that $Q = Q_0$ when $t = 0$.
   (b) A circuit possesses a resistance of $250 \times 10^3\,\Omega$ and a capacitance of 8.5×10^{-6} F, and after 0.32 seconds the charge falls to 8.0 C. Determine the initial charge and the charge after 1 second, each correct to 3 significant figures.
   [(a) $Q = Q_0 e^{\frac{-t}{CR}}$ (b) 9.30 C, 5.81 C]
7. A differential equation relating the difference in tension T, pulley contact angle θ and coefficient of friction μ is $\frac{dT}{d\theta} = \mu T$. When $\theta = 0$, $T = 150$ N, and $\mu = 0.30$ as slipping starts. Determine the tension at the point of slipping when $\theta = 2$ radians. Determine also the value of θ when T is 300 N. [273.3 N, 2.31 rads]

8. The rate of cooling of a body is given by $\frac{d\theta}{dt} = k\theta$, where k is a constant. If $\theta = 60°C$ when $t = 2$ minutes and $\theta = 50°C$ when $t = 5$ minutes, determine the time taken for θ to fall to 40°C, correct to the nearest second.

[8 m 40 s]

46.5 The solution of equations of the form $\frac{dy}{dx} = f(x) \cdot f(y)$

A differential equation of the form $\frac{dy}{dx} = f(x) \cdot f(y)$, where $f(x)$ is a function of x only and $f(y)$ is a function of y only, may be rearranged as $\frac{dy}{f(y)} = f(x)\,dx$, and then the solution is obtained by direct integration, i.e.

$$\int \frac{dy}{f(y)} = \int f(x)\,dx$$

Problem 9. Solve the equation $4xy\frac{dy}{dx} = y^2 - 1$

Separating the variables gives:

$$\left(\frac{4y}{y^2-1}\right)dy = \frac{1}{x}\,dx$$

Integrating both sides gives:

$$\int \left(\frac{4y}{y^2-1}\right)dy = \int \left(\frac{1}{x}\right)dx$$

Using the substitution $u = y^2 - 1$, the general solution is:

$$\mathbf{2\ln(y^2-1) = \ln x + c} \qquad (1)$$

or $\ln(y^2-1)^2 - \ln x = c$

from which, $\ln\left\{\frac{(y^2-1)^2}{x}\right\} = c$

and $$\frac{\mathbf{(y^2-1)^2}}{\mathbf{x}} = \mathbf{e^c} \qquad (2)$$

If in equation (1), $c = \ln A$, where A is a different constant,

then $\ln(y^2-1)^2 = \ln x + \ln A$

i.e. $\ln(y^2-1)^2 = \ln Ax$

i.e. $$\mathbf{(y^2-1)^2 = Ax} \qquad (3)$$

Equations (1) to (3) are thus three valid solutions of the differential equations

$$4xy\frac{dy}{dx} = y^2 - 1$$

Problem 10. Determine the particular solution of $\frac{d\theta}{dt} = 2e^{3t-2\theta}$, given that $t = 0$ when $\theta = 0$.

$$\frac{d\theta}{dt} = 2e^{3t-2\theta} = 2(e^{3t})(e^{-2\theta}),$$

by the laws of indices.

Separating the variables gives:

$$\frac{d\theta}{e^{-2\theta}} = 2e^{3t}dt,$$

i.e. $e^{2\theta}d\theta = 2e^{3t}dt$

Integrating both sides gives:

$$\int e^{2\theta}d\theta = \int 2e^{3t}dt$$

Thus the general solution is:

$$\frac{1}{2}e^{2\theta} = \frac{2}{3}e^{3t} + c$$

When $t = 0$, $\theta = 0$, thus:

$$\frac{1}{2}e^0 = \frac{2}{3}e^0 + c$$

from which, $c = \frac{1}{2} - \frac{2}{3} = -\frac{1}{6}$

Hence the particular solution is:

$$\frac{1}{2}e^{2\theta} = \frac{2}{3}e^{3t} - \frac{1}{6}$$

or $$\mathbf{3e^{2\theta} = 4e^{3t} - 1}$$

Problem 11. Find the curve which satisfies the equation $xy = (1+x^2)\frac{dy}{dx}$ and passes through the point (0, 1).

Separating the variables gives:

$$\frac{x}{(1+x^2)}dx = \frac{dy}{y}$$

Integrating both sides gives:

$$\tfrac{1}{2}\ln(1+x^2) = \ln y + c$$

When $x=0$, $y=1$ thus $\frac{1}{2}\ln 1=\ln 1+c$, from which, $c=0$.

Hence the particular solution is $\frac{1}{2}\ln(1+x^2)=\ln y$

i.e. $\ln(1+x^2)^{\frac{1}{2}}=\ln y$, from which, $(1+x^2)^{\frac{1}{2}}=y$

Hence the equation of the curve is $y=\sqrt{(1+x^2)}$.

Problem 12. The current i in an electric circuit containing resistance R and inductance L in series with a constant voltage source E is given by the differential equation $E-L\left(\frac{di}{dt}\right)=Ri$. Solve the equation and find i in terms of time t given that when $t=0$, $i=0$.

In the $R-L$ series circuit shown in Fig. 46.3, the supply p.d., E, is given by

$$E=V_R+V_L$$

$$V_R=iR \text{ and } V_L=L\frac{di}{dt}$$

Hence $$E=iR+L\frac{di}{dt}$$

from which $$E-L\frac{di}{dt}=Ri$$

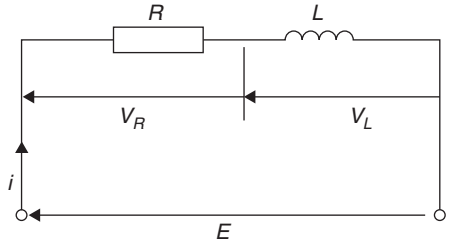


Figure 46.3

Most electrical circuits can be reduced to a differential equation.

Rearranging $E-L\frac{di}{dt}=Ri$ gives $\frac{di}{dt}=\frac{E-Ri}{L}$

and separating the variables gives:

$$\frac{di}{E-Ri}=\frac{dt}{L}$$

Integrating both sides gives:

$$\int\frac{di}{E-Ri}=\int\frac{dt}{L}$$

Hence the general solution is:

$$-\frac{1}{R}\ln(E-Ri)=\frac{t}{L}+c$$

(by making a substitution $u=E-Ri$, see Chapter 39).

When $t=0$, $i=0$, thus $-\frac{1}{R}\ln E=c$

Thus the particular solution is:

$$-\frac{1}{R}\ln(E-Ri)=\frac{t}{L}-\frac{1}{R}\ln E$$

Transposing gives:

$$-\frac{1}{R}\ln(E-Ri)+\frac{1}{R}\ln E=\frac{t}{L}$$

$$\frac{1}{R}[\ln E-\ln(E-Ri)]=\frac{t}{L}$$

$$\ln\left(\frac{E}{E-Ri}\right)=\frac{Rt}{L}$$

from which $\frac{E}{E-Ri}=e^{\frac{Rt}{L}}$

Hence $\frac{E-Ri}{E}=e^{\frac{-Rt}{L}}$ and $E-Ri=Ee^{\frac{-Rt}{L}}$ and $Ri=E-Ee^{\frac{-Rt}{L}}$.

Hence current,

$$\boldsymbol{i=\frac{E}{R}\left(1-e^{\frac{-Rt}{L}}\right)},$$

which represents the law of growth of current in an inductive circuit as shown in Fig. 46.4.

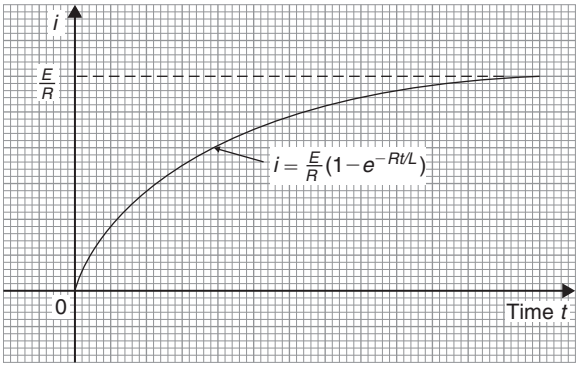


Figure 46.4

Problem 13. For an adiabatic expansion of a gas

$$C_v\frac{dp}{p}+C_p\frac{dV}{V}=0,$$

where C_p and C_v are constants. Given $n = \dfrac{C_p}{C_v}$, show that $pV^n =$ constant.

Separating the variables gives:

$$C_v \frac{dp}{p} = -C_p \frac{dV}{V}$$

Integrating both sides gives:

$$C_v \int \frac{dp}{p} = -C_p \int \frac{dV}{V}$$

i.e. $C_v \ln p = -C_p \ln V + k$

Dividing throughout by constant C_v gives:

$$\ln p = -\frac{C_p}{C_v} \ln V + \frac{k}{C_v}$$

Since $\dfrac{C_p}{C_v} = n$, then $\ln p + n \ln V = K$,

where $K = \dfrac{k}{C_v}$.

i.e. $\ln p + \ln V^n = K$ or $\ln pV^n = K$, by the laws of logarithms.

Hence $pV^n = e^K$, i.e. $\boldsymbol{pV^n = \text{constant}}$.

Now try the following exercise

Exercise 179 Further problems on equations of the form $\dfrac{dy}{dx} = f(x) \cdot f(y)$

In Problems 1 to 4, solve the differential equations.

1. $\dfrac{dy}{dx} = 2y \cos x$ $[\ln y = 2 \sin x + c]$

2. $(2y-1)\dfrac{dy}{dx} = (3x^2+1)$, given $x = 1$ when $y = 2$. $[y^2 - y = x^3 + x]$

3. $\dfrac{dy}{dx} = e^{2x-y}$, given $x = 0$ when $y = 0$. $\left[e^y = \dfrac{1}{2}e^{2x} + \dfrac{1}{2}\right]$

4. $2y(1-x) + x(1+y)\dfrac{dy}{dx} = 0$, given $x = 1$ when $y = 1$. $[\ln (x^2 y) = 2x - y - 1]$

5. Show that the solution of the equation $\dfrac{y^2+1}{x^2+1} = \dfrac{y}{x}\dfrac{dy}{dx}$ is of the form $\sqrt{\left(\dfrac{y^2+1}{x^2+1}\right)} =$ constant.

6. Solve $xy = (1-x^2)\dfrac{dy}{dx}$ for y, given $x = 0$ when $y = 1$. $\left[y = \dfrac{1}{\sqrt{(1-x^2)}}\right]$

7. Determine the equation of the curve which satisfies the equation $xy\dfrac{dy}{dx} = x^2 - 1$, and which passes through the point (1, 2). $[y^2 = x^2 - 2\ln x + 3]$

8. The p.d., V, between the plates of a capacitor C charged by a steady voltage E through a resistor R is given by the equation $CR\dfrac{dV}{dt} + V = E$.

   (a) Solve the equation for V given that at $t = 0$, $V = 0$.

   (b) Calculate V, correct to 3 significant figures, when $E = 25\text{V}$, $C = 20 \times 10^{-6}\text{F}$, $R = 200 \times 10^3\,\Omega$ and $t = 3.0\,\text{s}$.

   $\left[\text{(a) } V = E\left(1 - e^{\frac{-t}{CR}}\right)\right.$
   $\left.\text{(b) } 13.2\,\text{V}\right]$

9. Determine the value of p, given that $x^3\dfrac{dy}{dx} = p - x$, and that $y = 0$ when $x = 2$ and when $x = 6$. [3]

Chapter 47

Homogeneous first order differential equations

47.1 Introduction

Certain first order differential equations are not of the 'variable-separable' type, but can be made separable by changing the variable.

An equation of the form $P\dfrac{dy}{dx}=Q$, where P and Q are functions of both x and y of the same degree throughout, is said to be **homogeneous** in y and x. For example, $f(x, y)=x^2+3xy+y^2$ is a homogeneous function since each of the three terms are of degree 2. However, $f(x, y)=\dfrac{x^2-y}{2x^2+y^2}$ is not homogeneous since the term in y in the numerator is of degree 1 and the other three terms are of degree 2.

47.2 Procedure to solve differential equations of the form $P\dfrac{dy}{dx}=Q$

(i) Rearrange $P\dfrac{dy}{dx}=Q$ into the form $\dfrac{dy}{dx}=\dfrac{Q}{P}$.

(ii) Make the substitution $y=vx$ (where v is a function of x), from which, $\dfrac{dy}{dx}=v(1)+x\dfrac{dv}{dx}$, by the product rule.

(iii) Substitute for both y and $\dfrac{dy}{dx}$ in the equation $\dfrac{dy}{dx}=\dfrac{Q}{P}$. Simplify, by cancelling, and an equation results in which the variables are separable.

(iv) Separate the variables and solve using the method shown in Chapter 46.

(v) Substitute $v=\dfrac{y}{x}$ to solve in terms of the original variables.

47.3 Worked problems on homogeneous first order differential equations

Problem 1. Solve the differential equation: $y-x=x\dfrac{dy}{dx}$, given $x=1$ when $y=2$.

Using the above procedure:

(i) Rearranging $y-x=x\dfrac{dy}{dx}$ gives:

$$\frac{dy}{dx}=\frac{y-x}{x},$$

which is homogeneous in x and y.

(ii) Let $y=vx$, then $\dfrac{dy}{dx}=v+x\dfrac{dv}{dx}$

(iii) Substituting for y and $\dfrac{dy}{dx}$ gives:

$$v+x\frac{dv}{dx}=\frac{vx-x}{x}=\frac{x(v-1)}{x}=v-1$$

(iv) Separating the variables gives:

$$x\frac{dv}{dx}=v-1-v=-1, \text{ i.e. } dv=-\frac{1}{x}dx$$

Integrating both sides gives:

$$\int dv=\int -\frac{1}{x}dx$$

Hence, $v=-\ln x+c$

(v) Replacing v by $\frac{y}{x}$ gives: $\frac{y}{x}=-\ln x+c$, which is the general solution.

When $x=1$, $y=2$, thus: $\frac{2}{1}=-\ln 1+c$ from which, $c=2$

Thus, the particular solution is: $\frac{y}{x}=-\ln x+2$

or $\boldsymbol{y=-x(\ln x-2)}$ or $\boldsymbol{y=x(2-\ln x)}$

Problem 2. Find the particular solution of the equation: $x\frac{dy}{dx}=\frac{x^2+y^2}{y}$, given the boundary conditions that $y=4$ when $x=1$.

Using the procedure of section 47.2:

(i) Rearranging $x\frac{dy}{dx}=\frac{x^2+y^2}{y}$ gives: $\frac{dy}{dx}=\frac{x^2+y^2}{xy}$ which is homogeneous in x and y since each of the three terms on the right hand side are of the same degree (i.e. degree 2).

(ii) Let $y=vx$ then $\frac{dy}{dx}=v+x\frac{dv}{dx}$

(iii) Substituting for y and $\frac{dy}{dx}$ in the equation $\frac{dy}{dx}=\frac{x^2+y^2}{xy}$ gives:

$$v+x\frac{dv}{dx}=\frac{x^2+v^2x^2}{x(vx)}=\frac{x^2+v^2x^2}{vx^2}=\frac{1+v^2}{v}$$

(iv) Separating the variables gives:

$$x\frac{dv}{dx}=\frac{1+v^2}{v}-v=\frac{1+v^2-v^2}{v}=\frac{1}{v}$$

Hence, $v\,dv=\frac{1}{x}dx$

Integrating both sides gives:

$$\int v\,dv=\int\frac{1}{x}dx \text{ i.e. } \frac{v^2}{2}=\ln x+c$$

(v) Replacing v by $\frac{y}{x}$ gives: $\frac{y^2}{2x^2}=\ln x+c$, which is the general solution.

When $x=1$, $y=4$, thus: $\frac{16}{2}=\ln 1+c$ from which, $c=8$

Hence, the particular solution is: $\frac{y^2}{2x^2}=\ln x+8$

or $\boldsymbol{y^2=2x^2(8+\ln x)}$

Now try the following exercise

Exercise 180 Further problems on homogeneous first order differential equations

1. Find the general solution of: $x^2=y^2\frac{dy}{dx}$.
$$\left[-\frac{1}{3}\ln\left(\frac{x^3-y^3}{x^3}\right)=\ln x+c\right]$$

2. Find the general solution of:
$x-y+x\frac{dy}{dx}=0$. $[y=x(c-\ln x)]$

3. Find the particular solution of the differential equation: $(x^2+y^2)dy=xy\,dx$, given that $x=1$ when $y=1$.
$$\left[x^2=2y^2\left(\ln y+\frac{1}{2}\right)\right]$$

4. Solve the differential equation: $\frac{x+y}{y-x}=\frac{dy}{dx}$.
$$\left[-\frac{1}{2}\ln\left(1+\frac{2y}{x}-\frac{y^2}{x^2}\right)=\ln x+c \text{ or } x^2+2xy-y^2=k\right]$$

5. Find the particular solution of the differential equation: $\left(\frac{2y-x}{y+2x}\right)\frac{dy}{dx}=1$ given that $y=3$ when $x=2$. $[x^2+xy-y^2=1]$

47.4 Further worked problems on homogeneous first order differential equations

Problem 3. Solve the equation: $7x(x-y)\mathrm{d}y = 2(x^2+6xy-5y^2)\mathrm{d}x$ given that $x=1$ when $y=0$.

Using the procedure of section 47.2:

(i) Rearranging gives: $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{2x^2+12xy-10y^2}{7x^2-7xy}$ which is homogeneous in x and y since each of the terms on the right hand side is of degree 2.

(ii) Let $y=vx$ then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = v + x\dfrac{\mathrm{d}v}{\mathrm{d}x}$

(iii) Substituting for y and $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ gives:

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{2x^2+12x(vx)-10(vx)^2}{7x^2-7x(vx)} = \frac{2+12v-10v^2}{7-7v}$$

(iv) Separating the variables gives:

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{2+12v-10v^2}{7-7v} - v = \frac{(2+12v-10v^2)-v(7-7v)}{7-7v} = \frac{2+5v-3v^2}{7-7v}$$

Hence, $\dfrac{7-7v}{2+5v-3v^2}\mathrm{d}v = \dfrac{\mathrm{d}x}{x}$

Integrating both sides gives:

$$\int\left(\frac{7-7v}{2+5v-3v^2}\right)\mathrm{d}v = \int\frac{1}{x}\mathrm{d}x$$

Resolving $\dfrac{7-7v}{2+5v-3v^2}$ into partial fractions gives: $\dfrac{4}{(1+3v)} - \dfrac{1}{(2-v)}$ (see chapter 2)

Hence, $\displaystyle\int\left(\frac{4}{(1+3v)} - \frac{1}{(2-v)}\right)\mathrm{d}v = \int\frac{1}{x}\mathrm{d}x$

i.e. $\dfrac{4}{3}\ln(1+3v) + \ln(2-v) = \ln x + c$

(v) Replacing v by $\dfrac{y}{x}$ gives:

$$\frac{4}{3}\ln\left(1+\frac{3y}{x}\right) + \ln\left(2-\frac{y}{x}\right) = \ln + c$$

or $\dfrac{4}{3}\ln\left(\dfrac{x+3y}{x}\right) + \ln\left(\dfrac{2x-y}{x}\right) = \ln + c$

When $x=1$, $y=0$, thus: $\dfrac{4}{3}\ln 1 + \ln 2 = \ln 1 + c$ from which, $c = \ln 2$

Hence, the particular solution is:

$$\frac{4}{3}\ln\left(\frac{x+3y}{x}\right) + \ln\left(\frac{2x-y}{x}\right) = \ln + \ln 2$$

i.e. $\ln\left(\dfrac{x+3y}{x}\right)^{\frac{4}{3}}\left(\dfrac{2x-y}{x}\right) = \ln(2x)$

from the laws of logarithms

i.e. $\boldsymbol{\left(\dfrac{x+3y}{x}\right)^{\frac{4}{3}}\left(\dfrac{2x-y}{x}\right) = 2x}$

Problem 4. Show that the solution of the differential equation: $x^2 - 3y^2 + 2xy\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ is: $y = x\sqrt{(8x+1)}$, given that $y=3$ when $x=1$.

Using the procedure of section 47.2:

(i) Rearranging gives:

$$2xy\frac{\mathrm{d}y}{\mathrm{d}x} = 3y^2 - x^2 \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3y^2-x^2}{2xy}$$

(ii) Let $y=vx$ then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = v + x\dfrac{\mathrm{d}v}{\mathrm{d}x}$

(iii) Substituting for y and $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ gives:

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{3(vx)^2-x^2}{2x(vx)} = \frac{3v^2-1}{2v}$$

(iv) Separating the variables gives:

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{3v^2-1}{2v} - v = \frac{3v^2-1-2v^2}{2v} = \frac{v^2-1}{2v}$$

Hence, $\dfrac{2v}{v^2-1}\mathrm{d}v = \dfrac{1}{x}\mathrm{d}x$

Integrating both sides gives:

$$\int\frac{2v}{v^2-1}\mathrm{d}v = \int\frac{1}{x}\mathrm{d}x$$

i.e. $\ln(v^2-1) = \ln x + c$

(v) Replacing v by $\frac{y}{x}$ gives:

$$\ln\left(\frac{y^2}{x^2}-1\right)=\ln x+c,$$

which is the general solution.

When $y=3, x=1$, thus: $\ln\left(\frac{9}{1}-1\right)=\ln 1+c$ from which, $c=\ln 8$

Hence, the particular solution is:

$$\ln\left(\frac{y^2}{x^2}-1\right)=\ln x+\ln 8=\ln 8x$$

by the laws of logarithms

Hence, $\left(\frac{y^2}{x^2}-1\right)=8x$ i.e. $\frac{y^2}{x^2}=8x+1$ and $y^2=x^2(8x+1)$

i.e. $\boldsymbol{y=x\sqrt{(8x+1)}}$

Now try the following exercise

Exercise 181 Further problems on homogeneous first order differential equations

1. Solve the differential equation: $xy^3\,dy=(x^4+y^4)dx$.

$$\left[y^4=4x^4(\ln x+c)\right]$$

2. Solve: $(9xy-11xy)\frac{dy}{dx}=11y^2-16xy+3x^2$.

$$\left[\frac{1}{5}\left\{\frac{3}{13}\ln\left(\frac{13y-3x}{x}\right)-\ln\left(\frac{y-x}{x}\right)\right\}=\ln x+c\right]$$

3. Solve the differential equation: $2x\frac{dy}{dx}=x+3y$, given that when $x=1$, $y=1$.

$$\left[(x+y)^2=4x^3\right]$$

4. Show that the solution of the differential equation: $2xy\frac{dy}{dx}=x^2+y^2$ can be expressed as: $x=K(x^2-y^2)$, where K is a constant.

5. Determine the particular solution of $\frac{dy}{dx}=\frac{x^3+y^3}{xy^2}$, given that $x=1$ when $y=4$.

$$\left[y^3=x^3(3\ln x+64)\right]$$

6. Show that the solution of the differential equation: $\frac{dy}{dx}=\frac{y^3-xy^2-x^2y-5x^3}{xy^2-x^2y-2x^3}$ is of the form:

$$\frac{y^2}{2x^2}+\frac{4y}{x}+18\ln\left(\frac{y-5x}{x}\right)=\ln x+42,$$

when $x=1$ and $y=6$.

Chapter 48

Linear first order differential equations

48.1 Introduction

An equation of the form $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x only is called a **linear differential equation** since y and its derivatives are of the first degree.

(i) The solution of $\frac{dy}{dx} + Py = Q$ is obtained by multiplying throughout by what is termed an **integrating factor**.

(ii) Multiplying $\frac{dy}{dx} + Py = Q$ by say R, a function of x only, gives:

$$R\frac{dy}{dx} + RPy = RQ \qquad (1)$$

(iii) The differential coefficient of a product Ry is obtained using the product rule,

$$\text{i.e. } \frac{d}{dx}(Ry) = R\frac{dy}{dx} + y\frac{dR}{dx},$$

which is the same as the left hand side of equation (1), when R is chosen such that

$$RP = \frac{dR}{dx}$$

(iv) If $\frac{dR}{dx} = RP$, then separating the variables gives $\frac{dR}{R} = P\,dx$.

Integrating both sides gives:

$$\int \frac{dR}{R} = \int P\,dx \text{ i.e. } \ln R = \int P\,dx + c$$

from which,

$$R = e^{\int P\,dx + c} = e^{\int P\,dx}e^{c}$$

i.e. $R = Ae^{\int P\,dx}$, where $A = e^{c}$ = a constant.

(v) Substituting $R = Ae^{\int P\,dx}$ in equation (1) gives:

$$Ae^{\int P\,dx}\left(\frac{dy}{dx}\right) + Ae^{\int P\,dx}Py = Ae^{\int P\,dx}Q$$

$$\text{i.e. } e^{\int P\,dx}\left(\frac{dy}{dx}\right) + e^{\int P\,dx}Py = e^{\int P\,dx}Q \qquad (2)$$

(vi) The left hand side of equation (2) is

$$\frac{d}{dx}\left(ye^{\int P\,dx}\right)$$

which may be checked by differentiating $ye^{\int P\,dx}$ with respect to x, using the product rule.

(vii) From equation (2),

$$\frac{d}{dx}\left(ye^{\int P\,dx}\right) = e^{\int P\,dx}Q$$

Integrating both sides gives:

$$ye^{\int P\,dx} = \int e^{\int P\,dx}Q\,dx \qquad (3)$$

(viii) $e^{\int P\,dx}$ is the **integrating factor**.

48.2 Procedure to solve differential equations of the form $\frac{dy}{dx}+Py=Q$

(i) Rearrange the differential equation into the form $\frac{dy}{dx}+Py=Q$, where P and Q are functions of x.

(ii) Determine $\int P\,dx$.

(iii) Determine the integrating factor $e^{\int P\,dx}$.

(iv) Substitute $e^{\int P\,dx}$ into equation (3).

(v) Integrate the right hand side of equation (3) to give the general solution of the differential equation. Given boundary conditions, the particular solution may be determined.

48.3 Worked problems on linear first order differential equations

Problem 1. Solve $\frac{1}{x}\frac{dy}{dx}+4y=2$ given the boundary conditions $x=0$ when $y=4$.

Using the above procedure:

(i) Rearranging gives $\frac{dy}{dx}+4xy=2x$, which is of the form $\frac{dy}{dx}+Py=Q$ where $P=4x$ and $Q=2x$.

(ii) $\int P\,dx=\int 4x\,dx=2x^2$.

(iii) Integrating factor $e^{\int P\,dx}=e^{2x^2}$.

(iv) Substituting into equation (3) gives:

$$ye^{2x^2}=\int e^{2x^2}(2x)\,dx$$

(v) Hence the general solution is:

$$ye^{2x^2}=\tfrac{1}{2}e^{2x^2}+c,$$

by using the substitution $u=2x^2$ When $x=0$, $y=4$, thus $4e^0=\frac{1}{2}e^0+c$, from which, $c=\frac{7}{2}$.

Hence the particular solution is

$$ye^{2x^2}=\tfrac{1}{2}e^{2x^2}+\tfrac{7}{2}$$

or $\boldsymbol{y=\frac{1}{2}+\frac{7}{2}e^{-2x^2}}$ or $\boldsymbol{y=\frac{1}{2}\left(1+7e^{-2x^2}\right)}$

Problem 2. Show that the solution of the equation $\frac{dy}{dx}+1=-\frac{y}{x}$ is given by $y=\frac{3-x^2}{2x}$, given $x=1$ when $y=1$.

Using the procedure of Section 48.2:

(i) Rearranging gives: $\frac{dy}{dx}+\left(\frac{1}{x}\right)y=-1$, which is of the form $\frac{dy}{dx}+Py=Q$, where $P=\frac{1}{x}$ and $Q=-1$. (Note that Q can be considered to be $-1x^0$, i.e. a function of x).

(ii) $\int P\,dx=\int\frac{1}{x}\,dx=\ln x$.

(iii) Integrating factor $e^{\int P\,dx}=e^{\ln x}=x$ (from the definition of logarithm).

(iv) Substituting into equation (3) gives:

$$yx=\int x(-1)\,dx$$

(v) Hence the general solution is:

$$yx=\frac{-x^2}{2}+c$$

When $x=1$, $y=1$, thus $1=\frac{-1}{2}+c$, from which, $c=\frac{3}{2}$

Hence the particular solution is:

$$yx=\frac{-x^2}{2}+\frac{3}{2}$$

i.e. $2yx=3-x^2$ and $\boldsymbol{y=\frac{3-x^2}{2x}}$

Problem 3. Determine the particular solution of $\frac{dy}{dx}-x+y=0$, given that $x=0$ when $y=2$.

Using the procedure of Section 48.2:

(i) Rearranging gives $\frac{dy}{dx}+y=x$, which is of the form $\frac{dy}{dx}+P,=Q$, where $P=1$ and $Q=x$. (In this case P can be considered to be $1x^0$, i.e. a function of x).

(ii) $\int P\,dx=\int 1\,dx=x$.

(iii) Integrating factor $e^{\int P\,dx}=e^x$.

(iv) Substituting in equation (3) gives:

$$ye^x = \int e^x(x)\,dx \qquad (4)$$

(v) $\int e^x(x)\,dx$ is determined using integration by parts (see Chapter 43).

$$\int xe^x\,dx = xe^x - e^x + c$$

Hence from equation (4): $ye^x = xe^x - e^x + c$, which is the general solution.

When $x=0$, $y=2$ thus $2e^0 = 0 - e^0 + c$, from which, $c=3$.

Hence the particular solution is:

$$\boldsymbol{ye^x = xe^x - e^x + 3} \text{ or } \boldsymbol{y = x - 1 + 3e^{-x}}$$

Now try the following exercise

Exercise 182 Further problems on linear first order differential equations

Solve the following differential equations.

1. $x\dfrac{dy}{dx} = 3 - y$ $\left[y = 3 + \dfrac{c}{x}\right]$
2. $\dfrac{dy}{dx} = x(1-2y)$ $\left[y = \frac{1}{2} + ce^{-x^2}\right]$
3. $t\dfrac{dy}{dt} - 5t = -y$ $\left[y = \dfrac{5t}{2} + \dfrac{c}{t}\right]$
4. $x\left(\dfrac{dy}{dx} + 1\right) = x^3 - 2y$, given $x=1$ when $y=3$ $\left[y = \dfrac{x^3}{5} - \dfrac{x}{3} + \dfrac{47}{15x^2}\right]$
5. $\dfrac{1}{x}\dfrac{dy}{dx} + y = 1$ $\left[y = 1 + ce^{-x^2/2}\right]$
6. $\dfrac{dy}{dx} + x = 2y$ $\left[y = \dfrac{x}{2} + \dfrac{1}{4} + ce^{2x}\right]$

48.4 Further worked problems on linear first order differential equations

Problem 4. Solve the differential equation $\dfrac{dy}{d\theta} = \sec\theta + y\tan\theta$ given the boundary conditions $y=1$ when $\theta = 0$.

Using the procedure of Section 48.2:

(i) Rearranging gives $\dfrac{dy}{d\theta} - (\tan\theta)y = \sec\theta$, which is of the form $\dfrac{dy}{d\theta} + Py = Q$ where $P = -\tan\theta$ and $Q = \sec\theta$.

(ii) $\int P\,dx = \int -\tan\theta\,d\theta = -\ln(\sec\theta)$

$= \ln(\sec\theta)^{-1} = \ln(\cos\theta)$.

(iii) Integrating factor $e^{\int P\,d\theta} = e^{\ln(\cos\theta)} = \cos\theta$ (from the definition of a logarithm).

(iv) Substituting in equation (3) gives:

$$y\cos\theta = \int \cos\theta(\sec\theta)\,d\theta$$

i.e. $$y\cos\theta = \int d\theta$$

(v) Integrating gives: $y\cos\theta = \theta + c$, which is the general solution. When $\theta = 0$, $y = 1$, thus $1\cos 0 = 0 + c$, from which, $c = 1$.

Hence the particular solution is:

$$\boldsymbol{y\cos\theta = \theta + 1} \text{ or } \boldsymbol{y = (\theta + 1)\sec\theta}$$

Problem 5.

(a) Find the general solution of the equation

$$(x-2)\frac{dy}{dx} + \frac{3(x-1)}{(x+1)}y = 1$$

(b) Given the boundary conditions that $y=5$ when $x=-1$, find the particular solution of the equation given in (a).

(a) Using the procedure of Section 48.2:

(i) Rearranging gives:

$$\frac{dy}{dx} + \frac{3(x-1)}{(x+1)(x-2)}y = \frac{1}{(x-2)}$$

which is of the form $\dfrac{dy}{dx} + Py = Q$, where $P = \dfrac{3(x-1)}{(x+1)(x-2)}$ and $Q = \dfrac{1}{(x-2)}$

(ii) $\int P\,dx = \int \dfrac{3(x-1)}{(x+1)(x-2)}\,dx$, which is integrated using partial fractions.

Let $\dfrac{3x-3}{(x+1)(x-2)}$

$$\equiv \frac{A}{(x+1)}+\frac{B}{(x-2)}$$

$$\equiv \frac{A(x-2)+B(x+1)}{(x+1)(x-2)}$$

from which, $3x-3=A(x-2)+B(x+1)$

When $x=-1$,

$-6=-3A$, from which, $A=2$

When $x=2$,

$3=3B$, from which, $B=1$

Hence $\displaystyle\int \frac{3x-3}{(x+1)(x-2)}\,dx$

$$=\int\left[\frac{2}{x+1}+\frac{1}{x-2}\right]dx$$

$$=2\ln(x+1)+\ln(x-2)$$

$$=\ln[(x+1)^2(x-2)]$$

(iii) Integrating factor

$$e^{\int P\,dx}=e^{\ln[(x+1)^2(x-2)]}=(x+1)^2(x-2)$$

(iv) Substituting in equation (3) gives:

$$y(x+1)^2(x-2)$$

$$=\int (x+1)^2(x-2)\frac{1}{x-2}\,dx$$

$$=\int (x+1)^2\,dx$$

(v) **Hence the general solution is**:

$$\boldsymbol{y(x+1)^2(x-2)=\tfrac{1}{3}(x+1)^3+c}$$

(b) When $x=-1$, $y=5$ thus $5(0)(-3)=0+c$, from which, $c=0$.

Hence $y(x+1)^2(x-2)=\frac{1}{3}(x+1)^3$

i.e. $y=\dfrac{(x+1)^3}{3(x+1)^2(x-2)}$

and hence **the particular solution is**

$$\boldsymbol{y=\frac{(x+1)}{3(x-2)}}$$

Now try the following exercise

Exercise 183 Further problems on linear first order differential equations

In problems 1 and 2, solve the differential equations.

1. $\cot x\dfrac{dy}{dx}=1-2y$, given $y=1$ when $x=\dfrac{\pi}{4}$.
   $[y=\frac{1}{2}+\cos^2 x]$
2. $t\dfrac{d\theta}{dt}+\sec t(t\sin t+\cos t)\theta=\sec t$, given $t=\pi$ when $\theta=1$. $\left[\theta=\dfrac{1}{t}(\sin t-\pi\cos t)\right]$
3. Given the equation $x\dfrac{dy}{dx}=\dfrac{2}{x+2}-y$ show that the particular solution is $y=\dfrac{2}{x}\ln(x+2)$, given the boundary conditions that $x=-1$ when $y=0$.
4. Show that the solution of the differential equation
   $$\frac{dy}{dx}-2(x+1)^3=\frac{4}{(x+1)}y$$
   is $y=(x+1)^4\ln(x+1)^2$, given that $x=0$ when $y=0$.
5. Show that the solution of the differential equation
   $$\frac{dy}{dx}+ky=a\sin bx$$
   is given by:
   $$y=\left(\frac{a}{k^2+b^2}\right)(k\sin bx-b\cos bx)+\left(\frac{k^2+b^2+ab}{k^2+b^2}\right)e^{-kx},$$
   given $y=1$ when $x=0$.
6. The equation $\dfrac{dv}{dt}=-(av+bt)$, where a and b are constants, represents an equation of motion when a particle moves in a resisting medium. Solve the equation for v given that $v=u$ when $t=0$.
   $\left[v=\dfrac{b}{a^2}-\dfrac{bt}{a}+\left(u-\dfrac{b}{a^2}\right)e^{-at}\right]$

7. In an alternating current circuit containing resistance R and inductance L the current i is given by: $Ri + L\dfrac{\mathrm{d}i}{\mathrm{d}t} = E_0 \sin \omega t$. Given $i = 0$ when $t = 0$, show that the solution of the equation is given by:

$$i = \left(\frac{E_0}{R^2 + \omega^2 L^2}\right)(R \sin \omega t - \omega L \cos \omega t) + \left(\frac{E_0 \omega L}{R^2 + \omega^2 L^2}\right) \mathrm{e}^{-Rt/L}$$

8. The concentration, C, of impurities of an oil purifier varies with time t and is described by the equation $a\dfrac{\mathrm{d}C}{\mathrm{d}t} = b + \mathrm{d}m - Cm$, where a, b, d and m are constants. Given $C = c_0$ when $t = 0$, solve the equation and show that:

$$C = \left(\frac{b}{m} + d\right)(1 - \mathrm{e}^{-mt/a}) + c_0 \mathrm{e}^{-mt/a}$$

9. The equation of motion of a train is given by: $m\dfrac{\mathrm{d}v}{\mathrm{d}t} = mk(1 - \mathrm{e}^{-t}) - mcv$, where v is the speed, t is the time and m, k and c are constants. Determine the speed, v, given $v = 0$ at $t = 0$.

$$\left[v = k\left\{\frac{1}{c} - \frac{\mathrm{e}^{-t}}{c-1} + \frac{\mathrm{e}^{-ct}}{c(c-1)}\right\}\right]$$

Chapter 49

Numerical methods for first order differential equations

49.1 Introduction

Not all first order differential equations may be solved by separating the variables (as in Chapter 46) or by the integrating factor method (as in Chapter 48). A number of other analytical methods of solving differential equations exist. However the differential equations that can be solved by such analytical methods is fairly restricted. Where a differential equation and known boundary conditions are given, an approximate solution may be obtained by applying a **numerical method**. There are a number of such numerical methods available and the simplest of these is called **Euler's method**.

49.2 Euler's method

From Chapter 8, Maclaurin's series may be stated as:

$$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!}f''(0) + \cdots$$

Hence at some point $f(h)$ in Fig. 49.1:

$$f(h) = f(0) + h\,f'(0) + \frac{h^2}{2!}f''(0) + \cdots$$

If the y-axis and origin are moved a units to the left, as shown in Fig. 49.2, the equation of the same curve

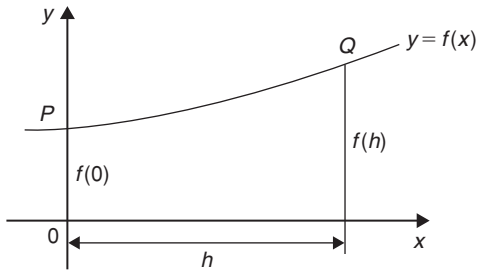


Figure 49.1

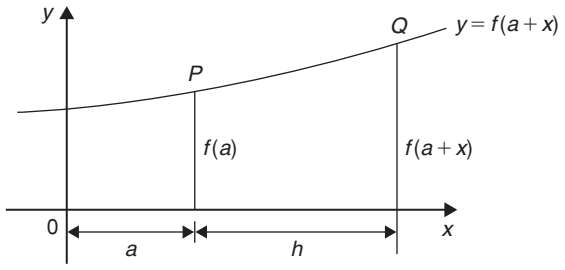


Figure 49.2

relative to the new axis becomes $y = f(a + x)$ and the function value at P is $f(a)$.

At point Q in Fig. 49.2:

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots \qquad (1)$$

which is a statement called **Taylor's series**.
If h is the interval between two new ordinates y_0 and y_1, as shown in Fig. 49.3, and if $f(a) = y_0$ and $y_1 = f(a+h)$, then Euler's method states:

$$f(a+h) = f(a) + hf'(a)$$

i.e. $$\boldsymbol{y_1 = y_0 + h(y')_0} \qquad (2)$$

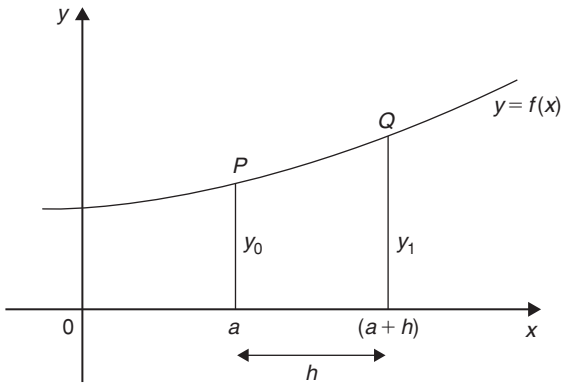


Figure 49.3

The approximation used with Euler's method is to take only the first two terms of Taylor's series shown in equation (1).
Hence if y_0, h and $(y')_0$ are known, y_1, which is an approximate value for the function at Q in Fig. 49.3, can be calculated.
Euler's method is demonstrated in the worked problems following.

49.3 Worked problems on Euler's method

Problem 1. Obtain a numerical solution of the differential equation

$$\frac{dy}{dx} = 3(1+x) - y$$

given the initial conditions that $x = 1$ when $y = 4$, for the range $x = 1.0$ to $x = 2.0$ with intervals of 0.2. Draw the graph of the solution.

$$\frac{dy}{dx} = y' = 3(1+x) - y$$

With $x_0 = 1$ and $y_0 = 4$, $\boldsymbol{(y')_0 = 3(1+1) - 4 = 2}$.

By Euler's method:

$$y_1 = y_0 + h(y')_0, \text{ from equation (2)}$$

Hence $\boldsymbol{y_1 = 4 + (0.2)(2) = 4.4}$, since $h = 0.2$

At point Q in Fig. 49.4, $x_1 = 1.2$, $y_1 = 4.4$

and $(y')_1 = 3(1+x_1) - y_1$

i.e. $\boldsymbol{(y')_1 = 3(1+1.2) - 4.4 = 2.2}$

If the values of x, y and y' found for point Q are regarded as new starting values of x_0, y_0 and $(y')_0$, the above process can be repeated and values found for the point R shown in Fig. 49.5.
Thus at point R,

$$\boldsymbol{y_1} = y_0 + h(y')_0 \quad \text{from equation (2)}$$

$$= 4.4 + (0.2)(2.2) = \boldsymbol{4.84}$$

When $x_1 = 1.4$ and $y_1 = 4.84$,
$\boldsymbol{(y')_1 = 3(1+1.4) - 4.84 = 2.36}$

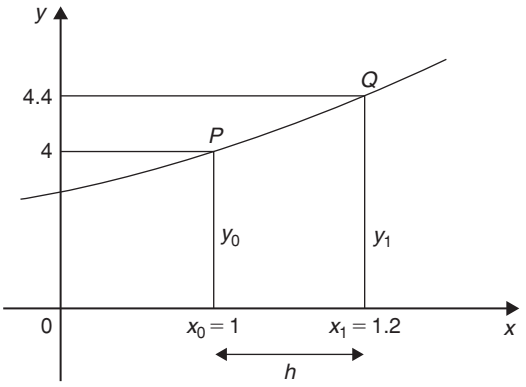


Figure 49.4

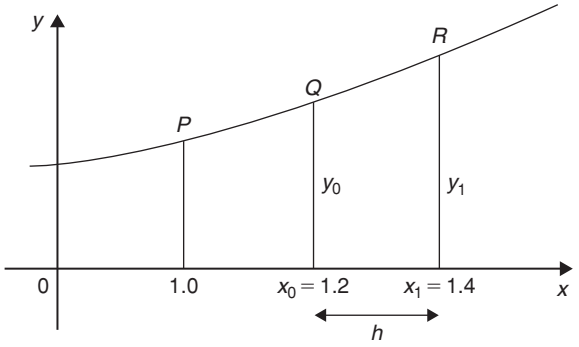


Figure 49.5

This step by step Euler's method can be continued and it is easiest to list the results in a table, as shown in Table 49.1. The results for lines 1 to 3 have been produced above.

Table 49.1

	x_0	y_0	$(y')_0$
1.	1	4	2
2.	1.2	4.4	2.2
3.	1.4	4.84	2.36
4.	1.6	5.312	2.488
5.	1.8	5.8096	2.5904
6.	2.0	6.32768	

For line 4, where $x_0 = 1.6$:

$$y_1 = y_0 + h(y')_0$$
$$= 4.84 + (0.2)(2.36) = \mathbf{5.312}$$

and $(y')_0 = 3(1 + 1.6) - 5.312 = \mathbf{2.488}$

For line 5, where $x_0 = 1.8$:

$$y_1 = y_0 + h(y')_0$$
$$= 5.312 + (0.2)(2.488) = \mathbf{5.8096}$$

and $(y')_0 = 3(1 + 1.8) - 5.8096 = \mathbf{2.5904}$

For line 6, where $x_0 = 2.0$:

$$y_1 = y_0 + h(y')_0$$
$$= 5.8096 + (0.2)(2.5904)$$
$$= \mathbf{6.32768}$$

(As the range is 1.0 to 2.0 there is no need to calculate $(y')_0$ in line 6). The particular solution is given by the value of y against x.

A graph of the solution of $\frac{dy}{dx} = 3(1 + x) - y$ with initial conditions $x = 1$ and $y = 4$ is shown in Fig. 49.6.

In practice it is probably best to plot the graph as each calculation is made, which checks that there is a smooth progression and that no calculation errors have occurred.

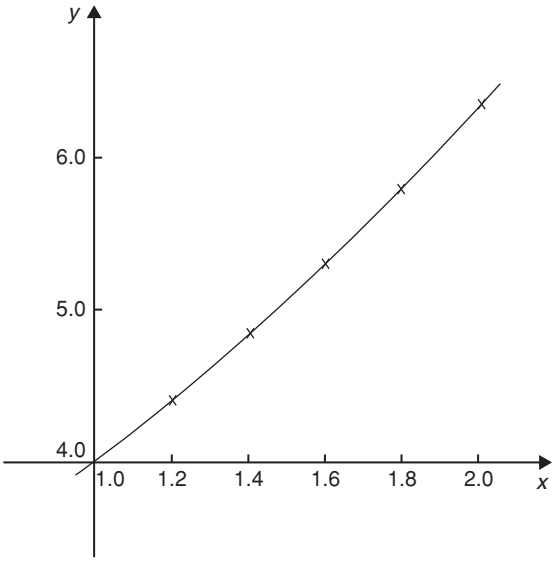


Figure 49.6

Problem 2. Use Euler's method to obtain a numerical solution of the differential equation $\frac{dy}{dx} + y = 2x$, given the initial conditions that at $x = 0$, $y = 1$, for the range $x = 0(0.2)1.0$. Draw the graph of the solution in this range.

$x = 0(0.2)1.0$ means that x ranges from 0 to 1.0 in equal intervals of 0.2 (i.e. $h = 0.2$ in Euler's method).

$$\frac{dy}{dx} + y = 2x,$$

hence $\frac{dy}{dx} = 2x - y$, i.e. $y' = 2x - y$

If initially $x_0 = 0$ and $y_0 = 1$, then $(y')_0 = 2(0) - 1 = \mathbf{-1}$.

Hence line 1 in Table 49.2 can be completed with $x = 0$, $y = 1$ and $y'(0) = -1$.

Table 49.2

	x_0	y_0	$(y')_0$
1.	0	1	−1
2.	0.2	0.8	−0.4
3.	0.4	0.72	0.08
4.	0.6	0.736	0.464
5.	0.8	0.8288	0.7712
6.	1.0	0.98304	

For line 2, where $x_0 = 0.2$ and $h = 0.2$:

$$y_1 = y_0 + h(y'), \quad \text{from equation (2)}$$
$$= 1 + (0.2)(-1) = \mathbf{0.8}$$

and $(y')_0 = 2x_0 - y_0 = 2(0.2) - 0.8 = \mathbf{-0.4}$

For line 3, where $x_0 = 0.4$:

$$y_1 = y_0 + h(y')_0$$
$$= 0.8 + (0.2)(-0.4) = \mathbf{0.72}$$

and $(y')_0 = 2x_0 - y_0 = 2(0.4) - 0.72 = \mathbf{0.08}$

For line 4, where $x_0 = 0.6$:

$$y_1 = y_0 + h(y')_0$$
$$= 0.72 + (0.2)(0.08) = \mathbf{0.736}$$

and $(y')_0 = 2x_0 - y_0 = 2(0.6) - 0.736 = \mathbf{0.464}$

For line 5, where $x_0 = 0.8$:

$$y_1 = y_0 + h(y')_0$$
$$= 0.736 + (0.2)(0.464) = \mathbf{0.8288}$$

and $(y')_0 = 2x_0 - y_0 = 2(0.8) - 0.8288 = \mathbf{0.7712}$

For line 6, where $x_0 = 1.0$:

$$y_1 = y_0 + h(y')_0$$
$$= 0.8288 + (0.2)(0.7712) = \mathbf{0.98304}$$

As the range is 0 to 1.0, $(y')_0$ in line 6 is not needed.

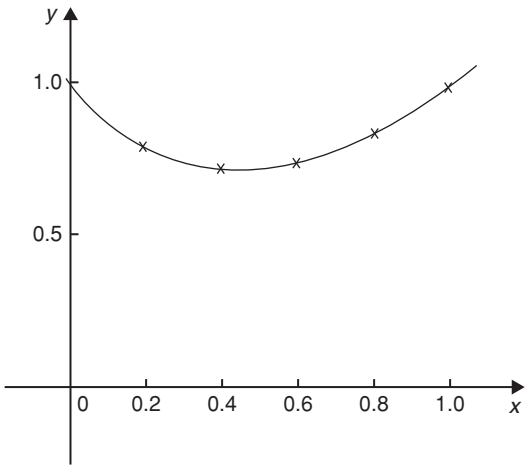


Figure 49.7

A graph of the solution of $\frac{dy}{dx} + y = 2x$, with initial conditions $x = 0$ and $y = 1$ is shown in Fig. 49.7.

Problem 3.

(a) Obtain a numerical solution, using Euler's method, of the differential equation $\frac{dy}{dx} = y - x$, with the initial conditions that at $x = 0$, $y = 2$, for the range $x = 0(0.1)0.5$. Draw the graph of the solution.

(b) By an analytical method (using the integrating factor method of Chapter 48), the solution of the above differential equation is given by $y = x + 1 + e^x$.

Determine the percentage error at $x = 0.3$

(a) $\frac{dy}{dx} = y' = y - x$.

If initially $x_0 = 0$ and $y_0 = 2$,
then $(y')_0 = y_0 - x_0 = 2 - 0 = \mathbf{2}$.
Hence line 1 of Table 49.3 is completed.

For line 2, where $x_0 = 0.1$:

$$y_1 = y_0 + h(y')_0, \quad \text{from equation (2)},$$
$$= 2 + (0.1)(2) = \mathbf{2.2}$$

and $(y')_0 = y_0 - x_0$
$$= 2.2 - 0.1 = \mathbf{2.1}$$

For line 3, where $x_0 = 0.2$:

$$y_1 = y_0 + h(y')_0$$
$$= 2.2 + (0.1)(2.1) = \mathbf{2.41}$$

and $(y')_0 = y_0 - x_0 = 2.41 - 0.2 = \mathbf{2.21}$

Table 49.3

	x_0	y_0	$(y')_0$
1.	0	2	2
2.	0.1	2.2	2.1
3.	0.2	2.41	2.21
4.	0.3	2.631	2.331
5.	0.4	2.8641	2.4641
6.	0.5	3.11051	

For line 4, where $x_0 = 0.3$:

$$y_1 = y_0 + h(y')_0$$
$$= 2.41 + (0.1)(2.21) = \mathbf{2.631}$$

and $(y')_0 = y_0 - x_0$
$$= 2.631 - 0.3 = \mathbf{2.331}$$

For line 5, where $x_0 = 0.4$:

$$y_1 = y_0 + h(y')_0$$
$$= 2.631 + (0.1)(2.331) = \mathbf{2.8641}$$

and $(y')_0 = y_0 - x_0$
$$= 2.8641 - 0.4 = \mathbf{2.4641}$$

For line 6, where $x_0 = 0.5$:

$$y_1 = y_0 + h(y')_0$$
$$= 2.8641 + (0.1)(2.4641) = \mathbf{3.11051}$$

A graph of the solution of $\frac{dy}{dx} = y - x$ with $x = 0$, $y = 2$ is shown in Fig. 49.8.

(b) If the solution of the differential equation $\frac{dy}{dx} = y - x$ is given by $y = x + 1 + e^x$, then when $x = 0.3$, $y = 0.3 + 1 + e^{0.3} = 2.649859$.

By Euler's method, when $x = 0.3$ (i.e. line 4 in Table 49.3), $y = 2.631$.

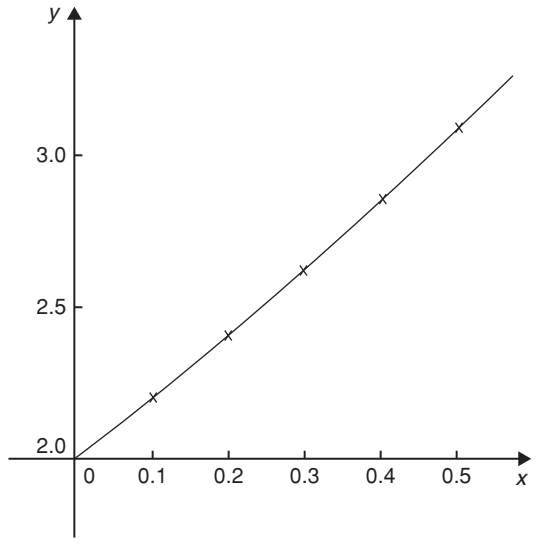


Figure 49.8

Percentage error

$$= \left(\frac{\text{actual} - \text{estimated}}{\text{actual}}\right) \times 100\%$$
$$= \left(\frac{2.649859 - 2.631}{2.649859}\right) \times 100\%$$
$$= \mathbf{0.712\%}$$

Euler's method of numerical solution of differential equations is simple, but approximate. The method is most useful when the interval h is small.

Now try the following exercise

Exercise 184 Further problems on Euler's method

1. Use Euler's method to obtain a numerical solution of the differential equation $\frac{dy}{dx} = 3 - \frac{y}{x}$, with the initial conditions that $x = 1$ when $y = 2$, for the range $x = 1.0$ to $x = 1.5$ with intervals of 0.1. Draw the graph of the solution in this range. [see Table 49.4]

Table 49.4

x	y
1.0	2
1.1	2.1
1.2	2.209091
1.3	2.325000
1.4	2.446154
1.5	2.571429

2. Obtain a numerical solution of the differential equation $\frac{1}{x}\frac{dy}{dx} + 2y = 1$, given the initial conditions that $x = 0$ when $y = 1$, in the range $x = 0(0.2)1.0$ [see Table 49.5]

3. (a) The differential equation $\frac{dy}{dx} + 1 = -\frac{y}{x}$ has the initial conditions that $y = 1$ at $x = 2$. Produce a numerical solution of the differential equation in the range $x = 2.0(0.1)2.5$

Table 49.5

x	y
0	1
0.2	1
0.4	0.96
0.6	0.8864
0.8	0.793664
1.0	0.699692

(b) If the solution of the differential equation by an analytical method is given by $y = \frac{4}{x} - \frac{x}{2}$, determine the percentage error at $x = 2.2$

[(a) see Table 49.6 (b) 1.206%]

Table 49.6

x	y
2.0	1
2.1	0.85
2.2	0.709524
2.3	0.577273
2.4	0.452174
2.5	0.333334

4. Use Euler's method to obtain a numerical solution of the differential equation $\frac{dy}{dx} = x - \frac{2y}{x}$, given the initial conditions that $y = 1$ when $x = 2$, in the range $x = 2.0(0.2)3.0$.

If the solution of the differential equation is given by $y = \frac{x^2}{4}$, determine the percentage error by using Euler's method when $x = 2.8$

[see Table 49.7, 1.596%]

Table 49.7

x	y
2.0	1
2.2	1.2
2.4	1.421818
2.6	1.664849
2.8	1.928718
3.0	2.213187

49.4 An improved Euler method

In Euler's method of Section 49, the gradient $(y')_0$ at $P_{(x_0, y_0)}$ in Fig. 49.9 across the whole interval h is used to obtain an approximate value of y_1 at point Q. QR in Fig. 49.9 is the resulting error in the result.

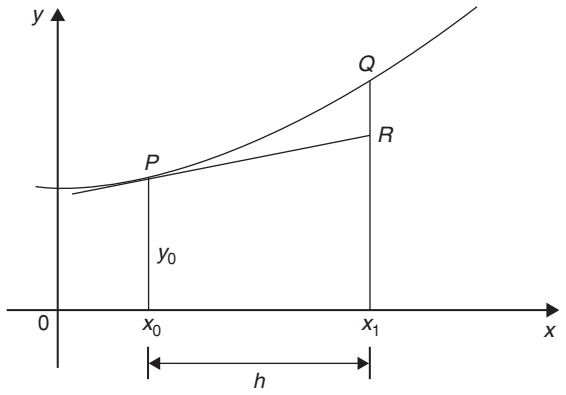


Figure 49.9

In an improved Euler method, called the **Euler-Cauchy method**, the gradient at $P_{(x_0, y_0)}$ across half the interval is used and then continues with a line whose gradient approximates to the gradient of the curve at x_1, shown in Fig. 49.10.

Let y_{P_1} be the predicted value at point R using Euler's method, i.e. length RZ, where

$$y_{P_1} = y_0 + h(y')_0 \tag{3}$$

The error shown as QT in Fig. 49.10 is now less than the error QR used in the basic Euler method and the calculated results will be of greater accuracy. The

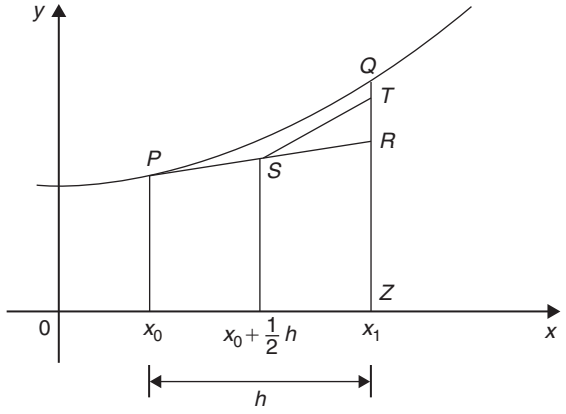


Figure 49.10

corrected value, y_{C_1} in the improved Euler method is given by:

$$y_{C_1} = y_0 + \tfrac{1}{2}h[(y')_0 + f(x_1, y_{P_1})] \quad (4)$$

The following worked problems demonstrate how equations (3) and (4) are used in the Euler-Cauchy method.

Problem 4. Apply the Euler-Cauchy method to solve the differential equation

$$\frac{dy}{dx} = y - x$$

in the range 0(0.1)0.5, given the initial conditions that at $x = 0$, $y = 2$.

$$\frac{dy}{dx} = y' = y - x$$

Since the initial conditions are $x_0 = 0$ and $y_0 = 2$ then $(y')_0 = 2 - 0 = 2$. Interval $h = 0.1$, hence $x_1 = x_0 + h = 0 + 0.1 = 0.1$.
From equation (3),

$$y_{P_1} = y_0 + h(y')_0 = 2 + (0.1)(2) = 2.2$$

From equation (4),

$$y_{C_1} = y_0 + \tfrac{1}{2}h[(y')_0 + f(x_1, y_{P_1})]$$
$$= y_0 + \tfrac{1}{2}h[(y')_0 + (y_{P_1} - x_1)], \quad \text{in this case}$$
$$= 2 + \tfrac{1}{2}(0.1)[2 + (2.2 - 0.1)] = \mathbf{2.205}$$
$$(y')_1 = y_{C_1} - x_1 = 2.205 - 0.1 = 2.105$$

If we produce a table of values, as in Euler's method, we have so far determined lines 1 and 2 of Table 49.8. The results in line 2 are now taken as x_0, y_0 and $(y')_0$ for the next interval and the process is repeated.

Table 49.8

	x	y	y'
1.	0	2	2
2.	0.1	2.205	2.105
3.	0.2	2.421025	2.221025
4.	0.3	2.649232625	2.349232625
5.	0.4	2.89090205	2.49090205
6.	0.5	3.147446765	

For line 3, $x_1 = 0.2$

$$y_{P_1} = y_0 + h(y')_0 = 2.205 + (0.1)(2.105)$$
$$= 2.4155$$
$$y_{C_1} = y_0 + \tfrac{1}{2}h[(y')_0 + f(x_1, y_{P_1})]$$
$$= 2.205 + \tfrac{1}{2}(0.1)[2.105 + (2.4155 - 0.2)]$$
$$= \mathbf{2.421025}$$
$$(y')_0 = y_{C_1} - x_1 = 2.421025 - 0.2 = 2.221025$$

For line 4, $x_1 = 0.3$

$$y_{P_1} = y_0 + h(y')_0$$
$$= 2.421025 + (0.1)(2.221025)$$
$$= 2.6431275$$
$$y_{C_1} = y_0 + \tfrac{1}{2}h[(y')_0 + f(x_1, y_{P_1})]$$
$$= 2.421025 + \tfrac{1}{2}(0.1)[2.221025 + (2.6431275 - 0.3)]$$
$$= \mathbf{2.649232625}$$
$$(y')_0 = y_{C_1} - x_1 = 2.649232625 - 0.3$$
$$= 2.349232625$$

For line 5, $x_1 = 0.4$

$$y_{P_1} = y_0 + h(y')_0$$
$$= 2.649232625 + (0.1)(2.349232625)$$
$$= 2.884155887$$
$$y_{C_1} = y_0 + \tfrac{1}{2}h[(y')_0 + f(x_1, y_{P_1})]$$
$$= 2.649232625 + \tfrac{1}{2}(0.1)[2.349232625 + (2.884155887 - 0.4)]$$
$$= \mathbf{2.89090205}$$
$$(y')_0 = y_{C_1} - x_1 = 2.89090205 - 0.4$$
$$= 2.49090205$$

For line 6, $x_1 = 0.5$

$$y_{P_1} = y_0 + h(y')_0$$
$$= 2.89090205 + (0.1)(2.49090205)$$
$$= 3.139992255$$

$$y_{C_1} = y_0 + \tfrac{1}{2}h[(y')_0 + f(x_1, y_{P_1})]$$
$$= 2.89090205 + \tfrac{1}{2}(0.1)[2.49090205 + (3.139992255 - 0.5)]$$
$$= \mathbf{3.147446765}$$

Problem 4 is the same example as Problem 3 and Table 49.9 shows a comparison of the results, i.e. it compares the results of Tables 49.3 and 49.8. $\frac{dy}{dx} = y - x$ may be solved analytically by the integrating factor method of Chapter 48 with the solution $y = x + 1 + e^x$. Substituting values of x of 0, 0.1, 0.2, ... give the exact values shown in Table 49.9.

The percentage error for each method for each value of x is shown in Table 49.10. For example when $x = 0.3$,

% error with Euler method

$$= \left(\frac{\text{actual} - \text{estimated}}{\text{actual}}\right) \times 100\%$$
$$= \left(\frac{2.649858808 - 2.631}{2.649858808}\right) \times 100\%$$
$$= \mathbf{0.712\%}$$

% error with Euler-Cauchy method

$$= \left(\frac{2.649858808 - 2.649232625}{2.649858808}\right) \times 100\%$$
$$= \mathbf{0.0236\%}$$

This calculation and the others listed in Table 49.10 show the Euler-Cauchy method to be more accurate than the Euler method.

Table 49.10

x	Error in Euler method	Error in Euler-Cauchy method
0	0	0
0.1	0.234%	0.00775%
0.2	0.472%	0.0156%
0.3	0.712%	0.0236%
0.4	0.959%	0.0319%
0.5	1.214%	0.0405%

Problem 5. Obtain a numerical solution of the differential equation

$$\frac{dy}{dx} = 3(1 + x) - y$$

in the range 1.0(0.2)2.0, using the Euler-Cauchy method, given the initial conditions that $x = 1$ when $y = 4$.

This is the same as Problem 1 on page 462, and a comparison of values may be made.

$$\frac{dy}{dx} = y' = 3(1 + x) - y \quad \text{i.e. } y' = 3 + 3x - y$$

$x_0 = 1.0$, $y_0 = 4$ and $h = 0.2$

$$(y')_0 = 3 + 3x_0 - y_0 = 3 + 3(1.0) - 4 = 2$$

$x_1 = 1.2$ and from equation (3),

Table 49.9

	x	Euler method y	Euler-Cauchy method y	Exact value $y = x + 1 + e^x$
1.	0	2	2	2
2.	0.1	2.2	2.205	2.205170918
3.	0.2	2.41	2.421025	2.421402758
4.	0.3	2.631	2.649232625	2.649858808
5.	0.4	2.8641	2.89090205	2.891824698
6.	0.5	3.11051	3.147446765	3.148721271

$$y_{P_1} = y_0 + h(y')_0 = 4 + 0.2(2) = 4.4$$

$$y_{C_1} = y_0 + \tfrac{1}{2}h[(y')_0 + f(x_1, y_{P_1})]$$

$$= y_0 + \tfrac{1}{2}h[(y')_0 + (3 + 3x_1 - y_{P_1})]$$

$$= 4 + \tfrac{1}{2}(0.2)[2 + (3 + 3(1.2) - 4.4)]$$

$$= \mathbf{4.42}$$

$$(y')_1 = 3 + 3x_1 - y_{P_1} = 3 + 3(1.2) - 4.42 = 2.18$$

Thus the first two lines of Table 49.11 have been completed.

For line 3, $x_1 = 1.4$

$$y_{P_1} = y_0 + h(y')_0 = 4.42 + 0.2(2.18) = 4.856$$

$$y_{C_1} = y_0 + \tfrac{1}{2}h[(y')_0 + (3 + 3x_1 - y_{P_1})]$$

$$= 4.42 + \tfrac{1}{2}(0.2)[2.18 + (3 + 3(1.4) - 4.856)]$$

$$= \mathbf{4.8724}$$

$$(y')_1 = 3 + 3x_1 - y_{P_1} = 3 + 3(1.4) - 4.8724$$

$$= 2.3276$$

For line 4, $x_1 = 1.6$

$$y_{P_1} = y_0 + h(y')_0 = 4.8724 + 0.2(2.3276)$$

$$= 5.33792$$

$$y_{C_1} = y_0 + \tfrac{1}{2}h[(y')_0 + (3 + 3x_1 - y_{P_1})]$$

$$= 4.8724 + \tfrac{1}{2}(0.2)[2.3276 + (3 + 3(1.6) - 5.33792)]$$

$$= \mathbf{5.351368}$$

Table 49.11

	x_0	y_0	y'_0
1.	1.0	4	2
2.	1.2	4.42	2.18
3.	1.4	4.8724	2.3276
4.	1.6	5.351368	2.448632
5.	1.8	5.85212176	2.54787824
6.	2.0	6.370739847	

$$(y')_1 = 3 + 3x_1 - y_{P_1}$$

$$= 3 + 3(1.6) - 5.351368$$

$$= 2.448632$$

For line 5, $x_1 = 1.8$

$$y_{P_1} = y_0 + h(y')_0 = 5.351368 + 0.2(2.448632)$$

$$= 5.8410944$$

$$y_{C_1} = y_0 + \tfrac{1}{2}h[(y')_0 + (3 + 3x_1 - y_{P_1})]$$

$$= 5.351368 + \tfrac{1}{2}(0.2)[2.448632 + (3 + 3(1.8) - 5.8410944)]$$

$$= \mathbf{5.85212176}$$

$$(y')_1 = 3 + 3x_1 - y_{P_1}$$

$$= 3 + 3(1.8) - 5.85212176$$

$$= 2.54787824$$

For line 6, $x_1 = 2.0$

$$y_{P_1} = y_0 + h(y')_0$$

$$= 5.85212176 + 0.2(2.54787824)$$

$$= 6.361697408$$

$$y_{C_1} = y_0 + \tfrac{1}{2}h[(y')_0 + (3 + 3x_1 - y_{P_1})]$$

$$= 5.85212176 + \tfrac{1}{2}(0.2)[2.54787824 + (3 + 3(2.0) - 6.361697408)]$$

$$= \mathbf{6.370739843}$$

Problem 6. Using the integrating factor method the solution of the differential equation $\frac{dy}{dx} = 3(1 + x) - y$ of Problem 5 is $y = 3x + e^{1-x}$. When $x = 1.6$, compare the accuracy, correct to 3 decimal places, of the Euler and the Euler-Cauchy methods.

When $x = 1.6$, $y = 3x + e^{1-x} = 3(1.6) + e^{1-1.6} = 4.8 + e^{-0.6} = 5.348811636$.

From Table 49.1, page 463, by Euler's method, when $x = 1.6$, $y = 5.312$

% error in the Euler method

$$= \left(\frac{5.348811636 - 5.312}{5.348811636}\right) \times 100\%$$

$$= \mathbf{0.688\%}$$

From Table 49.11 of Problem 5, by the Euler-Cauchy method, when $x = 1.6$, $y = 5.351368$

% error in the Euler-Cauchy method

$$= \left(\frac{5.348811636 - 5.351368}{5.348811636}\right) \times 100\%$$

$$= \mathbf{-0.048\%}$$

The Euler-Cauchy method is seen to be more accurate than the Euler method when $x = 1.6$.

Now try the following exercise

Exercise 185 Further problems on an improved Euler method

1. Apply the Euler-Cauchy method to solve the differential equation

$$\frac{dy}{dx} = 3 - \frac{y}{x}$$

for the range 1.0(0.1)1.5, given the initial conditions that $x = 1$ when $y = 2$.

[see Table 49.12]

Table 49.12

x	y	y'
1.0	2	1
1.1	2.10454546	1.08677686
1.2	2.216666672	1.152777773
1.3	2.33461539	1.204142008
1.4	2.457142859	1.2448987958
1.5	2.583333335	

2. Solving the differential equation in Problem 1 by the integrating factor method gives $y = \frac{3}{2}x + \frac{1}{2x}$. Determine the percentage error, correct to 3 significant figures, when $x = 1.3$ using (a) Euler's method (see Table 49.4, page 465), and (b) the Euler-Cauchy method.

[(a) 0.412% (b) 0.000000214%]

3. (a) Apply the Euler-Cauchy method to solve the differential equation

$$\frac{dy}{dx} - x = y$$

for the range $x = 0$ to $x = 0.5$ in increments of 0.1, given the initial conditions that when $x = 0$, $y = 1$

(b) The solution of the differential equation in part (a) is given by $y = 2e^x - x - 1$. Determine the percentage error, correct to 3 decimal places, when $x = 0.4$

[(a) see Table 49.13 (b) 0.117%]

Table 49.13

x	y	y'
0	1	1
0.1	1.11	1.21
0.2	1.24205	1.44205
0.3	1.398465	1.698465
0.4	1.581804	1.981804
0.5	1.794893	

4. Obtain a numerical solution of the differential equation

$$\frac{1}{x}\frac{dy}{dx} + 2y = 1$$

using the Euler-Cauchy method in the range $x = 0(0.2)1.0$, given the initial conditions that $x = 0$ when $y = 1$.

[see Table 49.14]

Table 49.14

x	y	y'
0	1	0
0.2	0.99	−0.196
0.4	0.958336	−0.3666688
0.6	0.875468851	−0.450562623
0.8	0.784755575	−0.45560892
1.0	0.700467925	

49.5 The Runge-Kutta method

The Runge-Kutta method for solving first order differential equations is widely used and provides a high degree of accuracy. Again, as with the two previous methods, the Runge-Kutta method is a step-by-step process where results are tabulated for a range of values of x. Although several intermediate calculations are needed at each stage, the method is fairly straightforward.

The 7 step **procedure for the Runge-Kutta method**, without proof, is as follows:

To solve the differential equation $\frac{dy}{dx} = f(x, y)$ given the initial condition $y = y_0$ at $x = x_0$ for a range of values of $x = x_0(h)x_n$:

1. Identify x_0, y_0 and h, and values of $x_1, x_2, x_3, \ldots$.
2. Evaluate $k_1 = f(x_n, y_n)$ starting with $n = 0$
3. Evaluate $k_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right)$
4. Evaluate $k_3 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_2\right)$
5. Evaluate $k_4 = f(x_n + h, y_n + hk_3)$
6. Use the values determined from steps 2 to 5 to evaluate:

$$y_{n+1} = y_n + \frac{h}{6}\{k_1 + 2k_2 + 2k_3 + k_4\}$$

7. Repeat steps 2 to 6 for $n = 1, 2, 3, \ldots$

Thus, step 1 is given, and steps 2 to 5 are intermediate steps leading to step 6. It is usually most convenient to construct a table of values.

The Runge-Kutta method is demonstrated in the following worked problems.

Problem 7. Use the Runge-Kutta method to solve the differential equation:

$$\frac{dy}{dx} = y - x$$

in the range 0(0.1)0.5, given the initial conditions that at $x = 0$, $y = 2$.

Using the above procedure:

1. $x_0 = 0$, $y_0 = 2$ and since $h = 0.1$, and the range is from $x = 0$ to $x = 0.5$, then $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$, $x_4 = 0.4$, and $x_5 = 0.5$

Let $n = 0$ to determine y_1:

2. $k_1 = f(x_0, y_0) = f(0, 2)$;
since $\frac{dy}{dx} = y - x$, $f(0, 2) = 2 - 0 = \mathbf{2}$

3. $$k_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_1\right)$$
$$= f\left(0 + \frac{0.1}{2}, 2 + \frac{0.1}{2}(2)\right)$$
$$= f(0.05, 2.1) = 2.1 - 0.05 = \mathbf{2.05}$$

4. $$k_3 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_2\right)$$
$$= f\left(0 + \frac{0.1}{2}, 2 + \frac{0.1}{2}(2.05)\right)$$
$$= f(0.05, 2.1025)$$
$$= 2.1025 - 0.05 = \mathbf{2.0525}$$

5. $$k_4 = f(x_0 + h, y_0 + hk_3)$$
$$= f(0 + 0.1, 2 + 0.1(2.0525))$$
$$= f(0.1, 2.20525)$$
$$= 2.20525 - 0.1 = \mathbf{2.10525}$$

6. $y_{n+1} = y_n + \frac{h}{6}\{k_1 + 2k_2 + 2k_3 + k_4\}$ and when $n = 0$:

$$\mathbf{y_1} = y_0 + \frac{h}{6}\{k_1 + 2k_2 + 2k_3 + k_4\}$$
$$= 2 + \frac{0.1}{6}\{2 + 2(2.05) + 2(2.0525) + 2.10525\}$$
$$= 2 + \frac{0.1}{6}\{12.31025\} = \mathbf{2.205171}$$

A table of values may be constructed as shown in Table 49.15. The working has been shown for the first two rows.

Table 49.15

n	x_n	k_1	k_2	k_3	k_4	y_n
0	**0**					**2**
1	**0.1**	2.0	2.05	2.0525	2.10525	**2.205171**
2	**0.2**	2.105171	2.160430	2.163193	2.221490	**2.421403**
3	**0.3**	2.221403	2.282473	2.285527	2.349956	**2.649859**
4	**0.4**	2.349859	2.417339	2.420726	2.491932	**2.891824**
5	**0.5**	2.491824	2.566415	2.570145	2.648838	**3.148720**

Let $n=1$ to determine y_2:

2. $k_1 = f(x_1, y_1) = f(0.1, 2.205171)$; since $\dfrac{dy}{dx} = y - x$, $f(0.1, 2.205171)$
$= 2.205171 - 0.1 = \mathbf{2.105171}$

3. $k_2 = f\left(x_1 + \dfrac{h}{2}, y_1 + \dfrac{h}{2}k_1\right)$
$= f\left(0.1 + \dfrac{0.1}{2}, 2.205171 + \dfrac{0.1}{2}(2.105171)\right)$
$= f(0.15, 2.31042955)$
$= 2.31042955 - 0.15 = \mathbf{2.160430}$

4. $k_3 = f\left(x_1 + \dfrac{h}{2}, y_1 + \dfrac{h}{2}k_2\right)$
$= f\left(0.1 + \dfrac{0.1}{2}, 2.205171 + \dfrac{0.1}{2}(2.160430)\right)$
$= f(0.15, 2.3131925) = 2.3131925 - 0.15$
$= \mathbf{2.163193}$

5. $k_4 = f(x_1 + h, y_1 + hk_3)$
$= f(0.1 + 0.1, 2.205171 + 0.1(2.163193))$
$= f(0.2, 2.421490)$
$= 2.421490 - 0.2 = \mathbf{2.221490}$

6. $y_{n+1} = y_n + \dfrac{h}{6}\{k_1 + 2k_2 + 2k_3 + k_4\}$
and when $n = 1$:

$$y_2 = y_1 + \frac{h}{6}\{k_1 + 2k_2 + 2k_3 + k_4\}$$
$$= 2.205171 + \frac{0.1}{6}\{2.105171 + 2(2.160430) + 2(2.163193) + 2.221490\}$$
$$= 2.205171 + \frac{0.1}{6}\{12.973907\} = \mathbf{2.421403}$$

This completes the third row of Table 49.15. In a similar manner y_3, y_4 and y_5 can be calculated and the results are as shown in Table 49.15. Such a table is best produced by using a **spreadsheet**, such as Microsoft Excel.

This problem is the same as problem 3, page 459 which used Euler's method, and problem 4, page 461 which used the improved Euler's method, and a comparison of results can be made.

The differential equation $\dfrac{dy}{dx} = y - x$ may be solved analytically using the integrating factor method of chapter 48, with the solution:

$$y = x + 1 + e^x$$

Substituting values of x of 0, 0.1, 0.2, …, 0.5 will give the exact values. A comparison of the results obtained by Euler's method, the Euler-Cauchy method and the Runga-Kutta method, together with the exact values is shown in Table 49.16.

It is seen from Table 49.16 that **the Runge-Kutta method is exact, correct to 5 decimal places**.

Problem 8. Obtain a numerical solution of the differential equation: $\dfrac{dy}{dx} = 3(1+x) - y$ in the range 1.0(0.2)2.0, using the Runge-Kutta method, given the initial conditions that $x = 1.0$ when $y = 4.0$.

Table 49.16

x	Euler's method y	Euler-Cauchy method y	Runge-Kutta method y	Exact value $y=x+1+e^x$
0	2	2	2	2
0.1	2.2	2.205	2.205171	2.205170918
0.2	2.41	2.421025	2.421403	2.421402758
0.3	2.631	2.649232625	2.649859	2.649858808
0.4	2.8641	2.89090205	2.891824	2.891824698
0.5	3.11051	3.147446765	3.148720	3.148721271

Using the above procedure:

1. $x_0=1.0$, $y_0=4.0$ and since $h=0.2$, and the range is from $x=1.0$ to $x=2.0$, then $x_1=1.2$, $x_2=1.4$, $x_3=1.6$, $x_4=1.8$, and $x_5=2.0$

Let $n=0$ to determine y_1:

2. $k_1=f(x_0, y_0)=f(1.0, 4.0)$; since $\dfrac{dy}{dx}=3(1+x)-y$,

$f(1.0, 4.0)=3(1+1.0)-4.0=\mathbf{2.0}$

3. $k_2=f\left(x_0+\dfrac{h}{2}, y_0+\dfrac{h}{2}k_1\right)$

$=f\left(1.0+\dfrac{0.2}{2}, 4.0+\dfrac{0.2}{2}(2)\right)$

$=f(1.1, 4.2)=3(1+1.1)-4.2=\mathbf{2.1}$

4. $k_3=f\left(x_0+\dfrac{h}{2}, y_0+\dfrac{h}{2}k_2\right)$

$=f\left(1.0+\dfrac{0.2}{2}, 4.0+\dfrac{0.2}{2}(2.1)\right)$

$=f(1.1, 4.21)$

$=3(1+1.1)-4.21=\mathbf{2.09}$

5. $k_4=f(x_0+h, y_0+hk_3)$

$=f(1.0+0.2, 4.1+0.2(2.09))$

$=f(1.2, 4.418)$

$=3(1+1.2)-4.418=\mathbf{2.182}$

6. $y_{n+1}=y_n+\dfrac{h}{6}\{k_1+2k_2+2k_3+k_4\}$ and when $n=0$:

$\mathbf{y_1}=y_0+\dfrac{h}{6}\{k_1+2k_2+2k_3+k_4\}$

$=4.0+\dfrac{0.2}{6}\{2.0+2(2.1)+2(2.09)+2.182\}$

$=4.0+\dfrac{0.2}{6}\{12.562\}=\mathbf{4.418733}$

A table of values is compiled in Table 49.17. The working has been shown for the first two rows.

Let $n=1$ to determine y_2:

2. $k_1=f(x_1, y_1)=f(1.2, 4.418733)$; since $\dfrac{dy}{dx}=3(1+x)-y$, $f(1.2, 4.418733)$

$=3(1+1.2)-4.418733=\mathbf{2.181267}$

3. $k_2=f\left(x_1+\dfrac{h}{2}, y_1+\dfrac{h}{2}k_1\right)$

$=f\left(1.2+\dfrac{0.2}{2}, 4.418733+\dfrac{0.2}{2}(2.181267)\right)$

$=f(1.3, 4.636860)$

$=3(1+1.3)-4.636860=\mathbf{2.263140}$

4. $k_3=f\left(x_1+\dfrac{h}{2}, y_1+\dfrac{h}{2}k_2\right)$

$=f\left(1.2+\dfrac{0.2}{2}, 4.418733+\dfrac{0.2}{2}(2.263140)\right)$

$=f(1.3, 4.645047)=3(1+1.3)-4.645047$

$=\mathbf{2.254953}$

Table 49.17

n	x_n	k_1	k_2	k_3	k_4	y_n
0	**1.0**					**4.0**
1	**1.2**	2.0	2.1	2.09	2.182	**4.418733**
2	**1.4**	2.181267	2.263140	2.254953	2.330276	**4.870324**
3	**1.6**	2.329676	2.396708	2.390005	2.451675	**5.348817**
4	**1.8**	2.451183	2.506065	2.500577	2.551068	**5.849335**
5	**2.0**	2.550665	2.595599	2.591105	2.632444	**6.367886**

5. $k_4 = f(x_1 + h,\ y_1 + hk_3)$

$= f(1.2 + 0.2,\ 4.418733 + 0.2(2.254953))$

$= f(1.4,\ 4.869724) = 3(1 + 1.4) - 4.869724$

$= \mathbf{2.330276}$

6. $y_{n+1} = y_n + \frac{h}{6}\{k_1 + 2k_2 + 2k_3 + k_4\}$ and when $n = 1$:

$$y_2 = y_1 + \frac{h}{6}\{k_1 + 2k_2 + 2k_3 + k_4\}$$

$$= 4.418733 + \frac{0.2}{6}\{2.181267 + 2(2.263140) + 2(2.254953) + 2.330276\}$$

$$= 4.418733 + \frac{0.2}{6}\{13.547729\} = \mathbf{4.870324}$$

This completes the third row of Table 49.17. In a similar manner y_3, y_4 and y_5 can be calculated and the results are as shown in Table 49.17. As in the previous problem such a table is best produced by using a **spreadsheet**.

This problem is the same as Problem 1, page 462 which used Euler's method, and Problem 5, page 468 which used the Euler-Cauchy method, and a comparison of results can be made.

The differential equation $\frac{dy}{dx} = 3(1 + x) - y$ may be solved analytically using the integrating factor method of chapter 48, with the solution:

$$y = 3x + e^{1-x}$$

Substituting values of x of 1.0, 1.2, 1.4, ..., 2.0 will give the exact values. A comparison of the results obtained by Euler's method, the Euler-Cauchy method and the Runga-Kutta method, together with the exact values is shown in Table 49.18.

It is seen from Table 49.18 that **the Runge-Kutta method is exact, correct to 4 decimal places**.

Table 49.18

x	Euler's method y	Euler-Cauchy method y	Runge-Kutta method y	Exact value $y = 3x + e^{1-x}$
1.0	4	4	4	4
1.2	4.4	4.42	4.418733	4.418730753
1.4	4.84	4.8724	4.870324	4.870320046
1.6	5.312	5.351368	5.348817	5.348811636
1.8	5.8096	5.85212176	5.849335	5.849328964
2.0	6.32768	6.370739847	6.367886	6.367879441

The percentage error in the Runge-Kutta method when, say, $x = 1.6$ is:

$$\left(\frac{5.348811636 - 5.348817}{5.348811636}\right) \times 100\% = \mathbf{-0.0001\%}$$

From Problem 6, page 469, when $x = 1.6$, the percentage error for the Euler method was 0.688%, and for the Euler-Cauchy method −0.048%. Clearly, the Runge-Kutta method is the most accurate of the three methods.

Now try the following exercise

Exercise 186 Further problems on the Runge-Kutta method

1. Apply the Runge-Kutta method to solve the differential equation: $\frac{dy}{dx} = 3 - \frac{y}{x}$ for the range 1.0(0.1)1.5, given that the initial conditions that $x = 1$ when $y = 2$.

   [see Table 49.19]

Table 49.19

n	x_n	y_n
0	1.0	2.0
1	1.1	2.104545
2	1.2	2.216667
3	1.3	2.334615
4	1.4	2.457143
5	1.5	2.533333

2. Obtain a numerical solution of the differential equation: $\frac{1}{x}\frac{dy}{dx} + 2y = 1$ using the Runge-Kutta method in the range $x = 0(0.2)1.0$, given the initial conditions that $x = 0$ when $y = 1$.

   [see Table 49.20]

Table 49.20

n	x_n	y_n
0	0	1.0
1	0.2	0.980395
2	0.4	0.926072
3	0.6	0.848838
4	0.8	0.763649
5	1.0	0.683952

3. (a) The differential equation: $\frac{dy}{dx} + 1 = -\frac{y}{x}$ has the initial conditions that $y = 1$ at $x = 2$. Produce a numerical solution of the differential equation, correct to 6 decimal places, using the Runge-Kutta method in the range $x = 2.0(0.1)2.5$

   (b) If the solution of the differential equation by an analytical method is given by: $y = \frac{4}{x} - \frac{x}{2}$ determine the percentage error at $x = 2.2$

   [(a) see Table 49.21 (b) no error]

Table 49.21

n	x_n	y_n
0	2.0	1.0
1	2.1	0.854762
2	2.2	0.718182
3	2.3	0.589130
4	2.4	0.466667
5	2.5	0.340000

Revision Test 14

This Revision Test covers the material contained in Chapters 46 to 49. *The marks for each question are shown in brackets at the end of each question.*

1. Solve the differential equation: $x\dfrac{dy}{dx}+x^2=5$ given that $y=2.5$ when $x=1$. (4)

2. Determine the equation of the curve which satisfies the differential equation $2xy\dfrac{dy}{dx}=x^2+1$ and which passes through the point (1, 2). (5)

3. A capacitor C is charged by applying a steady voltage E through a resistance R. The p.d. between the plates, V, is given by the differential equation:

$$CR\frac{dV}{dt}+V=E$$

   (a) Solve the equation for V given that when time $t=0$, $V=0$.

   (b) Evaluate voltage V when $E=50\,V$, $C=10\,\mu F$, $R=200\,k\Omega$ and $t=1.2\,s$. (14)

4. Show that the solution to the differential equation: $4x\dfrac{dy}{dx}=\dfrac{x^2+y^2}{y}$ is of the form $3y^2=\sqrt{x}\left(1-\sqrt{x^3}\right)$ given that $y=0$ when $x=1$. (12)

5. Show that the solution to the differential equation

$$x\cos x\frac{dy}{dx}+(x\sin x+\cos x)y=1$$

   is given by: $xy=\sin x+k\cos x$ where k is a constant. (11)

6. (a) Use Euler's method to obtain a numerical solution of the differential equation:

$$\frac{dy}{dx}=\frac{y}{x}+x^2-2$$

   given the initial conditions that $x=1$ when $y=3$, for the range $x=1.0\ (0.1)\ 1.5$

   (b) Apply the Euler-Cauchy method to the differential equation given in part (a) over the same range.

   (c) Apply the integrating factor method to solve the differential equation in part (a) analytically.

   (d) Determine the percentage error, correct to 3 significant figures, in each of the two numerical methods when $x=1.2$ (30)

7. Use the Runge-Kutta method to solve the differential equation: $\dfrac{dy}{dx}=\dfrac{y}{x}+x^2-2$ in the range 1.0(0.1)1.5, given the initial conditions that at $x=1$, $y=3$. Work to an accuracy of 6 decimal places. (24)

Chapter 50

Second order differential equations of the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$

50.1 Introduction

An equation of the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$, where a, b and c are constants, is called a **linear second order differential equation with constant coefficients**. When the right-hand side of the differential equation is zero, it is referred to as a **homogeneous differential equation**. When the right-hand side is not equal to zero (as in Chapter 51) it is referred to as a **non-homogeneous differential equation**.

There are numerous engineering examples of second order differential equations. Three examples are:

(i) $L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=0$, representing an equation for charge q in an electrical circuit containing resistance R, inductance L and capacitance C in series.

(ii) $m\frac{d^2s}{dt^2}+a\frac{ds}{dt}+ks=0$, defining a mechanical system, where s is the distance from a fixed point after t seconds, m is a mass, a the damping factor and k the spring stiffness.

(iii) $\frac{d^2y}{dx^2}+\frac{P}{EI}y=0$, representing an equation for the deflected profile y of a pin-ended uniform strut of length l subjected to a load P. E is Young's modulus and I is the second moment of area.

If D represents $\frac{d}{dx}$ and D^2 represents $\frac{d^2}{dx^2}$ then the above equation may be stated as $(aD^2+bD+c)y=0$. This equation is said to be in **'D-operator'** form.

If $y=Ae^{mx}$ then $\frac{dy}{dx}=Ame^{mx}$ and $\frac{d^2y}{dx^2}=Am^2e^{mx}$.

Substituting these values into $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$ gives:

$$a(Am^2e^{mx})+b(Ame^{mx})+c(Ae^{mx})=0$$

i.e. $$Ae^{mx}(am^2+bm+c)=0$$

Thus $y=Ae^{mx}$ is a solution of the given equation provided that $(am^2+bm+c)=0$. $am^2+bm+c=0$ is called the **auxiliary equation**, and since the equation is a quadratic, m may be obtained either by factorizing or by using the quadratic formula. Since, in the auxiliary equation, a, b and c are real values, then the equation may have either

(i) two different real roots (when $b^2>4ac$) or

(ii) two equal real roots (when $b^2=4ac$) or

(iii) two complex roots (when $b^2<4ac$).

50.2 Procedure to solve differential equations of the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$

(a) Rewrite the differential equation

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

as $(aD^2+bD+c)y=0$

(b) Substitute m for D and solve the auxiliary equation $am^2+bm+c=0$ for m.

(c) If the roots of the auxiliary equation are:

(i) **real and different**, say $m=\alpha$ and $m=\beta$, then the general solution is

$$\boldsymbol{y = Ae^{\alpha x} + Be^{\beta x}}$$

(ii) **real and equal, say** $m=\alpha$ twice, then the general solution is

$$\boldsymbol{y = (Ax + B)e^{\alpha x}}$$

(iii) **complex**, say $m=\alpha \pm j\beta$, then the general solution is

$$\boldsymbol{y = e^{\alpha x}\{A\cos\beta x + B\sin\beta x\}}$$

(d) Given boundary conditions, constants A and B, may be determined and the **particular solution** of the differential equation obtained.

The particular solutions obtained in the worked problems of Section 50.3 may each be verified by substituting expressions for y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ into the original equation.

50.3 Worked problems on differential equations of the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$

Problem 1. Determine the general solution of $2\frac{d^2y}{dx^2}+5\frac{dy}{dx}-3y=0$. Find also the particular solution given that when $x=0$, $y=4$ and $\frac{dy}{dx}=9$.

Using the above procedure:

(a) $2\frac{d^2y}{dx^2}+5\frac{dy}{dx}-3y=0$ in D-operator form is $(2D^2+5D-3)y=0$, where $D\equiv\frac{d}{dx}$

(b) Substituting m for D gives the auxiliary equation

$$2m^2+5m-3=0.$$

Factorising gives: $(2m-1)(m+3)=0$, from which, $m=\frac{1}{2}$ or $m=-3$.

(c) Since the roots are real and different the **general solution is** $\boldsymbol{y=Ae^{\frac{1}{2}x}+Be^{-3x}}$.

(d) When $x=0$, $y=4$,

hence $4=A+B$ (1)

Since $y=Ae^{\frac{1}{2}x}+Be^{-3x}$

then $\frac{dy}{dx}=\frac{1}{2}Ae^{\frac{1}{2}x}-3Be^{-3x}$

When $x=0$, $\frac{dy}{dx}=9$

thus $9=\frac{1}{2}A-3B$ (2)

Solving the simultaneous equations (1) and (2) gives $A=6$ and $B=-2$.

Hence the particular solution is

$$\boldsymbol{y=6e^{\frac{1}{2}x}-2e^{-3x}}$$

Problem 2. Find the general solution of $9\frac{d^2y}{dt^2}-24\frac{dy}{dt}+16y=0$ and also the particular solution given the boundary conditions that when $t=0$, $y=\frac{dy}{dt}=3$.

Using the procedure of Section 50.2:

(a) $9\frac{d^2y}{dt^2}-24\frac{dy}{dt}+16y=0$ in D-operator form is $(9D^2-24D+16)y=0$ where $D\equiv\frac{d}{dt}$

(b) Substituting m for D gives the auxiliary equation $9m^2-24m+16=0$.

Factorizing gives: $(3m-4)(3m-4)=0$, i.e. $m=\frac{4}{3}$ twice.

(c) Since the roots are real and equal, **the general solution is** $\boldsymbol{y=(At+B)e^{\frac{4}{3}t}}$.

(d) When $t=0$, $y=3$ hence $3=(0+B)e^0$, i.e. $B=3$.

Since $y=(At+B)e^{\frac{4}{3}t}$

then $\frac{dy}{dt}=(At+B)\left(\frac{4}{3}e^{\frac{4}{3}t}\right)+Ae^{\frac{4}{3}t}$, by the product rule.

When $t=0$, $\frac{dy}{dt}=3$

thus $3=(0+B)\frac{4}{3}e^0+Ae^0$

i.e. $3=\frac{4}{3}B+A$ from which, $A=-1$, since $B=3$.

Hence the particular solution is

$$y=(-t+3)e^{\frac{4}{3}t} \text{ or } y=(3-t)e^{\frac{4}{3}t}$$

Problem 3. Solve the differential equation $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+13y=0$, given that when $x=0$, $y=3$ and $\frac{dy}{dx}=7$.

Using the procedure of Section 50.2:

(a) $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+13y=0$ in D-operator form is $(D^2+6D+13)y=0$, where $D\equiv\frac{d}{dx}$

(b) Substituting m for D gives the auxiliary equation $m^2+6m+13=0$.

Using the quadratic formula:

$$m=\frac{-6\pm\sqrt{[(6)^2-4(1)(13)]}}{2(1)}$$

$$=\frac{-6\pm\sqrt{(-16)}}{2}$$

i.e. $m=\frac{-6\pm j4}{2}=-3\pm j2$

(c) Since the roots are complex, **the general solution is**

$$y=e^{-3x}(A\cos 2x+B\sin 2x)$$

(d) When $x=0$, $y=3$, hence $3=e^0(A\cos 0+B\sin 0)$, i.e. $A=3$.

Since $y=e^{-3x}(A\cos 2x+B\sin 2x)$

then $\frac{dy}{dx}=e^{-3x}(-2A\sin 2x+2B\cos 2x)$

$-3e^{-3x}(A\cos 2x+B\sin 2x)$,

by the product rule,

$=e^{-3x}[(2B-3A)\cos 2x$

$-(2A+3B)\sin 2x]$

When $x=0$, $\frac{dy}{dx}=7$,

hence $7=e^0[(2B-3A)\cos 0-(2A+3B)\sin 0]$

i.e. $7=2B-3A$, from which, $B=8$, since $A=3$.

Hence the particular solution is

$$y=e^{-3x}(3\cos 2x+8\sin 2x)$$

Since, from Chapter 17, page 165, $a\cos\omega t+b\sin\omega t=R\sin(\omega t+\alpha)$, where $R=\sqrt{(a^2+b^2)}$ and $\alpha=\tan^{-1}\frac{a}{b}$ then

$3\cos 2x+8\sin 2x$

$=\sqrt{(3^2+8^2)}\sin(2x+\tan^{-1}\frac{3}{8})$

$=\sqrt{73}\sin(2x+20.56°)$

$=\sqrt{73}\sin(2x+0.359)$

Thus the particular solution may also be expressed as

$$y=\sqrt{73}e^{-3x}\sin(2x+0.359)$$

Now try the following exercise

Exercise 187 Further problems on differential equations of the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$

In Problems 1 to 3, determine the general solution of the given differential equations.

1. $6\frac{d^2y}{dt^2}-\frac{dy}{dt}-2y=0$

$\left[y=Ae^{\frac{2}{3}t}+Be^{-\frac{1}{2}t}\right]$

2. $4\frac{d^2\theta}{dt^2}+4\frac{d\theta}{dt}+\theta=0$

$\left[\theta=(At+B)e^{-\frac{1}{2}t}\right]$

3. $\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}+5y=0$

$[y=e^{-x}(A\cos 2x+B\sin 2x)]$

In Problems 4 to 9, find the particular solution of the given differential equations for the stated boundary conditions.

4. $6\dfrac{d^2y}{dx^2}+5\dfrac{dy}{dx}-6y=0$; when $x=0$, $y=5$ and $\dfrac{dy}{dx}=-1$. $\left[y=3e^{\frac{2}{3}x}+2e^{-\frac{3}{2}x}\right]$

5. $4\dfrac{d^2y}{dt^2}-5\dfrac{dy}{dt}+y=0$; when $t=0$, $y=1$ and $\dfrac{dy}{dt}=-2$. $\left[y=4e^{\frac{1}{4}t}-3e^{t}\right]$

6. $(9D^2+30D+25)y=0$, where $D\equiv\dfrac{d}{dx}$; when $x=0$, $y=0$ and $\dfrac{dy}{dx}=2$. $\left[y=2xe^{-\frac{5}{3}x}\right]$

7. $\dfrac{d^2x}{dt^2}-6\dfrac{dx}{dt}+9x=0$; when $t=0$, $x=2$ and $\dfrac{dx}{dt}=0$. $[x=2(1-3t)e^{3t}]$

8. $\dfrac{d^2y}{dx^2}+6\dfrac{dy}{dx}+13y=0$; when $x=0$, $y=4$ and $\dfrac{dy}{dx}=0$. $[y=2e^{-3x}(2\cos 2x+3\sin 2x)]$

9. $(4D^2+20D+125)\theta=0$, where $D\equiv\dfrac{d}{dt}$; when $t=0$, $\theta=3$ and $\dfrac{d\theta}{dt}=2.5$. $[\theta=e^{-2.5t}(3\cos 5t+2\sin 5t)]$

50.4 Further worked problems on practical differential equations of the form $a\dfrac{d^2y}{dx^2}+b\dfrac{dy}{dx}+cy=0$

Problem 4. The equation of motion of a body oscillating on the end of a spring is

$$\frac{d^2x}{dt^2}+100x=0,$$

where x is the displacement in metres of the body from its equilibrium position after time t seconds. Determine x in terms of t given that at time $t=0$, $x=2m$ and $\dfrac{dx}{dt}=0$.

An equation of the form $\dfrac{d^2x}{dt^2}+m^2x=0$ is a differential equation representing simple harmonic motion (S.H.M.). Using the procedure of Section 50.2:

(a) $\dfrac{d^2x}{dt^2}+100x=0$ in D-operator form is $(D^2+100)x=0$.

(b) The auxiliary equation is $m^2+100=0$, i.e. $m^2=-100$ and $m=\sqrt{(-100)}$, i.e. $m=\pm j10$.

(c) Since the roots are complex, the general solution is $x=e^0(A\cos 10t+B\sin 10t)$,

i.e. $\boldsymbol{x=(A\cos 10t+B\sin 10t)}$ **metres**

(d) When $t=0$, $x=2$, thus $2=A$

$$\frac{dx}{dt}=-10A\sin 10t+10B\cos 10t$$

When $t=0$, $\dfrac{dx}{dt}=0$

thus $0=-10A\sin 0+10B\cos 0$, i.e. $B=0$

Hence the particular solution is

$$\boldsymbol{x=2\cos 10t}\ \textbf{metres}$$

Problem 5. Given the differential equation $\dfrac{d^2V}{dt^2}=\omega^2V$, where ω is a constant, show that its solution may be expressed as:

$$V=7\cosh\omega t+3\sinh\omega t$$

given the boundary conditions that when $t=0$, $V=7$ and $\dfrac{dV}{dt}=3\omega$.

Using the procedure of Section 50.2:

(a) $\dfrac{d^2V}{dt^2}=\omega^2V$, i.e. $\dfrac{d^2V}{dt^2}-\omega^2V=0$ in D-operator form is $(D^2-\omega^2)v=0$, where $D\equiv\dfrac{d}{dx}$.

(b) The auxiliary equation is $m^2-\omega^2=0$, from which, $m^2=\omega^2$ and $m=\pm\omega$.

(c) Since the roots are real and different, **the general solution is**

$$\boldsymbol{V=Ae^{\omega t}+Be^{-\omega t}}$$

(d) When $t=0$, $V=7$ hence $7=A+B$ (1)

$$\frac{dV}{dt}=A\omega e^{\omega t}-B\omega e^{-\omega t}$$

When $t=0$, $\frac{dV}{dt}=3\omega$,

thus $3\omega=A\omega-B\omega$,

i.e. $3=A-B$ (2)

From equations (1) and (2), $A=5$ and $B=2$

Hence **the particular solution is**

$$\boldsymbol{V=5e^{\omega t}+2e^{-\omega t}}$$

Since $\sinh\omega t=\frac{1}{2}(e^{\omega t}-e^{-\omega t})$

and $\cosh\omega t=\frac{1}{2}(e^{\omega t}+e^{-\omega t})$

then $\sinh\omega t+\cosh\omega t=e^{\omega t}$

and $\cosh\omega t-\sinh\omega t=e^{-\omega t}$ from Chapter 5.

Hence the particular solution may also be written as

$$V=5(\sinh\omega t+\cosh\omega t)+2(\cosh\omega t-\sinh\omega t)$$

i.e. $V=(5+2)\cosh\omega t+(5-2)\sinh\omega t$

i.e. $\boldsymbol{V=7\cosh\omega t+3\sinh\omega t}$

Problem 6. The equation

$$\frac{d^2i}{dt^2}+\frac{R}{L}\frac{di}{dt}+\frac{1}{LC}i=0$$

represents a current i flowing in an electrical circuit containing resistance R, inductance L and capacitance C connected in series. If $R=200$ ohms, $L=0.20$ henry and $C=20\times10^{-6}$ farads, solve the equation for i given the boundary conditions that when $t=0$, $i=0$ and $\frac{di}{dt}=100$.

Using the procedure of Section 50.2:

(a) $\frac{d^2i}{dt^2}+\frac{R}{L}\frac{di}{dt}+\frac{1}{LC}i=0$ in D-operator form is

$$\left(D^2+\frac{R}{L}D+\frac{1}{LC}\right)i=0 \text{ where } D\equiv\frac{d}{dt}$$

(b) The auxiliary equation is $m^2+\frac{R}{L}m+\frac{1}{LC}=0$

$$\text{Hence } m=\frac{-\frac{R}{L}\pm\sqrt{\left[\left(\frac{R}{L}\right)^2-4(1)\left(\frac{1}{LC}\right)\right]}}{2}$$

When $R=200$, $L=0.20$ and $C=20\times10^{-6}$, then

$$m=\frac{-\frac{200}{0.20}\pm\sqrt{\left[\left(\frac{200}{0.20}\right)^2-\frac{4}{(0.20)(20\times10^{-6})}\right]}}{2}$$

$$=\frac{-1000\pm\sqrt{0}}{2}=-500$$

(c) Since the two roots are real and equal (i.e. -500 twice, since for a second order differential equation there must be two solutions), **the general solution is $\boldsymbol{i=(At+B)e^{-500t}}$.**

(d) When $t=0$, $i=0$, hence $B=0$

$$\frac{di}{dt}=(At+B)(-500e^{-500t})+(e^{-500t})(A),$$

by the product rule

When $t=0$, $\frac{di}{dt}=100$, thus $100=-500B+A$

i.e. $A=100$, since $B=0$

Hence the particular solution is

$$\boldsymbol{i=100te^{-500t}}$$

Problem 7. The oscillations of a heavily damped pendulum satisfy the differential equation $\frac{d^2x}{dt^2}+6\frac{dx}{dt}+8x=0$, where x cm is the displacement of the bob at time t seconds. The initial displacement is equal to $+4$ cm and the initial velocity $\left(\text{i.e. } \frac{dx}{dt}\right)$ is 8 cm/s. Solve the equation for x.

Using the procedure of Section 50.2:

(a) $\frac{d^2x}{dt^2}+6\frac{dx}{dt}+8x=0$ in D-operator form is $(D^2+6D+8)x=0$, where $D\equiv\frac{d}{dt}$.

(b) The auxiliary equation is $m^2+6m+8=0$.

Factorising gives: $(m+2)(m+4)=0$, from which, $m=-2$ or $m=-4$.

(c) Since the roots are real and different, **the general solution is $x=Ae^{-2t}+Be^{-4t}$**.

(d) Initial displacement means that time $t=0$. At this instant, $x=4$.

Thus $4=A+B$ (1)

Velocity,

$$\frac{dx}{dt}=-2Ae^{-2t}-4Be^{-4t}$$

$\frac{dx}{dt}=8$ cm/s when $t=0$,

thus $8=-2A-4B$ (2)

From equations (1) and (2),

$A=12$ and $B=-8$

Hence the particular solution is

$$x=12e^{-2t}-8e^{-4t}$$

i.e. **displacement, $x=4(3e^{-2t}-2e^{-4t})$ cm**

Now try the following exercise

Exercise 188 Further problems on second order differential equations of the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$

1. The charge, q, on a capacitor in a certain electrical circuit satisfies the differential equation $\frac{d^2q}{dt^2}+4\frac{dq}{dt}+5q=0$. Initially (i.e. when $t=0$), $q=Q$ and $\frac{dq}{dt}=0$. Show that the charge in the circuit can be expressed as: $q=\sqrt{5}\,Qe^{-2t}\sin(t+0.464)$.

2. A body moves in a straight line so that its distance s metres from the origin after time t seconds is given by $\frac{d^2s}{dt^2}+a^2s=0$, where a is a constant. Solve the equation for s given that $s=c$ and $\frac{ds}{dt}=0$ when $t=\frac{2\pi}{a}$.
$[s=c\cos at]$

3. The motion of the pointer of a galvanometer about its position of equilibrium is represented by the equation
$$I\frac{d^2\theta}{dt^2}+K\frac{d\theta}{dt}+F\theta=0.$$
If I, the moment of inertia of the pointer about its pivot, is 5×10^{-3}, K, the resistance due to friction at unit angular velocity, is 2×10^{-2} and F, the force on the spring necessary to produce unit displacement, is 0.20, solve the equation for θ in terms of t given that when $t=0$, $\theta=0.3$ and $\frac{d\theta}{dt}=0$.
$[\theta=e^{-2t}(0.3\cos6t+0.1\sin6t)]$

4. Determine an expression for x for a differential equation $\frac{d^2x}{dt^2}+2n\frac{dx}{dt}+n^2x=0$ which represents a critically damped oscillator, given that at time $t=0$, $x=s$ and $\frac{dx}{dt}=u$.
$[x=\{s+(u+ns)t\}e^{-nt}]$

5. $L\frac{d^2i}{dt^2}+R\frac{di}{dt}+\frac{1}{C}i=0$ is an equation representing current i in an electric circuit. If inductance L is 0.25 henry, capacitance C is 29.76×10^{-6} farads and R is 250 ohms, solve the equation for i given the boundary conditions that when $t=0$, $i=0$ and $\frac{di}{dt}=34$.
$\left[i=\frac{1}{20}\left(e^{-160t}-e^{-840t}\right)\right]$

6. The displacement s of a body in a damped mechanical system, with no external forces, satisfies the following differential equation:
$$2\frac{d^2s}{dt^2}+6\frac{ds}{dt}+4.5s=0$$
where t represents time. If initially, when $t=0$, $s=0$ and $\frac{ds}{dt}=4$, solve the differential equation for s in terms of t. $[s=4te^{-\frac{3}{2}t}]$

Chapter 51

Second order differential equations of the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$

51.1 Complementary function and particular integral

If in the differential equation

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x) \qquad (1)$$

the substitution $y=u+v$ is made then:

$$a\frac{d^2(u+v)}{dx^2}+b\frac{d(u+v)}{dx}+c(u+v)=f(x)$$

Rearranging gives:

$$\left(a\frac{d^2u}{dx^2}+b\frac{du}{dx}+cu\right)+\left(a\frac{d^2v}{dx^2}+b\frac{dv}{dx}+cv\right)=f(x)$$

If we let

$$a\frac{d^2v}{dx^2}+b\frac{dv}{dx}+cv=f(x) \qquad (2)$$

then

$$a\frac{d^2u}{dx^2}+b\frac{du}{dx}+cu=0 \qquad (3)$$

The general solution, u, of equation (3) will contain two unknown constants, as required for the general solution of equation (1). The method of solution of equation (3) is shown in Chapter 50. The function u is called the **complementary function (C.F.)**.

If the particular solution, v, of equation (2) can be determined without containing any unknown constants then $y=u+v$ will give the general solution of equation (1). The function v is called the **particular integral (P.I.)**. Hence the general solution of equation (1) is given by:

$$\mathbf{y = C.F. + P.I.}$$

51.2 Procedure to solve differential equations of the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$

(i) Rewrite the given differential equation as $(aD^2+bD+c)y=f(x)$.

(ii) Substitute m for D, and solve the auxiliary equation $am^2+bm+c=0$ for m.

(iii) Obtain the complementary function, u, which is achieved using the same procedure as in Section 50.2(c), page 478.

Table 51.1 Form of particular integral for different functions

Type	*Straightforward cases* Try as particular integral:	*'Snag' cases* Try as particular integral:	*See problem*
(a) $f(x)=$ a constant	$v=k$	$v=kx$ (used when C.F. contains a constant)	1, 2
(b) $f(x)=$ polynomial (i.e. $f(x)=L+Mx+Nx^2+\cdots$ where any of the coefficients may be zero)	$v=a+bx+cx^2+\cdots$		3
(c) $f(x)=$ an exponential function (i.e. $f(x)=Ae^{ax}$)	$v=ke^{ax}$	(i) $v=kxe^{ax}$ (used when e^{ax} appears in the C.F.) (ii) $v=kx^2e^{ax}$ (used when e^{ax} **and** xe^{ax} both appear in the C.F.)	4, 5 6
(d) $f(x)=$ a sine or cosine function (i.e. $f(x)=a\sin px+b\cos px$, where a or b may be zero)	$v=A\sin px+B\cos px$	$v=x(A\sin px+B\cos px)$ (used when $\sin px$ and/or $\cos px$ appears in the C.F.)	7, 8
(e) $f(x)=$ a sum e.g. (i) $f(x)=4x^2-3\sin 2x$ (ii) $f(x)=2-x+e^{3x}$	 (i) $v=ax^2+bx+c+d\sin 2x+e\cos 2x$ (ii) $v=ax+b+ce^{3x}$		9
(f) $f(x)=$ a product e.g. $f(x)=2e^x\cos 2x$	$v=e^x(A\sin 2x+B\cos 2x)$		10

(iv) To determine the particular integral, v, firstly assume a particular integral which is suggested by $f(x)$, but which contains undetermined coefficients. Table 51.1 gives some suggested substitutions for different functions $f(x)$.

(v) Substitute the suggested P.I. into the differential equation $(aD^2+bD+c)v=f(x)$ and equate relevant coefficients to find the constants introduced.

(vi) The general solution is given by $y=$C.F.$+$P.I., i.e. $y=u+v$.

(vii) Given boundary conditions, arbitrary constants in the C.F. may be determined and the particular solution of the differential equation obtained.

51.3 Worked problems on differential equations of the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$ where $f(x)$ is a constant or polynomial

Problem 1. Solve the differential equation $\frac{d^2y}{dx^2}+\frac{dy}{dx}-2y=4$.

Using the procedure of Section 51.2:

(i) $\frac{d^2y}{dx^2}+\frac{dy}{dx}-2y=4$ in D-operator form is $(D^2+D-2)y=4$.

(ii) Substituting m for D gives the auxiliary equation $m^2+m-2=0$. Factorizing gives: $(m-1)(m+2)=0$, from which $m=1$ or $m=-2$.

(iii) Since the roots are real and different, the C.F., $\boldsymbol{u=Ae^{x}+Be^{-2x}}$.

(iv) Since the term on the right hand side of the given equation is a constant, i.e. $f(x)=4$, let the P.I. also be a constant, say $v=k$ (see Table 51.1(a)).

(v) Substituting $v=k$ into $(D^2+D-2)v=4$ gives $(D^2+D-2)k=4$. Since $D(k)=0$ and $D^2(k)=0$ then $-2k=4$, from which, $k=-2$. Hence the P.I., $\boldsymbol{v=-2}$.

(vi) The general solution is given by $y=u+v$, i.e. $\boldsymbol{y=Ae^{x}+Be^{-2x}-2}$.

Problem 2. Determine the particular solution of the equation $\frac{d^2y}{dx^2}-3\frac{dy}{dx}=9$, given the boundary conditions that when $x=0$, $y=0$ and $\frac{dy}{dx}=0$.

Using the procedure of Section 51.2:

(i) $\frac{d^2y}{dx^2}-3\frac{dy}{dx}=9$ in D-operator form is $(D^2-3D)y=9$.

(ii) Substituting m for D gives the auxiliary equation $m^2-3m=0$. Factorizing gives: $m(m-3)=0$, from which, $m=0$ or $m=3$.

(iii) Since the roots are real and different, the C.F., $u=Ae^{0}+Be^{3x}$, i.e. $\boldsymbol{u=A+Be^{3x}}$.

(iv) Since the C.F. contains a constant (i.e. A) then let the P.I., $v=kx$ (see Table 51.1(a)).

(v) Substituting $v=kx$ into $(D^2-3D)v=9$ gives $(D^2-3D)kx=9$.
$D(kx)=k$ and $D^2(kx)=0$.
Hence $(D^2-3D)kx=0-3k=9$, from which, $k=-3$.
Hence the P.I., $\boldsymbol{v=-3x}$.

(vi) The general solution is given by $y=u+v$, i.e. $\boldsymbol{y=A+Be^{3x}-3x}$.

(vii) When $x=0$, $y=0$, thus $0=A+Be^{0}-0$, i.e.
$0=A+B$ (1)
$\frac{dy}{dx}=3Be^{3x}-3$; $\frac{dy}{dx}=0$ when $x=0$, thus $0=3Be^{0}-3$ from which, $B=1$. From equation (1), $A=-1$.

Hence the particular solution is

$$y=-1+1e^{3x}-3x,$$

i.e. $\boldsymbol{y=e^{3x}-3x-1}$

Problem 3. Solve the differential equation $2\frac{d^2y}{dx^2}-11\frac{dy}{dx}+12y=3x-2$.

Using the procedure of Section 51.2:

(i) $2\frac{d^2y}{dx^2}-11\frac{dy}{dx}+12y=3x-2$ in D-operator form is

$$(2D^2-11D+12)y=3x-2.$$

(ii) Substituting m for D gives the auxiliary equation $2m^2-11m+12=0$. Factorizing gives: $(2m-3)(m-4)=0$, from which, $m=\frac{3}{2}$ or $m=4$.

(iii) Since the roots are real and different, the C.F., $\boldsymbol{u=Ae^{\frac{3}{2}x}+Be^{4x}}$

(iv) Since $f(x)=3x-2$ is a polynomial, let the P.I., $v=ax+b$ (see Table 51.1(b)).

(v) Substituting $v=ax+b$ into $(2D^2-11D+12)v=3x-2$ gives:

$$(2D^2-11D+12)(ax+b)=3x-2,$$

i.e. $2D^2(ax+b)-11D(ax+b)+12(ax+b)=3x-2$

i.e. $0-11a+12ax+12b=3x-2$

Equating the coefficients of x gives: $12a=3$, from which, $a=\frac{1}{4}$.

Equating the constant terms gives:

$$-11a+12b=-2.$$

i.e. $-11\left(\frac{1}{4}\right)+12b=-2$ from which,

$12b=-2+\frac{11}{4}=\frac{3}{4}$ i.e. $b=\frac{1}{16}$

Hence the P.I., $\boldsymbol{v=ax+b=\frac{1}{4}x+\frac{1}{16}}$

(vi) The general solution is given by $y=u+v$, i.e.

$$\boldsymbol{y=Ae^{\frac{3}{2}x}+Be^{4x}+\frac{1}{4}x+\frac{1}{16}}$$

Now try the following exercise

Exercise 189 Further problems on differential equations of the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$ where $f(x)$ is a constant or polynomial.

In Problems 1 and 2, find the general solutions of the given differential equations.

1. $2\frac{d^2y}{dx^2}+5\frac{dy}{dx}-3y=6$

$$\left[y=Ae^{\frac{1}{2}x}+Be^{-3x}-2\right]$$

2. $6\frac{d^2y}{dx^2}+4\frac{dy}{dx}-2y=3x-2$

$$\left[y=Ae^{\frac{1}{3}x}+Be^{-x}-2-\tfrac{3}{2}x\right]$$

In Problems 3 and 4 find the particular solutions of the given differential equations.

3. $3\frac{d^2y}{dx^2}+\frac{dy}{dx}-4y=8$; when $x=0$, $y=0$ and $\frac{dy}{dx}=0$.

$$\left[y=\tfrac{2}{7}(3e^{-\frac{4}{3}x}+4e^{x})-2\right]$$

4. $9\frac{d^2y}{dx^2}-12\frac{dy}{dx}+4y=3x-1$; when $x=0$, $y=0$ and $\frac{dy}{dx}=-\frac{4}{3}$

$$\left[y=-(2+\tfrac{3}{4}x)e^{\frac{2}{3}x}+2+\tfrac{3}{4}x\right]$$

5. The charge q in an electric circuit at time t satisfies the equation $L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=E$, where L, R, C and E are constants. Solve the equation given $L=2H$, $C=200\times10^{-6}$ F and $E=250$ V, when (a) $R=200\,\Omega$ and (b) R is negligible. Assume that when $t=0$, $q=0$ and $\frac{dq}{dt}=0$

$$\left[\begin{array}{ll}\text{(a)} & q=\frac{1}{20}-\left(\frac{5}{2}t+\frac{1}{20}\right)e^{-50t}\\ \text{(b)} & q=\frac{1}{20}(1-\cos 50t)\end{array}\right]$$

6. In a galvanometer the deflection θ satisfies the differential equation $\frac{d^2\theta}{dt^2}+4\frac{d\theta}{dt}+4\theta=8$. Solve the equation for θ given that when $t=0$, $\theta=\frac{d\theta}{dt}=2$. $[\theta=2(te^{-2t}+1)]$

51.4 Worked problems on differential equations of the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$ where $f(x)$ is an exponential function

Problem 4. Solve the equation $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=3e^{4x}$ given the boundary conditions that when $x=0$, $y=-\frac{2}{3}$ and $\frac{dy}{dx}=4\frac{1}{3}$

Using the procedure of Section 51.2:

(i) $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=3e^{4x}$ in D-operator form is $(D^2-2D+1)y=3e^{4x}$.

(ii) Substituting m for D gives the auxiliary equation $m^2-2m+1=0$. Factorizing gives: $(m-1)(m-1)=0$, from which, $m=1$ twice.

(iii) Since the roots are real and equal the C.F., $\boldsymbol{u=(Ax+B)e^{x}}$.

(iv) Let the particular integral, $v=ke^{4x}$ (see Table 51.1(c)).

(v) Substituting $v=ke^{4x}$ into $(D^2-2D+1)v=3e^{4x}$ gives:

$$(D^2-2D+1)ke^{4x}=3e^{4x}$$

i.e. $D^2(ke^{4x})-2D(ke^{4x})+1(ke^{4x})=3e^{4x}$

i.e. $16ke^{4x}-8ke^{4x}+ke^{4x}=3e^{4x}$

Hence $9ke^{4x}=3e^{4x}$, from which, $k=\frac{1}{3}$

Hence the P.I., $\boldsymbol{v=ke^{4x}=\frac{1}{3}e^{4x}}$.

(vi) The general solution is given by $y=u+v$, i.e. $\boldsymbol{y=(Ax+B)e^{x}+\frac{1}{3}e^{4x}}$.

(vii) When $x=0$, $y=-\frac{2}{3}$ thus

$-\frac{2}{3}=(0+B)e^0+\frac{1}{3}e^0$, from which, $B=-1$.

$\frac{dy}{dx}=(Ax+B)e^x+e^x(A)+\frac{4}{3}e^{4x}$.

When $x=0$, $\frac{dy}{dx}=4\frac{1}{3}$, thus $\frac{13}{3}=B+A+\frac{4}{3}$

from which, $A=4$, since $B=-1$.

Hence the particular solution is:

$$y=(4x-1)e^x+\tfrac{1}{3}e^{4x}$$

Problem 5. Solve the differential equation $2\frac{d^2y}{dx^2}-\frac{dy}{dx}-3y=5e^{\frac{3}{2}x}$.

Using the procedure of Section 51.2:

(i) $2\frac{d^2y}{dx^2}-\frac{dy}{dx}-3y=5e^{\frac{3}{2}x}$ in D-operator form is $(2D^2-D-3)y=5e^{\frac{3}{2}x}$.

(ii) Substituting m for D gives the auxiliary equation $2m^2-m-3=0$. Factorizing gives: $(2m-3)(m+1)=0$, from which, $m=\frac{3}{2}$ or $m=-1$. Since the roots are real and different then the C.F., $\boldsymbol{u=Ae^{\frac{3}{2}x}+Be^{-x}}$.

(iii) Since $e^{\frac{3}{2}x}$ appears in the C.F. **and** in the right hand side of the differential equation, let the P.I., $v=kxe^{\frac{3}{2}x}$ (see Table 51.1(c), snag case (i)).

(iv) Substituting $v=kxe^{\frac{3}{2}x}$ into $(2D^2-D-3)v=5e^{\frac{3}{2}x}$ gives: $(2D^2-D-3)kxe^{\frac{3}{2}x}=5e^{\frac{3}{2}x}$.

$$D\left(kxe^{\frac{3}{2}x}\right)=(kx)\left(\tfrac{3}{2}e^{\frac{3}{2}x}\right)+\left(e^{\frac{3}{2}x}\right)(k),$$

by the product rule,

$$=ke^{\frac{3}{2}x}\left(\tfrac{3}{2}x+1\right)$$

$$D^2\left(kxe^{\frac{3}{2}x}\right)=D\left[ke^{\frac{3}{2}x}\left(\tfrac{3}{2}x+1\right)\right]$$

$$=\left(ke^{\frac{3}{2}x}\right)\left(\tfrac{3}{2}\right)+\left(\tfrac{3}{2}x+1\right)\left(\tfrac{3}{2}ke^{\frac{3}{2}x}\right)$$

$$=ke^{\frac{3}{2}x}\left(\tfrac{9}{4}x+3\right)$$

Hence $(2D^2-D-3)\left(kxe^{\frac{3}{2}x}\right)$

$$=2\left[ke^{\frac{3}{2}x}\left(\tfrac{9}{4}x+3\right)\right]-\left[ke^{\frac{3}{2}x}\left(\tfrac{3}{2}x+1\right)\right]-3\left[kxe^{\frac{3}{2}x}\right]=5e^{\frac{3}{2}x}$$

i.e. $\frac{9}{2}kxe^{\frac{3}{2}x}+6ke^{\frac{3}{2}x}-\frac{3}{2}xke^{\frac{3}{2}x}-ke^{\frac{3}{2}x}-3kxe^{\frac{3}{2}x}=5e^{\frac{3}{2}x}$

Equating coefficients of $e^{\frac{3}{2}x}$ gives: $5k=5$, from which, $k=1$.

Hence the P.I., $\boldsymbol{v=kxe^{\frac{3}{2}x}=xe^{\frac{3}{2}x}}$.

(v) The general solution is $y=u+v$, i.e. $\boldsymbol{y=Ae^{\frac{3}{2}x}+Be^{-x}+xe^{\frac{3}{2}x}}$.

Problem 6. Solve $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=3e^{2x}$.

Using the procedure of Section 51.2:

(i) $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=3e^{2x}$ in D-operator form is $(D^2-4D+4)y=3e^{2x}$.

(ii) Substituting m for D gives the auxiliary equation $m^2-4m+4=0$. Factorizing gives: $(m-2)(m-2)=0$, from which, $m=2$ twice.

(iii) Since the roots are real and equal, the C.F., $\boldsymbol{u=(Ax+B)e^{2x}}$.

(iv) Since e^{2x} **and** xe^{2x} both appear in the C.F. let the P.I., $v=kx^2e^{2x}$ (see Table 51.1(c), snag case (ii)).

(v) Substituting $v=kx^2e^{2x}$ into $(D^2-4D+4)v=3e^{2x}$ gives: $(D^2-4D+4)(kx^2e^{2x})=3e^{2x}$

$$D(kx^2e^{2x})=(kx^2)(2e^{2x})+(e^{2x})(2kx)$$

$$=2ke^{2x}(x^2+x)$$

$$D^2(kx^2e^{2x})=D[2ke^{2x}(x^2+x)]$$

$$=(2ke^{2x})(2x+1)+(x^2+x)(4ke^{2x})$$

$$=2ke^{2x}(4x+1+2x^2)$$

Hence $(D^2-4D+4)(kx^2e^{2x})$

$$=[2ke^{2x}(4x+1+2x^2)]-4[2ke^{2x}(x^2+x)]+4[kx^2e^{2x}]$$

$$=3e^{2x}$$

from which, $2ke^{2x} = 3e^{2x}$ and $k = \frac{3}{2}$

Hence the P.I., $\boldsymbol{v = kx^2e^{2x} = \frac{3}{2}x^2e^{2x}}$.

(vi) The general solution, $y = u + v$, i.e.

$$\boldsymbol{y = (Ax + B)e^{2x} + \tfrac{3}{2}x^2e^{2x}}$$

Now try the following exercise

Exercise 190 Further problems on differential equations of the form $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$ where $f(x)$ is an exponential function

In Problems 1 to 4, find the general solutions of the given differential equations.

1. $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 2e^x$

$$\left[y = Ae^{3x} + Be^{-2x} - \tfrac{1}{3}e^x\right]$$

2. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 3e^{-x}$

$$\left[y = Ae^{4x} + Be^{-x} - \tfrac{3}{5}xe^{-x}\right]$$

3. $\frac{d^2y}{dx^2} + 9y = 26e^{2x}$

$$[y = A\cos 3x + B\sin 3x + 2e^{2x}]$$

4. $9\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + y = 12e^{\frac{t}{3}}$

$$\left[y = (At + B)e^{\frac{1}{3}t} + \tfrac{2}{3}t^2e^{\frac{1}{3}t}\right]$$

In problems 5 and 6 find the particular solutions of the given differential equations.

5. $5\frac{d^2y}{dx^2} + 9\frac{dy}{dx} - 2y = 3e^x$; when $x = 0$, $y = \frac{1}{4}$ and $\frac{dy}{dx} = 0$.

$$\left[y = \frac{5}{44}\left(e^{-2x} - e^{\frac{1}{5}x}\right) + \frac{1}{4}e^x\right]$$

6. $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 9y = 4e^{3t}$; when $t = 0$, $y = 2$ and $\frac{dy}{dt} = 0$ $\qquad [y = 2e^{3t}(1 - 3t + t^2)]$

51.5 Worked problems on differential equations of the form $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$ where $f(x)$ is a sine or cosine function

Problem 7. Solve the differential equation $2\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 5y = 6\sin 2x$.

Using the procedure of Section 51.2:

(i) $2\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 5y = 6\sin 2x$ in D-operator form is $(2D^2 + 3D - 5)y = 6\sin 2x$

(ii) The auxiliary equation is $2m^2 + 3m - 5 = 0$, from which,

$$(m - 1)(2m + 5) = 0,$$

$$\text{i.e.} \quad m = 1 \text{ or } m = -\tfrac{5}{2}$$

(iii) Since the roots are real and different the C.F., $\boldsymbol{u = Ae^x + Be^{-\frac{5}{2}x}}$.

(iv) Let the P.I., $v = A\sin 2x + B\cos 2x$ (see Table 51.1(d)).

(v) Substituting $v = A\sin 2x + B\cos 2x$ into $(2D^2 + 3D - 5)v = 6\sin 2x$ gives:
$(2D^2 + 3D - 5)(A\sin 2x + B\cos 2x) = 6\sin 2x$.

$$D(A\sin 2x + B\cos 2x)$$
$$= 2A\cos 2x - 2B\sin 2x$$
$$D^2(A\sin 2x + B\cos 2x)$$
$$= D(2A\cos 2x - 2B\sin 2x)$$
$$= -4A\sin 2x - 4B\cos 2x$$

Hence $(2D^2 + 3D - 5)(A\sin 2x + B\cos 2x)$
$$= -8A\sin 2x - 8B\cos 2x + 6A\cos 2x$$
$$- 6B\sin 2x - 5A\sin 2x - 5B\cos 2x$$
$$= 6\sin 2x$$

Equating coefficient of $\sin 2x$ gives:

$$-13A - 6B = 6 \qquad (1)$$

Equating coefficients of $\cos 2x$ gives:

$$6A-13B=0 \quad (2)$$

$6\times(1)$ gives : $-78A-36B=36$ (3)

$13\times(2)$ gives : $78A-169B=0$ (4)

$(3)+(4)$ gives : $-205B=36$

from which, $B=\dfrac{-36}{205}$

Substituting $B=\dfrac{-36}{205}$ into equation (1) or (2) gives $A=\dfrac{-78}{205}$

Hence the P.I., $\boldsymbol{v=\frac{-78}{205}\sin 2x-\frac{36}{205}\cos 2x}$.

(vi) The general solution, $y=u+v$, i.e.

$$\boldsymbol{y=Ae^{x}+Be^{-\frac{5}{2}x}-\frac{2}{205}(39\sin 2x+18\cos 2x)}$$

Problem 8. Solve $\dfrac{d^2y}{dx^2}+16y=10\cos 4x$ given $y=3$ and $\dfrac{dy}{dx}=4$ when $x=0$.

Using the procedure of Section 51.2:

(i) $\dfrac{d^2y}{dx^2}+16y=10\cos 4x$ in D-operator form is

$$(D^2+16)y=10\cos 4x$$

(ii) The auxiliary equation is $m^2+16=0$, from which $m=\sqrt{-16}=\pm j4$.

(iii) Since the roots are complex the C.F., $u=e^0(A\cos 4x+B\sin 4x)$

i.e. $\boldsymbol{u=A\cos 4x+B\sin 4x}$

(iv) Since $\sin 4x$ occurs in the C.F. **and** in the right hand side of the given differential equation, let the P.I., $v=x(C\sin 4x+D\cos 4x)$ (see Table 51.1(d), snag case—constants C and D are used since A and B have already been used in the C.F.).

(v) Substituting $v=x(C\sin 4x+D\cos 4x)$ into $(D^2+16)v=10\cos 4x$ gives:

$$(D^2+16)[x(C\sin 4x+D\cos 4x)]=10\cos 4x$$

$$D[x(C\sin 4x+D\cos 4x)]=x(4C\cos 4x-4D\sin 4x)+(C\sin 4x+D\cos 4x)(1),$$

by the product rule

$$D^2[x(C\sin 4x+D\cos 4x)]=x(-16C\sin 4x-16D\cos 4x)+(4C\cos 4x-4D\sin 4x)+(4C\cos 4x-4D\sin 4x)$$

Hence $(D^2+16)[x(C\sin 4x+D\cos 4x)]$

$$=-16Cx\sin 4x-16Dx\cos 4x+4C\cos 4x-4D\sin 4x+4C\cos 4x-4D\sin 4x+16Cx\sin 4x+16Dx\cos 4x$$

$$=10\cos 4x,$$

i.e. $-8D\sin 4x+8C\cos 4x=10\cos 4x$

Equating coefficients of $\cos 4x$ gives:
$8C=10$, from which, $C=\dfrac{10}{8}=\dfrac{5}{4}$

Equating coefficients of $\sin 4x$ gives:
$-8D=0$, from which, $D=0$.

Hence the P.I., $\boldsymbol{v=x\left(\frac{5}{4}\sin 4x\right)}$.

(vi) The general solution, $y=u+v$, i.e.

$$\boldsymbol{y=A\cos 4x+B\sin 4x+\tfrac{5}{4}x\sin 4x}$$

(vii) When $x=0$, $y=3$, thus
$3=A\cos 0+B\sin 0+0$, i.e. $A=3$.

$$\frac{dy}{dx}=-4A\sin 4x+4B\cos 4x+\tfrac{5}{4}x(4\cos 4x)+\tfrac{5}{4}\sin 4x$$

When $x=0$, $\dfrac{dy}{dx}=4$, thus

$$4=-4A\sin 0+4B\cos 0+0+\tfrac{5}{4}\sin 0$$

i.e. $4=4B$, from which, $B=1$

Hence the particular solution is

$$\boldsymbol{y=3\cos 4x+\sin 4x+\tfrac{5}{4}x\sin 4x}$$

Now try the following exercise

Exercise 191 Further problems on differential equations of the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$ where $f(x)$ is a sine or cosine function

In Problems 1 to 3, find the general solutions of the given differential equations.

1. $2\frac{d^2y}{dx^2}-\frac{dy}{dx}-3y=25\sin 2x$

$$\left[y=Ae^{\frac{3}{2}x}+Be^{-x}-\tfrac{1}{5}(11\sin 2x-2\cos 2x)\right]$$

2. $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=5\cos x$

$$\left[y=(Ax+B)e^{2x}-\tfrac{4}{5}\sin x+\tfrac{3}{5}\cos x\right]$$

3. $\frac{d^2y}{dx^2}+y=4\cos x$

$$[y=A\cos x+B\sin x+2x\sin x]$$

4. Find the particular solution of the differential equation $\frac{d^2y}{dx^2}-3\frac{dy}{dx}-4y=3\sin x$; when $x=0$, $y=0$ and $\frac{dy}{dx}=0$.

$$\left[y=\frac{1}{170}(6e^{4x}-51e^{-x})-\frac{1}{34}(15\sin x-9\cos x)\right]$$

5. A differential equation representing the motion of a body is $\frac{d^2y}{dt^2}+n^2y=k\sin pt$, where k, n and p are constants. Solve the equation (given $n\neq 0$ and $p^2\neq n^2$) given that when $t=0$, $y=\frac{dy}{dt}=0$.

$$\left[y=\frac{k}{n^2-p^2}\left(\sin pt-\frac{p}{n}\sin nt\right)\right]$$

6. The motion of a vibrating mass is given by $\frac{d^2y}{dt^2}+8\frac{dy}{dt}+20y=300\sin 4t$. Show that the general solution of the differential equation is given by:

$$y=e^{-4t}(A\cos 2t+B\sin 2t)+\frac{15}{13}(\sin 4t-8\cos 4t)$$

7. $L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=V_0\sin\omega t$ represents the variation of capacitor charge in an electric circuit. Determine an expression for q at time t seconds given that $R=40\,\Omega$, $L=0.02\,\text{H}$, $C=50\times10^{-6}\,\text{F}$, $V_0=540.8\,\text{V}$ and $\omega=200\,\text{rad/s}$ and given the boundary conditions that when $t=0$, $q=0$ and $\frac{dq}{dt}=4.8$

$$\left[q=(10t+0.01)e^{-1000t}+0.024\sin 200t-0.010\cos 200t\right]$$

51.6 Worked problems on differential equations of the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$ where $f(x)$ is a sum or a product

Problem 9. Solve $\frac{d^2y}{dx^2}+\frac{dy}{dx}-6y=12x-50\sin x$.

Using the procedure of Section 51.2:

(i) $\frac{d^2y}{dx^2}+\frac{dy}{dx}-6y=12x-50\sin x$ in D-operator form is

$$(D^2+D-6)y=12x-50\sin x$$

(ii) The auxiliary equation is $(m^2+m-6)=0$, from which,

$$(m-2)(m+3)=0,$$

i.e. $m=2$ or $m=-3$

(iii) Since the roots are real and different, the C.F., $\boldsymbol{u=Ae^{2x}+Be^{-3x}}$.

(iv) Since the right hand side of the given differential equation is the sum of a polynomial and a sine function let the P.I. $v=ax+b+c\sin x+d\cos x$ (see Table 51.1(e)).

(v) Substituting v into

$(D^2+D-6)v=12x-50\sin x$ gives:

$$(D^2+D-6)(ax+b+c\sin x+d\cos x)$$
$$=12x-50\sin x$$
$$D(ax+b+c\sin x+d\cos x)$$
$$=a+c\cos x-d\sin x$$
$$D^2(ax+b+c\sin x+d\cos x)$$
$$=-c\sin x-d\cos x$$

Hence $(D^2+D-6)(v)$

$$=(-c\sin x-d\cos x)+(a+c\cos x$$
$$-d\sin x)-6(ax+b+c\sin x+d\cos x)$$
$$=12x-50\sin x$$

Equating constant terms gives:

$$a-6b=0 \qquad (1)$$

Equating coefficients of x gives: $-6a=12$, from which, $a=-2$.

Hence, from (1), $b=-\frac{1}{3}$

Equating the coefficients of $\cos x$ gives:

$$-d+c-6d=0$$
i.e. $$c-7d=0 \qquad (2)$$

Equating the coefficients of $\sin x$ gives:

$$-c-d-6c=-50$$
i.e. $$-7c-d=-50 \qquad (3)$$

Solving equations (2) and (3) gives: $c=7$ and $d=1$.

Hence the P.I.,

$$\boldsymbol{v=-2x-\tfrac{1}{3}+7\sin x+\cos x}$$

(vi) The general solution, $y=u+v$,

i.e. $$\boldsymbol{y=Ae^{2x}+Be^{-3x}-2x}$$
$$\boldsymbol{-\tfrac{1}{3}+7\sin x+\cos x}$$

Problem 10. Solve the differential equation $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y=3e^x\cos 2x$, given that when $x=0$, $y=2$ and $\frac{dy}{dx}=3$.

Using the procedure of Section 51.2:

(i) $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y=3e^x\cos 2x$ in D-operator form is

$$(D^2-2D+2)y=3e^x\cos 2x$$

(ii) The auxiliary equation is $m^2-2m+2=0$

Using the quadratic formula,

$$m=\frac{2\pm\sqrt{[4-4(1)(2)]}}{2}$$
$$=\frac{2\pm\sqrt{-4}}{2}=\frac{2\pm j2}{2} \quad \text{i.e. } m=1\pm j1.$$

(iii) Since the roots are complex, the C.F., $\boldsymbol{u=e^x(A\cos x+B\sin x)}$.

(iv) Since the right hand side of the given differential equation is a product of an exponential and a cosine function, let the P.I., $v=e^x(C\sin 2x+D\cos 2x)$ (see Table 51.1(f) — again, constants C and D are used since A and B have already been used for the C.F.).

(v) Substituting v into $(D^2-2D+2)v=3e^x\cos 2x$ gives:

$$(D^2-2D+2)[e^x(C\sin 2x+D\cos 2x)]$$
$$=3e^x\cos 2x$$
$$D(v)=e^x(2C\cos 2x-2D\sin 2x)$$
$$+e^x(C\sin 2x+D\cos 2x)$$
$$(\equiv e^x\{(2C+D)\cos 2x$$
$$+(C-2D)\sin 2x\})$$
$$D^2(v)=e^x(-4C\sin 2x-4D\cos 2x)$$
$$+e^x(2C\cos 2x-2D\sin 2x)$$
$$+e^x(2C\cos 2x-2D\sin 2x)$$
$$+e^x(C\sin 2x+D\cos 2x)$$
$$\equiv e^x\{(-3C-4D)\sin 2x$$
$$+(4C-3D)\cos 2x\}$$

Hence $(D^2-2D+2)v$

$$=e^x\{(-3C-4D)\sin 2x$$
$$+(4C-3D)\cos 2x\}$$
$$-2e^x\{(2C+D)\cos 2x$$
$$+(C-2D)\sin 2x\}$$
$$+2e^x(C\sin 2x+D\cos 2x)$$
$$=3e^x\cos 2x$$

Equating coefficients of $e^x \sin 2x$ gives:

$$-3C - 4D - 2C + 4D + 2C = 0$$

i.e. $-3C = 0$, from which, $C = 0$.

Equating coefficients of $e^x \cos 2x$ gives:

$$4C - 3D - 4C - 2D + 2D = 3$$

i.e. $-3D = 3$, from which, $D = -1$.

Hence the P.I., $\boldsymbol{v = e^x(-\cos 2x)}$.

(vi) The general solution, $y = u + v$, i.e.

$$\boldsymbol{y = e^x(A\cos x + B\sin x) - e^x\cos 2x}$$

(vii) When $x = 0$, $y = 2$ thus

$$2 = e^0(A\cos 0 + B\sin 0) - e^0\cos 0$$

i.e. $2 = A - 1$, from which, $A = 3$

$$\frac{dy}{dx} = e^x(-A\sin x + B\cos x) + e^x(A\cos x + B\sin x) - [e^x(-2\sin 2x) + e^x\cos 2x]$$

When $x = 0$, $\dfrac{dy}{dx} = 3$

thus $3 = e^0(-A\sin 0 + B\cos 0) + e^0(A\cos 0 + B\sin 0) - e^0(-2\sin 0) - e^0\cos 0$

i.e. $3 = B + A - 1$, from which,

$B = 1$, since $A = 3$

Hence the particular solution is

$$\boldsymbol{y = e^x(3\cos x + \sin x) - e^x\cos 2x}$$

Now try the following exercise

Exercise 192 Further problems on second order differential equations of the form $a\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = f(x)$ where $f(x)$ is a sum or product

In Problems 1 to 4, find the general solutions of the given differential equations.

1. $8\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + y = 2x + 40\sin x$

$$\left[y = Ae^{\frac{x}{4}} + Be^{\frac{x}{2}} + 2x + 12 + \frac{8}{17}(6\cos x - 7\sin x)\right]$$

2. $\dfrac{d^2y}{d\theta^2} - 3\dfrac{dy}{d\theta} + 2y = 2\sin 2\theta - 4\cos 2\theta$

$$\left[y = Ae^{2\theta} + Be^{\theta} + \tfrac{1}{2}(\sin 2\theta + \cos 2\theta)\right]$$

3. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2y = x^2 + e^{2x}$

$$\left[y = Ae^{x} + Be^{-2x} - \tfrac{3}{4} - \tfrac{1}{2}x - \tfrac{1}{2}x^2 + \tfrac{1}{4}e^{2x}\right]$$

4. $\dfrac{d^2y}{dt^2} - 2\dfrac{dy}{dt} + 2y = e^t\sin t$

$$\left[y = e^t(A\cos t + B\sin t) - \tfrac{t}{2}e^t\cos t\right]$$

In Problems 5 to 6 find the particular solutions of the given differential equations.

5. $\dfrac{d^2y}{dx^2} - 7\dfrac{dy}{dx} + 10y = e^{2x} + 20$; when $x = 0$, $y = 0$ and $\dfrac{dy}{dx} = -\dfrac{1}{3}$

$$\left[y = \frac{4}{3}e^{5x} - \frac{10}{3}e^{2x} - \frac{1}{3}xe^{2x} + 2\right]$$

6. $2\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 6y = 6e^x\cos x$; when $x = 0$, $y = -\dfrac{21}{29}$ and $\dfrac{dy}{dx} = -6\dfrac{20}{29}$

$$\left[y = 2e^{-\frac{3}{2}x} - 2e^{2x} + \frac{3e^x}{29}(3\sin x - 7\cos x)\right]$$

Chapter 52

Power series methods of solving ordinary differential equations

52.1 Introduction

Second order ordinary differential equations that cannot be solved by analytical methods (as shown in Chapters 50 and 51), i.e. those involving variable coefficients, can often be solved in the form of an infinite series of powers of the variable. This chapter looks at some of the methods that make this possible—by the Leibniz–Maclaurin and Frobinius methods, involving Bessel's and Legendre's equations, Bessel and gamma functions and Legendre's polynomials. Before introducing Leibniz's theorem, some trends with higher differential coefficients are considered. To better understand this chapter it is necessary to be able to:

(i) differentiate standard functions (as explained in Chapters 27 and 32),

(ii) appreciate the binomial theorem (as explained in Chapters 7), and

(iii) use Maclaurins theorem (as explained in Chapter 8).

52.2 Higher order differential coefficients as series

The following is an extension of successive differentiation (see page 296), but looking for trends, or series, as the differential coefficient of common functions rises.

(i) If $\boldsymbol{y=e^{ax}}$, then $\dfrac{dy}{dx}=ae^{ax}$, $\dfrac{d^2y}{dx^2}=a^2e^{ax}$, and so on.

If we abbreviate $\dfrac{dy}{dx}$ as y', $\dfrac{d^2y}{dx^2}$ as y'', … and $\dfrac{d^ny}{dx^n}$ as $y^{(n)}$, then $y'=ae^{ax}$, $y''=a^2e^{ax}$, and the emerging pattern gives: $\boldsymbol{y^{(n)}=a^ne^{ax}}$ (1)

For example, if $\boldsymbol{y=3e^{2x}}$, then
$\dfrac{d^7y}{dx^7}=y^{(7)}=3(2^7)e^{2x}=\boldsymbol{384e^{2x}}$

(ii) If $\boldsymbol{y=\sin ax}$,

$$y'=a\cos ax=a\sin\left(ax+\frac{\pi}{2}\right)$$

$$y''=-a^2\sin ax=a^2\sin(ax+\pi)$$

$$=a^2\sin\left(ax+\frac{2\pi}{2}\right)$$

$$y'''=-a^3\cos x$$

$$=a^3\sin\left(ax+\frac{3\pi}{2}\right)\text{ and so on.}$$

In general, $\boldsymbol{y^{(n)} = a^n \sin\left(ax + \frac{n\pi}{2}\right)}$ (2)

For example, if

$\boldsymbol{y = \sin 3x}$, then $\frac{d^5y}{dx^5} = y^{(5)}$

$= 3^5 \sin\left(3x + \frac{5\pi}{2}\right) = 3^5 \sin\left(3x + \frac{\pi}{2}\right)$

$= \boldsymbol{243 \cos 3x}$

(iii) If $\boldsymbol{y = \cos ax}$,

$y' = -a \sin ax = a \cos\left(ax + \frac{\pi}{2}\right)$

$y'' = -a^2 \cos ax = a^2 \cos\left(ax + \frac{2\pi}{2}\right)$

$y''' = a^3 \sin ax = a^3 \cos\left(ax + \frac{3\pi}{2}\right)$ and so on.

In general, $\boldsymbol{y^{(n)} = a^n \cos\left(ax + \frac{n\pi}{2}\right)}$ (3)

For example, if $\boldsymbol{y = 4\cos 2x}$,

then $\frac{d^6y}{dx^6} = y^{(6)} = 4(2^6)\cos\left(2x + \frac{6\pi}{2}\right)$

$= 4(2^6)\cos(2x + 3\pi)$

$= 4(2^6)\cos(2x + \pi)$

$= \boldsymbol{-256 \cos 2x}$

(iv) If $\boldsymbol{y = x^a}$, $y' = a x^{a-1}$, $y'' = a(a-1)x^{a-2}$, $y''' = a(a-1)(a-2)x^{a-3}$, and $\boldsymbol{y^{(n)}} = a(a-1)(a-2)\ldots.(a-n+1)x^{a-n}$

or $\boldsymbol{y^{(n)} = \frac{a!}{(a-n)!} x^{a-n}}$ (4)

where a is a positive integer.

For example, if $\boldsymbol{y = 2x^6}$, then $\frac{d^4y}{dx^4} = y^{(4)}$

$= (2)\frac{6!}{(6-4)!}x^{6-4}$

$= (2)\frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1}x^2$

$= \boldsymbol{720x^2}$

(v) If $\boldsymbol{y = \sinh ax}$, $y' = a \cosh ax$

$y'' = a^2 \sinh ax$

$y''' = a^3 \cosh ax$, and so on

Since $\sinh ax$ is not periodic (see graph on page 43), it is more difficult to find a general statement for $y^{(n)}$. However, this is achieved with the following general series:

$$\boldsymbol{y^{(n)} = \frac{a^n}{2}\{[1 + (-1)^n]\sinh ax + [1 - (-1)^n]\cosh ax\}} \quad (5)$$

For example, if

$\boldsymbol{y = \sinh 2x}$, then $\frac{d^5y}{dx^5} = y^{(5)}$

$= \frac{2^5}{2}\{[1 + (-1)^5]\sinh 2x + [1 - (-1)^5]\cosh 2x\}$

$= \frac{2^5}{2}\{[0]\sinh 2x + [2]\cosh 2x\}$

$= \boldsymbol{32 \cosh 2x}$

(vi) If $\boldsymbol{y = \cosh ax}$,

$y' = a \sinh ax$

$y'' = a^2 \cosh ax$

$y''' = a^3 \sinh ax$, and so on

Since $\cosh ax$ is not periodic (see graph on page 43), again it is more difficult to find a general statement for $y^{(n)}$. However, this is achieved with the following general series:

$$\boldsymbol{y^{(n)} = \frac{a^n}{2}\{[1 - (-1)^n]\sinh ax + [1 + (-1)^n]\cosh ax\}} \quad (6)$$

For example, if $\boldsymbol{y = \frac{1}{9}\cosh 3x}$,

then $\frac{d^7y}{dx^7} = y^{(7)} = \left(\frac{1}{9}\right)\frac{3^7}{2}(2\sinh 3x)$

$= \boldsymbol{243 \sinh 3x}$

(vii) If $\boldsymbol{y = \ln ax}$, $y' = \frac{1}{x}$, $y'' = -\frac{1}{x^2}$, $y''' = \frac{2}{x^3}$, and so on.

In general, $\boldsymbol{y^{(n)} = (-1)^{n-1}\frac{(n-1)!}{x^n}}$ (7)

For example, if $\boldsymbol{y = \ln 5x}$, then

$\frac{d^6y}{dx^6} = y^{(6)} = (-1)^{6-1}\left(\frac{5!}{x^6}\right) = \boldsymbol{-\frac{120}{x^6}}$

Note that if $y = \ln x$, $y' = \dfrac{1}{x}$; if in equation (7), $n = 1$ then $y' = (-1)^0 \dfrac{(0)!}{x^1}$

$(-1)^0 = 1$ and if $y' = \dfrac{1}{x}$ then **(0)! = 1** (Check that $(-1)^0 = 1$ and $(0)! = 1$ on a calculator).

Now try the following exercise

Exercise 193 Further problems on higher order differential coefficients as series

Determine the following derivatives:

1. (a) $y^{(4)}$ when $y = e^{2x}$ (b) $y^{(5)}$ when $y = 8e^{\frac{t}{2}}$

   [(a) $16e^{2x}$ (b) $\dfrac{1}{4}e^{\frac{t}{2}}$]

2. (a) $y^{(4)}$ when $y = \sin 3t$
   (b) $y^{(7)}$ when $y = \dfrac{1}{50}\sin 5\theta$

   [(a) $81\sin 3t$ (b) $-1562.5\cos 5\theta$]

3. (a) $y^{(8)}$ when $y = \cos 2x$
   (b) $y^{(9)}$ when $y = 3\cos\dfrac{2}{3}t$

   $\left[\text{(a) } 256\cos 2x \quad \text{(b) } -\dfrac{2^9}{3^8}\sin\dfrac{2}{3}t\right]$

4. (a) $y^{(7)}$ when $y = 2x^9$ (b) $y^{(6)}$ when $y = \dfrac{t^7}{8}$

   [(a) $(9!)x^2$ (b) $630t$]

5. (a) $y^{(7)}$ when $y = \dfrac{1}{4}\sinh 2x$
   (b) $y^{(6)}$ when $y = 2\sinh 3x$

   [(a) $32\cosh 2x$ (b) $1458\sinh 3x$]

6. (a) $y^{(7)}$ when $y = \cosh 2x$
   (b) $y^{(8)}$ when $y = \dfrac{1}{9}\cosh 3x$

   [(a) $128\sinh 2x$ (b) $729\cosh 3x$]

7. (a) $y^{(4)}$ when $y = 2\ln 3\theta$
   (b) $y^{(7)}$ when $y = \dfrac{1}{3}\ln 2t$

   $\left[\text{(a) } -\dfrac{6}{\theta^4} \quad \text{(b) } \dfrac{240}{t^7}\right]$

52.3 Leibniz's theorem

If $y = uv$ (8)

where u and v are each functions of x, then by using the product rule,

$$y' = uv' + vu' \tag{9}$$

$$y'' = uv'' + v'u' + vu'' + u'v'$$
$$= u''v + 2u'v' + uv'' \tag{10}$$

$$y''' = u''v' + vu''' + 2u'v'' + 2v'u'' + uv''' + v''u'$$
$$= u'''v + 3u''v' + 3u'v'' + uv''' \tag{11}$$

$$y^{(4)} = u^{(4)}v + 4u^{(3)}v^{(1)} + 6u^{(2)}v^{(2)} + 4u^{(1)}v^{(3)} + uv^{(4)} \tag{12}$$

From equations (8) to (12) it is seen that

(a) the n'th derivative of u decreases by 1 moving from left to right,

(b) the n'th derivative of v increases by 1 moving from left to right,

(c) the coefficients 1, 4, 6, 4, 1 are the normal binomial coefficients (see page 58).

In fact, $(uv)^{(n)}$ may be obtained by expanding $(u+v)^{(n)}$ using the binomial theorem (see page 59), where the 'powers' are interpreted as derivatives. Thus, expanding $(u+v)^{(n)}$ gives:

$$\boldsymbol{y^{(n)} = (uv)^{(n)} = u^{(n)}v + nu^{(n-1)}v^{(1)} + \frac{n(n-1)}{2!}u^{(n-2)}v^{(2)} + \frac{n(n-1)(n-2)}{3!}u^{(n-3)}v^{(3)} + \cdots} \tag{13}$$

Equation (13) is a statement of **Leibniz's theorem**, which can be used to differentiate a product n times. The theorem is demonstrated in the following worked problems.

Problem 1. Determine $y^{(n)}$ when $y = x^2e^{3x}$.

For a product $y = uv$, the function taken as

(i) u is the one whose nth derivative can readily be determined (from equations (1) to (7)),

(ii) v is the one whose derivative reduces to zero after a few stages of differentiation.

Thus, when $y=x^2e^{3x}$, $v=x^2$, since its third derivative is zero, and $u=e^{3x}$ since the nth derivative is known from equation (1), i.e. $3^n e^{ax}$
Using Leinbiz's theorem (equation (13),

$$y^{(n)} = u^{(n)}v + nu^{(n-1)}v^{(1)} + \frac{n(n-1)}{2!}u^{(n-2)}v^{(2)} + \frac{n(n-1)(n-2)}{3!}u^{(n-3)}v^{(3)} + \cdots$$

where in this case $v=x^2$, $v^{(1)}=2x$, $v^{(2)}=2$ and $v^{(3)}=0$

Hence, $y^{(n)} = (3^n e^{3x})(x^2) + n(3^{n-1}e^{3x})(2x) + \frac{n(n-1)}{2!}(3^{n-2}e^{3x})(2) + \frac{n(n-1)(n-2)}{3!}(3^{n-3}e^{3x})(0)$

$= 3^{n-2}e^{3x}(3^2x^2 + n(3)(2x) + n(n-1) + 0)$

i.e. $\boldsymbol{y^{(n)} = e^{3x}3^{n-2}(9x^2 + 6nx + n(n-1))}$

Problem 2. If $x^2y'' + 2xy' + y = 0$ show that: $xy^{(n+2)} + 2(n+1)xy^{(n+1)} + (n^2+n+1)y^{(n)} = 0$

Differentiating each term of $x^2y'' + 2xy' + y = 0$ n times, using Leibniz's theorem of equation (13), gives:

$$\left\{y^{(n+2)}x^2 + n\,y^{(n+1)}(2x) + \frac{n(n-1)}{2!}y^{(n)}(2) + 0\right\} + \{y^{(n+1)}(2x) + n\,y^{(n)}(2) + 0\} + \{y^{(n)}\} = 0$$

i.e. $x^2y^{(n+2)} + 2nxy^{(n+1)} + n(n-1)y^{(n)} + 2xy^{(n+1)} + 2ny^{(n)} + y^{(n)} = 0$

i.e. $x^2y^{(n+2)} + 2(n+1)xy^{(n+1)} + (n^2 - n + 2n + 1)y^{(n)} = 0$

or $\boldsymbol{x^2y^{(n+2)} + 2(n+1)xy^{(n+1)} + (n^2+n+1)y^{(n)} = 0}$

Problem 3. Differentiate the following differential equation n times: $(1+x^2)y'' + 2xy' - 3y = 0$.

By Leibniz's equation, equation (13),

$$\left\{y^{(n+2)}(1+x^2) + n\,y^{(n+1)}(2x) + \frac{n(n-1)}{2!}y^{(n)}(2) + 0\right\} + 2\{y^{(n+1)}(x) + n\,y^{(n)}(1) + 0\} - 3\{y^{(n)}\} = 0$$

i.e. $(1+x^2)y^{(n+2)} + 2nxy^{(n+1)} + n(n-1)y^{(n)} + 2xy^{(n+1)} + 2ny^{(n)} - 3y^{(n)} = 0$

or $(1+x^2)y^{(n+2)} + 2(n+1)xy^{(n+1)} + (n^2 - n + 2n - 3)y^{(n)} = 0$

i.e. $\boldsymbol{(1+x^2)y^{(n+2)} + 2(n+1)xy^{(n+1)} + (n^2+n-3)y^{(n)} = 0}$

Problem 4. Find the 5th derivative of $y=x^4\sin x$.

If $y=x^4\sin x$, then using Leibniz's equation with $u=\sin x$ and $v=x^4$ gives:

$$y^{(n)} = \left[\sin\left(x+\frac{n\pi}{2}\right)x^4\right] + n\left[\sin\left(x+\frac{(n-1)\pi}{2}\right)4x^3\right] + \frac{n(n-1)}{2!}\left[\sin\left(x+\frac{(n-2)\pi}{2}\right)12x^2\right] + \frac{n(n-1)(n-2)}{3!}\left[\sin\left(x+\frac{(n-3)\pi}{2}\right)24x\right] + \frac{n(n-1)(n-2)(n-3)}{4!}\left[\sin\left(x+\frac{(n-4)\pi}{2}\right)24\right]$$

and $y^{(5)} = x^4\sin\left(x+\frac{5\pi}{2}\right) + 20x^3\sin(x+2\pi) + \frac{(5)(4)}{2}(12x^2)\sin\left(x+\frac{3\pi}{2}\right) + \frac{(5)(4)(3)}{(3)(2)}(24x)\sin(x+\pi) + \frac{(5)(4)(3)(2)}{(4)(3)(2)}(24)\sin\left(x+\frac{\pi}{2}\right)$

Since $\sin\left(x+\frac{5\pi}{2}\right) \equiv \sin\left(x+\frac{\pi}{2}\right) \equiv \cos x$,

$\sin(x+2\pi) \equiv \sin x$, $\sin\left(x+\frac{3\pi}{2}\right) \equiv -\cos x$,

and $\sin(x+\pi) \equiv -\sin x$,

then $y^{(5)} = x^4\cos x + 20x^3\sin x + 120x^2(-\cos x) + 240x(-\sin x) + 120\cos x$

i.e. $\boldsymbol{y^{(5)} = (x^4 - 120x^2 + 120)\cos x + (20x^3 - 240x)\sin x}$

Now try the following exercise

Exercise 194 Further problems on Leibniz's theorem

Use the theorem of Leibniz in the following problems:

1. Obtain the n'th derivative of: x^2y.
   $\left[x^2y^{(n)} + 2nxy^{(n-1)} + n(n-1)y^{(n-2)}\right]$

2. If $y = x^3e^{2x}$ find $y^{(n)}$ and hence $y^{(3)}$.
   $\left[\begin{array}{l} y^{(n)} = e^{2x}2^{n-3}\{8x^3 + 12nx^2 + n(n-1)(6x) + n(n-1)(n-2)\} \\ y^{(3)} = e^{2x}(8x^3 + 36x^2 + 36x + 6) \end{array}\right]$

3. Determine the 4th derivative of: $y = 2x^3e^{-x}$.
   $[y^{(4)} = 2e^{-x}(x^3 - 12x^2 + 36x - 24)]$

4. If $y = x^3\cos x$ determine the 5th derivative.
   $[y^{(5)} = (60x - x^3)\sin x + (15x^2 - 60)\cos x]$

5. Find an expression for $y^{(4)}$ if $y = e^{-t}\sin t$.
   $[y^{(4)} = -4e^{-t}\sin t]$

6. If $y = x^5\ln 2x$ find $y^{(3)}$.
   $[y^{(3)} = x^2(47 + 60\ln 2x)]$

7. Given $2x^2y'' + xy' + 3y = 0$ show that $2x^2y^{(n+2)} + (4n+1)xy^{(n+1)} + (2n^2 - n + 3)y^{(n)} = 0$.

8. If $y = (x^3 + 2x^2)e^{2x}$ determine an expansion for $y^{(5)}$.
   $[y^{(5)} = e^{2x}2^4(2x^3 + 19x^2 + 50x + 35)]$

52.4 Power series solution by the Leibniz–Maclaurin method

For second order differential equations that cannot be solved by algebraic methods, the **Leibniz–Maclaurin method** produces a solution in the form of infinite series of powers of the unknown variable. The following simple **5-step procedure** may be used in the Leibniz–Maclaurin method:

(i) Differentiate the given equation n times, using the Leibniz theorem of equation (13),

(ii) rearrange the result to obtain the recurrence relation at $x = 0$,

(iii) determine the values of the derivatives at $x = 0$, i.e. find $(y)_0$ and $(y')_0$,

(iv) substitute in the Maclaurin expansion for $y = f(x)$ (see page 69, equation (5)),

(v) simplify the result where possible and apply boundary condition (if given).

The Leibniz–Maclaurin method is demonstrated, using the above procedure, in the following worked problems.

Problem 5. Determine the power series solution of the differential equation: $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 2y = 0$ using Leibniz–Maclaurin's method, given the boundary conditions that at $x = 0$, $y = 1$ and $\frac{dy}{dx} = 2$.

Following the above procedure:

(i) The differential equation is rewritten as: $y'' + xy' + 2y = 0$ and from the Leibniz theorem of equation (13), each term is differentiated n times, which gives:

$$y^{(n+2)} + \{y^{(n+1)}(x) + n\,y^{(n)}(1) + 0\} + 2\,y^{(n)} = 0$$

i.e. $$y^{(n+2)} + xy^{(n+1)} + (n+2)\,y^{(n)} = 0 \qquad (14)$$

(ii) At $x = 0$, equation (14) becomes:

$$y^{(n+2)} + (n+2)\,y^{(n)} = 0$$

from which, $y^{(n+2)} = -(n+2)\,y^{(n)}$

This equation is called a **recurrence relation** or **recurrence formula**, because each recurring term depends on a previous term.

(iii) Substituting $n=0,\ 1,\ 2,\ 3,\ldots$ will produce a set of relationships between the various coefficients.

For $n=0$, $(y'')_0=-2(y)_0$

$n=1$, $(y''')_0=-3(y')_0$

$n=2$, $(y^{(4)})_0=-4(y'')_0=-4\{-2(y)_0\}$
$=2\times4(y)_0$

$n=3$, $(y^{(5)})_0=-5(y''')_0=-5\{-3(y')_0\}$
$=3\times5(y')_0$

$n=4$, $(y^{(6)})_0=-6(y^{(4)})_0=-6\{2\times4(y)_0\}$
$=-2\times4\times6(y)_0$

$n=5$, $(y^{(7)})_0=-7(y^{(5)})_0=-7\{3\times5(y')_0\}$
$=-3\times5\times7(y')_0$

$n=6$, $(y^{(8)})_0=-8(y^{(6)})_0=$
$-8\{-2\times4\times6(y)_0\}=2\times4\times6\times8(y)_0$

(iv) Maclaurin's theorem from page 69 may be written as:

$$y=(y)_0+x(y')_0+\frac{x^2}{2!}(y'')_0+\frac{x^3}{3!}(y''')_0+\frac{x^4}{4!}(y^{(4)})_0+\cdots$$

Substituting the above values into Maclaurin's theorem gives:

$$y=(y)_0+x(y')_0+\frac{x^2}{2!}\{-2(y)_0\}+\frac{x^3}{3!}\{-3(y')_0\}+\frac{x^4}{4!}\{2\times4(y)_0\}+\frac{x^5}{5!}\{3\times5(y')_0\}+\frac{x^6}{6!}\{-2\times4\times6(y)_0\}+\frac{x^7}{7!}\{-3\times5\times7(y')_0\}+\frac{x^8}{8!}\{2\times4\times6\times8(y)_0\}$$

(v) Collecting similar terms together gives:

$$y=(y)_0\left\{1-\frac{2x^2}{2!}+\frac{2\times4x^4}{4!}-\frac{2\times4\times6x^6}{6!}+\frac{2\times4\times6\times8x^8}{8!}-\cdots\right\}+(y')_0\left\{x-\frac{3x^3}{3!}+\frac{3\times5x^5}{5!}-\frac{3\times5\times7x^7}{7!}+\cdots\right\}$$

i.e. $$y=(y)_0\left\{1-\frac{x^2}{1}+\frac{x^4}{1\times3}-\frac{x^6}{3\times5}+\frac{x^8}{3\times5\times7}-\cdots\right\}+(y')_0\times\left\{\frac{x}{1}-\frac{x^3}{1\times2}+\frac{x^5}{2\times4}-\frac{x^7}{2\times4\times6}+\cdots\right\}$$

The boundary conditions are that at $x=0$, $y=1$ and $\frac{dy}{dx}=2$, i.e. $(y)_0=1$ and $(y')_0=2$.

Hence, the power series solution of the differential equation: $\frac{d^2y}{dx^2}+x\frac{dy}{dx}+2y=0$ is:

$$\boldsymbol{y=\left\{1-\frac{x^2}{1}+\frac{x^4}{1\times3}-\frac{x^6}{3\times5}+\frac{x^8}{3\times5\times7}-\cdots\right\}+2\left\{\frac{x}{1}-\frac{x^3}{1\times2}+\frac{x^5}{2\times4}-\frac{x^7}{2\times4\times6}+\cdots\right\}}$$

Problem 6. Determine the power series solution of the differential equation: $\frac{d^2y}{dx^2}+\frac{dy}{dx}+xy=0$ given the boundary conditions that at $x=0$, $y=0$ and $\frac{dy}{dx}=1$, using Leibniz–Maclaurin's method.

Following the above procedure:

(i) The differential equation is rewritten as: $y''+y'+xy=0$ and from the Leibniz theorem of

equation (13), each term is differentiated n times, which gives:

$$y^{(n+2)} + y^{(n+1)} + y^{(n)}(x) + n\,y^{(n-1)}(1) + 0 = 0$$

i.e. $$y^{(n+2)} + y^{(n+1)} + xy^{(n)} + n\,y^{(n-1)} = 0 \qquad (15)$$

(ii) At $x = 0$, equation (15) becomes:

$$y^{(n+2)} + y^{(n+1)} + n\,y^{(n-1)} = 0$$

from which, $y^{(n+2)} = -\{y^{(n+1)} + n\,y^{(n-1)}\}$

This is the **recurrence relation** and applies for $n \geq 1$

(iii) Substituting $n = 1, 2, 3, \ldots$ will produce a set of relationships between the various coefficients.

For $n = 1$, $(y''')_0 = -\{(y'')_0 + (y)_0\}$

$n = 2$, $(y^{(4)})_0 = -\{(y''')_0 + 2(y')_0\}$

$n = 3$, $(y^{(5)})_0 = -\{(y^{(4)})_0 + 3(y'')_0\}$

$n = 4$, $(y^{(6)})_0 = -\{(y^{(5)})_0 + 4(y''')_0\}$

$n = 5$, $(y^{(7)})_0 = -\{(y^{(6)})_0 + 5(y^{(4)})_0\}$

$n = 6$, $(y^{(8)})_0 = -\{(y^{(7)})_0 + 6(y^{(5)})_0\}$

From the given boundary conditions, at $x = 0$, $y = 0$, thus $(y)_0 = 0$, and at $x = 0$, $\dfrac{dy}{dx} = 1$, thus $(y')_0 = 1$

From the given differential equation, $y'' + y' + xy = 0$, and, at $x = 0$, $(y'')_0 + (y')_0 + (0)y = 0$ from which, $(y'')_0 = -(y')_0 = -1$

Thus, $(y)_0 = \mathbf{0}$, $(y')_0 = \mathbf{1}$, $(y'')_0 = \mathbf{-1}$,

$$(y''')_0 = -\{(y'')_0 + (y)_0\} = -(-1+0) = \mathbf{1}$$

$$(y^{(4)})_0 = -\{(y''')_0 + 2(y')_0\} = -[1+2(1)] = \mathbf{-3}$$

$$(y^{(5)})_0 = -\{(y^{(4)})_0 + 3(y'')_0\} = -[-3+3(-1)] = \mathbf{6}$$

$$(y^{(6)})_0 = -\{(y^{(5)})_0 + 4(y''')_0\} = -[6+4(1)] = \mathbf{-10}$$

$$(y^{(7)})_0 = -\{(y^{(6)})_0 + 5(y^{(4)})_0\} = -[-10+5(-3)] = \mathbf{25}$$

$$(y^{(8)})_0 = -\{(y^{(7)})_0 + 6(y^{(5)})_0\} = -[25+6(6)] = \mathbf{-61}$$

(iv) Maclaurin's theorem states:

$$y = (y)_0 + x(y')_0 + \frac{x^2}{2!}(y'')_0 + \frac{x^3}{3!}(y''')_0 + \frac{x^4}{4!}(y^{(4)})_0 + \cdots$$

and substituting the above values into Maclaurin's theorem gives:

$$y = 0 + x(1) + \frac{x^2}{2!}\{-1\} + \frac{x^3}{3!}\{1\} + \frac{x^4}{4!}\{-3\} + \frac{x^5}{5!}\{6\} + \frac{x^6}{6!}\{-10\} + \frac{x^7}{7!}\{25\} + \frac{x^8}{8!}\{-61\} + \cdots$$

(v) Simplifying, the power series solution of the differential equation: $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + xy = 0$ is given by:

$$\mathbf{y = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{3x^4}{4!} + \frac{6x^5}{5!} - \frac{10x^6}{6!} + \frac{25x^7}{7!} - \frac{61x^8}{8!} + \cdots}$$

Now try the following exercise

Exercise 195 Further problems on power series solutions by the Leibniz–Maclaurin method

1. Determine the power series solution of the differential equation: $\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} + y = 0$ using the Leibniz–Maclaurin method, given that at $x = 0$, $y = 1$ and $\dfrac{dy}{dx} = 2$.

$$\left[y = \left(1 - \frac{x^2}{2!} + \frac{5x^4}{4!} - \frac{5 \times 9x^6}{6!} + \frac{5 \times 9 \times 13x^8}{8!} - \cdots\right) + 2\left(x - \frac{3x^3}{3!} + \frac{3 \times 7x^5}{5!} - \frac{3 \times 7 \times 11x^7}{7!} + \cdots\right) \right]$$

2. Show that the power series solution of the differential equation: $(x+1)\dfrac{d^2y}{dx^2}+(x-1)\dfrac{dy}{dx}-2y=0$, using the Leibniz–Maclaurin method, is given by: $y=1+x^2+e^x$ given the boundary conditions that at $x=0$, $y=\dfrac{dy}{dx}=1$.

3. Find the particular solution of the differential equation: $(x^2+1)\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-4y=0$ using the Leibniz–Maclaurin method, given the boundary conditions that at $x=0$, $y=1$ and $\dfrac{dy}{dx}=1$.

$$\left[y=1+x+2x^2+\frac{x^3}{2}-\frac{x^5}{8}+\frac{x^7}{16}+\cdots\right]$$

4. Use the Leibniz–Maclaurin method to determine the power series solution for the differential equation: $x\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}+xy=1$ given that at $x=0$, $y=1$ and $\dfrac{dy}{dx}=2$.

$$\left[y=\left\{1-\frac{x^2}{2^2}+\frac{x^4}{2^2\times4^2}-\frac{x^6}{2^2\times4^2\times6^2}+\cdots\right\}+2\left\{x-\frac{x^3}{3^2}+\frac{x^5}{3^2\times5^2}-\frac{x^7}{3^2\times5^2\times7^2}+\cdots\right\}\right]$$

52.5 Power series solution by the Frobenius method

A differential equation of the form $y''+Py'+Qy=0$, where P and Q are both functions of x, such that the equation can be represented by a power series, may be solved by the **Frobenius method**.

The following **4-step procedure** may be used in the Frobenius method:

(i) Assume a trial solution of the form $\boldsymbol{y=x^c\{a_0+a_1x+a_2x^2+a_3x^3+\cdots+a_rx^r+\cdots\}}$

(ii) differentiate the trial series,

(iii) substitute the results in the given differential equation,

(iv) equate coefficients of corresponding powers of the variable on each side of the equation; this enables index c and coefficients a_1, a_2, a_3, … from the trial solution, to be determined.

This introductory treatment of the Frobenius method covering the simplest cases is demonstrated, using the above procedure, in the following worked problems.

Problem 7. Determine, using the Frobenius method, the general power series solution of the differential equation: $3x\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}-y=0$.

The differential equation may be rewritten as: $3xy''+y'-y=0$.

(i) Let a trial solution be of the form

$$y=x^c\{a_0+a_1x+a_2x^2+a_3x^3+\cdots+a_rx^r+\cdots\} \quad (16)$$

where $a_0\neq0$,

i.e. $$y=a_0x^c+a_1x^{c+1}+a_2x^{c+2}+a_3x^{c+3}+\cdots+a_rx^{c+r}+\cdots \quad (17)$$

(ii) Differentiating equation (17) gives:

$$y'=a_0cx^{c-1}+a_1(c+1)x^c+a_2(c+2)x^{c+1}+\cdots+a_r(c+r)x^{c+r-1}+\cdots$$

and $$y''=a_0c(c-1)x^{c-2}+a_1c(c+1)x^{c-1}+a_2(c+1)(c+2)x^c+\cdots+a_r(c+r-1)(c+r)x^{c+r-2}+\cdots$$

(iii) Substituting y, y' and y'' into each term of the given equation $3xy''+y'-y=0$ gives:

$$3xy''=3a_0c(c-1)x^{c-1}+3a_1c(c+1)x^c+3a_2(c+1)(c+2)x^{c+1}+\cdots+3a_r(c+r-1)(c+r)x^{c+r-1}+\cdots \quad (a)$$

$$y'=a_0cx^{c-1}+a_1(c+1)x^c+a_2(c+2)x^{c+1}+\cdots+a_r(c+r)x^{c+r-1}+\cdots \quad (b)$$

$$-y=-a_0x^c-a_1x^{c+1}-a_2x^{c+2}-a_3x^{c+3}-\cdots-a_rx^{c+r}-\cdots \quad (c)$$

(iv) The sum of these three terms forms the left-hand side of the equation. Since the right-hand side is zero, the coefficients of each power of x can be equated to zero.
For example, the coefficient of x^{c-1} is equated to zero giving: $3a_0c(c-1)+a_0c=0$

$$\text{or} \quad a_0c[3c-3+1]=\boldsymbol{a_0c(3c-2)=0} \qquad (18)$$

The coefficient of x^c is equated to zero giving: $3a_1c(c+1)+a_1(c+1)-a_0=0$

$$\text{i.e.} \quad a_1(3c^2+3c+c+1)-a_0$$
$$=a_1(3c^2+4c+1)-a_0=0$$

$$\text{or} \quad \boldsymbol{a_1(3c+1)(c+1)-a_0=0} \qquad (19)$$

In each of series (a), (b) and (c) an x^c term is involved, after which, a general relationship can be obtained for x^{c+r}, where $r \geq 0$.
In series (a) and (b), terms in x^{c+r-1} are present; replacing r by $(r+1)$ will give the corresponding terms in x^{c+r}, which occurs in all three equations, i.e.

in series (a), $3a_{r+1}(c+r)(c+r+1)x^{c+r}$

in series (b), $a_{r+1}(c+r+1)x^{c+r}$

in series (c), $-a_r x^{c+r}$

Equating the total coefficients of x^{c+r} to zero gives:

$$3a_{r+1}(c+r)(c+r+1)+a_{r+1}(c+r+1)$$
$$-a_r=0$$

which simplifies to:

$$\boldsymbol{a_{r+1}\{(c+r+1)(3c+3r+1)\}-a_r=0} \qquad (20)$$

Equation (18), which was formed from the coefficients of the lowest power of x, i.e. x^{c-1}, is called the **indicial equation**, from which, the value of c is obtained. From equation (18), since $a_0 \neq 0$, then $\boldsymbol{c=0}$ or $\boldsymbol{c=\frac{2}{3}}$

(a) When $c=0$:

From equation (19), if $c=0$, $a_1(1\times 1)-a_0=0$, i.e. $\boldsymbol{a_1=a_0}$

From equation (20), if $c=0$,
$a_{r+1}(r+1)(3r+1)-a_r=0$,
i.e. $\boldsymbol{a_{r+1}=\frac{a_r}{(r+1)(3r+1)}} \quad r\geq 0$

Thus, when $r=1$, $a_2=\frac{a_1}{(2\times 4)}=\frac{a_0}{(2\times 4)}$
since $a_1=a_0$

when $r=2$, $a_3=\frac{a_2}{(3\times 7)}=\frac{a_0}{(2\times 4)(3\times 7)}$
or $\frac{a_0}{(2\times 3)(4\times 7)}$

when $r=3$, $a_4=\frac{a_3}{(4\times 10)}$
$=\frac{a_0}{(2\times 3\times 4)(4\times 7\times 10)}$
and so on.

From equation (16), the trial solution was:

$$y=x^c\{a_0+a_1x+a_2x^2+a_3x^3+\cdots+a_rx^r+\cdots\}$$

Substituting $c=0$ and the above values of $a_1, a_2, a_3, \ldots$ into the trial solution gives:

$$y=x^0\left\{a_0+a_0x+\left(\frac{a_0}{(2\times 4)}\right)x^2\right.$$
$$+\left(\frac{a_0}{(2\times 3)(4\times 7)}\right)x^3$$
$$\left.+\left(\frac{a_0}{(2\times 3\times 4)(4\times 7\times 10)}\right)x^4+\cdots\right\}$$

$$\text{i.e.} \quad y=a_0\left\{1+x+\frac{x^2}{(2\times 4)}+\frac{x^3}{(2\times 3)(4\times 7)}\right.$$
$$\left.+\frac{x^4}{(2\times 3\times 4)(4\times 7\times 10)}+\cdots\right\} \qquad (21)$$

(b) When $c=\frac{2}{3}$:

From equation (19), if $c=\frac{2}{3}$, $a_1(3)\left(\frac{5}{3}\right)-a_0=0$, i.e. $\boldsymbol{a_1=\frac{a_0}{5}}$

From equation (20), if $c=\frac{2}{3}$

$$a_{r+1}\left(\frac{2}{3}+r+1\right)(2+3r+1)-a_r=0,$$

$$\text{i.e.} \quad a_{r+1}\left(r+\frac{5}{3}\right)(3r+3)-a_r$$
$$=a_{r+1}(3r^2+8r+5)-a_r=0,$$

$$\text{i.e.} \quad \boldsymbol{a_{r+1}=\frac{a_r}{(r+1)(3r+5)}} \quad r\geq 0$$

Thus, when $r=1$, $a_2=\dfrac{a_1}{(2\times 8)}=\dfrac{a_0}{(2\times 5\times 8)}$ since $a_1=\dfrac{a_0}{5}$

when $r=2$, $a_3=\dfrac{a_2}{(3\times 11)}=\dfrac{a_0}{(2\times 3)(5\times 8\times 11)}$

when $r=3$, $a_4=\dfrac{a_3}{(4\times 14)}=\dfrac{a_0}{(2\times 3\times 4)(5\times 8\times 11\times 14)}$

and so on.

From equation (16), the trial solution was:

$$y=x^c\{a_0+a_1x+a_2x^2+a_3x^3+\cdots+a_rx^r+\cdots\}$$

Substituting $c=\dfrac{2}{3}$ and the above values of a_1, a_2, a_3, … into the trial solution gives:

$$y=x^{\frac{2}{3}}\left\{a_0+\left(\frac{a_0}{5}\right)x+\left(\frac{a_0}{2\times 5\times 8}\right)x^2+\left(\frac{a_0}{(2\times 3)(5\times 8\times 11)}\right)x^3+\left(\frac{a_0}{(2\times 3\times 4)(5\times 8\times 11\times 14)}\right)x^4+\cdots\right\}$$

i.e.
$$y=a_0x^{\frac{2}{3}}\left\{1+\frac{x}{5}+\frac{x^2}{(2\times 5\times 8)}+\frac{x^3}{(2\times 3)(5\times 8\times 11)}+\frac{x^4}{(2\times 3\times 4)(5\times 8\times 11\times 14)}+\cdots\right\} \quad (22)$$

Since a_0 is an arbitrary (non-zero) constant in each solution, its value could well be different.
Let $a_0=A$ in equation (21), and $a_0=B$ in equation (22). Also, if the first solution is denoted by $u(x)$ and the second by $v(x)$, then the general solution of the given differential equation is $y=u(x)+v(x)$. Hence,

$$\mathbf{y=A\left\{1+x+\frac{x^2}{(2\times 4)}+\frac{x^3}{(2\times 3)(4\times 7)}+\frac{x^4}{(2\times 3\times 4)(4\times 7\times 10)}+\cdots\right\}}$$
$$\mathbf{+Bx^{\frac{2}{3}}\left\{1+\frac{x}{5}+\frac{x^2}{(2\times 5\times 8)}+\frac{x^3}{(2\times 3)(5\times 8\times 11)}+\frac{x^4}{(2\times 3\times 4)(5\times 8\times 11\times 14)}+\cdots\right\}}$$

Problem 8. Use the Frobenius method to determine the general power series solution of the differential equation:
$2x^2\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}+(1-x)y=0.$

The differential equation may be rewritten as: $2x^2y''-xy'+(1-x)y=0$.

(i) Let a trial solution be of the form

$$y=x^c\{a_0+a_1x+a_2x^2+a_3x^3+\cdots+a_rx^r+\cdots\} \quad (23)$$

where $a_0\neq 0$,

i.e. $y=a_0x^c+a_1x^{c+1}+a_2x^{c+2}+a_3x^{c+3}+\cdots+a_rx^{c+r}+\cdots$ (24)

(ii) Differentiating equation (24) gives:

$$y'=a_0cx^{c-1}+a_1(c+1)x^c+a_2(c+2)x^{c+1}+\cdots+a_r(c+r)x^{c+r-1}+\cdots$$

and $y''=a_0c(c-1)x^{c-2}+a_1c(c+1)x^{c-1}+a_2(c+1)(c+2)x^c+\cdots+a_r(c+r-1)(c+r)x^{c+r-2}+\cdots$

(iii) Substituting y, y' and y'' into each term of the given equation $2x^2y''-xy'+(1-x)y=0$ gives:

$$2x^2y''=2a_0c(c-1)x^c+2a_1c(c+1)x^{c+1}+2a_2(c+1)(c+2)x^{c+2}+\cdots+2a_r(c+r-1)(c+r)x^{c+r}+\cdots \quad \text{(a)}$$

$$-xy'=-a_0cx^c-a_1(c+1)x^{c+1}-a_2(c+2)x^{c+2}-\cdots-a_r(c+r)x^{c+r}-\cdots \quad \text{(b)}$$

$$(1-x)y=(1-x)(a_0x^c+a_1x^{c+1}+a_2x^{c+2}+a_3x^{c+3}+\cdots+a_rx^{c+r}+\cdots)=a_0x^c+a_1x^{c+1}+a_2x^{c+2}+a_3x^{c+3}+\cdots+a_rx^{c+r}+\cdots$$

$$- a_0x^{c+1} - a_1x^{c+2} - a_2x^{c+3} - a_3x^{c+4} - \cdots - a_rx^{c+r+1} - \cdots \qquad \text{(c)}$$

(iv) The **indicial equation**, which is obtained by equating the coefficient of the lowest power of x to zero, gives the value(s) of c. Equating the total coefficients of x^c (from equations (a) to (c)) to zero gives:

$$2a_0c(c-1) - a_0c + a_0 = 0$$

i.e. $a_0[2c(c-1) - c + 1] = 0$

i.e. $a_0[2c^2 - 2c - c + 1] = 0$

i.e. $a_0[2c^2 - 3c + 1] = 0$

i.e. $a_0[(2c-1)(c-1)] = 0$

from which, $\boldsymbol{c = 1}$ or $\boldsymbol{c = \frac{1}{2}}$

The coefficient of the general term, i.e. x^{c+r}, gives (from equations (a) to (c)):

$$2a_r(c+r-1)(c+r) - a_r(c+r) + a_r - a_{r-1} = 0$$

from which,

$$a_r[2(c+r-1)(c+r) - (c+r) + 1] = a_{r-1}$$

and $$a_r = \frac{a_{r-1}}{2(c+r-1)(c+r) - (c+r) + 1} \qquad (25)$$

(a) With $\boldsymbol{c = 1}$, $\boldsymbol{a_r} = \dfrac{a_{r-1}}{2(r)(1+r) - (1+r) + 1}$

$$= \frac{a_{r-1}}{2r + 2r^2 - 1 - r + 1}$$

$$= \frac{a_{r-1}}{2r^2 + r} = \frac{\boldsymbol{a_{r-1}}}{\boldsymbol{r(2r+1)}}$$

Thus, when $r = 1$,

$$a_1 = \frac{a_0}{1(2+1)} = \frac{a_0}{1 \times 3}$$

when $r = 2$,

$$a_2 = \frac{a_1}{2(4+1)} = \frac{a_1}{(2 \times 5)}$$

$$= \frac{a_0}{(1 \times 3)(2 \times 5)} \text{ or } \frac{a_0}{(1 \times 2) \times (3 \times 5)}$$

when $r = 3$,

$$a_3 = \frac{a_2}{3(6+1)} = \frac{a_2}{3 \times 7}$$

$$= \frac{a_0}{(1 \times 2 \times 3) \times (3 \times 5 \times 7)}$$

when $r = 4$,

$$a_4 = \frac{a_3}{4(8+1)} = \frac{a_3}{4 \times 9}$$

$$= \frac{a_0}{(1 \times 2 \times 3 \times 4) \times (3 \times 5 \times 7 \times 9)}$$

and so on.

From equation (23), the trial solution was:

$$y = x^c\{a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_rx^r + \cdots\}$$

Substituting $c = 1$ and the above values of $a_1, a_2, a_3, \ldots$ into the trial solution gives:

$$y = x^1\left\{a_0 + \frac{a_0}{(1 \times 3)}x + \frac{a_0}{(1 \times 2) \times (3 \times 5)}x^2 + \frac{a_0}{(1 \times 2 \times 3) \times (3 \times 5 \times 7)}x^3 + \frac{a_0}{(1 \times 2 \times 3 \times 4) \times (3 \times 5 \times 7 \times 9)}x^4 + \cdots\right\}$$

i.e. $$y = a_0x^1\left\{1 + \frac{x}{(1 \times 3)} + \frac{x^2}{(1 \times 2) \times (3 \times 5)} + \frac{x^3}{(1 \times 2 \times 3) \times (3 \times 5 \times 7)} + \frac{x^4}{(1 \times 2 \times 3 \times 4) \times (3 \times 5 \times 7 \times 9)} + \cdots\right\} \qquad (26)$$

(b) With $\boldsymbol{c = \frac{1}{2}}$

$$a_r = \frac{a_{r-1}}{2(c+r-1)(c+r) - (c+r) + 1}$$

from equation (25)

i.e. $$\boldsymbol{a_r} = \frac{a_{r-1}}{2\left(\frac{1}{2}+r-1\right)\left(\frac{1}{2}+r\right) - \left(\frac{1}{2}+r\right) + 1}$$

$$= \frac{a_{r-1}}{2\left(r-\frac{1}{2}\right)\left(r+\frac{1}{2}\right) - \frac{1}{2} - r + 1}$$

$$= \frac{a_{r-1}}{2\left(r^2 - \frac{1}{4}\right) - \frac{1}{2} - r + 1}$$

$$= \frac{a_{r-1}}{2r^2 - \frac{1}{2} - \frac{1}{2} - r + 1} = \frac{a_{r-1}}{2r^2 - r}$$

$$= \boldsymbol{\frac{a_{r-1}}{r(2r-1)}}$$

Thus, when $r=1$, $a_1 = \dfrac{a_0}{1(2-1)} = \dfrac{a_0}{1 \times 1}$

when $r=2$, $a_2 = \dfrac{a_1}{2(4-1)} = \dfrac{a_1}{(2 \times 3)}$

$$= \frac{a_0}{(2 \times 3)}$$

when $r=3$, $a_3 = \dfrac{a_2}{3(6-1)} = \dfrac{a_2}{3 \times 5}$

$$= \frac{a_0}{(2 \times 3) \times (3 \times 5)}$$

when $r=4$, $a_4 = \dfrac{a_3}{4(8-1)} = \dfrac{a_3}{4 \times 7}$

$$= \frac{a_0}{(2 \times 3 \times 4) \times (3 \times 5 \times 7)}$$

and so on.

From equation (23), the trial solution was:

$$y = x^c \left\{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_r x^r + \cdots \right\}$$

Substituting $c = \dfrac{1}{2}$ and the above values of $a_1, a_2, a_3, \ldots$ into the trial solution gives:

$$y = x^{\frac{1}{2}} \left\{ a_0 + a_0 x + \frac{a_0}{(2 \times 3)} x^2 + \frac{a_0}{(2 \times 3) \times (3 \times 5)} x^3 + \frac{a_0}{(2 \times 3 \times 4) \times (3 \times 5 \times 7)} x^4 + \cdots \right\}$$

i.e. $$y = a_0 x^{\frac{1}{2}} \left\{ 1 + x + \frac{x^2}{(2 \times 3)} + \frac{x^3}{(2 \times 3) \times (3 \times 5)} + \frac{x^4}{(2 \times 3 \times 4) \times (3 \times 5 \times 7)} + \cdots \right\} \quad (27)$$

Since a_0 is an arbitrary (non-zero) constant in each solution, its value could well be different. Let $a_0 = A$ in equation (26), and $a_0 = B$ in equation (27). Also, if the first solution is denoted by $u(x)$ and the second by $v(x)$, then the general solution of the given differential equation is $y = u(x) + v(x)$,

i.e. $$\boldsymbol{y = Ax\left\{1 + \frac{x}{(1 \times 3)} + \frac{x^2}{(1 \times 2) \times (3 \times 5)} + \frac{x^3}{(1 \times 2 \times 3) \times (3 \times 5 \times 7)} + \frac{x^4}{(1 \times 2 \times 3 \times 4) \times (3 \times 5 \times 7 \times 9)} + \cdots\right\} + Bx^{\frac{1}{2}}\left\{1 + x + \frac{x^2}{(2 \times 3)} + \frac{x^3}{(2 \times 3) \times (3 \times 5)} + \frac{x^4}{(2 \times 3 \times 4) \times (3 \times 5 \times 7)} + \cdots\right\}}$$

Problem 9. Use the Frobenius method to determine the general power series solution of the differential equation: $\dfrac{d^2y}{dx^2} - 2y = 0$.

The differential equation may be rewritten as: $y'' - 2y = 0$.

(i) Let a trial solution be of the form

$$y = x^c \left\{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_r x^r + \cdots \right\} \quad (28)$$

where $a_0 \neq 0$,

i.e. $$y = a_0 x^c + a_1 x^{c+1} + a_2 x^{c+2} + a_3 x^{c+3} + \cdots + a_r x^{c+r} + \cdots \quad (29)$$

(ii) Differentiating equation (29) gives:

$$y' = a_0 c x^{c-1} + a_1 (c+1) x^c + a_2 (c+2) x^{c+1} + \cdots + a_r (c+r) x^{c+r-1} + \cdots$$

and $$y'' = a_0 c(c-1) x^{c-2} + a_1 c(c+1) x^{c-1} + a_2 (c+1)(c+2) x^c + \cdots + a_r (c+r-1)(c+r) x^{c+r-2} + \cdots$$

(iii) Replacing r by $(r+2)$ in $a_r (c+r-1)(c+r)\, x^{c+r-2}$ gives: $a_{r+2}(c+r+1)(c+r+2)x^{c+r}$

Substituting y and y'' into each term of the given equation $y''-2y=0$ gives:

$$y''-2y = a_0c(c-1)x^{c-2}+a_1c(c+1)x^{c-1} + [a_2(c+1)(c+2)-2a_0]x^c+\cdots + [a_{r+2}(c+r+1)(c+r+2) - 2a_r]\,x^{c+r}+\cdots=0 \qquad (30)$$

(iv) The **indicial equation** is obtained by equating the coefficient of the lowest power of x to zero.

Hence, $a_0c(c-1)=0$ from which, $\boldsymbol{c=0}$ or $\boldsymbol{c=1}$ since $a_0\neq0$

For the term in x^{c-1}, i.e. $a_1c(c+1)=0$

With $\boldsymbol{c=1}$, $\boldsymbol{a_1=0}$; however, when $\boldsymbol{c=0}$, $\boldsymbol{a_1}$ **is indeterminate**, since any value of a_1 combined with the zero value of c would make the product zero.

For the term in x^c,

$$a_2(c+1)(c+2)-2a_0=0 \text{ from which,}$$
$$a_2=\frac{2a_0}{(c+1)(c+2)} \qquad (31)$$

For the term in x^{c+r},

$$a_{r+2}(c+r+1)(c+r+2)-2a_r=0$$

from which,

$$a_{r+2}=\frac{2a_r}{(c+r+1)(c+r+2)} \qquad (32)$$

(a) **When $\boldsymbol{c=0}$:** a_1 is indeterminate, and from equation (31)

$$a_2=\frac{2a_0}{(1\times2)}=\frac{2a_0}{2!}$$

In general, $a_{r+2}=\dfrac{2a_r}{(r+1)(r+2)}$ and

when $r=1$, $a_3=\dfrac{2a_1}{(2\times3)}=\dfrac{2a_1}{(1\times2\times3)}=\dfrac{2a_1}{3!}$

when $r=2$, $a_4=\dfrac{2a_2}{3\times4}=\dfrac{4a_0}{4!}$

Hence, $y=x^0\left\{a_0+a_1x+\dfrac{2a_0}{2!}x^2+\dfrac{2a_1}{3!}x^3+\dfrac{4a_0}{4!}x^4+\cdots\right\}$

from equation (28)

$$=a_0\left\{1+\frac{2x^2}{2!}+\frac{4x^4}{4!}+\cdots\right\} +a_1\left\{x+\frac{2x^3}{3!}+\frac{4x^5}{5!}+\cdots\right\}$$

Since a_0 and a_1 are arbitrary constants depending on boundary conditions, let $a_0=P$ and $a_1=Q$, then:

$$\boldsymbol{y=P\left\{1+\frac{2x^2}{2!}+\frac{4x^4}{4!}+\cdots\right\} +Q\left\{x+\frac{2x^3}{3!}+\frac{4x^5}{5!}+\cdots\right\}} \qquad (33)$$

(b) **When $\boldsymbol{c=1}$:** $a_1=0$, and from equation (31),

$$a_2=\frac{2a_0}{(2\times3)}=\frac{2a_0}{3!}$$

Since $c=1$, $a_{r+2}=\dfrac{2a_r}{(c+r+1)(c+r+2)}=\dfrac{2a_r}{(r+2)(r+3)}$

from equation (32) and when $r=1$,

$$a_3=\frac{2a_1}{(3\times4)}=0 \text{ since } a_1=0$$

when $r=2$,

$$a_4=\frac{2a_2}{(4\times5)}=\frac{2}{(4\times5)}\times\frac{2a_0}{3!}=\frac{4a_0}{5!}$$

when $r=3$,

$$a_5=\frac{2a_3}{(5\times6)}=0$$

Hence, when $c=1$,

$$y=x^1\left\{a_0+\frac{2a_0}{3!}x^2+\frac{4a_0}{5!}x^4+\cdots\right\}$$

from equation (28)

i.e. $y=a_0\left\{x+\dfrac{2x^3}{3!}+\dfrac{4x^5}{5!}+\ldots\right\}$

Again, a_0 is an arbitrary constant; let $a_0=K$,

then $\boldsymbol{y=K\left\{x+\frac{2x^3}{3!}+\frac{4x^5}{5!}+\cdots\right\}}$

However, this latter solution is not a separate solution, for it is the same form as the second series in equation (33). Hence, equation (33) with its two arbitrary constants P and Q gives the general solution. This is always

the case when the two values of c differ by an integer (i.e. whole number). From the above three worked problems, the following can be deduced, and in future assumed:

(i) if two solutions of the indicial equation differ by a quantity *not* an integer, then two independent solutions $y = u(x) + v(x)$ result, the general solution of which is $y = Au + Bv$ (note: Problem 7 had $c = 0$ and $\frac{2}{3}$ and Problem 8 had $c = 1$ and $\frac{1}{2}$; in neither case did c differ by an integer)

(ii) if two solutions of the indicial equation *do* differ by an integer, as in Problem 9 where $c = 0$ and 1, and if one coefficient is indeterminate, as with when $c = 0$, then the complete solution is always given by using this value of c. Using the second value of c, i.e. $c = 1$ in Problem 9, always gives a series which is one of the series in the first solution.

Now try the following exercise

Exercise 196 Further problems on power series solution by the Frobenius method

1. Produce, using Frobenius' method, a power series solution for the differential equation: $2x\frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0$.

$$\left[y = A\left\{1 + x + \frac{x^2}{(2\times 3)} + \frac{x^3}{(2\times 3)(3\times 5)} + \cdots\right\} + Bx^{\frac{1}{2}}\left\{1 + \frac{x}{(1\times 3)} + \frac{x^2}{(1\times 2)(3\times 5)} + \frac{x^3}{(1\times 2\times 3)(3\times 5\times 7)} + \cdots\right\}\right]$$

2. Use the Frobenius method to determine the general power series solution of the differential equation: $\frac{d^2y}{dx^2} + y = 0$.

$$\left[y = A\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + B\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) = P\cos x + Q\sin x\right]$$

3. Determine the power series solution of the differential equation: $3x\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - y = 0$ using the Frobenius method.

$$\left[y = A\left\{1 + \frac{x}{(1\times 4)} + \frac{x^2}{(1\times 2)(4\times 7)} + \frac{x^3}{(1\times 2\times 3)(4\times 7\times 10)} + \cdots\right\} + Bx^{-\frac{1}{3}}\left\{1 + \frac{x}{(1\times 2)} + \frac{x^2}{(1\times 2)(2\times 5)} + \frac{x^3}{(1\times 2\times 3)(2\times 5\times 8)} + \cdots\right\}\right]$$

4. Show, using the Frobenius method, that the power series solution of the differential equation: $\frac{d^2y}{dx^2} - y = 0$ may be expressed as $y = P\cosh x + Q\sinh x$, where P and Q are constants. [Hint: check the series expansions for $\cosh x$ and $\sinh x$ on page 47]

52.6 Bessel's equation and Bessel's functions

One of the most important differential equations in applied mathematics is **Bessel's equation** and is of the form:

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - v^2)y = 0$$

where v is a real constant. The equation, which has applications in electric fields, vibrations and heat conduction, may be solved using Frobenius' method of the previous section.

Problem 10. Determine the general power series solution of Bessels equation.

Bessel's equation $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - v^2)y = 0$ may be rewritten as: $x^2y'' + xy' + (x^2 - v^2)y = 0$

Using the Frobenius method from page 500:

(i) Let a trial solution be of the form

$$y = x^c\{a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_rx^r + \cdots\} \quad (34)$$

where $a_0 \neq 0$,

i.e. $y = a_0x^c + a_1x^{c+1} + a_2x^{c+2} + a_3x^{c+3} + \cdots + a_rx^{c+r} + \cdots$ (35)

(ii) Differentiating equation (35) gives:

$$y' = a_0cx^{c-1} + a_1(c+1)x^c + a_2(c+2)x^{c+1} + \cdots + a_r(c+r)x^{c+r-1} + \cdots$$

and $y'' = a_0c(c-1)x^{c-2} + a_1c(c+1)x^{c-1} + a_2(c+1)(c+2)x^c + \cdots + a_r(c+r-1)(c+r)x^{c+r-2} + \cdots$

(iii) Substituting y, y' and y'' into each term of the given equation: $x^2y'' + xy' + (x^2 - v^2)y = 0$ gives:

$$\begin{aligned}&a_0c(c-1)x^c + a_1c(c+1)x^{c+1} + a_2(c+1)(c+2)x^{c+2} + \cdots \\ &+ a_r(c+r-1)(c+r)x^{c+r} + \cdots + a_0cx^c \\ &+ a_1(c+1)x^{c+1} + a_2(c+2)x^{c+2} + \cdots \\ &+ a_r(c+r)x^{c+r} + \cdots + a_0x^{c+2} + a_1x^{c+3} \\ &+ a_2x^{c+4} + \cdots + a_rx^{c+r+2} + \cdots - a_0v^2x^c \\ &- a_1v^2x^{c+1} - \cdots - a_rv^2x^{c+r} + \cdots = 0\end{aligned}$$ (36)

(iv) The **indicial equation** is obtained by equating the coefficient of the lowest power of x to zero.

Hence, $a_0c(c-1) + a_0c - a_0v^2 = 0$

from which, $a_0[c^2 - c + c - v^2] = 0$

i.e. $a_0[c^2 - v^2] = 0$

from which, $\boldsymbol{c = +v}$ or $\boldsymbol{c = -v}$ since $a_0 \neq 0$

For the term in x^{c+r},

$$a_r(c+r-1)(c+r) + a_r(c+r) + a_{r-2} - a_rv^2 = 0$$

$$a_r[(c+r-1)(c+r) + (c+r) - v^2] = -a_{r-2}$$

i.e. $a_r[(c+r)(c+r-1+1) - v^2] = -a_{r-2}$

i.e. $a_r[(c+r)^2 - v^2] = -a_{r-2}$

i.e. the **recurrence relation** is:

$$\boldsymbol{a_r = \frac{a_{r-2}}{v^2 - (c+r)^2}} \quad \text{for } r \geq 2 \qquad (37)$$

For the term in x^{c+1},

$$a_1[c(c+1) + (c+1) - v^2] = 0$$

i.e. $a_1[(c+1)^2 - v^2] = 0$

but if $c = v$ $a_1[(v+1)^2 - v^2] = 0$

i.e. $a_1[2v+1] = 0$

Similarly, if $c = -v$ $a_1[1-2v] = 0$

The terms $(2v+1)$ *and* $(1-2v)$ cannot both be zero since v is a real constant, hence $a_1 = 0$.

Since $a_1 = 0$, then from equation (37) $a_3 = a_5 = a_7 = \ldots = 0$

and

$$a_2 = \frac{a_0}{v^2 - (c+2)^2}$$

$$a_4 = \frac{a_0}{[v^2 - (c+2)^2][v^2 - (c+4)^2]}$$

$$a_6 = \frac{a_0}{[v^2 - (c+2)^2][v^2 - (c+4)^2][v^2 - (c+6)^2]}$$

and so on.

When $\boldsymbol{c = +v}$,

$$a_2 = \frac{a_0}{v^2 - (v+2)^2} = \frac{a_0}{v^2 - v^2 - 4v - 4} = \frac{-a_0}{4 + 4v} = \frac{-a_0}{2^2(v+1)}$$

$$a_4 = \frac{a_0}{\left[v^2 - (v+2)^2\right]\left[v^2 - (v+4)^2\right]} = \frac{a_0}{[-2^2(v+1)][-2^3(v+2)]} = \frac{a_0}{2^5(v+1)(v+2)} = \frac{a_0}{2^4 \times 2(v+1)(v+2)}$$

$$a_6 = \frac{a_0}{[v^2 - (v+2)^2][v^2 - (v+4)^2][v^2 - (v+6)^2]} = \frac{a_0}{[2^4 \times 2(v+1)(v+2)][-12(v+3)]} = \frac{-a_0}{2^4 \times 2(v+1)(v+2) \times 2^2 \times 3(v+3)} = \frac{-a_0}{2^6 \times 3!\,(v+1)(v+2)(v+3)}$$ and so on.

The resulting solution for $c = +v$ is given by:

$$y = u = Ax^v\left\{1 - \frac{x^2}{2^2(v+1)} + \frac{x^4}{2^4 \times 2!\,(v+1)(v+2)} - \frac{x^6}{2^6 \times 3!\,(v+1)(v+2)(v+3)} + \ldots\right\}$$ (38)

which is valid provided v is not a negative integer and where A is an arbitrary constant.

When $c = -v$,

$$a_2 = \frac{a_0}{v^2 - (-v+2)^2} = \frac{a_0}{v^2 - (v^2 - 4v + 4)}$$

$$= \frac{-a_0}{4 - 4v} = \frac{-a_0}{2^2(v-1)}$$

$$a_4 = \frac{a_0}{[2^2(v-1)][v^2 - (-v+4)^2]}$$

$$= \frac{a_0}{[2^2(v-1)][2^3(v-2)]}$$

$$= \frac{a_0}{2^4 \times 2(v-1)(v-2)}$$

Similarly, $a_6 = \dfrac{a_0}{2^6 \times 3!\,(v-1)(v-2)(v-3)}$

Hence,

$$y = w = B x^{-v}\left\{1 + \frac{x^2}{2^2(v-1)} + \frac{x^4}{2^4 \times 2!\,(v-1)(v-2)} + \frac{x^6}{2^6 \times 3!\,(v-1)(v-2)(v-3)} + \cdots\right\}$$

which is valid provided v is not a positive integer and where B is an arbitrary constant.

The complete solution of Bessel's equation:

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - v^2)\,y = 0 \text{ is:}$$

$$\boldsymbol{y = u + w =}$$

$$\mathbf{A}\,x^v\left\{1 - \frac{x^2}{2^2(v+1)} + \frac{x^4}{2^4 \times 2!(v+1)(v+2)} - \frac{x^6}{2^6 \times 3!(v+1)(v+2)(v+3)} + \cdots\right\}$$

$$+ B x^{-v}\left\{1 + \frac{x^2}{2^2(v-1)} + \frac{x^4}{2^4 \times 2!(v-1)(v-2)} + \frac{x^6}{2^6 \times 3!(v-1)(v-2)(v-3)} + \cdots\right\} \quad (39)$$

The gamma function

The solution of the Bessel equation of Problem 10 may be expressed in terms of **gamma functions**. Γ is the upper case Greek letter gamma, and the gamma function $\Gamma(x)$ is defined by the integral

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt \quad (40)$$

and is convergent for $x > 0$

From equation (40), $\Gamma(x+1) = \int_0^\infty t^x e^{-t}\,dt$

and by using integration by parts (see page 420):

$$\Gamma(x+1) = \left[(t^x)\left(\frac{e^{-t}}{-1}\right)\right]_0^\infty - \int_0^\infty \left(\frac{e^{-t}}{-1}\right) x\,t^{x-1}\,dx$$

$$= (0 - 0) + x\int_0^\infty e^{-t}t^{x-1}\,dt$$

$$= x\Gamma(x) \quad \text{from equation (40)}$$

This is an important recurrence relation for gamma functions.

Thus, since $\Gamma(x+1) = x\Gamma(x)$

then similarly, $\Gamma(x+2) = (x+1)\Gamma(x+1)$

$$= (x+1)x\Gamma(x) \quad (41)$$

and $\Gamma(x+3) = (x+2)\Gamma(x+2)$

$$= (x+2)(x+1)x\Gamma(x),$$

and so on.

These relationships involving gamma functions are used with Bessel functions.

Bessel functions

The power series solution of the Bessel equation may be written in terms of gamma functions as shown in worked problem 11 below.

Problem 11. Show that the power series solution of the Bessel equation of worked problem 10 may be written in terms of the Bessel functions $J_v(x)$ and $J_{-v}(x)$ as:

$$AJ_v(x) + BJ_{-v}(x) = \left(\frac{x}{2}\right)^v\left\{\frac{1}{\Gamma(v+1)} - \frac{x^2}{2^2(1!)\Gamma(v+2)} + \frac{x^4}{2^4(2!)\Gamma(v+4)} - \cdots\right\}$$

$$+\left(\frac{x}{2}\right)^{-v}\left\{\frac{1}{\Gamma(1-v)}-\frac{x^2}{2^2(1!)\Gamma(2-v)}+\frac{x^4}{2^4(2!)\Gamma(3-v)}-\cdots\right\}$$

From Problem 10 above, **when $c=+v$**,

$a_2=\dfrac{-a_0}{2^2(v+1)}$

If we let $a_0=\dfrac{1}{2^v\Gamma(v+1)}$

then

$$a_2=\frac{-1}{2^2(v+1)2^v\Gamma(v+1)}=\frac{-1}{2^{v+2}(v+1)\Gamma(v+1)}$$

$$=\frac{-1}{2^{v+2}\Gamma(v+2)} \quad \text{from equation (41)}$$

Similarly, $a_4=\dfrac{a_2}{v^2-(c+4)^2}$ from equation (37)

$$=\frac{a_2}{(v-c-4)(v+c+4)}=\frac{a_2}{-4(2v+4)}$$

since $c=v$

$$=\frac{-a_2}{2^3(v+2)}=\frac{-1}{2^3(v+2)}\frac{-1}{2^{v+2}\Gamma(v+2)}$$

$$=\frac{1}{2^{v+4}(2!)\Gamma(v+3)}$$

since $(v+2)\Gamma(v+2)=\Gamma(v+3)$

and $a_6=\dfrac{-1}{2^{v+6}(3!)\Gamma(v+4)}$ and so on.

The **recurrence relation** is:

$$a_r=\frac{(-1)^{r/2}}{2^{v+r}\left(\frac{r}{2}!\right)\Gamma\left(v+\frac{r}{2}+1\right)}$$

And if we let $r=2k$, then

$$\boldsymbol{a_{2k}=\frac{(-1)^k}{2^{v+2k}(k!)\Gamma(v+k+1)}} \qquad (42)$$

for $k=1, 2, 3, \ldots$

Hence, it is possible to write the new form for equation (38) as:

$$y=Ax^v\left\{\frac{1}{2^v\Gamma(v+1)}-\frac{x^2}{2^{v+2}(1!)\Gamma(v+2)}+\frac{x^4}{2^{v+4}(2!)\Gamma(v+3)}-\cdots\right\}$$

This is called *the Bessel function of the first order kind, of order v,* and is denoted by $J_v(x)$,

i.e. $$\boldsymbol{J_v(x)=\left(\frac{x}{2}\right)^v\left\{\frac{1}{\Gamma(v+1)}-\frac{x^2}{2^2(1!)\Gamma(v+2)}+\frac{x^4}{2^4(2!)\Gamma(v+3)}-\cdots\right\}}$$

provided v is not a negative integer.

For the second solution, **when $c=-v$**, replacing v by $-v$ in equation (42) above gives:

$$a_{2k}=\frac{(-1)^k}{2^{2k-v}(k!)\,\Gamma(k-v+1)}$$

from which, when $k=0, a_0=\dfrac{(-1)^0}{2^{-v}(0!)\Gamma(1-v)}$

$=\dfrac{1}{2^{-v}\Gamma(1-v)}$ since $0!=1$ (see page 495)

when $k=1$, $a_2=\dfrac{(-1)^1}{2^{2-v}(1!)\Gamma(1-v+1)}$

$$=\frac{-1}{2^{2-v}(1!)\Gamma(2-v)}$$

when $k=2$, $a_4=\dfrac{(-1)^2}{2^{4-v}(2!)\Gamma(2-v+1)}$

$$=\frac{1}{2^{4-v}(2!)\Gamma(3-v)}$$

when $k=3$, $a_6=\dfrac{(-1)^3}{2^{6-v}(3!)\Gamma(3-v+1)}$

$=\dfrac{1}{2^{6-v}(3!)\Gamma(4-v)}$ and so on.

Hence, $y=Bx^{-v}\left\{\dfrac{1}{2^{-v}\Gamma(1-v)}-\dfrac{x^2}{2^{2-v}(1!)\Gamma(2-v)}+\dfrac{x^4}{2^{4-v}(2!)\Gamma(3-v)}-\cdots\right\}$

i.e. $$\boldsymbol{J_{-v}(x)=\left(\frac{x}{2}\right)^{-v}\left\{\frac{1}{\Gamma(1-v)}-\frac{x^2}{2^2(1!)\Gamma(2-v)}+\frac{x^4}{2^4(2!)\Gamma(3-v)}-\cdots\right\}}$$

provided v is not a positive integer.

$J_v(x)$ and $J_{-v}(x)$ are two independent solutions of the Bessel equation; the complete solution is:

$y=AJ_v(x)+BJ_{-v}(x)$ where A and B are constants

i.e. $y = AJ_v(x) + BJ_{-v}(x)$

$$= A\left(\frac{x}{2}\right)^v \left\{\frac{1}{\Gamma(v+1)} - \frac{x^2}{2^2(1!)\Gamma(v+2)} + \frac{x^4}{2^4(2!)\Gamma(v+4)} - \cdots\right\}$$

$$+ B\left(\frac{x}{2}\right)^{-v} \left\{\frac{1}{\Gamma(1-v)} - \frac{x^2}{2^2(1!)\Gamma(2-v)} + \frac{x^4}{2^4(2!)\Gamma(3-v)} - \cdots\right\}$$

In general terms: $J_v(x) = \left(\frac{x}{2}\right)^v \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)\Gamma(v+k+1)}$

and $J_{-v}(x) = \left(\frac{x}{2}\right)^{-v} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)\Gamma(k-v+1)}$

Another Bessel function

It may be shown that another series for $J_n(x)$ is given by:

$$J_n(x) = \left(\frac{x}{2}\right)^n \left\{\frac{1}{n!} - \frac{1}{(n+1)!}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)(n+2)!}\left(\frac{x}{2}\right)^4 - \cdots\right\}$$

From this series two commonly used function are derived,

i.e. $J_0(x) = \frac{1}{(0!)} - \frac{1}{(1!)^2}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \cdots$

$$= 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + \cdots$$

and $J_1(x) = \frac{x}{2}\left\{\frac{1}{(1!)} - \frac{1}{(1!)(2!)}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)(3!)}\left(\frac{x}{2}\right)^4 - \cdots\right\}$

$$= \frac{x}{2} - \frac{x^3}{2^3(1!)(2!)} + \frac{x^5}{2^5(2!)(3!)} - \frac{x^7}{2^7(3!)(4!)} + \cdots$$

Tables of Bessel functions are available for a range of values of n and x, and in these, $J_0(x)$ and $J_1(x)$ are most commonly used.

Graphs of $J_0(x)$, which looks similar to a cosine, and $J_1(x)$, which looks similar to a sine, are shown in Figure 52.1.

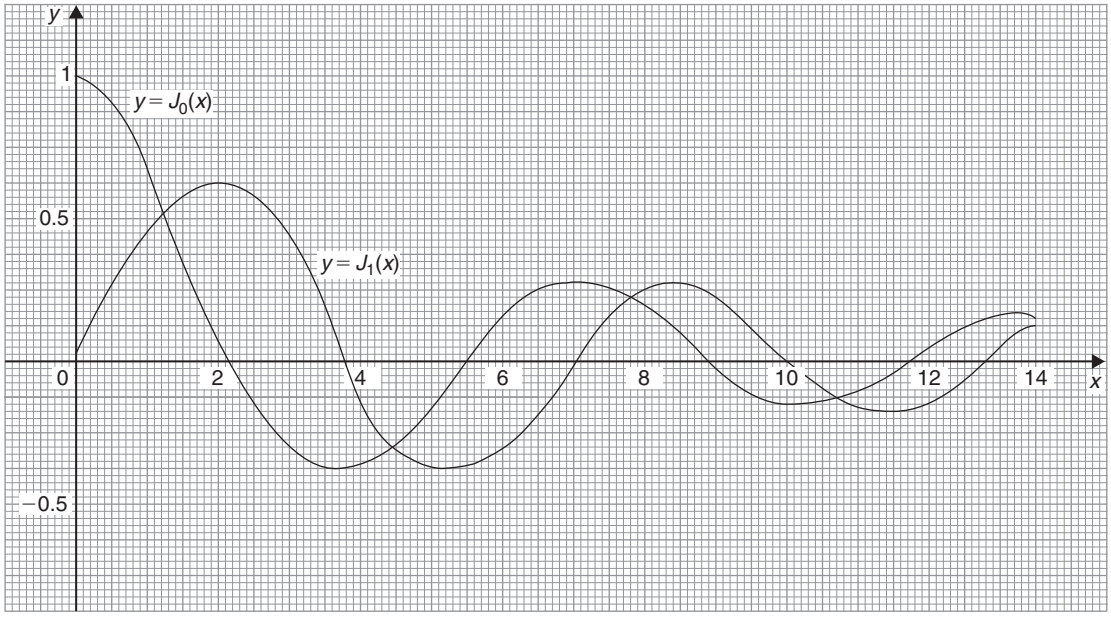


Figure 52.1

Now try the following exercise

Exercise 197 Further problems on Bessel's equation and Bessel's functions

1. Determine the power series solution of Bessel's equation: $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-v^2)y=0$ when $v=2$, up to and including the term in x^6.
$$\left[y=Ax^2\left\{1-\frac{x^2}{12}+\frac{x^4}{384}-\cdots\right\}\right]$$

2. Find the power series solution of the Bessel function: $x^2y''+xy'+\left(x^2-v^2\right)y=0$ in terms of the Bessel function $J_3(x)$ when $v=3$. Give the answer up to and including the term in x^7.
$$\left[y=AJ_3(x)=\left(\frac{x}{2}\right)^3\left\{\frac{1}{\Gamma 4}-\frac{x^2}{2^2\Gamma 5}+\frac{x^4}{2^5\Gamma 6}-\cdots\right\}\right]$$

3. Evaluate the Bessel functions $J_0(x)$ and $J_1(x)$ when $x=1$, correct to 3 decimal places.
$[J_0(x)=0.765,\ J_1(x)=0.440]$

52.7 Legendre's equation and Legendre polynomials

Another important differential equation in physics and engineering applications is Legendre's equation of the form: $(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+k(k+1)y=0$ or $(1-x^2)y''-2xy'+k(k+1)y=0$ where k is a real constant.

Problem 12. Determine the general power series solution of Legendre's equation.

To solve Legendre's equation $(1-x^2)y''-2xy'+k(k+1)y=0$ using the Frobenius method:

(i) Let a trial solution be of the form
$$y=x^c\left\{a_0+a_1x+a_2x^2+a_3x^3+\cdots+a_rx^r+\cdots\right\} \quad (43)$$
where $a_0\neq 0$,

i.e. $$y=a_0x^c+a_1x^{c+1}+a_2x^{c+2}+a_3x^{c+3}+\cdots+a_rx^{c+r}+\cdots \quad (44)$$

(ii) Differentiating equation (44) gives:
$$y'=a_0cx^{c-1}+a_1(c+1)x^c+a_2(c+2)x^{c+1}+\cdots+a_r(c+r)x^{c+r-1}+\cdots$$
and $$y''=a_0c(c-1)x^{c-2}+a_1c(c+1)x^{c-1}+a_2(c+1)(c+2)x^c+\cdots+a_r(c+r-1)(c+r)x^{c+r-2}+\cdots$$

(iii) Substituting y, y' and y'' into each term of the given equation:
$\left(1-x^2\right)y''-2xy'+k(k+1)y=0$ gives:
$$\begin{aligned}&a_0c(c-1)x^{c-2}+a_1c(c+1)x^{c-1}\\&+a_2(c+1)(c+2)x^c+\cdots\\&+a_r(c+r-1)(c+r)x^{c+r-2}+\cdots\\&-a_0c(c-1)x^c-a_1c(c+1)x^{c+1}\\&-a_2(c+1)(c+2)x^{c+2}-\cdots\\&-a_r(c+r-1)(c+r)x^{c+r}-\cdots-2a_0cx^c\\&-2a_1(c+1)x^{c+1}-2a_2(c+2)x^{c+2}-\cdots\\&-2a_r(c+r)x^{c+r}-\cdots+k^2a_0x^c\\&+k^2a_1x^{c+1}+k^2a_2x^{c+2}+\cdots+k^2a_rx^{c+r}\\&+\cdots+ka_0x^c+ka_1x^{c+1}+\cdots\\&+ka_rx^{c+r}+\cdots=0\end{aligned} \quad (45)$$

(iv) The **indicial equation** is obtained by equating the coefficient of the lowest power of x (i.e. x^{c-2}) to zero. Hence, $a_0c(c-1)=0$ from which, $\boldsymbol{c=0}$ or $\boldsymbol{c=1}$ since $a_0\neq 0$.

For the term in x^{c-1}, i.e. $a_1c(c+1)=0$ With $c=1$, $a_1=0$; however, when $c=0$, a_1 is indeterminate, since any value of a_1 combined with the zero value of c would make the product zero.

For the term in x^{c+r},
$a_{r+2}(c+r+1)(c+r+2)-a_r(c+r-1)(c+r)-2a_r(c+r)+k^2a_r+ka_r=0$

from which,

$$a_{r+2} = \frac{a_r\left[(c+r-1)(c+r)+2(c+r)-k^2-k\right]}{(c+r+1)(c+r+2)}$$

$$= \frac{a_r[(c+r)(c+r+1)-k(k+1)]}{(c+r+1)(c+r+2)} \quad (46)$$

When $c=0$,

$$a_{r+2} = \frac{a_r[r(r+1)-k(k+1)]}{(r+1)(r+2)}$$

For $r=0$,

$$a_2 = \frac{a_0[-k(k+1)]}{(1)(2)}$$

For $r=1$,

$$a_3 = \frac{a_1[(1)(2)-k(k+1)]}{(2)(3)}$$

$$= \frac{-a_1[k^2+k-2]}{3!} = \frac{-a_1(k-1)(k+2)}{3!}$$

For $r=2$,

$$a_4 = \frac{a_2[(2)(3)-k(k+1)]}{(3)(4)} = \frac{-a_2\left[k^2+k-6\right]}{(3)(4)}$$

$$= \frac{-a_2(k+3)(k-2)}{(3)(4)}$$

$$= \frac{-(k+3)(k-2)}{(3)(4)} \cdot \frac{a_0[-k(k+1)]}{(1)(2)}$$

$$= \frac{a_0k(k+1)(k+3)(k-2)}{4!}$$

For $r=3$,

$$a_5 = \frac{a_3[(3)(4)-k(k+1)]}{(4)(5)} = \frac{-a_3[k^2+k-12]}{(4)(5)}$$

$$= \frac{-a_3(k+4)(k-3)}{(4)(5)}$$

$$= \frac{-(k+4)(k-3)}{(4)(5)} \cdot \frac{-a_1(k-1)(k+2)}{(2)(3)}$$

$$= \frac{a_1(k-1)(k-3)(k+2)(k+4)}{5!}$$ and so on.

Substituting values into equation (43) gives:

$$y = x^0\left\{a_0 + a_1x - \frac{a_0k(k+1)}{2!}x^2 - \frac{a_1(k-1)(k+2)}{3!}x^3 + \frac{a_0k(k+1)(k-2)(k+3)}{4!}x^4 + \frac{a_1(k-1)(k-3)(k+2)(k+4)}{5!}x^5 + \cdots\right\}$$

i.e. $$\boldsymbol{y = a_0\left\{1 - \frac{k(k+1)}{2!}x^2 + \frac{k(k+1)(k-2)(k+3)}{4!}x^4 - \cdots\right\} + a_1\left\{x - \frac{(k-1)(k+2)}{3!}x^3 + \frac{(k-1)(k-3)(k+2)(k+4)}{5!}x^5 - \cdots\right\}} \quad (47)$$

From page 506, it was stated that if two solutions of the indicial equation differ by an integer, as in this case, where $c=0$ and 1, and if one coefficient is indeterminate, as with when $c=0$, then the complete solution is always given by using this value of c. Using the second value of c, i.e. $c=1$ in this problem, will give a series which is one of the series in the first solution. (This may be checked for $c=1$ and where $a_1=0$; the result will be the first part of equation (47) above).

Legendre's polynomials

(A **polynomial** is an expression of the form: $f(x)=a+bx+cx^2+dx^3+\cdots$). When k in equation (47) above is an integer, say, n, one of the solution series terminates after a finite number of terms. For example, if $k=2$, then the first series terminates after the term in x^2. The resulting polynomial in x, denoted by $P_n(x)$, is called a **Legendre polynomial**. Constants a_0 and a_1 are chosen so that $y=1$ when $x=1$. This is demonstrated in the following worked problems.

Problem 13. Determine the Legendre polynomial $P_2(x)$.

Since in $P_2(x)$, $n=k=2$, then from the first part of equation (47), i.e. the even powers of x:

$$y = a_0\left\{1 - \frac{2(3)}{2!}x^2 + 0\right\} = a_0\{1-3x^2\}$$

a_0 is chosen to make $y=1$ when $x=1$

i.e. $1 = a_0\{1-3(1)^2\} = -2a_0$, from which, $a_0 = -\frac{1}{2}$

Hence, $\boldsymbol{P_2(x)=-\frac{1}{2}\left(1-3x^2\right)=\frac{1}{2}(3x^2-1)}$

Problem 14. Determine the Legendre polynomial $P_3(x)$.

Since in $P_3(x)$, $n=k=3$, then from the second part of equation (47), i.e. the odd powers of x:

$$y=a_1\left\{x-\frac{(k-1)(k+2)}{3!}x^3+\frac{(k-1)(k-3)(k+2)(k+4)}{5!}x^5-\cdots\right\}$$

i.e. $$y=a_1\left\{x-\frac{(2)(5)}{3!}x^3+\frac{(2)(0)(5)(7)}{5!}x^5\right\}$$

$$=a_1\left\{x-\frac{5}{3}x^3+0\right\}$$

a_1 is chosen to make $y=1$ when $x=1$.

i.e. $1=a_1\left\{1-\frac{5}{3}\right\}=a_1\left(-\frac{2}{3}\right)$ from which, $a_1=-\frac{3}{2}$

Hence, $P_3(x)=-\frac{3}{2}\left(x-\frac{5}{3}x^3\right)$ or $\boldsymbol{P_3(x)=\frac{1}{2}(5x^3-3x)}$

Rodrigue's formula

An alternative method of determining Legendre polynomials is by using **Rodrigue's formula**, which states:

$$\boldsymbol{P_n(x)=\frac{1}{2^n n!}\frac{d^n\left(x^2-1\right)^n}{dx^n}} \tag{48}$$

This is demonstrated in the following worked problems.

Problem 15. Determine the Legendre polynomial $P_2(x)$ using Rodrigue's formula.

In Rodrigue's formula, $P_n(x)=\frac{1}{2^n n!}\frac{d^n\left(x^2-1\right)^n}{dx^n}$ and when $n=2$,

$$P_2(x)=\frac{1}{2^2 2!}\frac{d^2(x^2-1)^2}{dx^2}$$

$$=\frac{1}{2^3}\frac{d^2(x^4-2x^2+1)}{dx^2}$$

$$\frac{d}{dx}(x^4-2x^2+1)$$

$$=4x^3-4x$$

and $$\frac{d^2\left(x^4-2x^2+1\right)}{dx^2}=\frac{d(4x^3-4x)}{dx}=12x^2-4$$

Hence, $P_2(x)=\frac{1}{2^3}\frac{d^2\left(x^4-2x^2+1\right)}{dx^2}=\frac{1}{8}\left(12x^2-4\right)$

i.e. $\boldsymbol{P_2(x)=\frac{1}{2}(3x^2-1)}$ the same as in Problem 13.

Problem 16. Determine the Legendre polynomial $P_3(x)$ using Rodrigue's formula.

In Rodrigue's formula, $P_n(x)=\frac{1}{2^n n!}\frac{d^n\left(x^2-1\right)^n}{dx^n}$ and when $n=3$,

$$P_3(x)=\frac{1}{2^3 3!}\frac{d^3\left(x^2-1\right)^3}{dx^3}$$

$$=\frac{1}{2^3(6)}\frac{d^3\left(x^2-1\right)\left(x^4-2x^2+1\right)}{dx^3}$$

$$=\frac{1}{(8)(6)}\frac{d^3\left(x^6-3x^4+3x^2-1\right)}{dx^3}$$

$$\frac{d\left(x^6-3x^4+3x^2-1\right)}{dx}=6x^5-12x^3+6x$$

$$\frac{d\left(6x^5-12x^3+6x\right)}{dx}=30x^4-36x^2+6$$

and $$\frac{d\left(30x^4-36x^2+6\right)}{dx}=120x^3-72x$$

Hence, $P_3(x)=\frac{1}{(8)(6)}\frac{d^3\left(x^6-3x^4+3x^2-1\right)}{dx^3}$

$$=\frac{1}{(8)(6)}\left(120x^3-72x\right)=\frac{1}{8}\left(20x^3-12x\right)$$

i.e. $\boldsymbol{P_3(x)=\frac{1}{2}(5x^3-3x)}$ the same as in Problem 14.

Now try the following exercise

Exercise 198 Legendre's equation and Legendre polynomials

1. Determine the power series solution of the Legendre equation:
   $\left(1-x^2\right)y''-2xy'+k(k+1)y=0$ when
   (a) $k=0$ (b) $k=2$, up to and including the

term in x^5.

$$\left[\begin{array}{l}\text{(a) } y = a_0 + a_1\left(x + \dfrac{x^3}{3} + \dfrac{x^5}{5} + \cdots\right) \\ \text{(b) } y = a_0\left\{1 - 3x^2\right\} \\ \qquad\qquad + a_1\left\{x - \dfrac{2}{3}x^3 - \dfrac{1}{5}x^5\right\}\end{array}\right]$$

2. Find the following Legendre polynomials: (a) $P_1(x)$ (b) $P_4(x)$ (c) $P_5(x)$.

$$\left[\begin{array}{l}\text{(a) } x \text{ (b) } \dfrac{1}{8}\left(35x^4 - 30x^2 + 3\right) \\ \text{(c) } \dfrac{1}{8}\left(63x^5 - 70x^3 + 15x\right)\end{array}\right]$$

Chapter 53

An introduction to partial differential equations

53.1 Introduction

A partial differential equation is an equation that contains one or more partial derivatives. Examples include:

(i) $a\dfrac{\partial u}{\partial x}+b\dfrac{\partial u}{\partial y}=c$

(ii) $\dfrac{\partial^2 u}{\partial x^2}=\dfrac{1}{c^2}\dfrac{\partial u}{\partial t}$

(known as the heat conduction equation)

(iii) $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$

(known as Laplace's equation)

Equation (i) is a **first order partial differential equation**, and equations (ii) and (iii) are **second order partial differential equations** since the highest power of the differential is 2.

Partial differential equations occur in many areas of engineering and technology; electrostatics, heat conduction, magnetism, wave motion, hydrodynamics and aerodynamics all use models that involve partial differential equations. Such equations are difficult to solve, but techniques have been developed for the simpler types. In fact, for all but for the simplest cases, there are a number of numerical methods of solutions of partial differential equations available.

To be able to solve simple partial differential equations knowledge of the following is required:

(a) partial integration,

(b) first and second order partial differentiation — as explained in Chapter 34, and

(c) the solution of ordinary differential equations — as explained in Chapters 46–51.

It should be appreciated that whole books have been written on partial differential equations and their solutions. This chapter does no more than introduce the topic.

53.2 Partial integration

Integration is the reverse process of differentiation. Thus, if, for example, $\dfrac{\partial u}{\partial t}=5\cos x\sin t$ is integrated partially with respect to t, then the $5\cos x$ term is considered as a constant,

$$\text{and}\quad u=\int 5\cos x\sin t\,\mathrm{d}t=(5\cos x)\int \sin t\,\mathrm{d}t$$
$$=(5\cos x)(-\cos t)+c$$
$$=\mathbf{-5\cos x\cos t+f(x)}$$

Similarly, if $\dfrac{\partial^2 u}{\partial x\,\partial y}=6x^2\cos 2y$ is integrated partially with respect to y,

$$\text{then}\quad \frac{\partial u}{\partial x}=\int 6x^2\cos 2y\,\mathrm{d}y=\left(6x^2\right)\int \cos 2y\,\mathrm{d}y$$
$$=\left(6x^2\right)\left(\frac{1}{2}\sin 2y\right)+f(x)$$
$$=3x^2\sin 2y+f(x)$$

and integrating $\frac{\partial u}{\partial x}$ partially with respect to x gives:

$$u = \int [3x^2 \sin 2y + f(x)]\,dx$$

$$= \boldsymbol{x^3 \sin 2y + (x)f(x) + g(y)}$$

$f(x)$ and $g(y)$ are functions that may be determined if extra information, called **boundary conditions** or **initial conditions**, are known.

53.3 Solution of partial differential equations by direct partial integration

The simplest form of partial differential equations occurs when a solution can be determined by direct partial integration. This is demonstrated in the following worked problems.

Problem 1. Solve the differential equation $\frac{\partial^2 u}{\partial x^2} = 6x^2(2y-1)$ given the boundary conditions that at $x=0$, $\frac{\partial u}{\partial x} = \sin 2y$ and $u = \cos y$.

Since $\frac{\partial^2 u}{\partial x^2} = 6x^2(2y-1)$ then integrating partially with respect to x gives:

$$\frac{\partial u}{\partial x} = \int 6x^2(2y-1)\,dx = (2y-1)\int 6x^2\,dx$$

$$= (2y-1)\frac{6x^3}{3} + f(y)$$

$$= 2x^3(2y-1) + f(y)$$

where $f(y)$ is an arbitrary function.
From the boundary conditions, when $x=0$,

$$\frac{\partial u}{\partial x} = \sin 2y.$$

Hence, $\sin 2y = 2(0)^3(2y-1) + f(y)$

from which, $f(y) = \sin 2y$

Now $\frac{\partial u}{\partial x} = 2x^3(2y-1) + \sin 2y$

Integrating partially with respect to x gives:

$$u = \int [2x^3(2y-1) + \sin 2y]\,dx$$

$$= \frac{2x^4}{4}(2y-1) + x(\sin 2y) + F(y)$$

From the boundary conditions, when $x=0$, $u = \cos y$, hence

$$\cos y = \frac{(0)^4}{2}(2y-1) + (0)\sin 2y + F(y)$$

from which, $F(y) = \cos y$

Hence, the solution of $\frac{\partial^2 u}{\partial x^2} = 6x^2(2y-1)$ for the given boundary conditions is:

$$\boldsymbol{u = \frac{x^4}{2}(2y-1) + x\sin y + \cos y}$$

Problem 2. Solve the differential equation: $\frac{\partial^2 u}{\partial x\,\partial y} = \cos(x+y)$ given that $\frac{\partial u}{\partial x} = 2$ when $y=0$, and $u = y^2$ when $x=0$.

Since $\frac{\partial^2 u}{\partial x\,\partial y} = \cos(x+y)$ then integrating partially with respect to y gives:

$$\frac{\partial u}{\partial x} = \int \cos(x+y)\,dy = \sin(x+y) + f(x)$$

From the boundary conditions, $\frac{\partial u}{\partial x} = 2$ when $y=0$, hence

$$2 = \sin x + f(x)$$

from which, $f(x) = 2 - \sin x$

i.e. $\frac{\partial u}{\partial x} = \sin(x+y) + 2 - \sin x$

Integrating partially with respect to x gives:

$$u = \int [\sin(x+y) + 2 - \sin x]\,dx$$

$$= -\cos(x+y) + 2x + \cos x + f(y)$$

From the boundary conditions, $u = y^2$ when $x=0$, hence

$$y^2 = -\cos y + 0 + \cos 0 + f(y)$$

$$= 1 - \cos y + f(y)$$

from which, $f(y) = y^2 - 1 + \cos y$

Hence, the solution of $\frac{\partial^2 u}{\partial x\,\partial y} = \cos(x+y)$ is given by:

$$\boldsymbol{u = -\cos(x+y) + 2x + \cos x + y^2 - 1 + \cos y}$$

Problem 3. Verify that $\phi(x, y, z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$ satisfies the partial differential equation: $\dfrac{\partial^2\phi}{\partial x^2}+\dfrac{\partial^2\phi}{\partial y^2}+\dfrac{\partial^2\phi}{\partial z^2}=0$.

The partial differential equation $\dfrac{\partial^2\phi}{\partial x^2}+\dfrac{\partial^2\phi}{\partial y^2}+\dfrac{\partial^2\phi}{\partial z^2}=0$ is called **Laplace's equation**.

If $\phi(x, y, z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}} = (x^2+y^2+z^2)^{-\frac{1}{2}}$

then differentiating partially with respect to x gives:

$$\frac{\partial\phi}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}(2x)$$

$$= -x(x^2+y^2+z^2)^{-\frac{3}{2}}$$

and $$\frac{\partial^2\phi}{\partial x^2} = (-x)\left[-\frac{3}{2}(x^2+y^2+z^2)^{-\frac{5}{2}}(2x)\right] + (x^2+y^2+z^2)^{-\frac{3}{2}}(-1)$$

by the product rule

$$= \frac{3x^2}{(x^2+y^2+z^2)^{\frac{5}{2}}} - \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$$= \frac{(3x^2)-(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

Similarly, it may be shown that

$$\frac{\partial^2\phi}{\partial y^2} = \frac{(3y^2)-(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

and $$\frac{\partial^2\phi}{\partial z^2} = \frac{(3z^2)-(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

Thus,

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2} = \frac{(3x^2)-(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{\frac{5}{2}}} + \frac{(3y^2)-(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{\frac{5}{2}}} + \frac{(3z^2)-(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

$$= \frac{\begin{pmatrix}3x^2-(x^2+y^2+z^2)\\+3y^2-(x^2+y^2+z^2)\\+3z^2-(x^2+y^2+z^2)\end{pmatrix}}{(x^2+y^2+z^2)^{\frac{5}{2}}} = 0$$

Thus, $\dfrac{1}{\sqrt{x^2+y^2+z^2}}$ satisfies the Laplace equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0$$

Now try the following exercise

Exercise 199 Further problems on the solution of partial differential equations by direct partial integration

1. Determine the general solution of $\dfrac{\partial u}{\partial y}=4ty$. $[u=2ty^2+f(t)]$

2. Solve $\dfrac{\partial u}{\partial t}=2t\cos\theta$ given that $u=2t$ when $\theta=0$. $[u=t^2(\cos\theta-1)+2t]$

3. Verify that $u(\theta, t)=\theta^2+\theta t$ is a solution of $\dfrac{\partial u}{\partial\theta}-2\dfrac{\partial u}{\partial t}=t$.

4. Verify that $u=\mathrm{e}^{-y}\cos x$ is a solution of $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$.

5. Solve $\dfrac{\partial^2 u}{\partial x\,\partial y}=8\mathrm{e}^y\sin 2x$ given that at $y=0$, $\dfrac{\partial u}{\partial x}=\sin x$, and at $x=\dfrac{\pi}{2}$, $u=2y^2$. $[u=-4\mathrm{e}^y\cos 2x-\cos x+4\cos 2x+2y^2-4\mathrm{e}^y+4]$

6. Solve $\dfrac{\partial^2 u}{\partial x^2}=y(4x^2-1)$ given that at $x=0$, $u=\sin y$ and $\dfrac{\partial u}{\partial x}=\cos 2y$. $\left[u=y\left(\dfrac{x^4}{3}-\dfrac{x^2}{2}\right)+x\cos 2y+\sin y\right]$

7. Solve $\frac{\partial^2 u}{\partial x \partial t} = \sin(x+t)$ given that $\frac{\partial u}{\partial x} = 1$ when $t = 0$, and when $u = 2t$ when $x = 0$.

   $[u = -\sin(x+t) + x + \sin x + 2t + \sin t]$

8. Show that $u(x, y) = xy + \frac{x}{y}$ is a solution of $2x\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} = 2x$.

9. Find the particular solution of the differential equation $\frac{\partial^2 u}{\partial x \partial y} = \cos x \cos y$ given the initial conditions that when $y = \pi$, $\frac{\partial u}{\partial x} = x$, and when $x = \pi$, $u = 2\cos y$.

   $\left[u = \sin x \sin y + \frac{x^2}{2} + 2\cos y - \frac{\pi^2}{2}\right]$

10. Verify that $\phi(x, y) = x\cos y + e^x \sin y$ satisfies the differential equation

    $$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + x\cos y = 0.$$

53.4 Some important engineering partial differential equations

There are many types of partial differential equations. Some typically found in engineering and science include:

(a) The **wave equation**, where the equation of motion is given by:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

where $c^2 = \frac{T}{\rho}$, with T being the tension in a string and ρ being the mass/unit length of the string.

(b) The **heat conduction equation** is of the form:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial u}{\partial t}$$

where $c^2 = \frac{h}{\sigma\rho}$, with h being the thermal conductivity of the material, σ the specific heat of the material, and ρ the mass/unit length of material.

(c) **Laplace's equation**, used extensively with electrostatic fields is of the form:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

(d) The **transmission equation**, where the potential u in a transmission cable is of the form: $\frac{\partial^2 u}{\partial x^2} = A\frac{\partial^2 u}{\partial t^2} + B\frac{\partial u}{\partial t} + Cu$ where A, B and C are constants.

Some of these equations are used in the next sections.

53.5 Separating the variables

Let $u(x, t) = X(x)T(t)$, where $X(x)$ is a function of x only and $T(t)$ is a function of t only, be a trial solution to the wave equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$. If the trial solution is simplified to $u = XT$, then $\frac{\partial u}{\partial x} = X'T$ and $\frac{\partial^2 u}{\partial x^2} = X''T$. Also $\frac{\partial u}{\partial t} = XT'$ and $\frac{\partial^2 u}{\partial t^2} = XT''$.

Substituting into the partial differential equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$ gives:

$$X''T = \frac{1}{c^2}XT''$$

Separating the variables gives:

$$\frac{\boldsymbol{X''}}{\boldsymbol{X}} = \frac{\boldsymbol{1}}{\boldsymbol{c^2}}\frac{\boldsymbol{T''}}{\boldsymbol{T}}$$

Let $\mu = \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T}$ where μ is a constant.

Thus, since $\mu = \frac{X''}{X}$ (a function of x only), it must be independent of t; and, since $\mu = \frac{1}{c^2}\frac{T''}{T}$ (a function of t only), it must be independent of x.

If μ is independent of x *and* t, it can only be a constant. If $\mu = \frac{X''}{X}$ then $X'' = \mu X$ or $X'' - \mu X = 0$ and if $\mu = \frac{1}{c^2}\frac{T''}{T}$ then $T'' = c^2\mu T$ or $T'' - c^2\mu T = 0$.

Such ordinary differential equations are of the form found in Chapter 50, and their solutions will depend on whether $\mu > 0$, $\mu = 0$ or $\mu < 0$.

Worked Problem 4 will be a reminder of solving ordinary differential equations of this type.

Problem 4. Find the general solution of the following differential equations:

(a) $X''-4X=0$ (b) $T''+4T=0$.

(a) If $X''-4X=0$ then the auxiliary equation (see Chapter 50) is:

$m^2-4=0$ i.e. $m^2=4$ from which,

$m=+2$ or $m=-2$

Thus, the general solution is:

$$\boldsymbol{X=Ae^{2x}+Be^{-2x}}$$

(b) If $T''+4T=0$ then the auxiliary equation is:

$m^2+4=0$ i.e. $m^2=-4$ from which,

$m=\sqrt{-4}=\pm j2$

Thus, the general solution is:

$$\boldsymbol{T=e^0\{A\cos 2t+B\sin 2t\}=A\cos 2t+B\sin 2t}$$

Now try the following exercise

Exercise 200 Further problems on revising the solution of ordinary differential equation

1. Solve $T''=c^2\mu T$ given $c=3$ and $\mu=1$.
   $[T=Ae^{3t}+Be^{-3t}]$
2. Solve $T''-c^2\mu T=0$ given $c=3$ and $\mu=-1$.
   $[T=A\cos 3t+B\sin 3t]$
3. Solve $X''=\mu X$ given $\mu=1$.
   $[X=Ae^{x}+Be^{-x}]$
4. Solve $X''-\mu X=0$ given $\mu=-1$.
   $[X=A\cos x+B\sin x]$

53.6 The wave equation

An **elastic string** is a string with elastic properties, i.e. the string satisfies Hooke's law. Figure 53.1 shows a flexible elastic string stretched between two points at $x=0$ and $x=L$ with uniform tension T. The string will vibrate if the string is displace slightly from its initial position of rest and released, the end points remaining

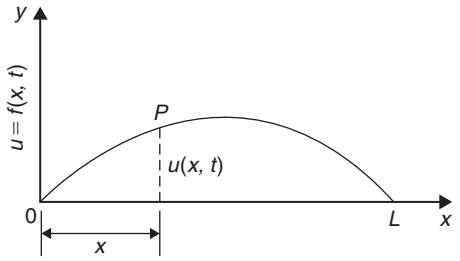


Figure 53.1

fixed. The position of any point P on the string depends on its distance from one end, and on the instant in time. Its displacement u at any time t can be expressed as $u=f(x,t)$, where x is its distance from 0.

The equation of motion is as stated in Section 53.4 (a), i.e. $\dfrac{\partial^2 u}{\partial x^2}=\dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2}$.

The boundary and initial conditions are:

(i) The string is fixed at both ends, i.e. $x=0$ and $x=L$ for all values of time t.

Hence, $u(x,t)$ becomes:

$$\left.\begin{aligned}u(0,t)&=0\\u(L,t)&=0\end{aligned}\right\}\ \text{for all values of } t\geq 0$$

(ii) If the initial deflection of P at $t=0$ is denoted by $f(x)$ then $u(x,0)=f(x)$

(iii) Let the initial velocity of P be $g(x)$, then

$$\left[\frac{\partial u}{\partial t}\right]_{t=0}=g(x)$$

Initially a **trial solution** of the form $u(x,t)=X(x)T(t)$ is assumed, where $X(x)$ is a function of x only and $T(t)$ is a function of t only. The trial solution may be simplified to $u=XT$ and the variables separated as explained in the previous section to give:

$$\frac{X''}{X}=\frac{1}{c^2}\frac{T''}{T}$$

When both sides are equated to a constant μ this results in two ordinary differential equations:

$$T''-c^2\mu T=0 \quad \text{and} \quad X''-\mu X=0$$

Three cases are possible, depending on the value of μ.

Case 1: $\mu>0$

For convenience, let $\mu=p^2$, where p is a real constant. Then the equations

$$X''-p^2X=0 \quad \text{and} \quad T''-c^2p^2T=0$$

have solutions: $X = Ae^{px} + Be^{-px}$ and $T = Ce^{cpt} + De^{-cpt}$ where A, B, C and D are constants.
But $X = 0$ at $x = 0$, hence $0 = A + B$ i.e. $B = -A$ and $X = 0$ at $x = L$, hence
$0 = Ae^{pL} + Be^{-pL} = A(e^{pL} - e^{-pL})$.
Assuming $(e^{pL} - e^{-pL})$ is not zero, then $A = 0$ and since $B = -A$, then $B = 0$ also.
This corresponds to the string being stationary; since it is non-oscillatory, this solution will be disregarded.

Case 2: $\mu = 0$

In this case, since $\mu = p^2 = 0$, $T'' = 0$ and $X'' = 0$. We will assume that $T(t) \neq 0$. Since $X'' = 0$, $X' = a$ and $X = ax + b$ where a and b are constants. But $X = 0$ at $x = 0$, hence $b = 0$ and $X = ax$ and $X = 0$ at $x = L$, hence $a = 0$. Thus, again, the solution is non-oscillatory and is also disregarded.

Case 3: $\mu < 0$

For convenience,
let $\mu = -p^2$ then $X'' + p^2X = 0$ from which,

$$X = A\cos px + B\sin px \tag{1}$$

and $T'' + c^2p^2T = 0$ from which,

$$T = C\cos cpt + D\sin cpt \tag{2}$$

(see worked Problem 4 above).

Thus, the suggested solution $u = XT$ now becomes:

$$u = \{A\cos px + B\sin px\}\{C\cos cpt + D\sin cpt\} \tag{3}$$

Applying the boundary conditions:

(i) $u = 0$ when $x = 0$ for all values of t,

thus $0 = \{A\cos 0 + B\sin 0\}\{C\cos cpt + D\sin cpt\}$

i.e. $0 = A\{C\cos cpt + D\sin cpt\}$

from which, $A = 0$, (since $\{C\cos cpt + D\sin cpt\} \neq 0$)

Hence, $u = \{B\sin px\}\{C\cos cpt + D\sin cpt\}$ (4)

(ii) $u = 0$ when $x = L$ for all values of t

Hence, $0 = \{B\sin pL\}\{C\cos cpt + D\sin cpt\}$

Now $B \neq 0$ or $u(x, t)$ would be identically zero.

Thus $\sin pL = 0$ i.e. $pL = n\pi$ or $p = \dfrac{n\pi}{L}$ for integer values of n.
Substituting in equation (4) gives:

$$u = \left\{B\sin\frac{n\pi x}{L}\right\}\left\{C\cos\frac{cn\pi t}{L} + D\sin\frac{cn\pi t}{L}\right\}$$

i.e. $u = \sin\dfrac{n\pi x}{L}\left\{A_n\cos\dfrac{cn\pi t}{L} + B_n\sin\dfrac{cn\pi t}{L}\right\}$

(where constant $A_n = BC$ and $B_n = BD$). There will be many solutions, depending on the value of n. Thus, more generally,

$$\boldsymbol{u_n(x,t) = \sum_{n=1}^{\infty}\left\{\sin\frac{n\pi x}{L}\left(A_n\cos\frac{cn\pi t}{L} + B_n\sin\frac{cn\pi t}{L}\right)\right\}} \tag{5}$$

To find A_n and B_n we put in the initial conditions not yet taken into account.

(i) At $t = 0$, $u(x, 0) = f(x)$ for $0 \leq x \leq L$
Hence, from equation (5),

$$\boldsymbol{u(x,0) = f(x) = \sum_{n=1}^{\infty}\left\{A_n\sin\frac{n\pi x}{L}\right\}} \tag{6}$$

(ii) Also at $t = 0$, $\left[\dfrac{\partial u}{\partial t}\right]_{t=0} = g(x)$ for $0 \leq x \leq L$

Differentiating equation (5) with respect to t gives:

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}\left\{\sin\frac{n\pi x}{L}\left(A_n\left(-\frac{cn\pi}{L}\sin\frac{cn\pi t}{L}\right) + B_n\left(\frac{cn\pi}{L}\cos\frac{cn\pi t}{L}\right)\right)\right\}$$

and when $t = 0$,

$$g(x) = \sum_{n=1}^{\infty}\left\{\sin\frac{n\pi x}{L}B_n\frac{cn\pi}{L}\right\}$$

i.e. $\boldsymbol{g(x) = \dfrac{c\pi}{L}\sum_{n=1}^{\infty}\left\{B_n n\sin\dfrac{n\pi x}{L}\right\}}$ (7)

From Fourier series (see page 638) it may be shown that: A_n is twice the mean value of $f(x)\sin\dfrac{n\pi x}{L}$ between $x = 0$ and $x = L$

i.e. $\boldsymbol{A_n = \dfrac{2}{L}\int_0^L f(x)\sin\dfrac{n\pi x}{L}dx}$

for $n = 1, 2, 3, \ldots$ (8)

and $B_n\left(\frac{cn\pi}{L}\right)$ is twice the mean value of $g(x)\sin\frac{n\pi x}{L}$ between $x=0$ and $x=L$

i.e. $$B_n=\frac{L}{cn\pi}\left(\frac{2}{L}\right)\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$$

or $$\boldsymbol{B_n=\frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx}$$

Summary of solution of the wave equation

The above may seem complicated; however a practical problem may be solved using the following **8-point procedure**:

1. Identify clearly the initial and boundary conditions.
2. Assume a solution of the form $u=XT$ and express the equations in terms of X and T and their derivatives.
3. Separate the variables by transposing the equation and equate each side to a constant, say, μ; two separate equations are obtained, one in x and the other in t.
4. Let $\mu=-p^2$ to give an oscillatory solution.
5. The two solutions are of the form:

   $$X=A\cos px+B\sin px$$

   and $$T=C\cos cpt+D\sin cpt.$$

   Then $u(x,t)=\{A\cos px+B\sin px\}\{C\cos cpt+D\sin cpt\}$.
6. Apply the boundary conditions to determine constants A and B.
7. Determine the general solution as an infinite sum.
8. Apply the remaining initial and boundary conditions and determine the coefficients A_n and B_n from equations (8) and (9), using Fourier series techniques.

Problem 5. Figure 53.2 shows a stretched string of length 50 cm which is set oscillating by displacing its mid-point a distance of 2 cm from its rest position and releasing it with zero velocity. Solve the wave equation: $\frac{\partial^2 u}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$ where $c^2=1$, to determine the resulting motion $u(x,t)$.

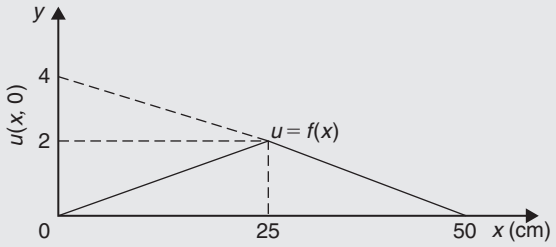


Figure 53.2

Following the above procedure,

1. The boundary and initial conditions given are:

   $$\left.\begin{aligned}u(0,t)&=0\\u(50,t)&=0\end{aligned}\right\}\ \text{i.e. fixed end points}$$

   $$u(x,0)=\boldsymbol{f(x)=\frac{2}{25}x}\quad 0\le x\le 25$$

   $$=-\frac{2}{25}x+4=\frac{\mathbf{100-2x}}{\mathbf{25}}\qquad 25\le x\le 50$$

   (Note: $y=mx+c$ is a straight line graph, so the gradient, m, between 0 and 25 is 2/25 and the y-axis intercept is zero, thus $y=f(x)=\frac{2}{25}x+0$; between 25 and 50, the gradient $=-2/25$ and the y-axis intercept is at 4, thus $f(x)=-\frac{2}{25}x+4$).

   $$\left[\frac{\partial u}{\partial t}\right]_{t=0}=0\quad\text{i.e. zero initial velocity.}$$

2. Assuming a solution $u=XT$, where X is a function of x only, and T is a function of t only, then $\frac{\partial u}{\partial x}=X'T$ and $\frac{\partial^2 u}{\partial x^2}=X''T$ and $\frac{\partial u}{\partial y}=XT'$ and $\frac{\partial^2 u}{\partial y^2}=XT''$. Substituting into the partial differential equation, $\frac{\partial^2 u}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$ gives:

   $$X''T=\frac{1}{c^2}XT''\quad\text{i.e.}\quad X''T=XT''\ \text{since}\ c^2=1.$$

3. Separating the variables gives: $\frac{X''}{X}=\frac{T''}{T}$

   Let constant,

   $$\mu=\frac{X''}{X}=\frac{T''}{T}\ \text{then}\ \mu=\frac{X''}{X}\ \text{and}\ \mu=\frac{T''}{T}$$

from which,

$$X'' - \mu X = 0 \quad \text{and} \quad T'' - \mu T = 0.$$

4. Letting $\mu = -p^2$ to give an oscillatory solution gives:

$$X'' + p^2 X = 0 \quad \text{and} \quad T'' + p^2 T = 0$$

The auxiliary equation for each is: $m^2 + p^2 = 0$ from which, $m = \sqrt{-p^2} = \pm jp$.

5. Solving each equation gives:
$X = A\cos px + B\sin px$, and
$T = C\cos pt + D\sin pt$.
Thus,
$u(x,t) = \{A\cos px + B\sin px\}\{C\cos pt + D\sin pt\}$.

6. Applying the boundary conditions to determine constants A and B gives:

(i) $u(0,t) = 0$, hence $0 = A\{C\cos pt + D\sin pt\}$ from which we conclude that $A = 0$.
Therefore,

$$u(x,t) = B\sin px\{C\cos pt + D\sin pt\} \quad \text{(a)}$$

(ii) $u(50,t) = 0$, hence
$0 = B\sin 50p\{C\cos pt + D\sin pt\}$. $B \neq 0$,
hence $\sin 50p = 0$ from which, $50p = n\pi$ and $p = \dfrac{n\pi}{50}$

7. Substituting in equation (a) gives:

$$u(x,t) = B\sin\frac{n\pi x}{50}\left\{C\cos\frac{n\pi t}{50} + D\sin\frac{n\pi t}{50}\right\}$$

or, more generally,

$$u_n(x,t) = \sum_{n=1}^{\infty}\sin\frac{n\pi x}{50}\left\{A_n\cos\frac{n\pi t}{50} + B_n\sin\frac{n\pi t}{50}\right\} \quad \text{(b)}$$

where $A_n = BC$ and $B_n = BD$.

8. From equation (8),

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$
$$= \frac{2}{50}\left[\int_0^{25}\left(\frac{2}{25}x\right)\sin\frac{n\pi x}{50}\,dx + \int_{25}^{50}\left(\frac{100-2x}{25}\right)\sin\frac{n\pi x}{50}\,dx\right]$$

Each integral is determined using integration by parts (see Chapter 43, page 420) with the result:

$$\boldsymbol{A_n = \frac{16}{n^2\pi^2}\sin\frac{n\pi}{2}}$$

From equation (9),

$$B_n = \frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$$

$$\left[\frac{\partial u}{\partial t}\right]_{t=0} = 0 = g(x) \text{ thus, } B_n = 0$$

Substituting into equation (b) gives:

$$u_n(x,t) = \sum_{n=1}^{\infty}\sin\frac{n\pi x}{50}\left\{A_n\cos\frac{n\pi t}{50} + B_n\sin\frac{n\pi t}{50}\right\}$$
$$= \sum_{n=1}^{\infty}\sin\frac{n\pi x}{50}\left\{\frac{16}{n^2\pi^2}\sin\frac{n\pi}{2}\cos\frac{n\pi t}{50} + (0)\sin\frac{n\pi t}{50}\right\}$$

Hence,

$$\boldsymbol{u(x,t) = \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi x}{50}\sin\frac{n\pi}{2}\cos\frac{n\pi t}{50}}$$

For stretched string problems as in problem 5 above, the main parts of the procedure are:

1. Determine A_n from equation (8).
Note that $\dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$ is **always** equal to $\dfrac{8d}{n^2\pi^2}\sin\dfrac{n\pi}{2}$ (see Fig. 53.3)
2. Determine B_n from equation (9)
3. Substitute in equation (5) to determine $u(x,t)$

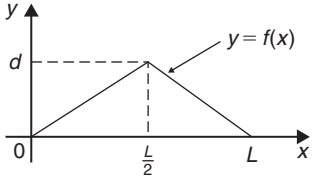


Figure 53.3

Now try the following exercise

Exercise 201 Further problems on the wave equation

1. An elastic string is stretched between two points 40 cm apart. Its centre point is displaced 1.5 cm from its position of rest at right angles to the original direction of the string and then released with zero velocity. Determine the subsequent motion $u(x,t)$ by applying the wave equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2}$ with $c^2 = 9$.

$$\left[u(x,t) = \frac{12}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{40}\cos\frac{3n\pi t}{40}\right]$$

2. The centre point of an elastic string between two points P and Q, 80 cm apart, is deflected a distance of 1 cm from its position of rest perpendicular to PQ and released initially with zero velocity. Apply the wave equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2}$ where $c = 8$, to determine the motion of a point distance x from P at time t.

$$\left[u(x,t) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{80}\cos\frac{n\pi t}{10}\right]$$

53.7 The heat conduction equation

The heat conduction equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial u}{\partial t}$ is solved in a similar manner to that for the wave equation; the equation differs only in that the right hand side contains a first partial derivative instead of the second.

The conduction of heat in a uniform bar depends on the initial distribution of temperature and on the physical properties of the bar, i.e. the thermal conductivity, h, the specific heat of the material, σ, and the mass per unit length, ρ, of the bar. In the above equation, $c^2 = \dfrac{h}{\sigma\rho}$

With a uniform bar insulated, except at its ends, any heat flow is along the bar and, at any instant, the temperature u at a point P is a function of its distance x from one end, and of the time t. Consider such a bar, shown in

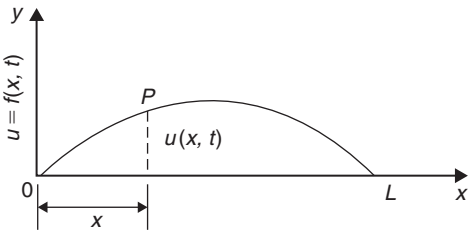


Figure 53.4

Fig. 53.4, where the bar extends from $x = 0$ to $x = L$, the temperature of the ends of the bar is maintained at zero, and the initial temperature distribution along the bar is defined by $f(x)$.

Thus, the boundary conditions can be expressed as:

$$\left.\begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0 \end{aligned}\right\} \quad \text{for all } t \geq 0$$

and $\quad u(x,0) = f(x) \quad \text{for } 0 \leq x \leq L$

As with the wave equation, a solution of the form $u(x,t) = X(x)T(t)$ is assumed, where X is a function of x only and T is a function of t only. If the trial solution is simplified to $u = XT$, then

$$\frac{\partial u}{\partial x} = X'T \quad \frac{\partial^2 u}{\partial x^2} = X''T \text{ and } \frac{\partial u}{\partial t} = XT'$$

Substituting into the partial differential equation, $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial u}{\partial t}$ gives:

$$X''T = \frac{1}{c^2}XT'$$

Separating the variables gives:

$$\boldsymbol{\frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T}}$$

Let $-p^2 = \dfrac{X''}{X} = \dfrac{1}{c^2}\dfrac{T'}{T}$ where $-p^2$ is a constant.

If $-p^2 = \dfrac{X''}{X}$ then $X'' = -p^2X$ or $X'' + p^2X = 0$, giving $\boldsymbol{X = A\cos px + B\sin px}$

and if $-p^2 = \dfrac{1}{c^2}\dfrac{T'}{T}$ then $\dfrac{T'}{T} = -p^2c^2$ and integrating with respect to t gives:

$$\int \frac{T'}{T}\,\mathrm{d}t = \int -p^2c^2\,\mathrm{d}t$$

from which, $\ln T = -p^2c^2t + c_1$

The left hand integral is obtained by an algebraic substitution (see Chapter 39).

If $\ln T = -p^2c^2t + c_1$ then
$T = e^{-p^2c^2t+c_1} = e^{-p^2c^2t}e^{c_1}$ i.e. $\boldsymbol{T = ke^{-p^2c^2t}}$ (where constant $k = e^{c_1}$).
Hence, $u(x,t) = XT = \{A\cos px + B\sin px\}ke^{-p^2c^2t}$ i.e. $u(x,t) = \{P\cos px + Q\sin px\}e^{-p^2c^2t}$ where $P = Ak$ and $Q = Bk$.
Applying the boundary conditions $u(0,t) = 0$ gives: $0 = \{P\cos 0 + Q\sin 0\}e^{-p^2c^2t} = Pe^{-p^2c^2t}$ from which, $P = 0$ and $u(x,t) = Q\sin px\, e^{-p^2c^2t}$.
Also, $u(L,t) = 0$ thus, $0 = Q\sin pL\, e^{-p^2c^2t}$ and since $Q \neq 0$ then $\sin pL = 0$ from which, $pL = n\pi$ or $p = \dfrac{n\pi}{L}$ where $n = 1, 2, 3, \ldots$
There are therefore many values of $u(x,t)$.
Thus, in general,

$$u(x,t) = \sum_{n=1}^{\infty}\left\{Q_n e^{-p^2c^2t}\sin\frac{n\pi x}{L}\right\}$$

Applying the remaining boundary condition, that when $t = 0$, $u(x,t) = f(x)$ for $0 \le x \le L$, gives:

$$f(x) = \sum_{n=1}^{\infty}\left\{Q_n \sin\frac{n\pi x}{L}\right\}$$

From Fourier series, $Q_n = 2 \times$ mean value of $f(x)\sin\dfrac{n\pi x}{L}$ from x to L.

Hence, $$Q_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

Thus, $u(x,t) =$

$$\frac{2}{L}\sum_{n=1}^{\infty}\left\{\left(\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx\right)e^{-p^2c^2t}\sin\frac{n\pi x}{L}\right\}$$

This method of solution is demonstrated in the following worked problem.

Problem 6. A metal bar, insulated along its sides, is 1 m long. It is initially at room temperature of 15°C and at time $t = 0$, the ends are placed into ice at 0°C. Find an expression for the temperature at a point P at a distance x m from one end at any time t seconds after $t = 0$.

The temperature u along the length of bar is shown in Fig. 53.5.

The heat conduction equation is $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial u}{\partial t}$ and the given boundary conditions are:

$$u(0,t) = 0,\ u(1,t) = 0 \quad \text{and} \quad u(x,0) = 15$$

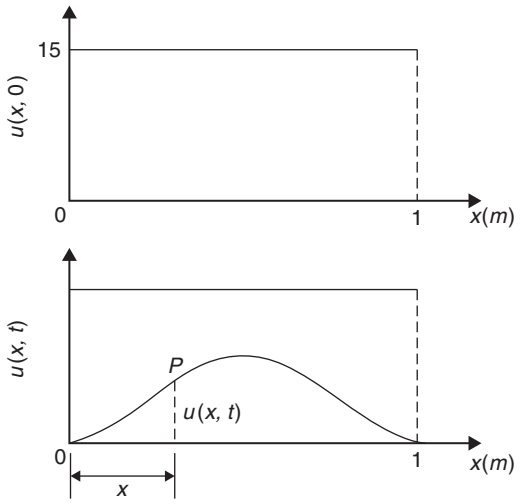


Figure 53.5

Assuming a solution of the form $u = XT$, then, from above,

$$X = A\cos px + B\sin px$$

and $$T = ke^{-p^2c^2t}.$$

Thus, the general solution is given by:

$$u(x,t) = \{P\cos px + Q\sin px\}e^{-p^2c^2t}$$

$$u(0,t) = 0 \text{ thus } 0 = Pe^{-p^2c^2t}$$

from which, $P = 0$ and $u(x,t) = \{Q\sin px\}e^{-p^2c^2t}$.
Also, $u(1,t) = 0$ thus $0 = \{Q\sin p\}e^{-p^2c^2t}$.
Since $Q \neq 0$, $\sin p = 0$ from which, $p = n\pi$ where $n = 1, 2, 3, \ldots$

Hence, $$u(x,t) = \sum_{n=1}^{\infty}\left\{Q_n e^{-p^2c^2t}\sin n\pi x\right\}$$

The final initial condition given was that at $t = 0$, $u = 15$, i.e. $u(x,0) = f(x) = 15$.

Hence, $15 = \sum_{n=1}^{\infty}\{Q_n \sin n\pi x\}$ where, from Fourier coefficients, $Q_n = 2 \times$ mean value of $15\sin n\pi x$ from $x = 0$ to $x = 1$,

i.e. $$Q_n = \frac{2}{1}\int_0^1 15\sin n\pi x\,dx = 30\left[-\frac{\cos n\pi x}{n\pi}\right]_0^1$$

$$= -\frac{30}{n\pi}[\cos n\pi - \cos 0]$$

$$= \frac{30}{n\pi}(1 - \cos n\pi)$$

$= 0$ (when n is even) and $\frac{60}{n\pi}$(when n is odd)

Hence, the required solution is:

$$u(x, t) = \sum_{n=1}^{\infty} \left\{ Q_n \, e^{-p^2 c^2 t} \sin n\pi x \right\}$$

$$= \frac{60}{\pi} \sum_{n(\text{odd})=1}^{\infty} \frac{1}{n} (\sin n\pi x) e^{-n^2\pi^2 c^2 t}$$

Now try the following exercise

Exercise 202 Further problems on the heat conduction equation

1. A metal bar, insulated along its sides, is 4 m long. It is initially at a temperature of 10°C and at time $t = 0$, the ends are placed into ice at 0°C. Find an expression for the temperature at a point P at a distance x m from one end at any time t seconds after $t = 0$.

$$\left[u(x,t) = \frac{40}{\pi} \sum_{n(\text{odd})=1}^{\infty} \frac{1}{n} e^{-\frac{n^2\pi^2 c^2 t}{16}} \sin \frac{n\pi x}{4} \right]$$

2. An insulated uniform metal bar, 8 m long, has the temperature of its ends maintained at 0°C, and at time $t = 0$ the temperature distribution $f(x)$ along the bar is defined by $f(x) = x(8 - x)$. If $c^2 = 1$, solve the heat conduction equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial u}{\partial t}$ to determine the temperature u at any point in the bar at time t.

$$\left[u(x,t) = \left(\frac{8}{\pi}\right)^3 \sum_{n(\text{odd})=1}^{\infty} \frac{1}{n^3} e^{-\frac{n^2\pi^2 t}{64}} \sin \frac{n\pi x}{8} \right]$$

3. The ends of an insulated rod PQ, 20 units long, are maintained at 0°C. At time $t = 0$, the temperature within the rod rises uniformly from each end reaching 4°C at the mid-point of PQ. Find an expression for the temperature $u(x, t)$ at any point in the rod, distant x from P at any time t after $t = 0$. Assume the heat conduction equation to be $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial u}{\partial t}$ and take $c^2 = 1$.

$$\left[u(x,t) = \frac{320}{\pi^2} \sum_{n(\text{odd})=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{20} e^{-\left(\frac{n^2\pi^2 t}{400}\right)} \right]$$

53.8 Laplace's equation

The distribution of electrical potential, or temperature, over a plane area subject to certain boundary conditions, can be described by Laplace's equation. The potential at a point P in a plane (see Fig. 53.6) can be indicated by an ordinate axis and is a function of its position, i.e. $z = u(x, y)$, where $u(x, y)$ is the solution of the Laplace two-dimensional equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

The method of solution of Laplace's equation is similar to the previous examples, as shown below.

Figure 53.7 shows a rectangle $OPQR$ bounded by the lines $x = 0$, $y = 0$, $x = a$, and $y = b$, for which we are required to find a solution of the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. The solution $z = (x, y)$ will give, say,

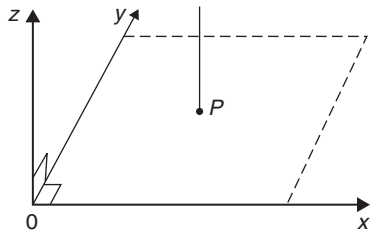


Figure 53.6

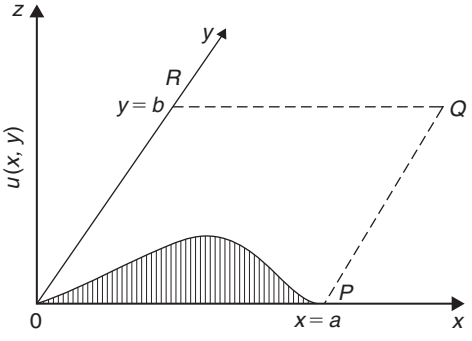


Figure 53.7

the potential at any point within the rectangle $OPQR$. The boundary conditions are:

$u=0$ when $x=0$ i.e. $u(0, y)=0$ for $0 \leq y \leq b$

$u=0$ when $x=a$ i.e. $u(a, y)=0$ for $0 \leq y \leq b$

$u=0$ when $y=b$ i.e. $u(x, b)=0$ for $0 \leq x \leq a$

$u=f(x)$ when $y=0$ i.e. $u(x, 0)=f(x)$ for $0 \leq x \leq a$

As with previous partial differential equations, a solution of the form $u(x, y)=X(x)Y(y)$ is assumed, where X is a function of x only, and Y is a function of y only. Simplifying to $u=XY$, determining partial derivatives, and substituting into $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$ gives:

$X''Y+XY''=0$

Separating the variables gives: $\dfrac{X''}{X}=-\dfrac{Y''}{Y}$

Letting each side equal a constant, $-p^2$, gives the two equations:

$$X''+p^2X=0 \quad \text{and} \quad Y''-p^2Y=0$$

from which, $X=A\cos px+B\sin px$ and $Y=C\,e^{py}+D\,e^{-py}$ or $Y=C\cosh py+D\sinh py$ (see Problem 5, page 480 for this conversion).

This latter form can also be expressed as:

$Y=E\sinh p(y+\phi)$ by using compound angles.

Hence $u(x, y)=XY$

$$=\{A\cos px+B\sin px\}\{E\sinh p(y+\phi)\}$$

or $u(x, y)$

$$=\{P\cos px+Q\sin px\}\{\sinh p(y+\phi)\}$$

where $P=AE$ and $Q=BE$.

The first boundary condition is: $u(0, y)=0$, hence $0=P\sinh p(y+\phi)$ from which, $P=0$. Hence, $u(x, y)=Q\sin px\sinh p(y+\phi)$.

The second boundary condition is: $u(a, y)=0$, hence $0=Q\sin pa\sinh p(y+\phi)$ from which, $\sin pa=0$, hence, $pa=n\pi$ or $p=\dfrac{n\pi}{a}$ for $n=1, 2, 3, \ldots$

The third boundary condition is: $u(x, b)=0$, hence, $0=Q\sin px\sinh p(b+\phi)$ from which, $\sinh p(b+\phi)=0$ and $\phi=-b$.

Hence, $u(x, y)=Q\sin px\sinh p(y-b)=Q_1\sin px\sinh p(b-y)$ where $Q_1=-Q$.

Since there are many solutions for integer values of n,

$$u(x, y)=\sum_{n=1}^{\infty} Q_n\sin px\sinh p(b-y)$$

$$=\sum_{n=1}^{\infty} Q_n\sin\frac{n\pi x}{a}\sinh\frac{n\pi}{a}(b-y)$$

The fourth boundary condition is: $u(x, 0)=f(x)$,

hence, $f(x)=\sum_{n=1}^{\infty} Q_n\sin\dfrac{n\pi x}{a}\sinh\dfrac{n\pi b}{a}$

i.e. $f(x)=\sum_{n=1}^{\infty}\left(Q_n\sinh\dfrac{n\pi b}{a}\right)\sin\dfrac{n\pi x}{a}$

From Fourier series coefficients,

$\left(Q_n\sinh\dfrac{n\pi b}{a}\right)=2\times$ the mean value of $f(x)\sin\dfrac{n\pi x}{a}$ from $x=0$ to $x=a$

i.e. $=\displaystyle\int_0^a f(x)\sin\frac{n\pi x}{a}\,dx$ from which, Q_n may be determined.

This is demonstrated in the following worked problem.

Problem 7. A square plate is bounded by the lines $x=0$, $y=0$, $x=1$ and $y=1$. Apply the Laplace equation $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$ to determine the potential distribution $u(x, y)$ over the plate, subject to the following boundary conditions:

$u=0$ when $x=0$ $0 \leq y \leq 1$,

$u=0$ when $x=1$ $0 \leq y \leq 1$,

$u=0$ when $y=0$ $0 \leq x \leq 1$,

$u=4$ when $y=1$ $0 \leq x \leq 1$.

Initially a solution of the form $u(x, y)=X(x)Y(y)$ is assumed, where X is a function of x only, and Y is a function of y only. Simplifying to $u=XY$, determining partial derivatives, and substituting into $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$ gives: $X''Y+XY''=0$

Separating the variables gives: $\dfrac{X''}{X}=-\dfrac{Y''}{Y}$

Letting each side equal a constant, $-p^2$, gives the two equations:

$$X''+p^2X=0 \quad \text{and} \quad Y''-p^2Y=0$$

from which, $X = A\cos px + B\sin px$

and $Y = Ce^{py} + De^{-py}$

or $Y = C\cosh py + D\sinh py$

or $Y = E\sinh p(y+\phi)$

Hence $u(x, y) = XY$

$= \{A\cos px + B\sin px\}\{E\sinh p(y+\phi)\}$

or $u(x, y)$

$= \{P\cos px + Q\sin px\}\{\sinh p(y+\phi)\}$

where $P = AE$ and $Q = BE$.

The first boundary condition is: $u(0, y) = 0$, hence $0 = P\sinh p(y+\phi)$ from which, $P = 0$.
Hence, $u(x, y) = Q\sin px\sinh p(y+\phi)$.
The second boundary condition is: $u(1, y) = 0$, hence $0 = Q\sin p(1)\sinh p(y+\phi)$ from which,
$\sin p = 0$, hence, $p = n\pi$ for $n = 1, 2, 3, \ldots$
The third boundary condition is: $u(x, 0) = 0$, hence, $0 = Q\sin px\sinh p(\phi)$ from which,
$\sinh p(\phi) = 0$ and $\phi = 0$.
Hence, $u(x, y) = Q\sin px\sinh py$.
Since there are many solutions for integer values of n,

$$u(x, y) = \sum_{n=1}^{\infty} Q_n \sin px \sinh py$$

$$= \sum_{n=1}^{\infty} Q_n \sin n\pi x \sinh n\pi y \qquad \text{(a)}$$

The fourth boundary condition is: $u(x, 1) = 4 = f(x)$, hence, $f(x) = \sum_{n=1}^{\infty} Q_n \sin n\pi x \sinh n\pi(1)$.
From Fourier series coefficients,

$Q_n \sinh n\pi = 2 \times$ the mean value of $f(x)\sin n\pi x$ from $x = 0$ to $x = 1$

i.e. $$= \frac{2}{1}\int_0^1 4\sin n\pi x\,\mathrm{d}x$$

$$= 8\left[-\frac{\cos n\pi x}{n\pi}\right]_0^1$$

$$= -\frac{8}{n\pi}(\cos n\pi - \cos 0)$$

$$= \frac{8}{n\pi}(1 - \cos n\pi)$$

$= 0$ (for even values of n),

$$= \frac{16}{n\pi} \text{ (for odd values of } n)$$

Hence, $Q_n = \dfrac{16}{n\pi(\sinh n\pi)} = \dfrac{16}{n\pi}\operatorname{cosech} n\pi$

Hence, from equation (a),

$$\boldsymbol{u(x,y)} = \sum_{n=1}^{\infty} Q_n \sin n\pi x \sinh n\pi y$$

$$= \frac{\mathbf{16}}{\boldsymbol{\pi}} \sum_{\boldsymbol{n(\text{odd})=1}}^{\infty} \frac{\mathbf{1}}{\boldsymbol{n}} \textbf{(cosech } \boldsymbol{n\pi} \textbf{ sin } \boldsymbol{n\pi x} \textbf{ sinh } \boldsymbol{n\pi y}\textbf{)}$$

Now try the following exercise

Exercise 203 Further problems on the Laplace equation

1. A rectangular plate is bounded by the lines $x = 0$, $y = 0$, $x = 1$ and $y = 3$. Apply the Laplace equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ to determine the potential distribution $u(x, y)$ over the plate, subject to the following boundary conditions:

   $u = 0$ when $x = 0$ $\quad 0 \le y \le 2$,
   $u = 0$ when $x = 1$ $\quad 0 \le y \le 2$,
   $u = 0$ when $y = 2$ $\quad 0 \le x \le 1$,
   $u = 5$ when $y = 3$ $\quad 0 \le x \le 1$.

   $$\left[u(x, y) = \frac{20}{\pi}\sum_{n(\text{odd})=1}^{\infty} \frac{1}{n}\operatorname{cosech} n\pi \sin n\pi x \sinh n\pi(y-2)\right]$$

2. A rectangular plate is bounded by the lines $x = 0$, $y = 0$, $x = 3$, $y = 2$. Determine the potential distribution $u(x, y)$ over the rectangle using the Laplace equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, subject to the following boundary conditions:

   $u(0, y) = 0$ $\quad 0 \le y \le 2$,
   $u(3, y) = 0$ $\quad 0 \le y \le 2$,
   $u(x, 2) = 0$ $\quad 0 \le x \le 3$,
   $u(x, 0) = x(3-x)$ $\quad 0 \le x \le 3$.

   $$\left[u(x, y) = \frac{216}{\pi^3}\sum_{n(\text{odd})=1}^{\infty} \frac{1}{n^3}\operatorname{cosech}\frac{2n\pi}{3}\sin\frac{n\pi x}{3}\sinh\frac{n\pi}{3}(2-y)\right]$$

Revision Test 15

This Revision Test covers the material contained in Chapters 50 to 53. *The marks for each question are shown in brackets at the end of each question.*

1. Find the particular solution of the following differential equations:
   (a) $12\dfrac{d^2y}{dt^2} - 3y = 0$ given that when $t = 0$, $y = 3$ and $\dfrac{dy}{dt} = \dfrac{1}{2}$
   (b) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 2y = 10e^x$ given that when $x = 0$, $y = 0$ and $\dfrac{dy}{dx} = 1$. (20)

2. In a galvanometer the deflection θ satisfies the differential equation:
$$\frac{d^2\theta}{dt^2} + 2\frac{d\theta}{dt} + \theta = 4$$
Solve the equation for θ given that when $t = 0$, $\theta = 0$ and $\dfrac{d\theta}{dt} = 0$. (12)

3. Determine $y^{(n)}$ when $y = 2x^3e^{4x}$. (10)

4. Determine the power series solution of the differential equation: $\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} + y = 0$ using Leibniz-Maclaurin's method, given the boundary conditions that at $x = 0$, $y = 2$ and $\dfrac{dy}{dx} = 1$. (20)

5. Use the Frobenius method to determine the general power series solution of the differential equation: $\dfrac{d^2y}{dx^2} + 4y = 0$. (21)

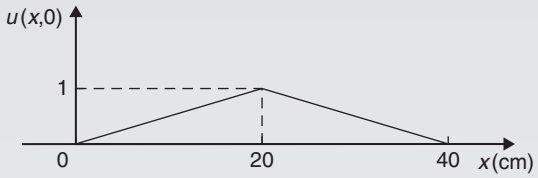


Figure RT15.1

6. Determine the general power series solution of Bessel's equation:
$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - v^2)y = 0$$
and hence state the series up to and including the term in x^6 when $v = +3$. (26)

7. Determine the general solution of $\dfrac{\partial u}{\partial x} = 5xy$ (2)

8. Solve the differential equation $\dfrac{\partial^2 u}{\partial x^2} = x^2(y - 3)$ given the boundary conditions that at $x = 0$, $\dfrac{\partial u}{\partial x} = \sin y$ and $u = \cos y$. (6)

9. Figure RT15.1 shows a stretched string of length 40 cm which is set oscillating by displacing its mid-point a distance of 1 cm from its rest position and releasing it with zero velocity. Solve the wave equation: $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2}$ where $c^2 = 1$, to determine the resulting motion $u(x, t)$. (23)

Chapter 54

Presentation of statistical data

54.1 Some statistical terminology

Data are obtained largely by two methods:

(a) by counting—for example, the number of stamps sold by a post office in equal periods of time, and

(b) by measurement—for example, the heights of a group of people.

When data are obtained by counting and only whole numbers are possible, the data are called **discrete**. Measured data can have any value within certain limits and are called **continuous** (see Problem 1).

A **set** is a group of data and an individual value within the set is called a **member** of the set. Thus, if the masses of five people are measured correct to the nearest 0.1 kg and are found to be 53.1 kg, 59.4 kg, 62.1 kg, 77.8 kg and 64.4 kg, then the set of masses in kilograms for these five people is:

$$\{53.1, 59.4, 62.1, 77.8, 64.4\}$$

and one of the members of the set is 59.4

A set containing all the members is called a **population**. Some members selected at random from a population are called a **sample**. Thus all car registration numbers form a population, but the registration numbers of, say, 20 cars taken at random throughout the country are a sample drawn from that population.

The number of times that the value of a member occurs in a set is called the **frequency** of that member. Thus in the set: $\{2, 3, 4, 5, 4, 2, 4, 7, 9\}$, member 4 has a frequency of three, member 2 has a frequency of two and the other members have a frequency of one.

The **relative frequency** with which any member of a set occurs is given by the ratio:

$$\frac{\text{frequency of member}}{\text{total frequency of all members}}$$

For the set: $\{2, 3, 5, 4, 7, 5, 6, 2, 8\}$, the relative frequency of member 5 is $\frac{2}{9}$

Often, relative frequency is expressed as a percentage and the **percentage relative frequency** is: (relative frequency $\times$ 100)%.

Problem 1. Data are obtained on the topics given below. State whether they are discrete or continuous data.

(a) The number of days on which rain falls in a month for each month of the year.

(b) The mileage travelled by each of a number of salesmen.

(c) The time that each of a batch of similar batteries lasts.

(d) The amount of money spent by each of several families on food.

(a) The number of days on which rain falls in a given month must be an integer value and is obtained by **counting** the number of days. Hence, these data are **discrete**.

(b) A salesman can travel any number of miles (and parts of a mile) between certain limits and these data are **measured**. Hence the data are **continuous**.

(c) The time that a battery lasts is **measured** and can have any value between certain limits. Hence these data are **continuous**.

(d) The amount of money spent on food can only be expressed correct to the nearest pence, the amount being **counted**. Hence, these data are **discrete**.

Now try the following exercise

Exercise 204 Further problems on discrete and continuous data

In Problems 1 and 2, state whether data relating to the topics given are discrete or continuous.

1. (a) The amount of petrol produced daily, for each of 31 days, by a refinery.
   (b) The amount of coal produced daily by each of 15 miners.
   (c) The number of bottles of milk delivered daily by each of 20 milkmen.
   (d) The size of 10 samples of rivets produced by a machine.

   [(a) continuous (b) continuous; (c) discrete (d) continuous]

2. (a) The number of people visiting an exhibition on each of 5 days.
   (b) The time taken by each of 12 athletes to run 100 metres.
   (c) The value of stamps sold in a day by each of 20 post offices.
   (d) The number of defective items produced in each of 10 one-hour periods by a machine.

   [(a) discrete (b) continuous; (c) discrete (d) discrete]

54.2 Presentation of ungrouped data

Ungrouped data can be presented diagrammatically in several ways and these include:

(a) **pictograms**, in which pictorial symbols are used to represent quantities (see Problem 2),

(b) **horizontal bar charts**, having data represented by equally spaced horizontal rectangles (see Problem 3), and

(c) **vertical bar charts**, in which data are represented by equally spaced vertical rectangles (see Problem 4).

Trends in ungrouped data over equal periods of time can be presented diagrammatically by a **percentage component bar chart**. In such a chart, equally spaced rectangles of any width, but whose height corresponds to 100%, are constructed. The rectangles are then subdivided into values corresponding to the percentage relative frequencies of the members (see Problem 5).

A **pie diagram** is used to show diagrammatically the parts making up the whole. In a pie diagram, the area of a circle represents the whole, and the areas of the sectors of the circle are made proportional to the parts which make up the whole (see Problem 6).

Problem 2. The number of television sets repaired in a workshop by a technician in six, one-month periods is as shown below. Present these data as a pictogram.

Month	Number repaired
January	11
February	6
March	15
April	9
May	13
June	8

Each symbol shown in Fig. 54.1 represents two television sets repaired. Thus, in January, $5\frac{1}{2}$ symbols are used to represent the 11 sets repaired, in February, 3 symbols are used to represent the 6 sets repaired, and so on.

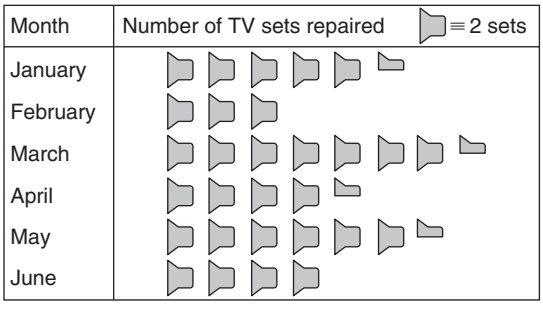


Figure 54.1

Problem 3. The distance in miles travelled by four salesmen in a week are as shown below.

Salesmen	P	Q	R	S
Distance travelled miles	413	264	597	143

Use a horizontal bar chart to represent these data diagrammatically.

Equally spaced horizontal rectangles of any width, but whose length is proportional to the distance travelled, are used. Thus, the length of the rectangle for salesman *P* is proportional to 413 miles, and so on. The horizontal bar chart depicting these data is shown in Fig. 54.2.

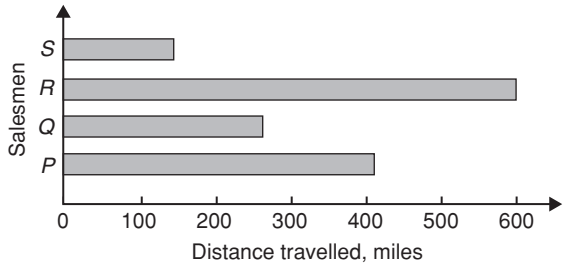


Figure 54.2

Problem 4. The number of issues of tools or materials from a store in a factory is observed for seven, one-hour periods in a day, and the results of the survey are as follows:

Period	1	2	3	4	5	6	7
Number of issues	34	17	9	5	27	13	6

Present these data on a vertical bar chart.

In a vertical bar chart, equally spaced vertical rectangles of any width, but whose height is proportional to the quantity being represented, are used. Thus the height of the rectangle for period 1 is proportional to 34 units, and so on. The vertical bar chart depicting these data is shown in Fig. 54.3.

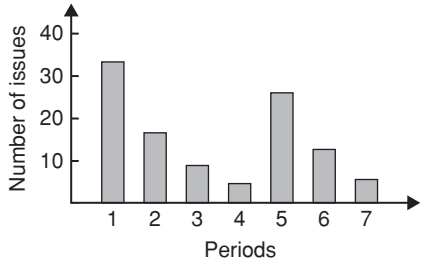


Figure 54.3

Problem 5. The numbers of various types of dwellings sold by a company annually over a three-year period are as shown below. Draw percentage component bar charts to present these data.

	Year 1	Year 2	Year 3
4-roomed bungalows	24	17	7
5-roomed bungalows	38	71	118
4-roomed houses	44	50	53
5-roomed houses	64	82	147
6-roomed houses	30	30	25

A table of percentage relative frequency values, correct to the nearest 1%, is the first requirement. Since,

$$\text{percentage relative frequency} = \frac{\text{frequency of member} \times 100}{\text{total frequency}}$$

then for 4-roomed bungalows in year 1:

$$\text{percentage relative frequency} = \frac{24 \times 100}{24+38+44+64+30} = 12\%$$

The percentage relative frequencies of the other types of dwellings for each of the three years are similarly calculated and the results are as shown in the table below.

	Year 1 (%)	Year 2 (%)	Year 3 (%)
4-roomed bungalows	12	7	2
5-roomed bungalows	19	28	34
4-roomed houses	22	20	15
5-roomed houses	32	33	42
6-roomed houses	15	12	7

The percentage component bar chart is produced by constructing three equally spaced rectangles of any width, corresponding to the three years. The heights of the rectangles correspond to 100% relative frequency, and are subdivided into the values in the table of percentages shown above. A key is used (different types of shading or different colour schemes) to indicate

corresponding percentage values in the rows of the table of percentages. The percentage component bar chart is shown in Fig. 54.4.

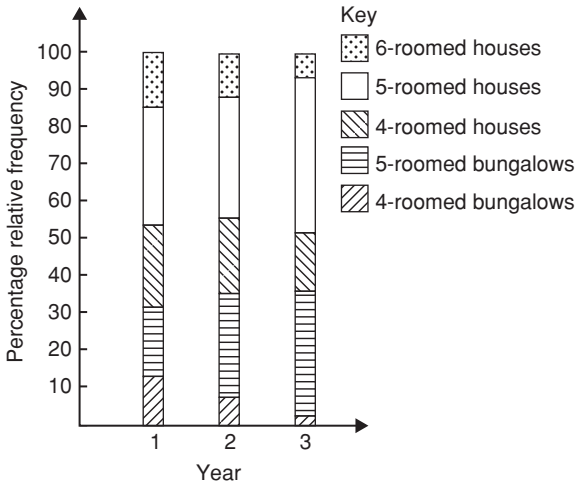


Figure 54.4

Problem 6. The retail price of a product costing £2 is made up as follows: materials 10 p, labour 20 p, research and development 40 p, overheads 70 p, profit 60 p. Present these data on a pie diagram.

A circle of any radius is drawn, and the area of the circle represents the whole, which in this case is £2. The circle is subdivided into sectors so that the areas of the sectors are proportional to the parts, i.e. the parts which make up the total retail price. For the area of a sector to be proportional to a part, the angle at the centre of the circle must be proportional to that part. The whole, £2 or 200 p, corresponds to 360°. Therefore,

$$10\,\text{p corresponds to } 360 \times \frac{10}{200} \text{ degrees, i.e. } 18°$$

$$20\,\text{p corresponds to } 360 \times \frac{20}{200} \text{ degrees, i.e. } 36°$$

and so on, giving the angles at the centre of the circle for the parts of the retail price as: 18°, 36°, 72°, 126° and 108°, respectively.
The pie diagram is shown in Fig. 54.5.

Problem 7.

(a) Using the data given in Fig. 54.2 only, calculate the amount of money paid to each salesman for travelling expenses, if they are paid an allowance of 37 p per mile.

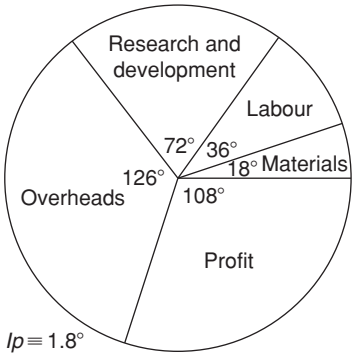


Figure 54.5

(b) Using the data presented in Fig. 54.4, comment on the housing trends over the three-year period.

(c) Determine the profit made by selling 700 units of the product shown in Fig. 54.5.

(a) By measuring the length of rectangle P the mileage covered by salesman P is equivalent to 413 miles. Hence salesman P receives a travelling allowance of

$$\frac{£413 \times 37}{100}, \text{ i.e. } \mathbf{£152.81}$$

Similarly, for salesman Q, the miles travelled are 264 and his allowance is

$$\frac{£264 \times 37}{100}, \text{ i.e. } \mathbf{£97.68}$$

Salesman R travels 597 miles and he receives

$$\frac{£597 \times 37}{100}, \text{ i.e. } \mathbf{£220.89}$$

Finally, salesman S receives

$$\frac{£143 \times 37}{100}, \text{ i.e. } \mathbf{£52.91}$$

(b) An analysis of Fig. 54.4 shows that 5-roomed bungalows and 5-roomed houses are becoming more popular, the greatest change in the three years being a 15% increase in the sales of 5-roomed bungalows.

(c) Since 1.8° corresponds to 1 p and the profit occupies 108° of the pie diagram, then the profit per unit is

$$\frac{108 \times 1}{1.8}, \text{ that is, } 60\,\text{p}$$

The profit when selling 700 units of the product is

$$£\frac{700 \times 60}{100}, \text{ that is, } \textbf{£420}$$

Now try the following exercise

Exercise 205 Further problems on presentation of ungrouped data

1. The number of vehicles passing a stationary observer on a road in six ten-minute intervals is as shown. Draw a pictogram to represent these data.

Period of Time	1	2	3	4	5	6
Number of Vehicles	35	44	62	68	49	41

[If one symbol is used to represent 10 vehicles, working correct to the nearest 5 vehicles, gives $3\frac{1}{2}, 4\frac{1}{2}$, 6, 7, 5 and 4 symbols respectively.]

2. The number of components produced by a factory in a week is as shown below:

Day	Number of Components
Mon	1580
Tues	2190
Wed	1840
Thur	2385
Fri	1280

Show these data on a pictogram.

[If one symbol represents 200 components, working correct to the nearest 100 components gives: Mon 8, Tues 11, Wed 9, Thurs 12 and Fri $6\frac{1}{2}$.]

3. For the data given in Problem 1 above, draw a horizontal bar chart.

[6 equally spaced horizontal rectangles, whose lengths are proportional to 35, 44, 62, 68, 49 and 41, respectively.]

4. Present the data given in Problem 2 above on a horizontal bar chart.

[5 equally spaced horizontal rectangles, whose lengths are proportional to 1580, 2190, 1840, 2385 and 1280 units, respectively.]

5. For the data given in Problem 1 above, construct a vertical bar chart.

[6 equally spaced vertical rectangles, whose heights are proportional to 35, 44, 62, 68, 49 and 41 units, respectively.]

6. Depict the data given in Problem 2 above on a vertical bar chart.

[5 equally spaced vertical rectangles, whose heights are proportional to 1580, 2190, 1840, 2385 and 1280 units, respectively.]

7. A factory produces three different types of components. The percentages of each of these components produced for three, one-month periods are as shown below. Show this information on percentage component bar charts and comment on the changing trend in the percentages of the types of component produced.

Month	1	2	3
Component P	20	35	40
Component Q	45	40	35
Component R	35	25	25

[Three rectangles of equal height, subdivided in the percentages shown in the columns above. P increases by 20% at the expense of Q and R]

8. A company has five distribution centres and the mass of goods in tonnes sent to each centre during four, one-week periods, is as shown.

Week	1	2	3	4
Centre A	147	160	174	158
Centre B	54	63	77	69
Centre C	283	251	237	211
Centre D	97	104	117	144
Centre E	224	218	203	194

Use a percentage component bar chart to present these data and comment on any trends.

[Four rectangles of equal heights, subdivided as follows: week 1: 18%, 7%, 35%, 12%, 28% week 2: 20%, 8%, 32%, 13%, 27% week 3: 22%, 10%, 29%, 14%, 25% week 4: 20%, 9%, 27%, 19%, 25%. Little change in centres A and B, a reduction of about 8% in C, an increase of about 7% in D and a reduction of about 3% in E.]

9. The employees in a company can be split into the following categories: managerial 3, supervisory 9, craftsmen 21, semi-skilled 67, others 44. Show these data on a pie diagram.

[A circle of any radius, subdivided into sectors having angles of $7\frac{1}{2}^\circ$, $22\frac{1}{2}^\circ$, $52\frac{1}{2}^\circ$, $167\frac{1}{2}^\circ$ and 110°, respectively.]

10. The way in which an apprentice spent his time over a one-month period is as follows: drawing office 44 hours, production 64 hours, training 12 hours, at college 28 hours.

Use a pie diagram to depict this information.

[A circle of any radius, subdivided into sectors having angles of 107°, 156°, 29° and 68°, respectively.]

11. (a) With reference to Fig. 54.5, determine the amount spent on labour and materials to produce 1650 units of the product.

(b) If in year 2 of Fig. 54.4, 1% corresponds to 2.5 dwellings, how many bungalows are sold in that year. [(a) £ 495, (b) 88]

12. (a) If the company sell 23500 units per annum of the product depicted in Fig. 54.5, determine the cost of their overheads per annum.

(b) If 1% of the dwellings represented in year 1 of Fig. 54.4 corresponds to 2 dwellings, find the total number of houses sold in that year.

[(a) £ 16450, (b) 138]

54.3 Presentation of grouped data

When the number of members in a set is small, say ten or less, the data can be represented diagrammatically without further analysis, by means of pictograms, bar charts, percentage components bar charts or pie diagrams (as shown in Section 54.2).

For sets having more than ten members, those members having similar values are grouped together in **classes** to form a **frequency distribution**. To assist in accurately counting members in the various classes, a **tally diagram** is used (see Problems 8 and 12).

A frequency distribution is merely a table showing classes and their corresponding frequencies (see Problems 8 and 12).

The new set of values obtained by forming a frequency distribution is called **grouped data**.

The terms used in connection with grouped data are shown in Fig. 54.6(a). The size or range of a class is given by the **upper class boundary value** minus the **lower class boundary value**, and in Fig. 54.6 is $7.65 - 7.35$, i.e. 0.30. The **class interval** for the class shown in Fig. 54.6(b) is 7.4 to 7.6 and the class mid-point value is given by,

$$\frac{\left(\begin{array}{c}\text{upper class}\\ \text{boundary value}\end{array}\right)+\left(\begin{array}{c}\text{lower class}\\ \text{boundary value}\end{array}\right)}{2}$$

and in Fig. 54.6 is $\dfrac{7.65+7.35}{2}$, i.e. 7.5.

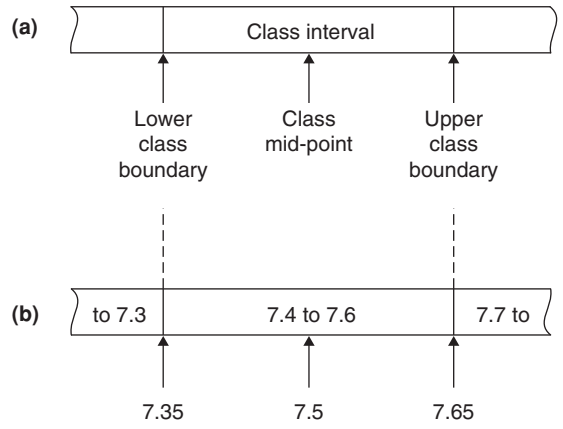


Figure 54.6

One of the principal ways of presenting grouped data diagrammatically is by using a **histogram**, in which the **areas** of vertical, adjacent rectangles are made proportional to frequencies of the classes (see Problem 9). When class intervals are equal, the heights of the rectangles of a histogram are equal to the frequencies of the classes. For histograms having unequal class intervals, the area must be proportional to the frequency. Hence, if the class interval of class A is twice the class interval of class B, then for equal frequencies, the height of the rectangle representing A is half that of B (see Problem 11). Another method of presenting grouped data diagrammatically is by using a **frequency polygon**, which is the graph produced by plotting frequency against class mid-point values and joining the co-ordinates with straight lines (see Problem 12).

A **cumulative frequency distribution** is a table showing the cumulative frequency for each value of upper class boundary. The cumulative frequency for a particular value of upper class boundary is obtained by adding the frequency of the class to the sum of the previous frequencies. A cumulative frequency distribution is formed in Problem 13.

The curve obtained by joining the co-ordinates of cumulative frequency (vertically) against upper class boundary (horizontally) is called an **ogive** or a **cumulative frequency distribution curve** (see Problem 13).

Problem 8. The data given below refer to the gain of each of a batch of 40 transistors, expressed correct to the nearest whole number. Form a frequency distribution for these data having seven classes.

81	83	87	74	76	89	82	84
86	76	77	71	86	85	87	88
84	81	80	81	73	89	82	79
81	79	78	80	85	77	84	78
83	79	80	83	82	79	80	77

The **range** of the data is the value obtained by taking the value of the smallest member from that of the largest member. Inspection of the set of data shows that, range $= 89 - 71 = 18$. The size of each class is given approximately by range divided by the number of classes. Since 7 classes are required, the size of each class is 18/7, that is, approximately 3. To achieve seven equal classes spanning a range of values from 71 to 89, the class intervals are selected as: 70–72, 73–75, and so on.

To assist with accurately determining the number in each class, a **tally diagram** is produced, as shown in Table 54.1(a). This is obtained by listing the classes in the left-hand column, and then inspecting each of the 40 members of the set in turn and allocating them to the appropriate classes by putting '1s' in the appropriate rows. Every fifth '1' allocated to the particular row is shown as an oblique line crossing the four previous '1s', to help with final counting.

A **frequency distribution** for the data is shown in Table 54.1(b) and lists classes and their corresponding frequencies, obtained from the tally diagram. (Class mid-point value are also shown in the table, since they are used for constructing the histogram for these data (see Problem 9)).

Problem 9. Construct a histogram for the data given in Table 54.1(b).

The histogram is shown in Fig. 54.7. The width of the rectangles correspond to the upper class boundary values minus the lower class boundary values and the heights of the rectangles correspond to the class frequencies. The easiest way to draw a histogram is to mark the class mid-point values on the horizontal scale and draw the rectangles symmetrically about the appropriate class mid-point values and touching one another.

Table 54.1(a)

Class	Tally
70–72	1
73–75	11
76–78	𝍸 11
79–81	𝍸 𝍸 11
82–84	𝍸 1111
85–87	𝍸 1
88–90	111

Table 54.1(b)

Class	Class mid-point	Frequency
70–72	71	1
73–75	74	2
76–78	77	7
79–81	80	12
82–84	83	9
85–87	86	6
88–90	89	3

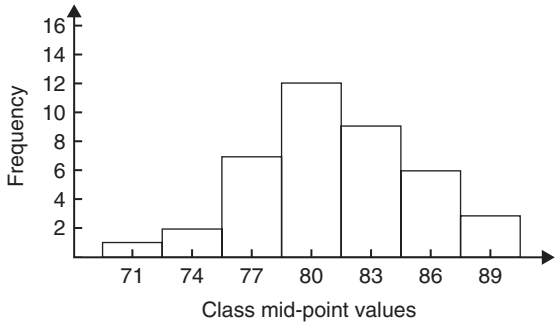


Figure 54.7

Problem 10. The amount of money earned weekly by 40 people working part-time in a factory, correct to the nearest £10, is shown below. Form a frequency distribution having 6 classes for these data.

80	90	70	110	90	160	110	80
140	30	90	50	100	110	60	100
80	90	110	80	100	90	120	70
130	170	80	120	100	110	40	110
50	100	110	90	100	70	110	80

Inspection of the set given shows that the majority of the members of the set lie between £80 and £110 and that there are a much smaller number of extreme values ranging from £30 to £170. If equal class intervals are selected, the frequency distribution obtained does not give as much information as one with unequal class intervals. Since the majority of members are between £80 and £100, the class intervals in this range are selected to be smaller than those outside of this range. There is no unique solution and one possible solution is shown in Table 54.2.

Table 54.2

Class	Frequency
20–40	2
50–70	6
80–90	12
100–110	14
120–140	4
150–170	2

Problem 11. Draw a histogram for the data given in Table 54.2.

When dealing with unequal class intervals, the histogram must be drawn so that the areas, (and not the heights), of the rectangles are proportional to the frequencies of the classes. The data given are shown in columns 1 and 2 of Table 54.3. Columns 3 and 4 give the upper and lower class boundaries, respectively. In column 5, the class ranges (i.e. upper class boundary minus lower class boundary values) are listed. The heights of the rectangles are proportional to the ratio $\frac{\text{frequency}}{\text{class range}}$, as shown in column 6. The histogram is shown in Fig. 54.8.

Table 54.3

1 Class	2 Frequency	3 Upper class boundary	4 Lower class boundary	5 Class range	6 Height of rectangle
20–40	2	45	15	30	$\frac{2}{30}=\frac{1}{15}$
50–70	6	75	45	30	$\frac{6}{30}=\frac{3}{15}$
80–90	12	95	75	20	$\frac{12}{20}=\frac{9}{15}$
100–110	14	115	95	20	$\frac{14}{20}=\frac{10\frac{1}{2}}{15}$
120–140	4	145	115	30	$\frac{4}{30}=\frac{2}{15}$
150–170	2	175	145	30	$\frac{2}{30}=\frac{1}{15}$

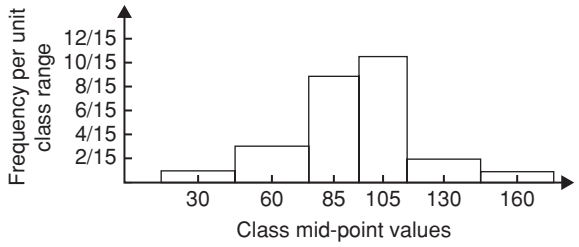


Figure 54.8

Problem 12. The masses of 50 ingots in kilograms are measured correct to the nearest 0.1 kg and the results are as shown below. Produce a frequency distribution having about 7 classes for these data and then present the grouped data as (a) a frequency polygon and (b) a histogram.

8.0	8.6	8.2	7.5	8.0	9.1	8.5	7.6	8.2	7.8
8.3	7.1	8.1	8.3	8.7	7.8	8.7	8.5	8.4	8.5
7.7	8.4	7.9	8.8	7.2	8.1	7.8	8.2	7.7	7.5
8.1	7.4	8.8	8.0	8.4	8.5	8.1	7.3	9.0	8.6
7.4	8.2	8.4	7.7	8.3	8.2	7.9	8.5	7.9	8.0

The **range** of the data is the member having the largest value minus the member having the smallest value. Inspection of the set of data shows that:

$$\text{range} = 9.1 - 7.1 = 2.0$$

The size of each class is given approximately by

$$\frac{\text{range}}{\text{number of classes}}$$

Since about seven classes are required, the size of each class is 2.0/7, that is approximately 0.3, and thus the **class limits** are selected as 7.1 to 7.3, 7.4 to 7.6, 7.7 to 7.9, and so on.

The **class mid-point** for the 7.1 to 7.3 class is $\frac{7.35+7.05}{2}$, i.e. 7.2, for the 7.4 to 7.6 class is $\frac{7.65+7.35}{2}$, i.e. 7.5, and so on.

To assist with accurately determining the number in each class, a **tally diagram** is produced as shown in Table 54.4. This is obtained by listing the classes in the left-hand column and then inspecting each of the 50 members of the set of data in turn and allocating it

Table 54.4

Class	Tally
7.1 to 7.3	111
7.4 to 7.6	𝍸 1
7.7 to 7.9	𝍸 1111
8.0 to 8.2	𝍸 𝍸 1111
8.3 to 8.5	𝍸 𝍸 1
8.6 to 8.8	𝍸 1
8.9 to 9.1	11

to the appropriate class by putting a '1' in the appropriate row. Each fifth '1' allocated to a particular row is marked as an oblique line to help with final counting.

A **frequency distribution** for the data is shown in Table 54.5 and lists classes and their corresponding frequencies. Class mid-points are also shown in this table, since they are used when constructing the frequency polygon and histogram.

Table 54.5

Class	Class mid-point	Frequency
7.1 to 7.3	7.2	3
7.4 to 7.6	7.5	5
7.5 to 7.9	7.8	9
8.0 to 8.2	8.1	14
8.1 to 8.5	8.4	11
8.2 to 8.8	8.7	6
8.9 to 9.1	9.0	2

A **frequency polygon** is shown in Fig. 54.9, the co-ordinates corresponding to the class mid-point/frequency values, given in Table 54.5. The co-ordinates are joined by straight lines and the polygon is 'anchored-down' at each end by joining to the next class mid-point value and zero frequency.

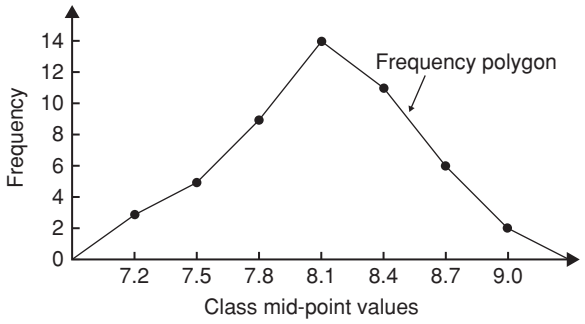


Figure 54.9

A **histogram** is shown in Fig. 54.10, the width of a rectangle corresponding to (upper class boundary value—lower class boundary value) and height corresponding to the class frequency. The easiest way to draw a histogram is to mark class mid-point values on the horizontal scale and to draw the rectangles symmetrically about the appropriate class mid-point values and touching one another. A histogram for the data given in Table 54.5 is shown in Fig. 54.10.

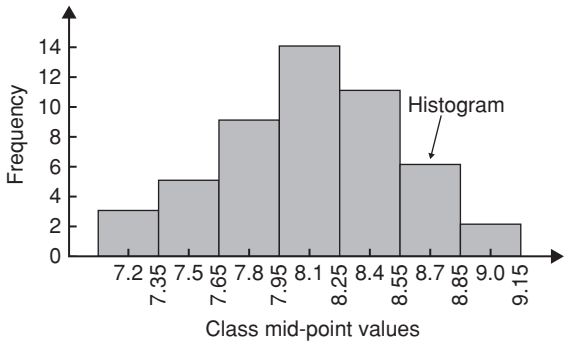


Figure 54.10

Problem 13. The frequency distribution for the masses in kilograms of 50 ingots is:

7.1 to 7.3 3, 7.4 to 7.6 5, 7.7 to 7.9 9,
8.0 to 8.2 14, 8.3 to 8.5 11, 8.6 to 8.8, 6,
8.9 to 9.1 2,

Form a cumulative frequency distribution for these data and draw the corresponding ogive.

A **cumulative frequency distribution** is a table giving values of cumulative frequency for the value of upper class boundaries, and is shown in Table 54.6. Columns 1 and 2 show the classes and their frequencies. Column 3 lists the upper class boundary values for the classes given in column 1. Column 4 gives the cumulative frequency values for all frequencies less than the upper class boundary values given in column 3. Thus, for example, for the 7.7 to 7.9 class

Table 54.6

1 Class	2 Frequency	3 Upper Class boundary	4 Cumulative frequency
		Less than	
7.1–7.3	3	7.35	3
7.4–7.6	5	7.65	8
7.7–7.9	9	7.95	17
8.0–8.2	14	8.25	31
8.3–8.5	11	8.55	42
8.6–8.8	6	8.85	48
8.9–9.1	2	9.15	50

shown in row 3, the cumulative frequency value is the sum of all frequencies having values of less than 7.95, i.e. 3+5+9=17, and so on. The **ogive** for the cumulative frequency distribution given in Table 54.6 is shown in Fig. 54.11. The co-ordinates corresponding to each upper class boundary/cumulative frequency value are plotted and the co-ordinates are joined by straight lines (—not the best curve drawn through the co-ordinates as in experimental work.) The ogive is 'anchored' at its start by adding the co-ordinate (7.05, 0).

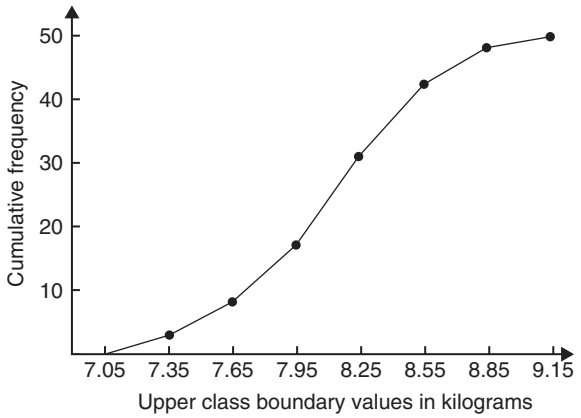


Figure 54.11

Now try the following exercise

Exercise 206 Further problems on presentation of grouped data

1. The mass in kilograms, correct to the nearest one-tenth of a kilogram, of 60 bars of metal are as shown. Form a frequency distribution of about 8 classes for these data.

39.8	40.3	40.6	40.0	39.6
39.6	40.2	40.3	40.4	39.8
40.2	40.3	39.9	39.9	40.0
40.1	40.0	40.1	40.1	40.2
39.7	40.4	39.9	40.1	39.9
39.5	40.0	39.8	39.5	39.9
40.1	40.0	39.7	40.4	39.3
40.7	39.9	40.2	39.9	40.0
40.1	39.7	40.5	40.5	39.9
40.8	40.0	40.2	40.0	39.9
39.8	39.7	39.5	40.1	40.2
40.6	40.1	39.7	40.2	40.3

[There is no unique solution, but one solution is:
39.3–39.4 1; 39.5–39.6 5;
39.7–39.8 9; 39.9–40.0 17;
40.1–40.2 15; 40.3–40.4 7;
40.5–40.6 4; 40.7–40.8 2]

2. Draw a histogram for the frequency distribution given in the solution of Problem 1.

[Rectangles, touching one another, having mid-points of 39.35, 39.55, 39.75, 39.95, . . . and heights of 1, 5, 9, 17, . . .]

3. The information given below refers to the value of resistance in ohms of a batch of 48 resistors of similar value. Form a frequency distribution for the data, having about 6 classes, and draw a frequency polygon and histogram to represent these data diagramatically.

21.0	22.4	22.8	21.5	22.6	21.1	21.6	22.3
22.9	20.5	21.8	22.2	21.0	21.7	22.5	20.7
23.2	22.9	21.7	21.4	22.1	22.2	22.3	21.3
22.1	21.8	22.0	22.7	21.7	21.9	21.1	22.6
21.4	22.4	22.3	20.9	22.8	21.2	22.7	21.6
22.2	21.6	21.3	22.1	21.5	22.0	23.4	21.2

[There is no unique solution, but one solution is:
20.5–20.9 3; 21.0–21.4 10;
21.5–21.9 11; 22.0–22.4 13;
22.5–22.9 9; 23.0–23.4 2]

4. The time taken in hours to the failure of 50 specimens of a metal subjected to fatigue failure tests are as shown. Form a frequency distribution, having about 8 classes and unequal class intervals, for these data.

28	22	23	20	12	24	37	28	21	25
21	14	30	23	27	13	23	7	26	19
24	22	26	3	21	24	28	40	27	24
20	25	23	26	47	21	29	26	22	33
27	9	13	35	20	16	20	25	18	22

[There is no unique solution, but one solution is: 1–10 3; 11–19 7; 20–22 12; 23–25 11; 26–28 10; 29–38 5; 39–48 2]

5. Form a cumulative frequency distribution and hence draw the ogive for the frequency distribution given in the solution to Problem 3.

[20.95 3; 21.45 13; 21.95 24; 22.45 37; 22.95 46; 23.45 48]

6. Draw a histogram for the frequency distribution given in the solution to Problem 4.

[Rectangles, touching one another, having mid-points of 5.5, 15, 21, 24, 27, 33.5 and 43.5. The heights of the rectangles (frequency per unit class range) are 0.3, 0.78, 4. 4.67, 2.33, 0.5 and 0.2]

7. The frequency distribution for a batch of 50 capacitors of similar value, measured in microfarads, is:

10.5–10.9	2,	11.0–11.4	7,
11.5–11.9	10,	12.0–12.4	12,
12.5–12.9	11,	13.0–13.4	8

Form a cumulative frequency distribution for these data.

[(10.95 2), (11.45 9), (11.95 11), (12.45 31), (12.95 42), (13.45 50)]

8. Draw an ogive for the data given in the solution of Problem 7.

9. The diameter in millimetres of a reel of wire is measured in 48 places and the results are as shown.

2.10	2.29	2.32	2.21	2.14	2.22
2.28	2.18	2.17	2.20	2.23	2.13
2.26	2.10	2.21	2.17	2.28	2.15
2.16	2.25	2.23	2.11	2.27	2.34
2.24	2.05	2.29	2.18	2.24	2.16
2.15	2.22	2.14	2.27	2.09	2.21
2.11	2.17	2.22	2.19	2.12	2.20
2.23	2.07	2.13	2.26	2.16	2.12

(a) Form a frequency distribution of diameters having about 6 classes.

(b) Draw a histogram depicting the data.

(c) Form a cumulative frequency distribution.

(d) Draw an ogive for the data.

[(a) There is no unique solution, but one solution is:

2.05–2.09 3; 2.10–21.4 10; 2.15–2.19 11; 2.20–2.24 13; 2.25–2.29 9; 2.30–2.34 2

(b) Rectangles, touching one another, having mid-points of 2.07, 2.12 . . . and heights of 3, 10, . . .

(c) Using the frequency distribution given in the solution to part (a) gives:

2.095 3; 2.145 13; 2.195 24; 2.245 37; 2.295 46; 2.345 48

(d) A graph of cumulative frequency against upper class boundary having the coordinates given in part (c).]

Chapter 55

Measures of central tendency and dispersion

55.1 Measures of central tendency

A single value, which is representative of a set of values, may be used to give an indication of the general size of the members in a set, the word '**average**' often being used to indicate the single value.

The statistical term used for 'average' is the arithmetic mean or just the **mean**.

Other measures of central tendency may be used and these include the **median** and the **modal** values.

55.2 Mean, median and mode for discrete data

Mean

The **arithmetic mean value** is found by adding together the values of the members of a set and dividing by the number of members in the set. Thus, the mean of the set of numbers: {4, 5, 6, 9} is:

$$\frac{4+5+6+9}{4}, \text{ i.e. } 6$$

In general, the mean of the set: $\{x_1, x_2, x_3, \ldots, x_n\}$ is

$$\overline{x} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}, \text{ written as } \frac{\sum x}{n}$$

where $\sum$ is the Greek letter 'sigma' and means 'the sum of', and $\overline{x}$ (called x-bar) is used to signify a mean value.

Median

The **median value** often gives a better indication of the general size of a set containing extreme values. The set: {7, 5, 74, 10} has a mean value of 24, which is not really representative of any of the values of the members of the set. The median value is obtained by:

(a) **ranking** the set in ascending order of magnitude, and

(b) selecting the value of the **middle member** for sets containing an odd number of members, or finding the value of the mean of the two middle members for sets containing an even number of members.

For example, the set: {7, 5, 74, 10} is ranked as {5, 7, 10, 74}, and since it contains an even number of members (four in this case), the mean of 7 and 10 is taken, giving a median value of 8.5. Similarly, the set: {3, 81, 15, 7, 14} is ranked as {3, 7, 14, 15, 81} and the median value is the value of the middle member, i.e. 14.

Mode

The **modal value**, or **mode**, is the most commonly occurring value in a set. If two values occur with the same frequency, the set is 'bi-modal'. The set: {5, 6, 8, 2, 5, 4, 6, 5, 3} has a model value of 5, since the member having a value of 5 occurs three times.

Problem 1. Determine the mean, median and mode for the set:

{2, 3, 7, 5, 5, 13, 1, 7, 4, 8, 3, 4, 3}

The mean value is obtained by adding together the values of the members of the set and dividing by the number of members in the set.

Thus, **mean value**,

$$\bar{x} = \frac{2+3+7+5+5+13+1+7+4+8+3+4+3}{13} = \frac{65}{13} = \mathbf{5}$$

To obtain the median value the set is ranked, that is, placed in ascending order of magnitude, and since the set contains an odd number of members the value of the middle member is the median value. Ranking the set gives:

$$\{1, 2, 3, 3, 3, 4, 4, 5, 5, 7, 7, 8, 13\}$$

The middle term is the seventh member, i.e. 4, thus the **median value is 4**. The **modal value** is the value of the most commonly occurring member and is **3**, which occurs three times, all other members only occurring once or twice.

Problem 2. The following set of data refers to the amount of money in £s taken by a news vendor for 6 days. Determine the mean, median and modal values of the set:

$$\{27.90, 34.70, 54.40, 18.92, 47.60, 39.68\}$$

$$\textbf{Mean value} = \frac{27.90+34.70+54.40+18.92+47.60+39.68}{6} = \textbf{£37.20}$$

The ranked set is:

$$\{18.92, 27.90, 34.70, 39.68, 47.60, 54.40\}$$

Since the set has an even number of members, the mean of the middle two members is taken to give the median value, i.e.

$$\textbf{Median value} = \frac{34.70+39.68}{2} = \textbf{£37.19}$$

Since no two members have the same value, this set has **no mode**.

Now try the following exercise

Exercise 207 Further problems on mean, median and mode for discrete data

In Problems 1 to 4, determine the mean, median and modal values for the sets given.

1. {3, 8, 10, 7, 5, 14, 2, 9, 8}
   [mean $7\frac{1}{3}$, median 8, mode 8]
2. {26, 31, 21, 29, 32, 26, 25, 28}
   [mean 27.25, median 27, mode 26]
3. {4.72, 4.71, 4.74, 4.73, 4.72, 4.71, 4.73, 4.72}
   [mean 4.7225, median 4.72, mode 4.72]
4. {73.8, 126.4, 40.7, 141.7, 28.5, 237.4, 157.9}
   [mean 115.2, median 126.4, no mode]

55.3 Mean, median and mode for grouped data

The mean value for a set of grouped data is found by determining the sum of the (frequency × class mid-point values) and dividing by the sum of the frequencies,

i.e. mean value $\bar{x} = \dfrac{f_1x_1 + f_2x_2 + \cdots + f_nx_n}{f_1 + f_2 + \cdots + f_n}$

$$= \frac{\sum(fx)}{\sum f}$$

where f is the frequency of the class having a mid-point value of x, and so on.

Problem 3. The frequency distribution for the value of resistance in ohms of 48 resistors is as shown. Determine the mean value of resistance.

20.5–20.9	3,	21.0–21.4	10,
21.5–21.9	11,	22.0–22.4	13,
22.5–22.9	9,	23.0–23.4	2

The class mid-point/frequency values are:

20.7 3, 21.2 10, 21.7 11, 22.2 13, 22.7 9 and 23.2 2

For grouped data, the mean value is given by:

$$\bar{x} = \frac{\sum(fx)}{\sum f}$$

where f is the class frequency and x is the class mid-point value. Hence mean value,

$$\bar{x} = \frac{(3 \times 20.7) + (10 \times 21.2) + (11 \times 21.7) + (13 \times 22.2) + (9 \times 22.7) + (2 \times 23.2)}{48}$$

$$= \frac{1052.1}{48} = 21.919.$$

i.e. **the mean value is 21.9 ohms**, correct to 3 significant figures.

Histogram

The mean, median and modal values for grouped data may be determined from a **histogram**. In a histogram, frequency values are represented vertically and variable values horizontally. The mean value is given by the value of the variable corresponding to a vertical line drawn through the centroid of the histogram. The median value is obtained by selecting a variable value such that the area of the histogram to the left of a vertical line drawn through the selected variable value is equal to the area of the histogram on the right of the line. The modal value is the variable value obtained by dividing the width of the highest rectangle in the histogram in proportion to the heights of the adjacent rectangles. The method of determining the mean, median and modal values from a histogram is shown in Problem 4.

Problem 4. The time taken in minutes to assemble a device is measured 50 times and the results are as shown. Draw a histogram depicting this data and hence determine the mean, median and modal values of the distribution.

14.5–15.5	5,	16.5–17.5	8,
18.5–19.5	16,	20.5–21.5	12,
22.5–23.5	6,	24.5–25.5	3

The histogram is shown in Fig. 55.1. The mean value lies at the centroid of the histogram. With reference to any arbitrary axis, say YY shown at a time of 14 minutes, the position of the horizontal value of the centroid can be obtained from the relationship $AM = \sum(am)$, where A is the area of the histogram, M is the horizontal distance of the centroid from the axis YY, a is the area of a rectangle of the histogram and m is the distance of the centroid of the rectangle from YY. The areas of the individual rectangles are shown circled on the histogram giving a

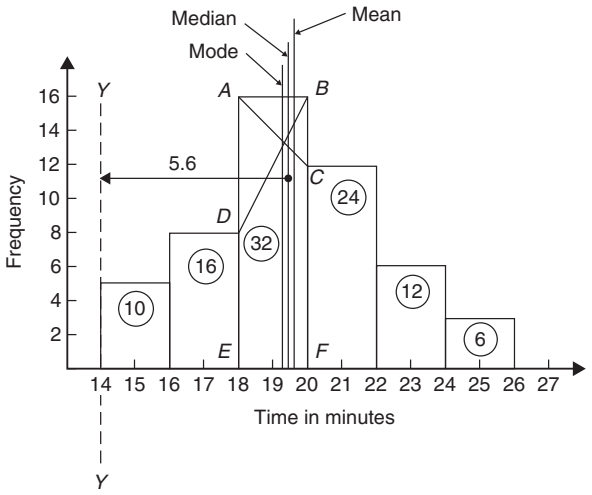


Figure 55.1

total area of 100 square units. The positions, m, of the centroids of the individual rectangles are 1, 3, 5, ... units from YY. Thus

$$100M = (10 \times 1) + (16 \times 3) + (32 \times 5) + (24 \times 7) + (12 \times 9) + (6 \times 11)$$

i.e. $$M = \frac{560}{100} = 5.6 \text{ units from } YY$$

Thus the position of the **mean** with reference to the time scale is 14 + 5.6, i.e. **19.6 minutes**.

The median is the value of time corresponding to a vertical line dividing the total area of the histogram into two equal parts. The total area is 100 square units, hence the vertical line must be drawn to give 50 units of area on each side. To achieve this with reference to Fig. 55.1, rectangle $ABFE$ must be split so that 50 − (10 + 16) units of area lie on one side and 50 − (24 + 12 + 6) units of area lie on the other. This shows that the area of $ABFE$ is split so that 24 units of area lie to the left of the line and 8 units of area lie to the right, i.e. the vertical line must pass through 19.5 minutes. Thus the **median value** of the distribution is **19.5 minutes**.

The mode is obtained by dividing the line AB, which is the height of the highest rectangle, proportionally to the heights of the adjacent rectangles. With reference to Fig. 55.1, this is done by joining AC and BD and drawing a vertical line through the point of intersection of these two lines. This gives the **mode** of the distribution and is **19.3 minutes**.

Now try the following exercise

Exercise 208 Further problems on mean, median and mode for grouped data

1. The frequency distribution given below refers to the heights in centimetres of 100 people. Determine the mean value of the distribution, correct to the nearest millimetre.

   150–156 5, 157–163 18,
   164–170 20, 171–177 27,
   178–184 22, 185–191 8

   [171.7 cm]

2. The gain of 90 similar transistors is measured and the results are as shown.

   83.5–85.5 6, 86.5–88.5 39,
   89.5–91.5 27, 92.5–94.5 15,
   95.5–97.5 3

   By drawing a histogram of this frequency distribution, determine the mean, median and modal values of the distribution.

   [mean 89.5, median 89, mode 88.2]

3. The diameters, in centimetres, of 60 holes bored in engine castings are measured and the results are as shown. Draw a histogram depicting these results and hence determine the mean, median and modal values of the distribution.

   2.011–2.014 7, 2.016–2.019 16,
   2.021–2.024 23, 2.026–2.029 9,
   2.031–2.034 5

   [mean 2.02158 cm, median 2.02152 cm, mode 2.02167 cm]

55.4 Standard deviation

(a) Discrete data

The standard deviation of a set of data gives an indication of the amount of dispersion, or the scatter, of members of the set from the measure of central tendency. Its value is the root-mean-square value of the members of the set and for discrete data is obtained as follows:

(a) determine the measure of central tendency, usually the mean value, (occasionally the median or modal values are specified),

(b) calculate the deviation of each member of the set from the mean, giving

$$(x_1 - \bar{x}), (x_2 - \bar{x}), (x_3 - \bar{x}), \ldots,$$

(c) determine the squares of these deviations, i.e.

$$(x_1 - \bar{x})^2, (x_2 - \bar{x})^2, (x_3 - \bar{x})^2, \ldots,$$

(d) find the sum of the squares of the deviations, that is

$$(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + (x_3 - \bar{x})^2, \ldots,$$

(e) divide by the number of members in the set, n, giving

$$\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + (x_3 - \bar{x})^2 + \cdots}{n}$$

(f) determine the square root of (e).

The standard deviation is indicated by σ (the Greek letter small 'sigma') and is written mathematically as:

$$\textbf{Standard deviation}, \sigma = \sqrt{\left\{\frac{\sum(x - \bar{x})^2}{n}\right\}}$$

where x is a member of the set, $\bar{x}$ is the mean value of the set and n is the number of members in the set. The value of standard deviation gives an indication of the distance of the members of a set from the mean value. The set: {1, 4, 7, 10, 13} has a mean value of 7 and a standard deviation of about 4.2. The set {5, 6, 7, 8, 9} also has a mean value of 7, but the standard deviation is about 1.4. This shows that the members of the second set are mainly much closer to the mean value than the members of the first set. The method of determining the standard deviation for a set of discrete data is shown in Problem 5.

Problem 5. Determine the standard deviation from the mean of the set of numbers: {5, 6, 8, 4, 10, 3} correct to 4 significant figures.

The arithmetic mean,

$$\bar{x} = \frac{\sum x}{n} = \frac{5+6+8+4+10+3}{6} = 6$$

Standard deviation, $\sigma = \sqrt{\left\{\frac{\sum(x-\bar{x})^2}{n}\right\}}$

The $(x-\bar{x})^2$ values are: $(5-6)^2$, $(6-6)^2$, $(8-6)^2$, $(4-6)^2$, $(10-6)^2$ and $(3-6)^2$.

The sum of the $(x-\bar{x})^2$ values,

i.e. $$\sum(x-\bar{x})^2 = 1+0+4+4+16+9 = 34$$

and $$\frac{\sum(x-\bar{x})^2}{n} = \frac{34}{6} = 5.\dot{6}$$

since there are 6 members in the set.

Hence, **standard deviation**,

$$\sigma = \sqrt{\left\{\frac{\sum(x-\bar{x})^2}{n}\right\}} = \sqrt{5.6}$$

$$= \mathbf{2.380}, \text{ correct to 4 significant figures.}$$

(b) Grouped data

For **grouped data, standard deviation**

$$\sigma = \sqrt{\left\{\frac{\sum\{f(x-\bar{x})^2\}}{\sum f}\right\}}$$

where f is the class frequency value, x is the class mid-point value and $\bar{x}$ is the mean value of the grouped data. The method of determining the standard deviation for a set of grouped data is shown in Problem 6.

Problem 6. The frequency distribution for the values of resistance in ohms of 48 resistors is as shown. Calculate the standard deviation from the mean of the resistors, correct to 3 significant figures.

20.5–20.9	3,	21.0–21.4	10,
21.5–21.9	11,	22.0–22.4	13,
22.5–22.9	9,	23.0–23.4	2

The standard deviation for grouped data is given by:

$$\sigma = \sqrt{\left\{\frac{\sum\{f(x-\bar{x})^2\}}{\sum f}\right\}}$$

From Problem 3, the distribution mean value, $\bar{x} = 21.92$, correct to 4 significant figures.

The 'x-values' are the class mid-point values, i.e. 20.7, 21.2, 21.7, ...

Thus the $(x-\bar{x})^2$ values are $(20.7-21.92)^2$, $(21.2-21.92)^2$, $(21.7-21.92)^2$, ...

and the $f(x-\bar{x})^2$ values are $3(20.7-21.92)^2$, $10(21.2-21.92)^2$, $11(21.7-21.92)^2$, ...

The $\sum f(x-\bar{x})^2$ values are

$$4.4652 + 5.1840 + 0.5324 + 1.0192 + 5.4756 + 3.2768 = 19.9532$$

$$\frac{\sum\{f(x-\bar{x})^2\}}{\sum f} = \frac{19.9532}{48} = 0.41569$$

and **standard deviation**,

$$\sigma = \sqrt{\left\{\frac{\sum\{f(x-\bar{x})^2\}}{\sum f}\right\}} = \sqrt{0.41569}$$

$$= \mathbf{0.645}, \text{ correct to 3 significant figures.}$$

Now try the following exercise

Exercise 209 Further problems on standard deviation

1. Determine the standard deviation from the mean of the set of numbers:

   {35, 22, 25, 23, 28, 33, 30}

   correct to 3 significant figures. [4.60]

2. The values of capacitances, in microfarads, of ten capacitors selected at random from a large batch of similar capacitors are:

   34.3, 25.0, 30.4, 34.6, 29.6, 28.7, 33.4,

   32.7, 29.0 and 31.3

   Determine the standard deviation from the mean for these capacitors, correct to 3 significant figures. [2.83 μF]

3. The tensile strength in megapascals for 15 samples of tin were determined and found to be:

   34.61, 34.57, 34.40, 34.63, 34.63,

   34.51, 34.49, 34.61, 34.52, 34.55,

   34.58, 34.53, 34.44, 34.48 and 34.40

   Calculate the mean and standard deviation from the mean for these 15 values, correct to 4 significant figures.

   [mean 34.53 MPa, standard deviation 0.07474 MPa]

4. Determine the standard deviation from the mean, correct to 4 significant figures, for the heights of the 100 people given in Problem 1 of Exercise 208, page 544. [9.394 cm]

5. Calculate the standard deviation from the mean for the data given in Problem 3 of Exercise 208, page 544, correct to 3 significant figures. [0.00544 cm]

55.5 Quartiles, deciles and percentiles

Other measures of dispersion which are sometimes used are the quartile, decile and percentile values. The **quartile values** of a set of discrete data are obtained by selecting the values of members which divide the set into four equal parts. Thus for the set: {2, 3, 4, 5, 5, 7, 9, 11, 13, 14, 17} there are 11 members and the values of the members dividing the set into four equal parts are 4, 7, and 13. These values are signified by Q_1, Q_2 and Q_3 and called the first, second and third quartile values, respectively. It can be seen that the second quartile value, Q_2, is the value of the middle member and hence is the median value of the set.

For grouped data the ogive may be used to determine the quartile values. In this case, points are selected on the vertical cumulative frequency values of the ogive, such that they divide the total value of cumulative frequency into four equal parts. Horizontal lines are drawn from these values to cut the ogive. The values of the variable corresponding to these cutting points on the ogive give the quartile values (see Problem 7).

When a set contains a large number of members, the set can be split into ten parts, each containing an equal number of members. These ten parts are then called **deciles**. For sets containing a very large number of members, the set may be split into one hundred parts, each containing an equal number of members. One of these parts is called a **percentile**.

Problem 7. The frequency distribution given below refers to the overtime worked by a group of craftsmen during each of 48 working weeks in a year.

25–29 5, 30–34 4, 35–39 7,
40–44 11, 45–49 12, 50–54 8,
55–59 1

Draw an ogive for this data and hence determine the quartile values.

The cumulative frequency distribution (i.e. upper class boundary/cumulative frequency values) is:

29.5 5, 34.5 9, 39.5 16, 44.5 27,
49.5 39, 54.5 47, 59.5 48

The ogive is formed by plotting these values on a graph, as shown in Fig. 55.2. The total frequency is divided into four equal parts, each having a range of 48/4, i.e. 12. This gives cumulative frequency values of 0 to 12 corresponding to the first quartile, 12 to 24 corresponding to the second quartile, 24 to 36 corresponding to the third quartile and 36 to 48 corresponding to the fourth quartile of the distribution, i.e. the distribution is divided into four equal parts. The quartile values are those of the variable corresponding to cumulative frequency values of 12, 24 and 36, marked Q_1, Q_2 and Q_3 in Fig. 55.2. These values, correct to the nearest hour, are **37 hours, 43 hours and 48 hours**, respectively. The Q_2 value is also equal to the median value of the distribution. One measure of the dispersion of a distribution is called the **semi-interquartile range** and is given by $(Q_3 - Q_1)/2$, and is $(48 - 37)/2$ in this case, i.e. $\mathbf{5\frac{1}{2}}$ **hours**.

Problem 8. Determine the numbers contained in the (a) 41st to 50th percentile group, and (b) 8th decile group of the set of numbers shown below:

14 22 17 21 30 28 37 7 23 32
24 17 20 22 27 19 26 21 15 29

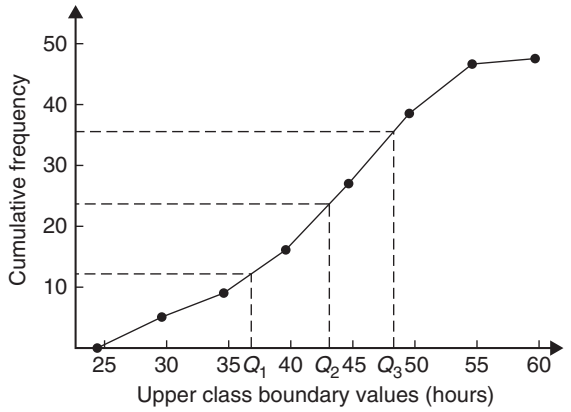


Figure 55.2

The set is ranked, giving:

7 14 15 17 17 19 20 21 21 22 22 23

24 26 27 28 29 30 32 37

(a) There are 20 numbers in the set, hence the first 10% will be the two numbers 7 and 14, the second 10% will be 15 and 17, and so on. Thus the 41st to 50th percentile group will be the numbers **21 and 22**.

(b) The first decile group is obtained by splitting the ranked set into 10 equal groups and selecting the first group, i.e. the numbers 7 and 14. The second decile group are the numbers 15 and 17, and so on. Thus the 8th decile group contains the numbers **27 and 28**.

Now try the following exercise

Exercise 210 Further problems on quartiles, deciles and percentiles

1. The number of working days lost due to accidents for each of 12 one-monthly periods are as shown. Determine the median and first and third quartile values for this data.

   27 37 40 28 23 30 35 24 30 32 31 2

   [30, 25.5, 33.5 days]

2. The number of faults occurring on a production line in a nine-week period are as shown below. Determine the median and quartile values for the data.

   30 27 25 24 27 37 31 27 35

   [27, 26, 33 faults]

3. Determine the quartile values and semi-interquartile range for the frequency distribution given in Problem 1 of Exercise 208, page 544.

   $\left[\begin{array}{l} Q_1 = 164.5\,\text{cm},\ Q_2 = 172.5\,\text{cm}, \\ Q_3 = 179\,\text{cm},\ 7.25\,\text{cm} \end{array}\right]$

4. Determine the numbers contained in the 5th decile group and in the 61st to 70th percentile groups for the set of numbers:

   40 46 28 32 37 42 50 31 48 45
   32 38 27 33 40 35 25 42 38 41

   [37 and 38; 40 and 41]

5. Determine the numbers in the 6th decile group and in the 81st to 90th percentile group for the set of numbers:

   43 47 30 25 15 51 17 21
   36 44 33 17 35 58 51 35

   37 33 44 56 40 49 22
   44 40 31 41 55 50 16

   [40, 40, 41; 50, 51, 51]

Chapter 56

Probability

56.1 Introduction to probability

The **probability** of something happening is the likelihood or chance of it happening. Values of probability lie between 0 and 1, where 0 represents an absolute impossibility and 1 represents an absolute certainty. The probability of an event happening usually lies somewhere between these two extreme values and is expressed either as a proper or decimal fraction. Examples of probability are:

that a length of copper wire has zero resistance at 100°C	0
that a fair, six-sided dice will stop with a 3 upwards	$\frac{1}{6}$ or 0.1667
that a fair coin will land with a head upwards	$\frac{1}{2}$ or 0.5
that a length of copper wire has some resistance at 100°C	1

If p is the probability of an event happening and q is the probability of the same event not happening, then the total probability is $p+q$ and is equal to unity, since it is an absolute certainty that the event either does or does not occur, i.e. $\boldsymbol{p+q=1}$

Expectation

The **expectation**, E, of an event happening is defined in general terms as the product of the probability p of an event happening and the number of attempts made, n, i.e. $\boldsymbol{E=pn}$.

Thus, since the probability of obtaining a 3 upwards when rolling a fair dice is $\frac{1}{6}$, the expectation of getting a 3 upwards on four throws of the dice is $\frac{1}{6} \times 4$, i.e. $\frac{2}{3}$

Thus expectation is the average occurrence of an event.

Dependent event

A **dependent event** is one in which the probability of an event happening affects the probability of another event happening. Let 5 transistors be taken at random from a batch of 100 transistors for test purposes, and the probability of there being a defective transistor, p_1, be determined. At some later time, let another 5 transistors be taken at random from the 95 remaining transistors in the batch and the probability of there being a defective transistor, p_2, be determined. The value of p_2 is different from p_1 since batch size has effectively altered from 100 to 95, i.e. probability p_2 is dependent on probability p_1. Since 5 transistors are drawn, and then another 5 transistors drawn without replacing the first 5, the second random selection is said to be **without replacement**.

Independent event

An independent event is one in which the probability of an event happening does not affect the probability of another event happening. If 5 transistors are taken at random from a batch of transistors and the probability of a defective transistor p_1 is determined and the process is repeated after the original 5 have been replaced in the batch to give p_2, then p_1 is equal to p_2. Since the 5 transistors are replaced between draws, the second selection is said to be **with replacement**.

Conditional probability

Conditional probability is concerned with the probability of say event B occurring, given that event A has already taken place.

If A and B are independent events, then the fact that event A has already occurred will not affect the probability of event B.

If A and B are dependent events, then event A having occurred will effect the probability of event B.

56.2 Laws of probability

The addition law of probability

The addition law of probability is recognized by the word **'or'** joining the probabilities. If p_A is the probability of event A happening and p_B is the probability of event B happening, the probability of **event A or event B** happening is given by $\boldsymbol{p_A + p_B}$ (provided events A and B are **mutually exclusive**, i.e. A and B are events which cannot occur together). Similarly, the probability of events **A or B or C or ... N** happening is given by

$$p_A + p_B + p_C + \cdots + p_N$$

The multiplication law of probability

The multiplication law of probability is recognized by the word **'and'** joining the probabilities. If p_A is the probability of event A happening and p_B is the probability of event B happening, the probability of **event A and event B** happening is given by $p_A \times p_B$. Similarly, the probability of events **A and B and C and ... N** happening is given by

$$p_A \times p_B \times p_C \times \cdots \times p_N$$

56.3 Worked problems on probability

Problem 1. Determine the probabilities of selecting at random (a) a man, and (b) a woman from a crowd containing 20 men and 33 women.

(a) The probability of selecting at random a man, p, is given by the ratio

$$\frac{\text{number of men}}{\text{number in crowd}}$$

i.e. $p = \dfrac{20}{20+33} = \dfrac{\mathbf{20}}{\mathbf{53}}$ or **0.3774**

(b) The probability of selecting at random a women, q, is given by the ratio

$$\frac{\text{number of women}}{\text{number in crowd}}$$

i.e. $q = \dfrac{33}{20+33} = \dfrac{\mathbf{33}}{\mathbf{53}}$ or **0.6226**

(Check: the total probability should be equal to 1;

$p = \dfrac{20}{53}$ and $q = \dfrac{33}{53}$

thus the total probability,

$$p + q = \frac{20}{53} + \frac{33}{53} = 1$$

hence no obvious error has been made).

Problem 2. Find the expectation of obtaining a 4 upwards with 3 throws of a fair dice.

Expectation is the average occurrence of an event and is defined as the probability times the number of attempts. The probability, p, of obtaining a 4 upwards for one throw of the dice is $\frac{1}{6}$

Also, 3 attempts are made, hence $n = 3$ and the expectation, E, is pn, i.e. $\boldsymbol{E = \frac{1}{6} \times 3 = \frac{1}{2}}$ or **0.50**

Problem 3. Calculate the probabilities of selecting at random:

(a) the winning horse in a race in which 10 horses are running,

(b) the winning horses in both the first and second races if there are 10 horses in each race.

(a) Since only one of the ten horses can win, the probability of selecting at random the winning horse is $\dfrac{\text{number of winners}}{\text{number of horses}}$, i.e. $\dfrac{\mathbf{1}}{\mathbf{10}}$ or **0.10**

(b) The probability of selecting the winning horse in the first race is $\frac{1}{10}$. The probability of selecting the winning horse in the second race is $\frac{1}{10}$. The probability of selecting the winning horses in the first **and** second race is given by the multiplication law of probability, i.e.

$$\textbf{probability} = \frac{1}{10} \times \frac{1}{10}$$

$$= \frac{\mathbf{1}}{\mathbf{100}} \text{ or } \mathbf{0.01}$$

Problem 4. The probability of a component failing in one year due to excessive temperature is $\dfrac{1}{20}$, due to excessive vibration is $\dfrac{1}{25}$ and due to excessive humidity is $\dfrac{1}{50}$. Determine the probabilities that during a one-year period a component: (a) fails due to excessive temperature and excessive vibration, (b) fails due to excessive vibration or excessive humidity, and (c) will not fail because of both excessive temperature and excessive humidity.

Let p_A be the probability of failure due to excessive temperature, then

$$p_A = \frac{1}{20} \text{ and } \overline{p_A} = \frac{19}{20}$$

(where $\overline{p_A}$ is the probability of not failing).

Let p_B be the probability of failure due to excessive vibration, then

$$p_B = \frac{1}{25} \text{ and } \overline{p_B} = \frac{24}{25}$$

Let p_C be the probability of failure due to excessive humidity, then

$$p_C = \frac{1}{50} \text{ and } \overline{p_C} = \frac{49}{50}$$

(a) The probability of a component failing due to excessive temperature **and** excessive vibration is given by:

$$p_A \times p_B = \frac{1}{20} \times \frac{1}{25} = \mathbf{\frac{1}{500}} \text{ or } \mathbf{0.002}$$

(b) The probability of a component failing due to excessive vibration **or** excessive humidity is:

$$p_B + p_C = \frac{1}{25} + \frac{1}{50} = \mathbf{\frac{3}{50}} \text{ or } \mathbf{0.06}$$

(c) The probability that a component will not fail due to excessive temperature **and** will not fail due to excess humidity is:

$$\overline{p_A} \times \overline{p_C} = \frac{19}{20} \times \frac{49}{50} = \mathbf{\frac{931}{1000}} \text{ or } \mathbf{0.931}$$

Problem 5. A batch of 100 capacitors contains 73 which are within the required tolerance values, 17 which are below the required tolerance values, and the remainder are above the required tolerance values. Determine the probabilities that when randomly selecting a capacitor and then a second capacitor: (a) both are within the required tolerance values when selecting with replacement, and (b) the first one drawn is below and the second one drawn is above the required tolerance value, when selection is without replacement.

(a) The probability of selecting a capacitor within the required tolerance values is $\frac{73}{100}$. The first capacitor drawn is now replaced and a second one is drawn from the batch of 100. The probability of this capacitor being within the required tolerance values is also $\frac{73}{100}$.

Thus, the probability of selecting a capacitor within the required tolerance values for both the first **and** the second draw is

$$\frac{73}{100} \times \frac{73}{100} = \mathbf{\frac{5329}{10000}} \text{ or } \mathbf{0.5329}$$

(b) The probability of obtaining a capacitor below the required tolerance values on the first draw is $\frac{17}{100}$. There are now only 99 capacitors left in the batch, since the first capacitor is not replaced. The probability of drawing a capacitor above the required tolerance values on the second draw is $\frac{10}{99}$, since there are $(100-73-17)$, i.e. 10 capacitors above the required tolerance value. Thus, the probability of randomly selecting a capacitor below the required tolerance values and followed by randomly selecting a capacitor above the tolerance' values is

$$\frac{17}{100} \times \frac{10}{99} = \frac{170}{9900} = \mathbf{\frac{17}{990}} \text{ or } \mathbf{0.0172}$$

Now try the following exercise

Exercise 211 Further problems on probability

1. In a batch of 45 lamps there are 10 faulty lamps. If one lamp is drawn at random, find the probability of it being (a) faulty and (b) satisfactory.

$$\begin{bmatrix} \text{(a) } \frac{2}{9} \text{ or } 0.2222 \\ \text{(b) } \frac{7}{9} \text{ or } 0.7778 \end{bmatrix}$$

2. A box of fuses are all of the same shape and size and comprises 23 2 A fuses, 47 5 A fuses and 69 13 A fuses. Determine the probability of selecting at random (a) a 2 A fuse, (b) a 5 A fuse and (c) a 13 A fuse.

$$\begin{bmatrix} \text{(a) } \frac{23}{139} \text{ or } 0.1655 \\ \text{(b) } \frac{47}{139} \text{ or } 0.3381 \\ \text{(c) } \frac{69}{139} \text{ or } 0.4964 \end{bmatrix}$$

3. (a) Find the probability of having a 2 upwards when throwing a fair 6-sided dice. (b) Find the probability of having a 5 upwards when throwing a fair 6-sided dice. (c) Determine the probability of having a 2 and then a 5 on two successive throws of a fair 6-sided dice.

$$\left[\text{(a) } \frac{1}{6} \text{ (b) } \frac{1}{6} \text{ (c) } \frac{1}{36}\right]$$

4. Determine the probability that the total score is 8 when two like dice are thrown.

$$\left[\frac{5}{36}\right]$$

5. The probability of event A happening is $\frac{3}{5}$ and the probability of event B happening is $\frac{2}{3}$. Calculate the probabilities of (a) both A and B happening, (b) only event A happening, i.e. event A happening and event B not happening, (c) only event B happening, and (d) either A, or B, or A and B happening.

$$\left[\text{(a) } \frac{2}{5} \text{ (b) } \frac{1}{5} \text{ (c) } \frac{4}{15} \text{ (d) } \frac{13}{15}\right]$$

6. When testing 1000 soldered joints, 4 failed during a vibration test and 5 failed due to having a high resistance. Determine the probability of a joint failing due to (a) vibration, (b) high resistance, (c) vibration or high resistance and (d) vibration and high resistance.

$$\left[\text{(a) } \frac{1}{250} \text{ (b) } \frac{1}{200} \text{ (c) } \frac{9}{1000} \text{ (d) } \frac{1}{50000}\right]$$

56.4 Further worked problems on probability

Problem 6. A batch of 40 components contains 5 which are defective. A component is drawn at random from the batch and tested and then a second component is drawn. Determine the probability that neither of the components is defective when drawn (a) with replacement, and (b) without replacement.

(a) With replacement

The probability that the component selected on the first draw is satisfactory is $\frac{35}{40}$, i.e. $\frac{7}{8}$. The component is now replaced and a second draw is made. The probability that this component is also satisfactory is $\frac{7}{8}$. Hence, the probability that both the first component drawn **and** the second component drawn are satisfactory is:

$$\frac{7}{8} \times \frac{7}{8} = \mathbf{\frac{49}{64}} \text{ or } \mathbf{0.7656}$$

(b) Without replacement

The probability that the first component drawn is satisfactory is $\frac{7}{8}$. There are now only 34 satisfactory components left in the batch and the batch number is 39. Hence, the probability of drawing a satisfactory component on the second draw is $\frac{34}{39}$. Thus the probability that the first component drawn **and** the second component drawn are satisfactory, i.e. neither is defective, is:

$$\frac{7}{8} \times \frac{34}{39} = \mathbf{\frac{238}{312}} \text{ or } \mathbf{0.7628}$$

Problem 7. A batch of 40 components contains 5 which are defective. If a component is drawn at random from the batch and tested and then a second component is drawn at random, calculate the probability of having one defective component, both with and without replacement.

The probability of having one defective component can be achieved in two ways. If p is the probability of drawing a defective component and q is the probability of drawing a satisfactory component, then the probability of having one defective component is given by drawing a satisfactory component and then a defective component **or** by drawing a defective component and then a satisfactory one, i.e. by $q \times p + p \times q$

With replacement:

$$p = \frac{5}{40} = \frac{1}{8}$$

and

$$q = \frac{35}{40} = \frac{7}{8}$$

Hence, probability of having one defective component is:

$$\frac{1}{8}\times\frac{7}{8}+\frac{7}{8}\times\frac{1}{8}$$

i.e.

$$\frac{7}{64}+\frac{7}{64}=\frac{7}{32} \text{ or } \mathbf{0.2188}$$

Without replacement:

$p_1=\frac{1}{8}$ and $q_1=\frac{7}{8}$ on the first of the two draws. The batch number is now 39 for the second draw, thus,

$$p_2=\frac{5}{39} \text{ and } q_2=\frac{35}{39}$$

$$p_1q_2+q_1p_2=\frac{1}{8}\times\frac{35}{39}+\frac{7}{8}\times\frac{5}{39}$$

$$=\frac{35+35}{312}$$

$$=\frac{70}{312} \text{ or } \mathbf{0.2244}$$

Problem 8. A box contains 74 brass washers, 86 steel washers and 40 aluminium washers. Three washers are drawn at random from the box without replacement. Determine the probability that all three are steel washers.

Assume, for clarity of explanation, that a washer is drawn at random, then a second, then a third (although this assumption does not affect the results obtained). The total number of washers is $74+86+40$, i.e. 200. The probability of randomly selecting a steel washer on the first draw is $\frac{86}{200}$. There are now 85 steel washers in a batch of 199. The probability of randomly selecting a steel washer on the second draw is $\frac{85}{199}$. There are now 84 steel washers in a batch of 198. The probability of randomly selecting a steel washer on the third draw is $\frac{84}{198}$. Hence the probability of selecting a steel washer on the third draw is $\frac{84}{198}$. Hence the probability of selecting a steel washer on the first draw **and** the second draw **and** the third draw is:

$$\frac{86}{200}\times\frac{85}{199}\times\frac{84}{198}=\frac{614040}{7880400}=\mathbf{0.0779}$$

Problem 9. For the box of washers given in Problem 8 above, determine the probability that there are no aluminium washers drawn, when three washers are drawn at random from the box without replacement.

The probability of not drawing an aluminium washer on the first draw is $1-\left(\frac{40}{200}\right)$, i.e. $\frac{160}{200}$. There are now 199 washers in the batch of which 159 are not aluminium washers. Hence, the probability of not drawing an aluminium washer on the second draw is $\frac{159}{199}$. Similarly, the probability of not drawing an aluminium washer on the third draw is $\frac{158}{198}$. Hence the probability of not drawing an aluminium washer on the first **and** second **and** third draws is

$$\frac{160}{200}\times\frac{159}{199}\times\frac{158}{198}=\frac{4019520}{7880400}=\mathbf{0.5101}$$

Problem 10. For the box of washers in Problem 8 above, find the probability that there are two brass washers and either a steel or an aluminium washer when three are drawn at random, without replacement.

Two brass washers (A) and one steel washer (B) can be obtained in any of the following ways:

1st draw	2nd draw	3rd draw
A	A	B
A	B	A
B	A	A

Two brass washers and one aluminium washer (C) can also be obtained in any of the following ways:

1st draw	2nd draw	3rd draw
A	A	C
A	C	A
C	A	A

Thus there are six possible ways of achieving the combinations specified. If A represents a brass washer,

B a steel washer and C an aluminium washer, then the combinations and their probabilities are as shown:

Draw			Probability
First	Second	Third	
A	A	B	$\frac{74}{200}\times\frac{73}{199}\times\frac{86}{198}=0.0590$
A	B	A	$\frac{74}{200}\times\frac{86}{199}\times\frac{73}{198}=0.0590$
B	A	A	$\frac{86}{200}\times\frac{74}{199}\times\frac{73}{198}=0.0590$
A	A	C	$\frac{74}{200}\times\frac{73}{199}\times\frac{40}{198}=0.0274$
A	C	A	$\frac{74}{200}\times\frac{40}{199}\times\frac{73}{198}=0.0274$
C	A	A	$\frac{40}{200}\times\frac{74}{199}\times\frac{73}{198}=0.0274$

The probability of having the first combination **or** the second, **or** the third, and so on, is given by the sum of the probabilities,

i.e. by $3\times0.0590+3\times0.0274$, that is, **0.2592**

Now try the following exercise

Exercise 212 Further problems on probability

1. The probability that component A will operate satisfactorily for 5 years is 0.8 and that B will operate satisfactorily over that same period of time is 0.75. Find the probabilities that in a 5 year period: (a) both components operate satisfactorily, (b) only component A will operate satisfactorily, and (c) only component B will operate satisfactorily.

   [(a) 0.6 (b) 0.2 (c) 0.15]

2. In a particular street, 80% of the houses have telephones. If two houses selected at random are visited, calculate the probabilities that (a) they both have a telephone and (b) one has a telephone but the other does not have telephone.

   [(a) 0.64 (b) 0.32]

3. Veroboard pins are packed in packets of 20 by a machine. In a thousand packets, 40 have less than 20 pins. Find the probability that if 2 packets are chosen at random, one will contain less than 20 pins and the other will contain 20 pins or more. [0.0768]

4. A batch of 1 kW fire elements contains 16 which are within a power tolerance and 4 which are not. If 3 elements are selected at random from the batch, calculate the probabilities that (a) all three are within the power tolerance and (b) two are within but one is not within the power tolerance.

   [(a) 0.4912 (b) 0.4211]

5. An amplifier is made up of three transistors, A, B and C. The probabilities of A, B or C being defective are $\frac{1}{20}$, $\frac{1}{25}$ and $\frac{1}{50}$, respectively. Calculate the percentage of amplifiers produced (a) which work satisfactorily and (b) which have just one defective transistor.

   $\begin{bmatrix}\text{(a) } 89.38\% \\ \text{(b) } 10.25\%\end{bmatrix}$

6. A box contains 14 40 W lamps, 28 60 W lamps and 58 25 W lamps, all the lamps being of the same shape and size. Three lamps are drawn at random from the box, first one, then a second, then a third. Determine the probabilities of: (a) getting one 25 W, one 40 W and one 60 W lamp, with replacement, (b) getting one 25 W, one 40 W and one 60 W lamp without replacement, and (c) getting either one 25 W and two 40 W or one 60 W and two 40 W lamps with replacement.

   [(a) 0.0227 (b) 0.0234 (c) 0.0169]

Revision Test 16

This Revision Test covers the material contained in Chapters 54 to 56. *The marks for each question are shown in brackets at the end of each question.*

1. A company produces five products in the following proportions:

   Product A 24 Product B 16 Product C 15
   Product D 11 Product E 6

   Present these data visually by drawing (a) a vertical bar chart, (b) a percentage component bar chart, (c) a pie diagram. (13)

2. The following lists the diameters of 40 components produced by a machine, each measured correct to the nearest hundredth of a centimetre:

1.39	1.36	1.38	1.31	1.33	1.40	1.28
1.40	1.24	1.28	1.42	1.34	1.43	1.35
1.36	1.36	1.35	1.45	1.29	1.39	1.38
1.38	1.35	1.42	1.30	1.26	1.37	1.33
1.37	1.34	1.34	1.32	1.33	1.30	1.38
1.41	1.35	1.38	1.27	1.37		

   (a) Using 8 classes form a frequency distribution and a cumulative frequency distribution.
   (b) For the above data draw a histogram, a frequency polygon and an ogive. (21)

3. Determine for the 10 measurements of lengths shown below:

   (a) the arithmetic mean, (b) the median, (c) the mode, and (d) the standard deviation.

   28 m, 20 m, 32 m, 44 m, 28 m, 30 m, 30 m, 26 m, 28 m and 34 m (10)

4. The heights of 100 people are measured correct to the nearest centimetre with the following results:

150–157 cm	5	158–165 cm	18
166–173 cm	42	174–181 cm	27
182–189 cm	8		

   Determine for the data (a) the mean height and (b) the standard deviation. (12)

5. Draw an ogive for the data of component measurements given below, and hence determine the median and the first and third quartile values for this distribution.

Class intervals (mm)	Frequency	Cumulative frequency
1.24–1.26	2	2
1.27–1.29	4	6
1.30–1.32	4	10
1.33–1.35	10	20
1.36–1.38	11	31
1.39–1.41	5	36
1.42–1.44	3	39
1.45–1.47	1	40

   (10)

6. Determine the probabilities of:

   (a) drawing a white ball from a bag containing 6 black and 14 white balls,
   (b) winning a prize in a raffle by buying 6 tickets when a total of 480 tickets are sold,
   (c) selecting at random a female from a group of 12 boys and 28 girls,
   (d) winning a prize in a raffle by buying 8 tickets when there are 5 prizes and a total of 800 tickets are sold. (8)

7. The probabilities of an engine failing are given by: p_1, failure due to overheating; p_2, failure due to ignition problems; p_3, failure due to fuel blockage. When $p_1 = \frac{1}{8}$, $p_2 = \frac{1}{5}$ and $p_3 = \frac{2}{7}$, determine the probabilities of:

   (a) all three failures occurring,
   (b) the first and second but not the third failure occurring,
   (c) only the second failure occurring,
   (d) the first or the second failure occurring but not the third. (12)

8. In a box containing 120 similar transistors 70 are satisfactory, 37 give too high a gain under normal

operating conditions and the remainder give too low a gain.

Calculate the probability that when drawing two transistors in turn, at random, **with replacement**, of having

(a) two satisfactory,

(b) none with low gain,

(c) one with high gain and one satisfactory,

(d) one with low gain and none satisfactory.

Determine the probabilities in (a), (b) and (c) above if the transistors are drawn **without replacement**. (14)

Chapter 57

The binomial and Poisson distributions

57.1 The binomial distribution

The binomial distribution deals with two numbers only, these being the probability that an event will happen, p, and the probability that an event will not happen, q. Thus, when a coin is tossed, if p is the probability of the coin landing with a head upwards, q is the probability of the coin landing with a tail upwards. $p + q$ must always be equal to unity. A binomial distribution can be used for finding, say, the probability of getting three heads in seven tosses of the coin, or in industry for determining defect rates as a result of sampling. One way of defining a binomial distribution is as follows:

> *'if* p *is the probability that an event will happen and* q *is the probability that the event will not happen, then the probabilities that the event will happen 0, 1, 2, 3, . . . ,* n *times in* n *trials are given by the successive terms of the expansion of* $(q + p)^n$*, taken from left to right'.*

The binomial expansion of $(q + p)^n$ is:

$$q^n + nq^{n-1}p + \frac{n(n-1)}{2!}q^{n-2}p^2 + \frac{n(n-1)(n-2)}{3!}q^{n-3}p^3 + \cdots$$

from Chapter 7.

This concept of a binomial distribution is used in Problems 1 and 2.

Problem 1. Determine the probabilities of having (a) at least 1 girl and (b) at least 1 girl and 1 boy in a family of 4 children, assuming equal probability of male and female birth.

The probability of a girl being born, p, is 0.5 and the probability of a girl not being born (male birth), q, is also 0.5. The number in the family, n, is 4. From above, the probabilities of 0, 1, 2, 3, 4 girls in a family of 4 are given by the successive terms of the expansion of $(q + p)^4$ taken from left to right. From the binomial expansion:

$$(q + p)^4 = q^4 + 4q^3p + 6q^2p^2 + 4qp^3 + p^4$$

Hence the probability of no girls is q^4,

i.e. $0.5^4 = 0.0625$

the probability of 1 girl is $4q^3p$,

i.e. $4 \times 0.5^3 \times 0.5 = 0.2500$

the probability of 2 girls is $6q^2p^2$,

i.e. $6 \times 0.5^2 \times 0.5^2 = 0.3750$

the probability of 3 girls is $4qp^3$,

i.e. $4 \times 0.5 \times 0.5^3 = 0.2500$

the probability of 4 girls is p^4,

i.e. $0.5^4 = 0.0625$

Total probability, $(q + p)^4 = 1.0000$

(a) The probability of having at least one girl is the sum of the probabilities of having 1, 2, 3 and 4 girls, i.e.

$0.2500 + 0.3750 + 0.2500 + 0.0625 =$ **0.9375**

(Alternatively, the probability of having at least 1 girl is: 1 − (the probability of having no girls), i.e. 1 − 0.0625, giving **0.9375**, as obtained previously.)

(b) The probability of having at least 1 girl and 1 boy is given by the sum of the probabilities of having: 1 girl and 3 boys, 2 girls and 2 boys and 3 girls and 2 boys, i.e.

$$0.2500 + 0.3750 + 0.2500 = \mathbf{0.8750}$$

(Alternatively, this is also the probability of having 1 − (probability of having no girls + probability of having no boys), i.e. $1 - 2 \times 0.0625 = \mathbf{0.8750}$, as obtained previously.)

Problem 2. A dice is rolled 9 times. Find the probabilities of having a 4 upwards (a) 3 times and (b) less than 4 times.

Let p be the probability of having a 4 upwards. Then $p = 1/6$, since dice have six sides.
Let q be the probability of not having a 4 upwards. Then $q = 5/6$. The probabilities of having a 4 upwards $0, 1, 2, \ldots, n$ times are given by the successive terms of the expansion of $(q + p)^n$, taken from left to right. From the binomial expansion:

$$(q + p)^9 = q^9 + 9q^8p + 36q^7p^2 + 84q^6p^3 + \cdots$$

The probability of having a 4 upwards no times is

$$q^9 = (5/6)^9 = 0.1938$$

The probability of having a 4 upwards once is

$$9q^8p = 9(5/6)^8(1/6) = 0.3489$$

The probability of having a 4 upwards twice is

$$36q^7p^2 = 36(5/6)^7(1/6)^2 = 0.2791$$

The probability of having a 4 upwards 3 times is

$$84q^6p^3 = 84(5/6)^6(1/6)^3 = 0.1302$$

(a) The probability of having a 4 upwards 3 times is **0.1302**

(b) The probability of having a 4 upwards less than 4 times is the sum of the probabilities of having a 4 upwards 0, 1, 2, and 3 times, i.e.

$$0.1938 + 0.3489 + 0.2791 + 0.1302 = \mathbf{0.9520}$$

Industrial inspection

In industrial inspection, p is often taken as the probability that a component is defective and q is the probability that the component is satisfactory. In this case, a binomial distribution may be defined as:

'the probabilities that 0, 1, 2, 3,…, n *components are defective in a sample of* n *components, drawn at random from a large batch of components, are given by the successive terms of the expansion of* $(q + p)^n$*, taken from left to right'.*

This definition is used in Problems 3 and 4.

Problem 3. A machine is producing a large number of bolts automatically. In a box of these bolts, 95% are within the allowable tolerance values with respect to diameter, the remainder being outside of the diameter tolerance values. Seven bolts are drawn at random from the box. Determine the probabilities that (a) two and (b) more than two of the seven bolts are outside of the diameter tolerance values.

Let p be the probability that a bolt is outside of the allowable tolerance values, i.e. is defective, and let q be the probability that a bolt is within the tolerance values, i.e. is satisfactory. Then $p = 5\%$, i.e. 0.05 per unit and $q = 95\%$, i.e. 0.95 per unit. The sample number is 7.
The probabilities of drawing $0, 1, 2, \ldots, n$ defective bolts are given by the successive terms of the expansion of $(q + p)^n$, taken from left to right. In this problem

$$(q + p)^n = (0.95 + 0.05)^7$$

$$= 0.95^7 + 7 \times 0.95^6 \times 0.05 + 21 \times 0.95^5 \times 0.05^2 + \cdots$$

Thus the probability of no defective bolts is

$$0.95^7 = 0.6983$$

The probability of 1 defective bolt is

$$7 \times 0.95^6 \times 0.05 = 0.2573$$

The probability of 2 defective bolts is

$$21 \times 0.95^5 \times 0.05^2 = 0.0406, \text{ and so on.}$$

(a) The probability that two bolts are outside of the diameter tolerance values is **0.0406**

(b) To determine the probability that more than two bolts are defective, the sum of the probabilities of 3 bolts, 4 bolts, 5 bolts, 6 bolts and 7 bolts being defective can be determined. An easier way to find this sum is to find 1 − (sum of 0 bolts, 1 bolt and 2 bolts being defective), since the sum of all the terms is unity. Thus, the probability of there being more than two bolts outside of the tolerance values is:

$$1 - (0.6983 + 0.2573 + 0.0406), \text{ i.e. } \mathbf{0.0038}$$

Problem 4. A package contains 50 similar components and inspection shows that four have been damaged during transit. If six components are drawn at random from the contents of the package determine the probabilities that in this sample (a) one and (b) less than three are damaged.

The probability of a component being damaged, p, is 4 in 50, i.e. 0.08 per unit. Thus, the probability of a component not being damaged, q, is $1 - 0.08$, i.e. 0.92. The probability of there being $0, 1, 2, \ldots, 6$ damaged components is given by the successive terms of $(q + p)^6$, taken from left to right.

$$(q + p)^6 = q^6 + 6q^5p + 15q^4p^2 + 20q^3p^3 + \cdots$$

(a) The probability of one damaged component is

$$6q^5p = 6 \times 0.92^5 \times 0.08 = \mathbf{0.3164}$$

(b) The probability of less than three damaged components is given by the sum of the probabilities of 0, 1 and 2 damaged components.

$$\begin{aligned} q^6 + 6q^5p + 15q^4p^2 &= 0.92^6 + 6 \times 0.92^5 \times 0.08 \\ &\quad + 15 \times 0.92^4 \times 0.08^2 \\ &= 0.6064 + 0.3164 + 0.0688 = \mathbf{0.9916} \end{aligned}$$

Histogram of probabilities

The terms of a binomial distribution may be represented pictorially by drawing a histogram, as shown in Problem 5.

Problem 5. The probability of a student successfully completing a course of study in three years is 0.45. Draw a histogram showing the probabilities of $0, 1, 2, \ldots, 10$ students successfully completing the course in three years.

Let p be the probability of a student successfully completing a course of study in three years and q be the probability of not doing so. Then $p = 0.45$ and $q = 0.55$. The number of students, n, is 10.

The probabilities of $0, 1, 2, \ldots, 10$ students successfully completing the course are given by the successive terms of the expansion of $(q + p)^{10}$, taken from left to right.

$$\begin{aligned} (q + p)^{10} &= q^{10} + 10q^9p + 45q^8p^2 + 120q^7p^3 \\ &\quad + 210q^6p^4 + 252q^5p^5 + 210q^4p^6 \\ &\quad + 120q^3p^7 + 45q^2p^8 + 10qp^9 + p^{10} \end{aligned}$$

Substituting $q = 0.55$ and $p = 0.45$ in this expansion gives the values of the successive terms as: 0.0025, 0.0207, 0.0763, 0.1665, 0.2384, 0.2340, 0.1596, 0.0746, 0.0229, 0.0042 and 0.0003. The histogram depicting these probabilities is shown in Fig. 57.1.

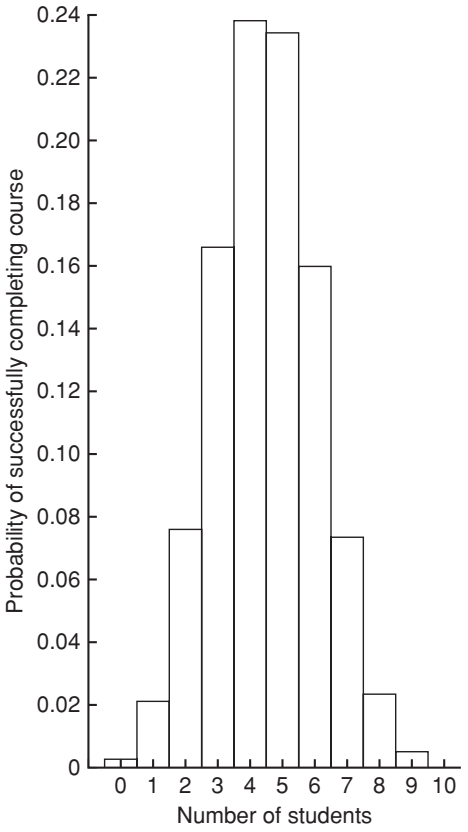


Figure 57.1

Now try the following exercise

Exercise 213 Further problems on the binomial distribution

1. Concrete blocks are tested and it is found that, on average, 7% fail to meet the required specification. For a batch of 9 blocks, determine the probabilities that (a) three blocks and (b) less than four blocks will fail to meet the specification. [(a) 0.0186 (b) 0.9976]
2. If the failure rate of the blocks in Problem 1 rises to 15%, find the probabilities that (a) no blocks and (b) more than two blocks will fail to meet the specification in a batch of 9 blocks. [(a) 0.2316 (b) 0.1408]
3. The average number of employees absent from a firm each day is 4%. An office within the firm has seven employees. Determine the probabilities that (a) no employee and (b) three employees will be absent on a particular day. [(a) 0.7514 (b) 0.0019]
4. A manufacturer estimates that 3% of his output of a small item is defective. Find the probabilities that in a sample of 10 items (a) less than two and (b) more than two items will be defective. [(a) 0.9655 (b) 0.0028]
5. Five coins are tossed simultaneously. Determine the probabilities of having 0, 1, 2, 3, 4 and 5 heads upwards, and draw a histogram depicting the results.

   [Vertical adjacent rectangles, whose heights are proportional to 0.0313, 0.1563, 0.3125, 0.3125, 0.1563 and 0.0313]
6. If the probability of rain falling during a particular period is 2/5, find the probabilities of having 0, 1, 2, 3, 4, 5, 6 and 7 wet days in a week. Show these results on a histogram.

   [Vertical adjacent rectangles, whose heights are proportional to 0.0280, 0.1306, 0.2613, 0.2903, 0.1935, 0.0774, 0.0172 and 0.0016]
7. An automatic machine produces, on average, 10% of its components outside of the tolerance required. In a sample of 10 components from this machine, determine the probability of having three components outside of the tolerance required by assuming a binomial distribution. [0.0574]

57.2 The Poisson distribution

When the number of trials, n, in a binomial distribution becomes large (usually taken as larger than 10), the calculations associated with determining the values of the terms becomes laborious. If n is large and p is small, and the product np is less than 5, a very good approximation to a binomial distribution is given by the corresponding Poisson distribution, in which calculations are usually simpler.

The Poisson approximation to a binomial distribution may be defined as follows:

> *'the probabilities that an event will happen 0, 1, 2, 3, …,* n *times in* n *trials are given by the successive terms of the expression*
>
> $$e^{-\lambda}\left(1+\lambda+\frac{\lambda^2}{2!}+\frac{\lambda^3}{3!}+\cdots\right)$$
>
> *taken from left to right'.*

The symbol λ is the expectation of an event happening and is equal to np.

Problem 6. If 3% of the gearwheels produced by a company are defective, determine the probabilities that in a sample of 80 gearwheels (a) two and (b) more than two will be defective.

The sample number, n, is large, the probability of a defective gearwheel, p, is small and the product np is 80×0.03, i.e. 2.4, which is less than 5.

Hence a Poisson approximation to a binomial distribution may be used. The expectation of a defective gearwheel, $\lambda = np = 2.4$

The probabilities of 0, 1, 2, … defective gearwheels are given by the successive terms of the expression

$$e^{-\lambda}\left(1+\lambda+\frac{\lambda^2}{2!}+\frac{\lambda^3}{3!}+\cdots\right)$$

taken from left to right, i.e. by

$$e^{-\lambda}, \lambda e^{-\lambda}, \frac{\lambda^2 e^{-\lambda}}{2!}, \ldots$$

Thus probability of no defective gearwheels is

$$e^{-\lambda} = e^{-2.4} = 0.0907$$

probability of 1 defective gearwheel is

$$\lambda e^{-\lambda} = 2.4e^{-2.4} = 0.2177$$

probability of 2 defective gearwheels is

$$\frac{\lambda^2 e^{-\lambda}}{2!} = \frac{2.4^2 e^{-2.4}}{2 \times 1} = 0.2613$$

(a) The probability of having 2 defective gearwheels is **0.2613**

(b) The probability of having more than 2 defective gearwheels is 1 − (the sum of the probabilities of having 0, 1, and 2 defective gearwheels), i.e.

$$1 - (0.0907 + 0.2177 + 0.2613),$$

that is, **0.4303**

The principal use of a Poisson distribution is to determine the theoretical probabilities when p, the probability of an event happening, is known, but q, the probability of the event not happening is unknown. For example, the average number of goals scored per match by a football team can be calculated, but it is not possible to quantify the number of goals which were not scored. In this type of problem, a Poisson distribution may be defined as follows:

'the probabilities of an event occurring 0, 1, 2, 3, ... times are given by the successive terms of the expression

$$e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots\right),$$

taken from left to right'

The symbol λ is the value of the average occurrence of the event.

Problem 7. A production department has 35 similar milling machines. The number of breakdowns on each machine averages 0.06 per week. Determine the probabilities of having (a) one, and (b) less than three machines breaking down in any week.

Since the average occurrence of a breakdown is known but the number of times when a machine did not break down is unknown, a Poisson distribution must be used. The expectation of a breakdown for 35 machines is 35×0.06, i.e. 2.1 breakdowns per week. The probabilities of a breakdown occurring $0, 1, 2, \ldots$ times are given by the successive terms of the expression

$$e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots\right),$$

taken from left to right.

Hence probability of no breakdowns

$$e^{-\lambda} = e^{-2.1} = 0.1225$$

probability of 1 breakdown is

$$\lambda e^{-\lambda} = 2.1e^{-2.1} = 0.2572$$

probability of 2 breakdowns is

$$\frac{\lambda^2 e^{-\lambda}}{2!} = \frac{2.1^2 e^{-2.1}}{2 \times 1} = 0.2700$$

(a) The probability of 1 breakdown per week is **0.2572**

(b) The probability of less than 3 breakdowns per week is the sum of the probabilities of 0, 1, and 2 breakdowns per week,

i.e. $0.1225 + 0.2572 + 0.2700$, i.e. **0.6497**

Histogram of probabilities

The terms of a Poisson distribution may be represented pictorially by drawing a histogram, as shown in Problem 8.

Problem 8. The probability of a person having an accident in a certain period of time is 0.0003. For a population of 7500 people, draw a histogram showing the probabilities of 0, 1, 2, 3, 4, 5 and 6 people having an accident in this period.

The probabilities of $0, 1, 2, \ldots$ people having an accident are given by the terms of expression

$$e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots\right),$$

taken from left to right.

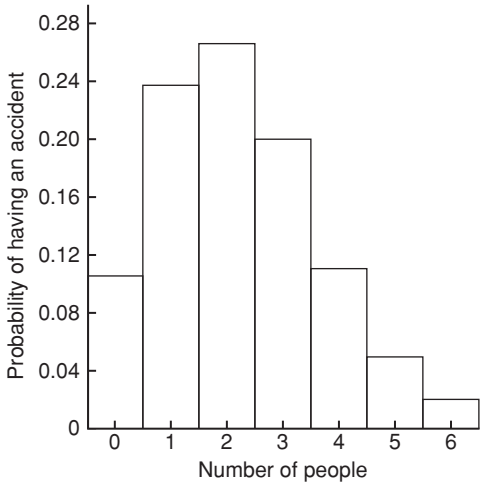


Figure 57.2

The average occurrence of the event, λ, is 7500×0.0003, i.e. 2.25

The probability of no people having an accident is

$$e^{-\lambda} = e^{-2.25} = 0.1054$$

The probability of 1 person having an accident is

$$\lambda e^{-\lambda} = 2.25e^{-2.25} = 0.2371$$

The probability of 2 people having an accident is

$$\frac{\lambda^2 e^{-\lambda}}{2!} = \frac{2.25^2 e^{-2.25}}{2!} = 0.2668$$

and so on, giving probabilities of 0.2001, 0.1126, 0.0506 and 0.0190 for 3, 4, 5 and 6 respectively having an accident. The histogram for these probabilities is shown in Fig. 57.2.

Now try the following exercise

Exercise 214 Further problems on the Poisson distribution

1. In problem 7 of Exercise 213, page 559, determine the probability of having three components outside of the required tolerance using the Poisson distribution. [0.0613]

2. The probability that an employee will go to hospital in a certain period of time is 0.0015. Use a Poisson distribution to determine the probability of more than two employees going to hospital during this period of time if there are 2000 employees on the payroll. [0.5768]

3. When packaging a product, a manufacturer finds that one packet in twenty is underweight. Determine the probabilities that in a box of 72 packets (a) two and (b) less than four will be underweight. [(a) 0.1771 (b) 0.5153]

4. A manufacturer estimates that 0.25% of his output of a component are defective. The components are marketed in packets of 200. Determine the probability of a packet containing less than three defective components. [0.9856]

5. The demand for a particular tool from a store is, on average, five times a day and the demand follows a Poisson distribution. How many of these tools should be kept in the stores so that the probability of there being one available when required is greater than 10%?

   [The probabilities of the demand for 0, 1, 2, … tools are 0.0067, 0.0337, 0.0842, 0.1404, 0.1755, 0.1755, 0.1462, 0.1044, 0.0653, … This shows that the probability of wanting a tool 8 times a day is 0.0653, i.e. less than 10%. Hence 7 should be kept in the store]

6. Failure of a group of particular machine tools follows a Poisson distribution with a mean value of 0.7. Determine the probabilities of 0, 1, 2, 3, 4 and 5 failures in a week and present these results on a histogram.

   [Vertical adjacent rectangles having heights proportional to 0.4966, 0.3476, 0.1217, 0.0284, 0.0050 and 0.0007]

Chapter 58

The normal distribution

58.1 Introduction to the normal distribution

When data is obtained, it can frequently be considered to be a sample (i.e. a few members) drawn at random from a large population (i.e. a set having many members). If the sample number is large, it is theoretically possible to choose class intervals which are very small, but which still have a number of members falling within each class. A frequency polygon of this data then has a large number of small line segments and approximates to a continuous curve. Such a curve is called a **frequency or a distribution curve**.

An extremely important symmetrical distribution curve is called the **normal curve** and is as shown in Fig. 58.1. This curve can be described by a mathematical equation and is the basis of much of the work done in more advanced statistics. Many natural occurrences such as the heights or weights of a group of people, the sizes of components produced by a particular machine and the life length of certain components approximate to a normal distribution.

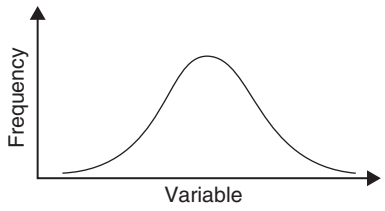


Figure 58.1

Normal distribution curves can differ from one another in the following four ways:

(a) by having different mean values

(b) by having different values of standard deviations

(c) the variables having different values and different units and

(d) by having different areas between the curve and the horizontal axis.

A normal distribution curve is **standardized** as follows:

(a) The mean value of the unstandardized curve is made the origin, thus making the mean value, $\overline{x}$, zero.

(b) The horizontal axis is scaled in standard deviations. This is done by letting $z = \dfrac{x - \overline{x}}{\sigma}$, where z is called the **normal standard variate**, x is the value of the variable, $\overline{x}$ is the mean value of the distribution and σ is the standard deviation of the distribution.

(c) The area between the normal curve and the horizontal axis is made equal to unity.

When a normal distribution curve has been standardized, the normal curve is called a **standardized normal curve** or a **normal probability curve**, and any normally distributed data may be represented by the **same** normal probability curve.

The area under part of a normal probability curve is directly proportional to probability and the value of the shaded area shown in Fig. 58.2 can be determined by evaluating:

$$\int \frac{1}{\sqrt{(2\pi)}} \mathrm{e}^{\left(\frac{z^2}{2}\right)} \mathrm{d}z, \quad \text{where } z = \frac{x - \overline{x}}{\sigma}$$

To save repeatedly determining the values of this function, tables of partial areas under the standardized normal curve are available in many mathematical formulae books, and such a table is shown in Table 58.1, on page 564.

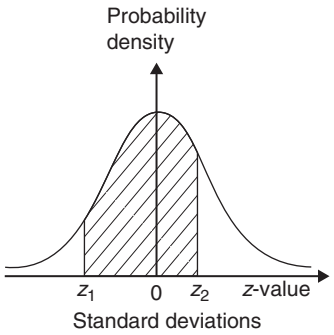


Figure 58.2

Problem 1. The mean height of 500 people is 170 cm and the standard deviation is 9 cm. Assuming the heights are normally distributed, determine the number of people likely to have heights between 150 cm and 195 cm.

The mean value, $\bar{x}$, is 170 cm and corresponds to a normal standard variate value, z, of zero on the standardized normal curve. A height of 150 cm has a z-value given by $z = \dfrac{x - \bar{x}}{\sigma}$ standard deviations, i.e. $\dfrac{150 - 170}{9}$ or -2.22 standard deviations. Using a table of partial areas beneath the standardized normal curve (see Table 58.1), a z-value of -2.22 corresponds to an area of 0.4868 between the mean value and the ordinate $z = -2.22$. The negative z-value shows that it lies to the left of the $z = 0$ ordinate.

This area is shown shaded in Fig. 58.3(a). Similarly, 195 cm has a z-value of $\dfrac{195 - 170}{9}$ that is 2.78 standard deviations. From Table 58.1, this value of z corresponds to an area of 0.4973, the positive value of z showing that it lies to the right of the $z = 0$ ordinate. This area is shown shaded in Fig. 58.3(b). The total area shaded in Figs. 58.3(a) and (b) is shown in Fig. 58.3(c) and is $0.4868 + 0.4973$, i.e. 0.9841 of the total area beneath the curve.

However, the area is directly proportional to probability. Thus, the probability that a person will have a height of between 150 and 195 cm is 0.9841. For a group of 500 people, 500×0.9841, i.e. **492 people are likely to have heights in this range**. The value of 500×0.9841 is 492.05, but since answers based on a normal probability distribution can only be approximate, results are usually given correct to the nearest whole number.

Problem 2. For the group of people given in Problem 1, find the number of people likely to have heights of less than 165 cm.

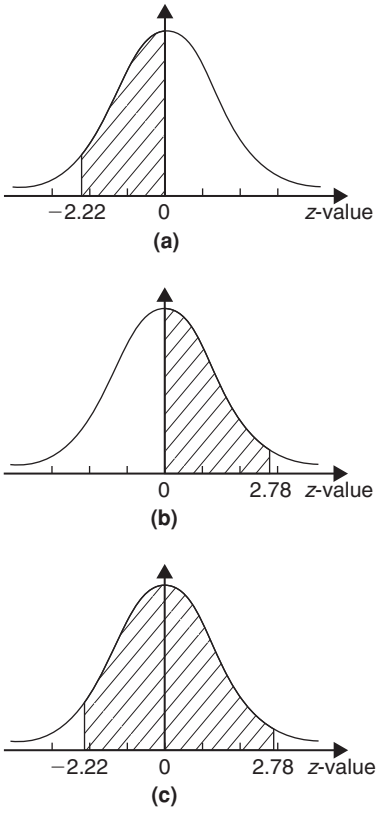


Figure 58.3

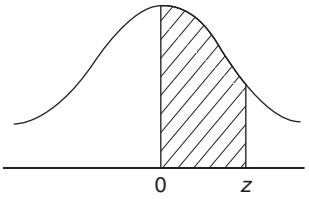
A height of 165 cm corresponds to $\dfrac{165 - 170}{9}$ i.e. -0.56 standard deviations.

The area between $z = 0$ and $z = -0.56$ (from Table 58.1) is 0.2123, shown shaded in Fig. 58.4(a). The total area under the standardized normal curve is unity and since the curve is symmetrical, it follows that the total area to the left of the $z = 0$ ordinate is 0.5000. Thus the area to the left of the $z = -0.56$ ordinate ('left' means 'less than', 'right' means 'more than') is $0.5000 - 0.2123$, i.e. 0.2877 of the total area, which is shown shaded in Fig 58.4(b). The area is directly proportional to probability and since the total area beneath the standardized normal curve is unity, the probability of a person's height being less than 165 cm is 0.2877. For a group of 500 people, 500×0.2877, i.e. **144 people are likely to have heights of less than 165 cm**.

Problem 3. For the group of people given in Problem 1 find how many people are likely to have heights of more than 194 cm.

194 cm corresponds to a z-value of $\dfrac{194 - 170}{9}$ that is, 2.67 standard deviations. From Table 58.1, the area

Table 58.1 Partial areas under the standardized normal curve



$z = \frac{x - \bar{x}}{\sigma}$	0	1	2	3	4	5	6	7	8	9
0.0	0.0000	0.0040	0.0080	0.0120	0.0159	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0678	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1388	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2086	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2760	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3451	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4430	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4762	0.4767
2.0	0.4772	0.4778	0.4783	0.4785	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4980	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990
3.1	0.4990	0.4991	0.4991	0.4991	0.4992	0.4992	0.4992	0.4992	0.4993	0.4993
3.2	0.4993	0.4993	0.4994	0.4994	0.4994	0.4994	0.4994	0.4995	0.4995	0.4995
3.3	0.4995	0.4995	0.4995	0.4996	0.4996	0.4996	0.4996	0.4996	0.4996	0.4997
3.4	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4998
3.5	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998
3.6	0.4998	0.4998	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999
3.7	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999
3.8	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999
3.9	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000

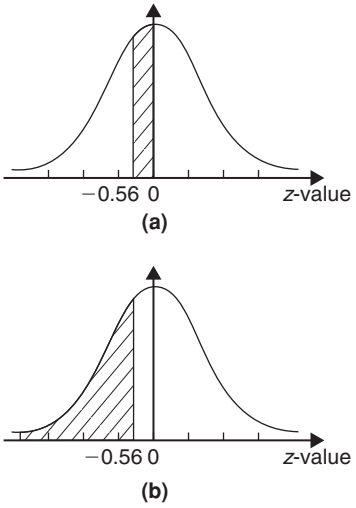


Figure 58.4

between $z=0$, $z=2.67$ and the standardized normal curve is 0.4962, shown shaded in Fig. 58.5(a). Since the standardized normal curve is symmetrical, the total area to the right of the $z=0$ ordinate is 0.5000, hence the shaded area shown in Fig. 58.5(b) is $0.5000-0.4962$, i.e. 0.0038. This area represents the probability of a person having a height of more than 194 cm, and for 500 people, the number of people likely to have a height of more than 194 cm is 0.0038×500, i.e. **2 people**.

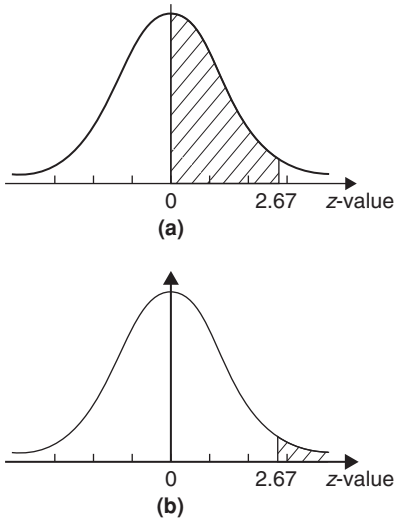


Figure 58.5

Problem 4. A batch of 1500 lemonade bottles have an average contents of 753 ml and the standard deviation of the contents is 1.8 ml. If the volumes of the contents are normally distributed, find

(a) the number of bottles likely to contain less than 750 ml,

(b) the number of bottles likely to contain between 751 and 754 ml,

(c) the number of bottles likely to contain more than 757 ml, and

(d) the number of bottles likely to contain between 750 and 751 ml.

(a) The z-value corresponding to 750 ml is given by $\dfrac{x-\bar{x}}{\sigma}$ i.e. $\dfrac{750-753}{1.8}=-1.67$ standard deviations. From Table 58.1, the area between $z=0$ and $z=-1.67$ is 0.4525. Thus the area to the left of the $z=-1.67$ ordinate is $0.5000-0.4525$ (see Problem 2), i.e. 0.0475. This is the probability of a bottle containing less than 750 ml. Thus, for a batch of 1500 bottles, it is likely that 1500×0.0475, i.e. **71 bottles will contain less than 750 ml**.

(b) The z-value corresponding to 751 and 754 ml are $\dfrac{751-753}{1.8}$ and $\dfrac{754-753}{1.8}$ i.e. -1.11 and 0.56 respectively. From Table 58.1, the areas corresponding to these values are 0.3665 and 0.2123 respectively. Thus the probability of a bottle containing between 751 and 754 ml is $0.3665+0.2123$ (see Problem 1), i.e. 0.5788. For 1500 bottles, it is likely that 1500×0.5788, i.e. **868 bottles will contain between 751 and 754 ml**.

(c) The z-value corresponding to 757 ml is $\dfrac{757-753}{1.8}$, i.e. 2.22 standard deviations. From Table 58.1, the area corresponding to a z-value of 2.22 is 0.4868. The area to the right of the $z=2.22$ ordinate is $0.5000-0.4868$ (see Problem 3), i.e. 0.0132. Thus, for 1500 bottles, it is likely that 1500×0.0132, i.e. **20 bottles will have contents of more than 757 ml**.

(d) The z-value corresponding to 750 ml is -1.67 (see part (a)), and the z-value corresponding to 751 ml is -1.11 (see part (b)). The areas corresponding to these z-values are 0.4525 and 0.3665 respectively, and both these areas lie on the left of the $z=0$ ordinate. The area between $z=-1.67$ and $z=-1.11$ is $0.4525-0.3665$, i.e. 0.0860 and this is the probability of a bottle having contents between 750 and 751 ml. For 1500 bottles, it is

likely that 1500×0.0860, i.e. **129 bottles will be in this range**.

Now try the following exercise

Exercise 215 Further problems on the introduction to the normal distribution

1. A component is classed as defective if it has a diameter of less than 69 mm. In a batch of 350 components, the mean diameter is 75 mm and the standard deviation is 2.8 mm. Assuming the diameters are normally distributed, determine how many are likely to be classed as defective. [6]

2. The masses of 800 people are normally distributed, having a mean value of 64.7 kg and a standard deviation of 5.4 kg. Find how many people are likely to have masses of less than 54.4 kg. [22]

3. 500 tins of paint have a mean content of 1010 ml and the standard deviation of the contents is 8.7 ml. Assuming the volumes of the contents are normally distributed, calculate the number of tins likely to have contents whose volumes are less than (a) 1025 ml (b) 1000 ml and (c) 995 ml.

   [(a) 479 (b) 63 (c) 21]

4. For the 350 components in Problem 1, if those having a diameter of more than 81.5 mm are rejected, find, correct to the nearest component, the number likely to be rejected due to being oversized. [4]

5. For the 800 people in Problem 2, determine how many are likely to have masses of more than (a) 70 kg and (b) 62 kg.

   [(a) 131 (b) 553]

6. The mean diameter of holes produced by a drilling machine bit is 4.05 mm and the standard deviation of the diameters is 0.0028 mm. For twenty holes drilled using this machine, determine, correct to the nearest whole number, how many are likely to have diameters of between (a) 4.048 and 4.0553 mm and (b) 4.052 and 4.056 mm, assuming the diameters are normally distributed.

   [(a) 15 (b) 4]

7. The intelligence quotients of 400 children have a mean value of 100 and a standard deviation of 14. Assuming that I.Q.'s are normally distributed, determine the number of children likely to have I.Q.'s of between (a) 80 and 90, (b) 90 and 110 and (c) 110 and 130.

   [(a) 65 (b) 209 (c) 89]

8. The mean mass of active material in tablets produced by a manufacturer is 5.00 g and the standard deviation of the masses is 0.036 g. In a bottle containing 100 tablets, find how many tablets are likely to have masses of (a) between 4.88 and 4.92 g, (b) between 4.92 and 5.04 g and (c) more than 5.04 g.

   [(a) 1 (b) 85 (c) 13]

58.2 Testing for a normal distribution

It should never be assumed that because data is continuous it automatically follows that it is normally distributed. One way of checking that data is normally distributed is by using **normal probability paper**, often just called **probability paper**. This is special graph paper which has linear markings on one axis and percentage probability values from 0.01 to 99.99 on the other axis (see Figs. 58.6 and 58.7). The divisions on the probability axis are such that a straight line graph results for normally distributed data when percentage cumulative frequency values are plotted against upper class boundary values. If the points do not lie in a reasonably straight line, then the data is not normally distributed. The method used to test the normality of a distribution is shown in Problems 5 and 6. The mean value and standard deviation of normally distributed data may be determined using normal probability paper. For normally distributed data, the area beneath the standardized normal curve and a z-value of unity (i.e. one standard deviation) may be obtained from Table 58.1. For one standard deviation, this area is 0.3413, i.e. 34.13%. An area of ± 1 standard deviation is symmetrically placed on either side of the $z = 0$ value, i.e. is symmetrically placed on either side of the 50% cumulative frequency value. Thus an area corresponding to ± 1 standard deviation extends from percentage cumulative frequency values of $(50 + 34.13)\%$ to $(50 - 34.13)\%$, i.e. from 84.13% to 15.87%. For most purposes, these values are taken as 84% and 16%. Thus, when using normal probability

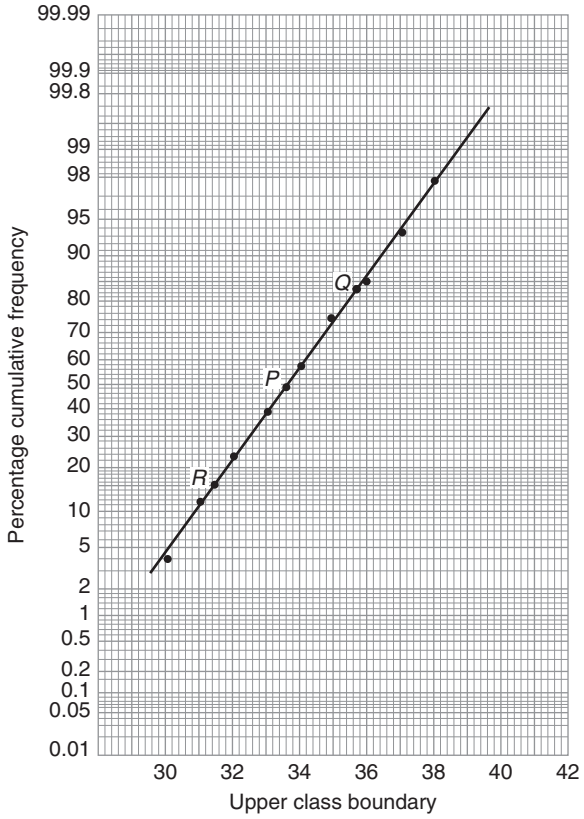


Figure 58.6

paper, the standard deviation of the distribution is given by:

$$\frac{\left(\begin{array}{l}\text{variable value for 84\% cumulative frequency} - \\ \text{variable value for 16\% cumulative frequency}\end{array}\right)}{2}$$

Problem 5. Use normal probability paper to determine whether the data given below, which refers to the masses of 50 copper ingots, is approximately normally distributed. If the data is normally distributed, determine the mean and standard deviation of the data from the graph drawn.

Class mid-point value (kg)	Frequency
29.5	2
30.5	4
31.5	6
32.5	8
33.5	9
34.5	8

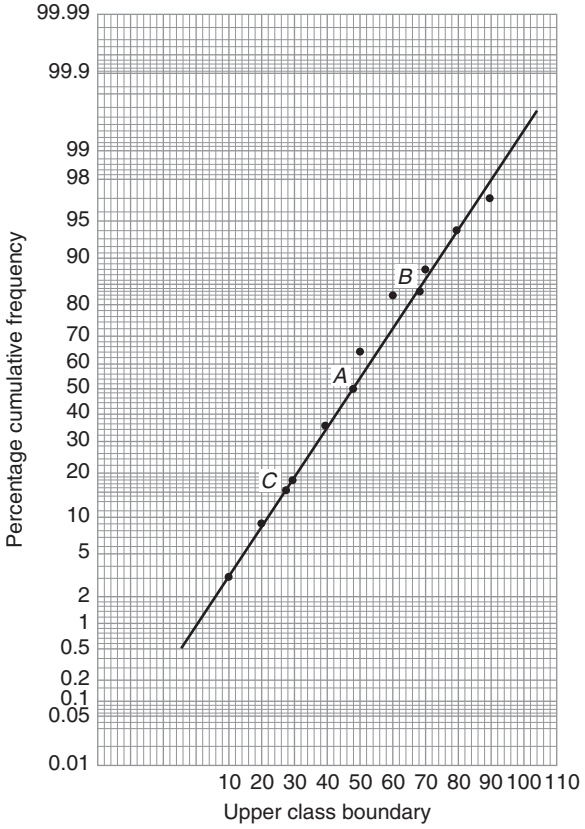


Figure 58.7

Class mid-point value (kg)	Frequency
35.5	6
36.5	4
37.5	2
38.5	1

To test the normality of a distribution, the upper class boundary/percentage cumulative frequency values are plotted on normal probability paper. The upper class boundary values are: 30, 31, 32,…, 38, 39. The corresponding cumulative frequency values (for 'less than' the upper class boundary values) are: 2, $(4+2)=6$, $(6+4+2)=12$, 20, 29, 37, 43, 47, 49 and 50. The corresponding percentage cumulative frequency values are $\frac{2}{50}\times 100=4$, $\frac{6}{50}\times 100=12$, 24, 40, 58, 74, 86, 94, 98 and 100%.

The co-ordinates of upper class boundary/percentage cumulative frequency values are plotted as shown

in Fig. 58.6. When plotting these values, it will always be found that the co-ordinate for the 100% cumulative frequency value cannot be plotted, since the maximum value on the probability scale is 99.99. **Since the points plotted in Fig. 58.6 lie very nearly in a straight line, the data is approximately normally distributed**.

The mean value and standard deviation can be determined from Fig. 58.6. Since a normal curve is symmetrical, the mean value is the value of the variable corresponding to a 50% cumulative frequency value, shown as point P on the graph. This shows that **the mean value is 33.6 kg**. The standard deviation is determined using the 84% and 16% cumulative frequency values, shown as Q and R in Fig. 58.6. The variable values for Q and R are 35.7 and 31.4 respectively; thus two standard deviations correspond to $35.7-31.4$, i.e. 4.3, showing that the standard deviation of the distribution is approximately $\frac{4.3}{2}$ i.e. **2.15 standard deviations**.

The mean value and standard deviation of the distribution can be calculated using

$$\text{mean, } \bar{x} = \frac{(\sum fx)}{(\sum f)}$$

and standard deviation,

$$\sigma = \sqrt{\left\{\frac{(\sum[f(x-\bar{x})^2])}{(\sum f)}\right\}}$$

where f is the frequency of a class and x is the class mid-point value. Using these formulae gives a mean value of the distribution of 33.6 (as obtained graphically) and a standard deviation of 2.12, showing that the graphical method of determining the mean and standard deviation give quite realistic results.

Problem 6. Use normal probability paper to determine whether the data given below is normally distributed. Use the graph and assume a normal distribution whether this is so or not, to find approximate values of the mean and standard deviation of the distribution.

Class mid-point values	Frequency
5	1
15	2
25	3
35	6
45	9
55	6
65	2
75	2
85	1
95	1

To test the normality of a distribution, the upper class boundary/percentage cumulative frequency values are plotted on normal probability paper. The upper class boundary values are: 10, 20, 30, …, 90 and 100. The corresponding cumulative frequency values are 1, $1+2=3$, $1+2+3=6$, 12, 21, 27, 29, 31, 32 and 33. The percentage cumulative frequency values are $\frac{1}{33}\times 100=3$, $\frac{3}{33}\times 100=9$, 18, 36, 64, 82, 88, 94, 97 and 100.

The co-ordinates of upper class boundary values/percentage cumulative frequency values are plotted as shown in Fig. 58.7. Although six of the points lie approximately in a straight line, three points corresponding to upper class boundary values of 50, 60 and 70 are not close to the line and indicate that **the distribution is not normally** distributed. However, if a normal distribution is assumed, the mean value corresponds to the variable value at a cumulative frequency of 50% and, from Fig. 58.7, point A is **48**. The value of the standard deviation of the distribution can be obtained from the variable values corresponding to the 84% and 16% cumulative frequency values, shown as B and C in Fig. 58.7 and give: $2\sigma = 69-28$, i.e. the standard deviation $\sigma = \mathbf{20.5}$. The calculated values of the mean and standard deviation of the distribution are 45.9 and 19.4 respectively, showing that errors are introduced if the graphical method of determining these values is used for data which is not normally distributed.

Now try the following exercise

Exercise 216 Further problems on testing for a normal distribution

1. A frequency distribution of 150 measurements is as shown:

Class mid-point value	Frequency
26.4	5
26.6	12
26.8	24
27.0	36
27.2	36
27.4	25
27.6	12

Use normal probability paper to show that this data approximates to a normal distribution and hence determine the approximate values of the mean and standard deviation of the distribution. Use the formula for mean and standard deviation to verify the results obtained.

$$\left[\begin{array}{ll} \text{Graphically,} & \bar{x} = 27.1, \sigma = 0.3; \\ \text{by calculation,} & \bar{x} = 27.079, \\ & \sigma = 0.3001 \end{array}\right]$$

2. A frequency distribution of the class mid-point values of the breaking loads for 275 similar fibres is as shown below:

Load (kN)	17	19	21	23	25	27	29	31
Frequency	9	23	55	78	64	28	14	4

Use normal probability paper to show that this distribution is approximately normally distributed and determine the mean and standard deviation of the distribution (a) from the graph and (b) by calculation.

$$\left[\begin{array}{ll} \text{(a)} \ \bar{x} = 23.5\,\text{kN}, & \sigma = 2.9\,\text{kN} \\ \text{(b)} \ \bar{x} = 23.364\,\text{kN}, & \sigma = 2.917\,\text{kN} \end{array}\right]$$

Chapter 59

Linear correlation

59.1 Introduction to linear correlation

Correlation is a measure of the amount of association existing between two variables. For linear correlation, if points are plotted on a graph and all the points lie on a straight line, then **perfect linear correlation** is said to exist. When a straight line having a positive gradient can reasonably be drawn through points on a graph **positive or direct linear correlation** exists, as shown in Fig. 59.1(a). Similarly, when a straight line having a negative gradient can reasonably be drawn through points on a graph, **negative or inverse linear correlation** exists, as shown in Fig. 59.1(b). When there is no apparent relationship between co-ordinate values plotted on a graph then **no correlation** exists between the points, as shown in Fig. 59.1(c). In statistics, when two variables are being investigated, the location of the co-ordinates on a rectangular co-ordinate system is called a **scatter diagram**—as shown in Fig. 59.1.

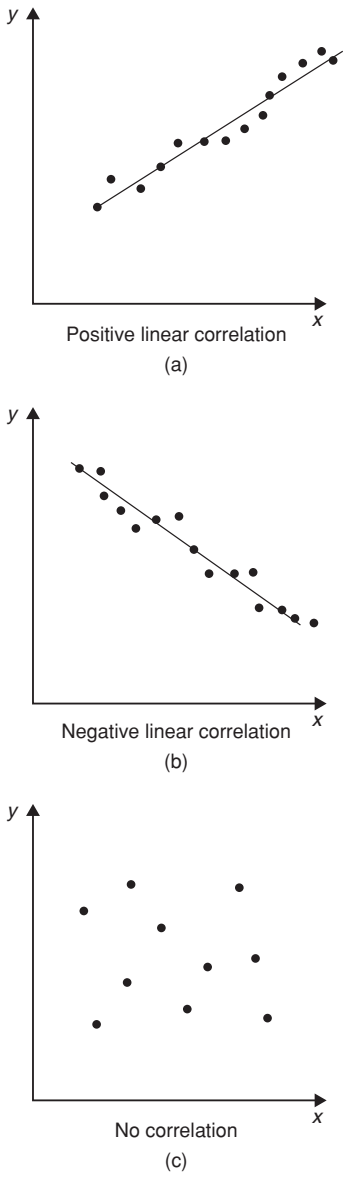


Figure 59.1

59.2 The product-moment formula for determining the linear correlation coefficient

The amount of linear correlation between two variables is expressed by a **coefficient of correlation**, given the symbol r. This is defined in terms of the deviations of the co-ordinates of two variables from their mean values and is given by the **product-moment formula** which states:

coefficient of correlation,

$$r = \frac{\sum xy}{\sqrt{\{(\sum x^2)(\sum y^2)\}}} \qquad (1)$$

where the x-values are the values of the deviations of co-ordinates X from $\overline{X}$, their mean value and the y-values

are the values of the deviations of co-ordinates Y from $\overline{Y}$, their mean value. That is, $x=(X-\overline{X})$ and $y=(Y-\overline{Y})$. The results of this determination give values of r lying between $+1$ and -1, where $+1$ indicates perfect direct correlation, -1 indicates perfect inverse correlation and 0 indicates that no correlation exists. Between these values, the smaller the value of r, the less is the amount of correlation which exists. Generally, values of r in the ranges 0.7 to 1 and -0.7 to -1 show that a fair amount of correlation exists.

59.3 The significance of a coefficient of correlation

When the value of the coefficient of correlation has been obtained from the product moment formula, some care is needed before coming to conclusions based on this result. Checks should be made to ascertain the following two points:

(a) that a 'cause and effect' relationship exists between the variables; it is relatively easy, mathematically, to show that some correlation exists between, say, the number of ice creams sold in a given period of time and the number of chimneys swept in the same period of time, although there is no relationship between these variables;

(b) that a linear relationship exists between the variables; the product-moment formula given in Section 59.2 is based on linear correlation. Perfect non-linear correlation may exist (for example, the co-ordinates exactly following the curve $y=x^3$), but this gives a low value of coefficient of correlation since the value of r is determined using the product-moment formula, based on a linear relationship.

59.4 Worked problems on linear correlation

Problem 1. In an experiment to determine the relationship between force on a wire and the resulting extension, the following data is obtained:

Force (N)	10	20	30	40	50	60	70
Extension (mm)	0.22	0.40	0.61	0.85	1.20	1.45	1.70

Determine the linear coefficient of correlation for this data.

Let X be the variable force values and Y be the dependent variable extension values. The coefficient of correlation is given by:

$$r=\frac{\sum xy}{\sqrt{\{(\sum x^2)(\sum y^2)\}}}$$

where $x=(X-\overline{X})$ and $y=(Y-\overline{Y})$, $\overline{X}$ and $\overline{Y}$ being the mean values of the X and Y values respectively. Using a tabular method to determine the quantities of this formula gives:

X	Y	$x=(X-\overline{X})$	$y=(Y-\overline{Y})$
10	0.22	−30	−0.699
20	0.40	−20	−0.519
30	0.61	−10	−0.309
40	0.85	0	−0.069
50	1.20	10	0.281
60	1.45	20	0.531
70	1.70	30	0.781
$\sum X=280$, $\overline{X}=\frac{280}{7}=40$			
$\sum Y=6.43$, $\overline{Y}=\frac{6.43}{7}=0.919$			

xy	x^2	y^2
20.97	900	0.489
10.38	400	0.269
3.09	100	0.095
0	0	0.005
2.81	100	0.079
10.62	400	0.282
23.43	900	0.610
$\sum xy=71.30$	$\sum x^2=2800$	$\sum y^2=1.829$

Thus $r=\frac{71.3}{\sqrt{[2800\times 1.829]}}=\mathbf{0.996}$

This shows that a **very good direct correlation exists** between the values of force and extension.

Problem 2. The relationship between expenditure on welfare services and absenteeism for similar periods of time is shown below for a small company.

Expenditure (£′000)	3.5	5.0	7.0	10	12	15	18
Days lost	241	318	174	110	147	122	86

Determine the coefficient of linear correlation for this data.

Let X be the expenditure in thousands of pounds and Y be the days lost.

The coefficient of correlation,

$$r = \frac{\sum xy}{\sqrt{\{(\sum x^2)(\sum y^2)\}}}$$

where $x = (X - \overline{X})$ and $y = (Y - \overline{Y})$, $\overline{X}$ and $\overline{Y}$ being the mean values of X and Y respectively. Using a tabular approach:

X	Y	$x = (X - \overline{X})$	$y = (Y - \overline{Y})$
3.5	241	−6.57	69.9
5.0	318	−5.07	146.9
7.0	174	−3.07	2.9
10	110	−0.07	−61.1
12	147	1.93	−24.1
15	122	4.93	−49.1
18	86	7.93	−85.1

$\sum X = 70.5$, $\overline{X} = \frac{70.5}{7} = 10.07$

$\sum Y = 1198$, $\overline{Y} = \frac{1198}{7} = 171.1$

xy	x^2	y^2
−459.2	43.2	4886
−744.8	25.7	21580
−8.9	9.4	8
4.3	0	3733
−46.5	3.7	581
−242.1	24.3	2411
−674.8	62.9	7242
$\sum xy = -2172$	$\sum x^2 = 169.2$	$\sum y^2 = 40441$

Thus

$$r = \frac{-2172}{\sqrt{[169.2 \times 40441]}} = \mathbf{-0.830}$$

This shows that there is **fairly good inverse correlation** between the expenditure on welfare and days lost due to absenteeism.

Problem 3. The relationship between monthly car sales and income from the sale of petrol for a garage is as shown:

Cars sold	2	5	3	12	14	7	3	28	14	7	3	13
Income from petrol sales (£′000)	12	9	13	21	17	22	31	47	17	10	9	11

Determine the linear coefficient of correlation between these quantities.

Let X represent the number of cars sold and Y the income, in thousands of pounds, from petrol sales. Using the tabular approach:

X	Y	$x = (X - \overline{X})$	$y = (Y - \overline{Y})$
2	12	−7.25	−6.25
5	9	−4.25	−9.25
3	13	−6.25	−5.25
12	21	2.75	2.75
14	17	4.75	−1.25
7	22	−2.25	3.75
3	31	−6.25	12.75
28	47	18.75	28.75
14	17	4.75	−1.25
7	10	−2.25	−8.25
3	9	−6.25	−9.25

X	Y	$x=(X-\overline{X})$	$y=(Y-\overline{Y})$
13	11	3.75	−7.25

$\sum X = 111, \quad \overline{X} = \frac{111}{12} = 9.25$

$\sum Y = 219, \quad \overline{Y} = \frac{219}{12} = 18.25$

xy	x^2	y^2
45.3	52.6	39.1
39.3	18.1	85.6
32.8	39.1	27.6
7.6	7.6	7.6
−5.9	22.6	1.6
−8.4	5.1	14.1
−79.7	39.1	162.6
539.1	351.6	826.6
−5.9	22.6	1.6
18.6	5.1	68.1
57.8	39.1	85.6
−27.2	14.1	52.6
$\sum xy = 613.4$	$\sum x^2 = 616.7$	$\sum y^2 = 1372.7$

The coefficient of correlation,

$$r = \frac{\sum xy}{\sqrt{\{(\sum x^2)(\sum y^2)\}}}$$

$$= \frac{613.4}{\sqrt{\{(616.7)(1372.7)\}}} = \mathbf{0.667}$$

Thus, there is **no appreciable correlation** between petrol and car sales.

Now try the following exercise

Exercise 217 Further problems on linear correlation

In Problems 1 to 3, determine the coefficient of correlation for the data given, correct to 3 decimal places.

1. X 14 18 23 30 50
   Y 900 1200 1600 2100 3800
   [0.999]

2. X 2.7 4.3 1.2 1.4 4.9
   Y 11.9 7.10 33.8 25.0 7.50
   [−0.916]

3. X 24 41 9 18 73
   Y 39 46 90 30 98
   [0.422]

4. In an experiment to determine the relationship between the current flowing in an electrical circuit and the applied voltage, the results obtained are:

Current (mA)	5	11	15	19	24	28	33
Applied voltage (V)	2	4	6	8	10	12	14

   Determine, using the product-moment formula, the coefficient of correlation for these results. [0.999]

5. A gas is being compressed in a closed cylinder and the values of pressures and corresponding volumes at constant temperature are as shown:

Pressure (kPa)	Volume (m^3)
160	0.034
180	0.036
200	0.030
220	0.027
240	0.024
260	0.025
280	0.020
300	0.019

   Find the coefficient of correlation for these values. [−0.962]

6. The relationship between the number of miles travelled by a group of engineering salesmen in ten equal time periods and the corresponding value of orders taken is given below. Calculate the coefficient of correlation using the product-moment formula for these values.

Miles travelled	Orders taken (£'000)
1370	23
1050	17
980	19
1770	22
1340	27
1560	23
2110	30
1540	23
1480	25
1670	19

[0.632]

7. The data shown below refers to the number of times machine tools had to be taken out of service, in equal time periods, due to faults occurring and the number of hours worked by maintenance teams. Calculate the coefficient of correlation for this data.

Machines out of service:	4	13	2	9	16	8	7
Maintenance hours:	400	515	360	440	570	380	415

[0.937]

Chapter 60

Linear regression

60.1 Introduction to linear regression

Regression analysis, usually termed **regression**, is used to draw the line of 'best fit' through co-ordinates on a graph. The techniques used enable a mathematical equation of the straight line form $y = mx + c$ to be deduced for a given set of co-ordinate values, the line being such that the sum of the deviations of the co-ordinate values from the line is a minimum, i.e. it is the line of 'best fit'. When a regression analysis is made, it is possible to obtain two lines of best fit, depending on which variable is selected as the dependent variable and which variable is the independent variable. For example, in a resistive electrical circuit, the current flowing is directly proportional to the voltage applied to the circuit. There are two ways of obtaining experimental values relating the current and voltage. Either, certain voltages are applied to the circuit and the current values are measured, in which case the voltage is the independent variable and the current is the dependent variable; or, the voltage can be adjusted until a desired value of current is flowing and the value of voltage is measured, in which case the current is the independent value and the voltage is the dependent value.

60.2 The least-squares regression lines

For a given set of co-ordinate values, (X_1, Y_1), $(X_2, Y_2), \ldots, (X_n, Y_n)$ let the X values be the independent variables and the Y-values be the dependent values. Also let $D_1, \ldots, D_n$ be the vertical distances between the line shown as PQ in Fig. 60.1 and the points representing the co-ordinate values. The least-squares regression line, i.e. the line of best fit, is the line which makes the value of $D_1^2 + D_2^2 + \cdots + D_n^2$ a minimum value.

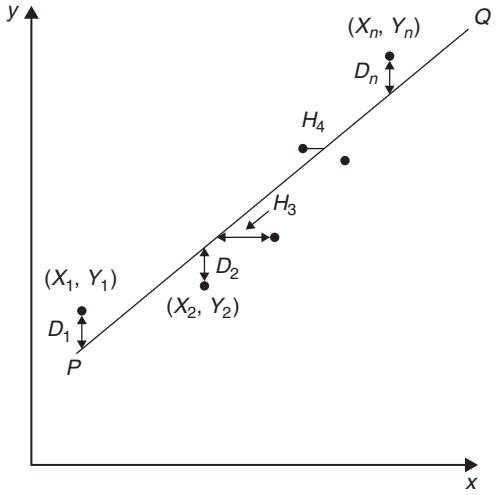


Figure 60.1

The equation of the least-squares regression line is usually written as $Y = a_0 + a_1 X$, where a_0 is the Y-axis intercept value and a_1 is the gradient of the line (analogous to c and m in the equation $y = mx + c$). The values of a_0 and a_1 to make the sum of the 'deviations squared' a minimum can be obtained from the two equations:

$$\sum Y = a_0 N + a_1 \sum X \qquad (1)$$

$$\sum (XY) = a_0 \sum X + a_1 \sum X^2 \qquad (2)$$

where X and Y are the co-ordinate values, N is the number of co-ordinates and a_0 and a_1 are called the **regression coefficients** of Y on X. Equations (1) and (2) are called the **normal equations** of the regression lines of Y on X. The regression line of Y on X is used to estimate values of Y for given values of X. If the Y-values (vertical-axis) are selected as the independent variables, the horizontal distances between the line shown as PQ

in Fig. 60.1 and the co-ordinate values (H_3, H_4, etc.) are taken as the deviations. The equation of the regression line is of the form: $X = b_0 + b_1 Y$ and the normal equations become:

$$\sum X = b_0 N + b_1 \sum Y \tag{3}$$

$$\sum (XY) = b_0 \sum Y + b_1 \sum Y^2 \tag{4}$$

where X and Y are the co-ordinate values, b_0 and b_1 are the regression coefficients of X on Y and N is the number of co-ordinates. These normal equations are of the regression line of X on Y, which is slightly different to the regression line of Y on X. The regression line of X on Y is used to estimated values of X for given values of Y. The regression line of Y on X is used to determine any value of Y corresponding to a given value of X. If the value of Y lies within the range of Y-values of the extreme co-ordinates, the process of finding the corresponding value of X is called **linear interpolation**. If it lies outside of the range of Y-values of the extreme co-ordinates than the process is called **linear extrapolation** and the assumption must be made that the line of best fit extends outside of the range of the co-ordinate values given.

By using the regression line of X on Y, values of X corresponding to given values of Y may be found by either interpolation or extrapolation.

60.3 Worked problems on linear regression

Problem 1. In an experiment to determine the relationship between frequency and the inductive reactance of an electrical circuit, the following results were obtained:

Frequency (Hz)	Inductive reactance (ohms)
50	30
100	65
150	90
200	130
250	150
300	190
350	200

Determine the equation of the regression line of inductive reactance on frequency, assuming a linear relationship.

Since the regression line of inductive reactance on frequency is required, the frequency is the independent variable, X, and the inductive reactance is the dependent variable, Y. The equation of the regression line of Y on X is:

$$Y = a_0 + a_1 X$$

and the regression coefficients a_0 and a_1 are obtained by using the normal equations

$$\sum Y = a_0 N + a_1 \sum X$$

and $$\sum XY = a_0 \sum X + a_1 \sum X^2$$

(from equations (1) and (2))

A tabular approach is used to determine the summed quantities.

Frequency, X	Inductive reactance, Y	X^2	XY	Y^2
50	30	2500	1500	900
100	65	10000	6500	4225
150	90	22500	13500	8100
200	130	40000	26000	16900
250	150	62500	37500	22500
300	190	90000	57000	36100
350	200	122500	70000	40000
$\sum X = 1400$	$\sum Y = 855$	$\sum X^2 = 350000$	$\sum XY = 212000$	$\sum Y^2 = 128725$

The number of co-ordinate values given, N is 7. Substituting in the normal equations gives:

$$855 = 7a_0 + 1400a_1 \quad (1)$$

$$212000 = 1400a_0 + 350000a_1 \quad (2)$$

$1400 \times (1)$ gives:

$$1197000 = 9800a_0 + 1960000a_1 \quad (3)$$

$7 \times (2)$ gives:

$$1484000 = 9800a_0 + 2450000a_1 \quad (4)$$

$(4) - (3)$ gives:

$$287000 = 0 + 490000a_1$$

from which, $a_1 = \dfrac{287000}{490000} = 0.586$

Substituting $a_1 = 0.586$ in equation (1) gives:

$$855 = 7a_0 + 1400(0.586)$$

i.e. $$a_0 = \frac{855 - 820.4}{7} = 4.94$$

Thus the equation of the regression line of inductive reactance on frequency is:

$$\boldsymbol{Y = 4.94 + 0.586X}$$

Problem 2. For the data given in Problem 1, determine the equation of the regression line of frequency on inductive reactance, assuming a linear relationship.

In this case, the inductive reactance is the independent variable X and the frequency is the dependent variable Y. From equations 3 and 4, the equation of the regression line of X on Y is:

$$X = b_0 + b_1 Y$$

and the normal equations are

$$\sum X = b_0 N + b_1 \sum Y$$

and $$\sum XY = b_0 \sum Y + b_1 \sum Y^2$$

From the table shown in Problem 1, the simultaneous equations are:

$$1400 = 7b_0 + 855b_1$$

$$212000 = 855b_0 + 128725b_1$$

Solving these equations in a similar way to that in Problem 1 gives:

$$b_0 = -6.15$$

and $b_1 = 1.69$, correct to 3 significant figures.

Thus the equation of the regression line of frequency on inductive reactance is:

$$\boldsymbol{X = -6.15 + 1.69Y}$$

Problem 3. Use the regression equations calculated in Problems 1 and 2 to find (a) the value of inductive reactance when the frequency is 175 Hz and (b) the value of frequency when the inductive reactance is 250 ohms, assuming the line of best fit extends outside of the given co-ordinate values. Draw a graph showing the two regression lines.

(a) From Problem 1, the regression equation of inductive reactance on frequency is $Y = 4.94 + 0.586X$. When the frequency, X, is 175 Hz, $Y = 4.94 + 0.586(175) = 107.5$, correct to 4 significant figures, i.e. the inductive reactance is **107.5 ohms** when the frequency is 175 Hz.

(b) From Problem 2, the regression equation of frequency on inductive reactance is $X = -6.15 + 1.69Y$. When the inductive reactance, Y, is 250 ohms, $X = -6.15 + 1.69(250) = 416.4$ Hz, correct to 4 significant figures, i.e. the frequency is **416.4 Hz** when the inductive reactance is 250 ohms.

The graph depicting the two regression lines is shown in Fig. 60.2. To obtain the regression line of inductive reactance on frequency the regression line equation $Y = 4.94 + 0.586X$ is used, and X (frequency) values of 100 and 300 have been selected in order to find the corresponding Y values. These values gave the co-ordinates as (100, 63.5) and (300, 180.7), shown as points A and B in Fig. 60.2. Two co-ordinates for the regression line of frequency on inductive reactance are calculated using the equation $X = -6.15 + 1.69Y$, the values of inductive reactance of 50 and 150 being used to obtain the co-ordinate values. These values gave co-ordinates (78.4, 50) and (247.4, 150), shown as points C and D in Fig. 60.2.

It can be seen from Fig. 60.2 that to the scale drawn, the two regression lines coincide. Although it is not necessary to do so, the co-ordinate values are also shown to indicate that the regression lines do appear to be the

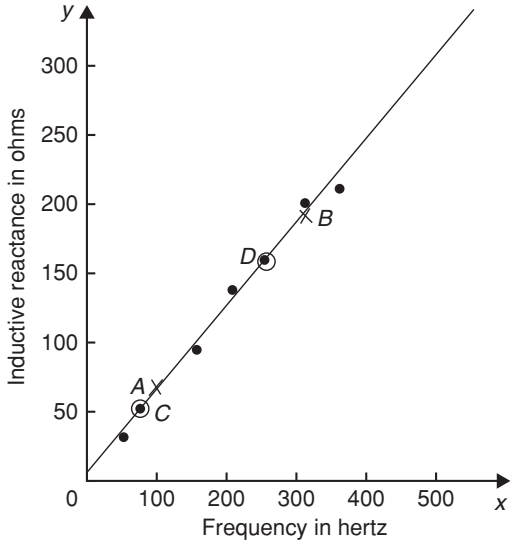


Figure 60.2

lines of best fit. A graph showing co-ordinate values is called a **scatter diagram** in statistics.

Problem 4. The experimental values relating centripetal force and radius, for a mass travelling at constant velocity in a circle, are as shown:

Force (N)	5	10	15	20	25	30	35	40
Radius (cm)	55	30	16	12	11	9	7	5

Determine the equations of (a) the regression line of force on radius and (b) the regression line of radius on force. Hence, calculate the force at a radius of 40 cm and the radius corresponding to a force of 32 newtons.

Let the radius be the independent variable X, and the force be the dependent variable Y. (This decision is usually based on a 'cause' corresponding to X and an 'effect' corresponding to Y.)

(a) The equation of the regression line of force on radius is of the form $Y = a_0 + a_1 X$ and the constants a_0 and a_1 are determined from the normal equations:

$$\sum Y = a_0 N + a_1 \sum X$$

and $$\sum XY = a_0 \sum X + a_1 \sum X^2$$

(from equations (1) and (2))

Using a tabular approach to determine the values of the summations gives:

Radius, X	Force, Y	X^2
55	5	3025
30	10	900
16	15	256
12	20	144
11	25	121
9	30	81
7	35	49
5	40	25
$\sum X = 145$	$\sum Y = 180$	$\sum X^2 = 4601$

XY	Y^2
275	25
300	100
240	225
240	400
275	625
270	900
245	1225
200	1600
$\sum XY = 2045$	$\sum Y^2 = 5100$

Thus $180 = 8a_0 + 145a_1$

and $2045 = 145a_0 + 4601a_1$

Solving these simultaneous equations gives $a_0 = 33.7$ and $a_1 = -0.617$, correct to 3 significant figures. Thus the equation of the regression line of force on radius is:

$$\boldsymbol{Y = 33.7 - 0.617X}$$

(b) The equation of the regression line of radius on force is of the form $X = b_0 + b_1 Y$ and the constants b_0 and b_1 are determined from the normal equations:

$$\sum X = b_0 N + b_1 \sum Y$$

and $$\sum XY = b_0 \sum Y + b_1 \sum Y^2$$

(from equations (3) and (4))

The values of the summations have been obtained in part (a) giving:

$$145 = 8b_0 + 180b_1$$

and $$2045 = 180b_0 + 5100b_1$$

Solving these simultaneous equations gives $b_0 = 44.2$ and $b_1 = -1.16$, correct to 3 significant figures. Thus the equation of the regression line of radius on force is:

$$\boldsymbol{X = 44.2 - 1.16Y}$$

The force, Y, at a radius of 40 cm, is obtained from the regression line of force on radius, i.e. $y = 33.7 - 0.617(40) = 9.02$,

i.e. **the force at a radius of 40 cm is 9.02 N**.

The radius, X, when the force is 32 newtons is obtained from the regression line of radius on force, i.e. $X = 44.2 - 1.16(32) = 7.08$,

i.e. **the radius when the force is 32 N is 7.08 cm**.

Now try the following exercise

Exercise 218 Further problems on linear regression

In Problems 1 and 2, determine the equation of the regression line of Y on X, correct to 3 significant figures.

1.

X	14	18	23	30	50
Y	900	1200	1600	2100	3800

[$Y = -256 + 80.6X$]

2.

X	6	3	9	15	2	14	21	13
Y	1.3	0.7	2.0	3.7	0.5	2.9	4.5	2.7

[$Y = 0.0477 + 0.216X$]

In Problems 3 and 4, determine the equations of the regression lines of X on Y for the data stated, correct to 3 significant figures.

3. The data given in Problem 1

[$X = 3.20 + 0.0124Y$]

4. The data given in Problem 2

[$X = -0.056 + 4.56Y$]

5. The relationship between the voltage applied to an electrical circuit and the current flowing is as shown:

Current (mA)	Applied voltage (V)
2	5
4	11
6	15
8	19
10	24
12	28
14	33

Assuming a linear relationship, determine the equation of the regression line of applied voltage, Y, on current, X, correct to 4 significant figures.

[$Y = 1.142 + 2.268X$]

6. For the data given in Problem 5, determine the equation of the regression line of current on applied voltage, correct to 3 significant figures.

[$X = -0.483 + 0.440Y$]

7. Draw the scatter diagram for the data given in Problem 5 and show the regression lines of applied voltage on current and current on applied voltage. Hence determine the values of (a) the applied voltage needed to give a current of 3 mA and (b) the current flowing when the applied voltage is 40 volts, assuming the regression lines are still true outside of the range of values given.

[(a) 7.92 V (b) 17.1 mA]

8. In an experiment to determine the relationship between force and momentum, a force X, is applied to a mass, by placing the mass on an inclined plane, and the time, Y, for the velocity

to change from u m/s to v m/s is measured. The results obtained are as follows:

Force (N)	Time (s)
11.4	0.56
18.7	0.35
11.7	0.55
12.3	0.52
14.7	0.43
18.8	0.34
19.6	0.31

Determine the equation of the regression line of time on force, assuming a linear relationship between the quantities, correct to 3 significant figures.

$[Y = 0.881 - 0.0290X]$

9. Find the equation for the regression line of force on time for the data given in Problem 8, correct to 3 decimal places.

$[X = 30.194 - 34.039Y]$

10. Draw a scatter diagram for the data given in Problem 8 and show the regression lines of time on force and force on time. Hence find (a) the time corresponding to a force of 16 N, and (b) the force at a time of 0.25 s, assuming the relationship is linear outside of the range of values given. [(a) 0.417 s (b) 21.7 N]

Revision Test 17

This Revision Test covers the material contained in chapters 57 to 60. *The marks for each question are shown in brackets at the end of each question.*

1. A machine produces 15% defective components. In a sample of 5, drawn at random, calculate, using the binomial distribution, the probability that:

   (a) there will be 4 defective items,

   (b) there will be not more than 3 defective items,

   (c) all the items will be non-defective.

   Draw a histogram showing the probabilities of 0, 1, 2, . . . , 5 defective items. (20)

2. 2% of the light bulbs produced by a company are defective. Determine, using the Poisson distribution, the probability that in a sample of 80 bulbs: (a) 3 bulbs will be defective, (b) not more than 3 bulbs will be defective, (c) at least 2 bulbs will be defective. (13)

3. Some engineering components have a mean length of 20 mm and a standard deviation of 0.25 mm. Assume that the data on the lengths of the components is normally distributed.

   In a batch of 500 components, determine the number of components likely to:

   (a) have a length of less than 19.95 mm,

   (b) be between 19.95 mm and 20.15 mm,

   (c) be longer than 20.54 mm. (15)

4. In a factory, cans are packed with an average of 1.0 kg of a compound and the masses are normally distributed about the average value. The standard deviation of a sample of the contents of the cans is 12 g. Determine the percentage of cans containing (a) less than 985 g, (b) more than 1030 g, (c) between 985 g and 1030 g. (10)

5. The data given below gives the experimental values obtained for the torque output, X, from an electric motor and the current, Y, taken from the supply.

Torque X	Current Y
0	3
1	5
2	6
3	6
4	9
5	11
6	12
7	12
8	14
9	13

   Determine the linear coefficient of correlation for this data. (18)

6. Some results obtained from a tensile test on a steel specimen are shown below:

Tensile force (kN)	Extension (mm)
4.8	3.5
9.3	8.2
12.8	10.1
17.7	15.6
21.6	18.4
26.0	20.8

   Assuming a linear relationship:

   (a) determine the equation of the regression line of extension on force,

   (b) determine the equation of the regression line of force on extension,

   (c) estimate (i) the value of extension when the force is 16 kN, and (ii) the value of force when the extension is 17 mm. (24)

Chapter 61

Introduction to Laplace transforms

61.1 Introduction

The solution of most electrical circuit problems can be reduced ultimately to the solution of differential equations. The use of **Laplace transforms** provides an alternative method to those discussed in Chapters 46 to 51 for solving linear differential equations.

61.2 Definition of a Laplace transform

The Laplace transform of the function $f(t)$ is defined by the integral $\int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t$, where s is a parameter assumed to be a real number.

Common notations used for the Laplace transform

There are various commonly used notations for the Laplace transform of $f(t)$ and these include:

(i) $\mathcal{L}\{f(t)\}$ or $L\{f(t)\}$

(ii) $\mathcal{L}(f)$ or Lf

(iii) $\overline{f}(s)$ or $f(s)$

Also, the letter p is sometimes used instead of s as the parameter. The notation adopted in this book will be $f(t)$ for the original function and $\mathcal{L}\{f(t)\}$ for its Laplace transform.

Hence, from above:

$$\mathcal{L}\{f(t)\} = \int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t \qquad (1)$$

61.3 Linearity property of the Laplace transform

From equation (1),

$$\mathcal{L}\{kf(t)\} = \int_0^\infty \mathrm{e}^{-st} k f(t)\,\mathrm{d}t$$

$$= k\int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t$$

i.e. $\mathcal{L}\{k f(t)\} = k\mathcal{L}\{f(t)\}$ (2)

where k is any constant.

Similarly,

$$\mathcal{L}\{a f(t) + bg(t)\} = \int_0^\infty \mathrm{e}^{-st}(a f(t) + bg(t))\,\mathrm{d}t$$

$$= a\int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t + b\int_0^\infty \mathrm{e}^{-st} g(t)\,\mathrm{d}t$$

i.e. $\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$, (3)

where a and b are any real constants.

The Laplace transform is termed a **linear operator** because of the properties shown in equations (2) and (3).

61.4 Laplace transforms of elementary functions

Using the definition of the Laplace transform in equation (1) a number of elementary functions may be transformed. For example:

(a) $f(t)=1$. From equation (1),

$$\mathcal{L}\{1\}=\int_0^\infty e^{-st}(1)\,dt=\left[\frac{e^{-st}}{-s}\right]_0^\infty$$
$$=-\frac{1}{s}[e^{-s(\infty)}-e^0]=-\frac{1}{s}[0-1]$$
$$=\frac{\mathbf{1}}{\boldsymbol{s}}\ \text{(provided } s>0)$$

(b) $f(t)=k$. From equation (2),

$$\mathcal{L}\{k\}=k\mathcal{L}\{1\}$$

Hence $\mathcal{L}\{k\}=k\left(\frac{1}{s}\right)=\frac{\boldsymbol{k}}{\boldsymbol{s}}$, from (a) above.

(c) $f(t)=e^{at}$ (where a is a real constant $\neq 0$).
From equation (1),

$$\mathcal{L}\{e^{at}\}=\int_0^\infty e^{-st}(e^{at})\,dt=\int_0^\infty e^{-(s-a)t}\,dt,$$

from the laws of indices,

$$=\left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty=\frac{1}{-(s-a)}(0-1)$$
$$=\frac{\mathbf{1}}{\boldsymbol{s-a}}$$

(provided $(s-a)>0$, i.e. $s>a$)

(d) $f(t)=\cos at$ (where a is a real constant).
From equation (1),

$$\mathcal{L}\{\cos at\}=\int_0^\infty e^{-st}\cos at\,dt$$
$$=\left[\frac{e^{-st}}{s^2+a^2}(a\sin at-s\cos at)\right]_0^\infty$$

by integration by parts twice (see page 423),

$$=\left[\frac{e^{-s(\infty)}}{s^2+a^2}(a\sin a(\infty)-s\cos a(\infty))\right.$$
$$\left.-\frac{e^0}{s^2+a^2}(a\sin 0-s\cos 0)\right]$$
$$=\frac{\boldsymbol{s}}{\boldsymbol{s^2+a^2}}\ (\text{provided } s>0)$$

(e) $f(t)=t$. From equation (1),

$$\mathcal{L}\{t\}=\int_0^\infty e^{-st}t\,dt=\left[\frac{te^{-st}}{-s}-\int\frac{e^{-st}}{-s}\,dt\right]_0^\infty$$
$$=\left[\frac{te^{-st}}{-s}-\frac{e^{-st}}{s^2}\right]_0^\infty$$

by integration by parts,

$$=\left[\frac{\infty e^{-s(\infty)}}{-s}-\frac{e^{-s(\infty)}}{s^2}\right]-\left[0-\frac{e^0}{s^2}\right]$$
$$=(0-0)-\left(0-\frac{1}{s^2}\right)$$

since $(\infty\times 0)=0$,

$$=\frac{\mathbf{1}}{\boldsymbol{s^2}}\ (\text{provided } s>0)$$

(f) $f(t)=t^n$ (where $n=0, 1, 2, 3, \ldots$).

By a similar method to (e) it may be shown that $\mathcal{L}\{t^2\}=\frac{2}{s^3}$ and $\mathcal{L}\{t^3\}=\frac{(3)(2)}{s^4}=\frac{3!}{s^4}$. These results can be extended to n being any positive integer.

Thus $\boldsymbol{\mathcal{L}\{t^n\}=\frac{n!}{s^{n+1}}}$ provided $s>0$)

(g) $f(t)=\sinh at$. From Chapter 5,
$\sinh at=\frac{1}{2}(e^{at}-e^{-at})$. Hence,

$$\mathcal{L}\{\sinh at\}=\mathcal{L}\left\{\frac{1}{2}e^{at}-\frac{1}{2}e^{-at}\right\}$$
$$=\frac{1}{2}\mathcal{L}\{e^{at}\}-\frac{1}{2}\mathcal{L}\{e^{-at}\}$$

from equations (2) and (3),

$$=\frac{1}{2}\left[\frac{1}{s-a}\right]-\frac{1}{2}\left[\frac{1}{s+a}\right]$$

from (c) above,

$$=\frac{1}{2}\left[\frac{1}{s-a}-\frac{1}{s+a}\right]$$
$$=\frac{\boldsymbol{a}}{\boldsymbol{s^2-a^2}}\ (\text{provided } s>a)$$

A list of elementary standard Laplace transforms are summarized in Table 61.1.

61.5 Worked problems on standard Laplace transforms

Problem 1. Using a standard list of Laplace transforms determine the following:
(a) $\mathcal{L}\left\{1+2t-\frac{1}{3}t^4\right\}$ (b) $\mathcal{L}\{5e^{2t}-3e^{-t}\}$.

Table 61.1 Elementary standard Laplace transforms

	Function $f(t)$	Laplace transforms $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$
(i)	1	$\frac{1}{s}$
(ii)	k	$\frac{k}{s}$
(iii)	e^{at}	$\frac{1}{s-a}$
(iv)	$\sin at$	$\frac{a}{s^2+a^2}$
(v)	$\cos at$	$\frac{s}{s^2+a^2}$
(vi)	t	$\frac{1}{s^2}$
(vii)	t^2	$\frac{2!}{s^3}$
(viii)	t^n $(n = 1, 2, 3, \ldots)$	$\frac{n!}{s^{n+1}}$
(ix)	$\cosh at$	$\frac{s}{s^2-a^2}$
(x)	$\sinh at$	$\frac{a}{s^2-a^2}$

(a) $\mathcal{L}\left\{1 + 2t - \frac{1}{3}t^4\right\}$

$= \mathcal{L}\{1\} + 2\mathcal{L}\{t\} - \frac{1}{3}\mathcal{L}\{t^4\}$,

from equations (2) and (3)

$= \frac{1}{s} + 2\left(\frac{1}{s^2}\right) - \frac{1}{3}\left(\frac{4!}{s^{4+1}}\right)$,

from (i), (vi) and (viii) of Table 61.1

$= \frac{1}{s} + \frac{2}{s^2} - \frac{1}{3}\left(\frac{4.3.2.1}{s^5}\right)$

$= \mathbf{\frac{1}{s} + \frac{2}{s^2} - \frac{8}{s^5}}$

(b) $\mathcal{L}\{5e^{2t} - 3e^{-t}\} = 5\mathcal{L}(e^{2t}) - 3\mathcal{L}\{e^{-t}\}$,

from equations (2) and (3)

$= 5\left(\frac{1}{s-2}\right) - 3\left(\frac{1}{s-(-1)}\right)$,

from (iii) of Table 61.1

$= \frac{5}{s-2} - \frac{3}{s+1}$

$= \frac{5(s+1) - 3(s-2)}{(s-2)(s+1)}$

$= \mathbf{\frac{2s+11}{s^2-s-2}}$

Problem 2. Find the Laplace transforms of:
(a) $6\sin 3t - 4\cos 5t$ (b) $2\cosh 2\theta - \sinh 3\theta$.

(a) $\mathcal{L}\{6\sin 3t - 4\cos 5t\}$

$= 6\mathcal{L}\{\sin 3t\} - 4\mathcal{L}\{\cos 5t\}$

$= 6\left(\frac{3}{s^2+3^2}\right) - 4\left(\frac{s}{s^2+5^2}\right)$,

from (iv) and (v) of Table 61.1

$= \mathbf{\frac{18}{s^2+9} - \frac{4s}{s^2+25}}$

(b) $\mathcal{L}\{2\cosh 2\theta - \sinh 3\theta\}$

$= 2\mathcal{L}\{\cosh 2\theta\} - \mathcal{L}\{\sinh 3\theta\}$

$= 2\left(\frac{s}{s^2-2^2}\right) - \left(\frac{3}{s^2-3^2}\right)$

from (ix) and (x) of Table 61.1

$= \mathbf{\frac{2s}{s^2-4} - \frac{3}{s^2-9}}$

Problem 3. Prove that
(a) $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$ (b) $\mathcal{L}\{t^2\} = \frac{2}{s^3}$
(c) $\mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}$

(a) From equation (1),

$\mathcal{L}\{\sin at\} = \int_0^\infty e^{-st}\sin at\,dt$

$= \left[\frac{e^{-st}}{s^2+a^2}(-s\sin at - a\cos at)\right]_0^\infty$

by integration by parts,

$= \frac{1}{s^2+a^2}[e^{-s(\infty)}(-s\sin a(\infty)$

$- a\cos a(\infty)) - e^0(-s\sin 0$

$- a\cos 0)]$

$$= \frac{1}{s^2+a^2}[(0) - 1(0-a)]$$

$$= \frac{\boldsymbol{a}}{\boldsymbol{s^2+a^2}} \text{ (provided } s>0)$$

(b) From equation (1),

$$\mathcal{L}\{t^2\} = \int_0^\infty \mathrm{e}^{-st}t^2\,\mathrm{d}t$$

$$= \left[\frac{t^2\mathrm{e}^{-st}}{-s} - \frac{2t\mathrm{e}^{-st}}{s^2} - \frac{2\mathrm{e}^{-st}}{s^3}\right]_0^\infty$$

by integration by parts twice,

$$= \left[(0-0-0) - \left(0-0-\frac{2}{s^3}\right)\right]$$

$$= \frac{\boldsymbol{2}}{\boldsymbol{s^3}} \text{ (provided } s>0)$$

(c) From equation (1),

$$\mathcal{L}\{\cosh at\} = \mathcal{L}\left\{\frac{1}{2}(\mathrm{e}^{at}+\mathrm{e}^{-at})\right\},$$

from Chapter 5

$$= \frac{1}{2}\mathcal{L}\{\mathrm{e}^{at}\} + \frac{1}{2}\mathcal{L}\{\mathrm{e}^{-at}\},$$

equations (2) and (3)

$$= \frac{1}{2}\left(\frac{1}{s-a}\right) + \frac{1}{2}\left(\frac{1}{s-(-a)}\right)$$

from (iii) of Table 61.1

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{(s+a)+(s-a)}{(s-a)(s+a)}\right]$$

$$= \frac{\boldsymbol{s}}{\boldsymbol{s^2-a^2}} \text{ (provided } s>a)$$

Problem 4. Determine the Laplace transforms of: (a) $\sin^2 t$ (b) $\cosh^2 3x$.

(a) Since $\cos 2t = 1 - 2\sin^2 t$ then $\sin^2 t = \frac{1}{2}(1-\cos 2t)$. Hence,

$$\mathcal{L}\{\sin^2 t\} = \mathcal{L}\left\{\frac{1}{2}(1-\cos 2t)\right\}$$

$$= \frac{1}{2}\mathcal{L}\{1\} - \frac{1}{2}\mathcal{L}\{\cos 2t\}$$

$$= \frac{1}{2}\left(\frac{1}{s}\right) - \frac{1}{2}\left(\frac{s}{s^2+2^2}\right)$$

from (i) and (v) of Table 61.1

$$= \frac{(s^2+4)-s^2}{2s(s^2+4)} = \frac{4}{2s(s^2+4)}$$

$$= \frac{\boldsymbol{2}}{\boldsymbol{s(s^2+4)}}$$

(b) Since $\cosh 2x = 2\cosh^2 x - 1$ then $\cosh^2 x = \frac{1}{2}(1+\cosh 2x)$ from Chapter 5.

Hence $\cosh^2 3x = \frac{1}{2}(1+\cosh 6x)$

Thus $\mathcal{L}\{\cosh^2 3x\} = \mathcal{L}\left\{\frac{1}{2}(1+\cosh 6x)\right\}$

$$= \frac{1}{2}\mathcal{L}\{1\} + \frac{1}{2}\mathcal{L}\{\cosh 6x\}$$

$$= \frac{1}{2}\left(\frac{1}{s}\right) + \frac{1}{2}\left(\frac{s}{s^2-6^2}\right)$$

$$= \frac{2s^2-36}{2s(s^2-36)} = \frac{\boldsymbol{s^2-18}}{\boldsymbol{s(s^2-36)}}$$

Problem 5. Find the Laplace transform of $3\sin(\omega t+\alpha)$, where ω and α are constants.

Using the compound angle formula for $\sin(A+B)$, from Chapter 17, $\sin(\omega t+\alpha)$ may be expanded to $(\sin\omega t\cos\alpha + \cos\omega t\sin\alpha)$. Hence,

$$\mathcal{L}\{3\sin(\omega t+\alpha)\}$$

$$= \mathcal{L}\{3(\sin\omega t\cos\alpha + \cos\omega t\sin\alpha)\}$$

$$= 3\cos\alpha\,\mathcal{L}\{\sin\omega t\} + 3\sin\alpha\,\mathcal{L}\{\cos\omega t\},$$

since α is a constant

$$= 3\cos\alpha\left(\frac{\omega}{s^2+\omega^2}\right) + 3\sin\alpha\left(\frac{s}{s^2+\omega^2}\right)$$

from (iv) and (v) of Table 61.1

$$= \frac{\boldsymbol{3}}{\boldsymbol{(s^2+\omega^2)}}\boldsymbol{(\omega\cos\alpha + s\sin\alpha)}$$

Now try the following exercise

Exercise 219 Further problems on an introduction to Laplace transforms

Determine the Laplace transforms in Problems 1 to 9.

1. (a) $2t-3$ (b) $5t^2+4t-3$

$$\left[\text{(a) } \frac{2}{s^2}-\frac{3}{s} \text{ (b) } \frac{10}{s^3}+\frac{4}{s^2}-\frac{3}{s}\right]$$

2. (a) $\frac{t^3}{24}-3t+2$ (b) $\frac{t^5}{15}-2t^4+\frac{t^2}{2}$

$$\left[\text{(a) } \frac{1}{4s^4}-\frac{3}{s^2}+\frac{2}{s} \text{ (b) } \frac{8}{s^6}-\frac{48}{s^5}+\frac{1}{s^3}\right]$$

3. (a) $5e^{3t}$ (b) $2e^{-2t}$ $\left[\text{(a) } \frac{5}{s-3} \text{ (b) } \frac{2}{s+2}\right]$

4. (a) $4\sin 3t$ (b) $3\cos 2t$

$$\left[\text{(a) } \frac{12}{s^2+9} \text{ (b) } \frac{3s}{s^2+4}\right]$$

5. (a) $7\cosh 2x$ (b) $\frac{1}{3}\sinh 3t$

$$\left[\text{(a) } \frac{7s}{s^2-4} \text{ (b) } \frac{1}{s^2-9}\right]$$

6. (a) $2\cos^2 t$ (b) $3\sin^2 2x$

$$\left[\text{(a) } \frac{2(s^2+2)}{s(s^2+4)} \text{ (b) } \frac{24}{s(s^2+16)}\right]$$

7. (a) $\cosh^2 t$ (b) $2\sinh^2 2\theta$

$$\left[\text{(a) } \frac{s^2-2}{s(s^2-4)} \text{ (b) } \frac{16}{s(s^2-16)}\right]$$

8. $4\sin(at+b)$, where a and b are constants.

$$\left[\frac{4}{s^2+a^2}(a\cos b+s\sin b)\right]$$

9. $3\cos(\omega t-\alpha)$, where ω and α are constants.

$$\left[\frac{3}{s^2+\omega^2}(s\cos\alpha+\omega\sin\alpha)\right]$$

10. Show that $\mathcal{L}(\cos^2 3t-\sin^2 3t)=\frac{s}{s^2+36}$

Chapter 62

Properties of Laplace transforms

62.1 The Laplace transform of $e^{at}f(t)$

From Chapter 61, the definition of the Laplace transform of $f(t)$ is:

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt \qquad (1)$$

$$\text{Thus } \mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{-st}(e^{at} f(t))\,dt$$

$$= \int_0^\infty e^{-(s-a)} f(t)\,dt \qquad (2)$$

(where a is a real constant)

Hence the substitution of $(s-a)$ for s in the transform shown in equation (1) corresponds to the multiplication of the original function $f(t)$ by e^{at}. This is known as a shift theorem.

62.2 Laplace transforms of the form $e^{at}f(t)$

From equation (2), Laplace transforms of the form $e^{at}f(t)$ may be deduced. For example:

(i) $\mathcal{L}\{e^{at}t^n\}$

Since $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$ from (viii) of Table 61.1, page 584.

then $\mathcal{L}\{e^{at}\ t^n\} = \dfrac{n!}{(s-a)^{n+1}}$ from equation (2) above (provided $s > a$).

(ii) $\mathcal{L}\{e^{at}\sin\omega t\}$

Since $\mathcal{L}\{\sin\omega t\} = \dfrac{\omega}{s^2+\omega^2}$ from (iv) of Table 61.1, page 584.

then $\mathcal{L}\{e^{at}\sin\omega t\} = \dfrac{\omega}{(s-a)^2+\omega^2}$ from equation (2) (provided $s > a$).

(iii) $\mathcal{L}\{e^{at}\cosh\omega t\}$

Since $\mathcal{L}\{\cosh\omega t\} = \dfrac{s}{s^2-\omega^2}$ from (ix) of Table 61.1, page 584.

then $\mathcal{L}\{e^{at}\cosh\omega t\} = \dfrac{s-a}{(s-a)^2-\omega^2}$ from equation (2) (provided $s > a$).

A summary of Laplace transforms of the form $e^{at}f(t)$ is shown in Table 62.1.

Table 62.1 Laplace transforms of the form $e^{at}f(t)$

Function $e^{at}f(t)$ (a is a real constant)	Laplace transform $\mathcal{L}\{e^{at}f(t)\}$
(i) $e^{at}t^n$	$\dfrac{n!}{(s-a)^{n+1}}$
(ii) $e^{at}\sin\omega t$	$\dfrac{\omega}{(s-a)^2+\omega^2}$
(iii) $e^{at}\cos\omega t$	$\dfrac{s-a}{(s-a)^2+\omega^2}$
(iv) $e^{at}\sinh\omega t$	$\dfrac{\omega}{(s-a)^2-\omega^2}$
(v) $e^{at}\cosh\omega t$	$\dfrac{s-a}{(s-a)^2-\omega^2}$

Problem 1. Determine (a) $\mathcal{L}\{2t^4e^{3t}\}$ (b) $\mathcal{L}\{4e^{3t}\cos 5t\}$.

(a) From (i) of Table 62.1,

$$\mathcal{L}\{2t^4e^{3t}\} = 2\mathcal{L}\{t^4e^{3t}\} = 2\left(\frac{4!}{(s-3)^{4+1}}\right)$$

$$= \frac{2(4)(3)(2)}{(s-3)^5} = \frac{\mathbf{48}}{\mathbf{(s-3)^5}}$$

(b) From (iii) of Table 62.1,

$$\mathcal{L}\{4e^{3t}\cos 5t\} = 4\mathcal{L}\{e^{3t}\cos 5t\}$$

$$= 4\left(\frac{s-3}{(s-3)^2+5^2}\right)$$

$$= \frac{4(s-3)}{s^2-6s+9+25}$$

$$= \frac{\mathbf{4(s-3)}}{\mathbf{s^2-6s+34}}$$

Problem 2. Determine (a) $\mathcal{L}\{e^{-2t}\sin 3t\}$ (b) $\mathcal{L}\{3e^{\theta}\cosh 4\theta\}$.

(a) From (ii) of Table 62.1,

$$\mathcal{L}\{e^{-2t}\sin 3t\} = \frac{3}{(s-(-2))^2+3^2} = \frac{3}{(s+2)^2+9}$$

$$= \frac{3}{s^2+4s+4+9} = \frac{\mathbf{3}}{\mathbf{s^2+4s+13}}$$

(b) From (v) of Table 62.1,

$$\mathcal{L}\{3e^{\theta}\cosh 4\theta\} = 3\mathcal{L}\{e^{\theta}\cosh 4\theta\} = \frac{3(s-1)}{(s-1)^2-4^2}$$

$$= \frac{3(s-1)}{s^2-2s+1-16} = \frac{\mathbf{3(s-1)}}{\mathbf{s^2-2s-15}}$$

Problem 3. Determine the Laplace transforms of (a) $5e^{-3t}\sinh 2t$ (b) $2e^{3t}(4\cos 2t - 5\sin 2t)$.

(a) From (iv) of Table 62.1,

$$\mathcal{L}\{5e^{-3t}\sinh 2t\} = 5\mathcal{L}\{e^{-3t}\sinh 2t\}$$

$$= 5\left(\frac{2}{(s-(-3))^2-2^2}\right)$$

$$= \frac{10}{(s+3)^2-2^2} = \frac{10}{s^2+6s+9-4}$$

$$= \frac{\mathbf{10}}{\mathbf{s^2+6s+5}}$$

(b) $\mathcal{L}\{2e^{3t}(4\cos 2t - 5\sin 2t)\}$

$$= 8\mathcal{L}\{e^{3t}\cos 2t\} - 10\mathcal{L}\{e^{3t}\sin 2t\}$$

$$= \frac{8(s-3)}{(s-3)^2+2^2} - \frac{10(2)}{(s-3)^2+2^2}$$

from (iii) and (ii) of Table 62.1

$$= \frac{8(s-3)-10(2)}{(s-3)^2+2^2} = \frac{\mathbf{8s-44}}{\mathbf{s^2-6s+13}}$$

Problem 4. Show that

$$\mathcal{L}\left\{3e^{-\frac{1}{2}x}\sin^2 x\right\} = \frac{48}{(2s+1)(4s^2+4s+17)}$$

Since $\cos 2x = 1-2\sin^2 x$, $\sin^2 x = \frac{1}{2}(1-\cos 2x)$.

Hence,

$$\mathcal{L}\left\{3e^{-\frac{1}{2}x}\sin^2 x\right\}$$

$$= \mathcal{L}\left\{3e^{-\frac{1}{2}x}\frac{1}{2}(1-\cos 2x)\right\}$$

$$= \frac{3}{2}\mathcal{L}\left\{e^{-\frac{1}{2}x}\right\} - \frac{3}{2}\mathcal{L}\left\{e^{-\frac{1}{2}x}\cos 2x\right\}$$

$$= \frac{3}{2}\left(\frac{1}{s-\left(-\frac{1}{2}\right)}\right) - \frac{3}{2}\left(\frac{\left(s-\left(-\frac{1}{2}\right)\right)}{\left(s-\left(-\frac{1}{2}\right)\right)^2+2^2}\right)$$

from (iii) of Table 61.1 (page 584) and (iii) of Table 62.1 above,

$$= \frac{3}{2\left(s+\frac{1}{2}\right)} - \frac{3\left(s+\frac{1}{2}\right)}{2\left[\left(s+\frac{1}{2}\right)^2+2^2\right]}$$

$$= \frac{3}{2s+1} - \frac{6s+3}{4\left(s^2+s+\frac{1}{4}+4\right)}$$

$$= \frac{3}{2s+1} - \frac{6s+3}{4s^2+4s+17}$$

$$= \frac{3(4s^2+4s+17) - (6s+3)(2s+1)}{(2s+1)(4s^2+4s+17)}$$

$$= \frac{12s^2+12s+51-12s^2-6s-6s-3}{(2s+1)(4s^2+4s+17)}$$

$$= \frac{\mathbf{48}}{\mathbf{(2s+1)(4s^2+4s+17)}}$$

Now try the following exercise

Exercise 220 Further problems on Laplace transforms of the form $e^{at}f(t)$

Determine the Laplace transforms of the following functions:

1. (a) $2te^{2t}$ (b) t^2e^t

$$\left[\text{(a) } \frac{2}{(s-2)^2} \quad \text{(b) } \frac{2}{(s-1)^3}\right]$$

2. (a) $4t^3e^{-2t}$ (b) $\frac{1}{2}t^4e^{-3t}$

$$\left[\text{(a) } \frac{24}{(s+2)^4} \quad \text{(b) } \frac{12}{(s+3)^5}\right]$$

3. (a) $e^t\cos t$ (b) $3e^{2t}\sin 2t$

$$\left[\text{(a) } \frac{s-1}{s^2-2s+2} \quad \text{(b) } \frac{6}{s^2-4s+8}\right]$$

4. (a) $5e^{-2t}\cos 3t$ (b) $4e^{-5t}\sin t$

$$\left[\text{(a) } \frac{5(s+2)}{s^2+4s+13} \quad \text{(b) } \frac{4}{s^2+10s+26}\right]$$

5. (a) $2e^t\sin^2 t$ (b) $\frac{1}{2}e^{3t}\cos^2 t$

$$\left[\begin{array}{l}\text{(a) } \dfrac{1}{s-1} - \dfrac{s-1}{s^2-2s+5} \\ \text{(b) } \dfrac{1}{4}\left(\dfrac{1}{s-3} + \dfrac{s-3}{s^2-6s+13}\right)\end{array}\right]$$

6. (a) $e^t\sinh t$ (b) $3e^{2t}\cosh 4t$

$$\left[\text{(a) } \frac{1}{s(s-2)} \quad \text{(b) } \frac{3(s-2)}{s^2-4s-12}\right]$$

7. (a) $2e^{-t}\sinh 3t$ (b) $\frac{1}{4}e^{-3t}\cosh 2t$

$$\left[\text{(a) } \frac{6}{s^2+2s-8} \quad \text{(b) } \frac{s+3}{4(s^2+6s+5)}\right]$$

8. (a) $2e^t(\cos 3t - 3\sin 3t)$

(b) $3e^{-2t}(\sinh 2t - 2\cosh 2t)$

$$\left[\text{(a) } \frac{2(s-10)}{s^2-2s+10} \quad \text{(b) } \frac{-6(s+1)}{s(s+4)}\right]$$

62.3 The Laplace transforms of derivatives

(a) First derivative

Let the first derivative of $f(t)$ be $f'(t)$ then, from equation (1),

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st}f'(t)\,dt$$

From Chapter 43, when integrating by parts

$$\int u\frac{dv}{dt}\,dt = uv - \int v\frac{du}{dt}\,dt$$

When evaluating $\int_0^\infty e^{-st}f'(t)\,dt$,

$$\text{let } u = e^{-st} \text{ and } \frac{dv}{dt} = f'(t)$$

from which,

$$\frac{du}{dt} = -se^{-st} \text{ and } v = \int f'(t)\,dt = f(t)$$

Hence $\int_0^\infty e^{-st}f'(t)\,dt$

$$= \left[e^{-st}f(t)\right]_0^\infty - \int_0^\infty f(t)(-se^{-st})\,dt$$

$$= [0 - f(0)] + s\int_0^\infty e^{-st}f(t)\,dt$$

$$= -f(0) + s\mathcal{L}\{f(t)\}$$

assuming $e^{-st}f(t) \to 0$ as $t \to \infty$, and $f(0)$ is the value of $f(t)$ at $t = 0$. Hence,

$$\left.\begin{array}{c}\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) \\ \text{or} \quad \mathcal{L}\left\{\dfrac{dy}{dx}\right\} = s\mathcal{L}\{y\} - y(0)\end{array}\right\} \tag{3}$$

where $y(0)$ is the value of y at $x = 0$.

(b) Second derivative

Let the second derivative of $f(t)$ be $f''(t)$, then from equation (1),

$$\mathcal{L}\{f''(t)\} = \int_0^\infty e^{-st} f''(t)\,dt$$

Integrating by parts gives:

$$\int_0^\infty e^{-st} f''(t)\,dt = \left[e^{-st} f'(t)\right]_0^\infty + s\int_0^\infty e^{-st} f'(t)\,dt$$

$$= [0 - f'(0)] + s\mathcal{L}\{f'(t)\}$$

assuming $e^{-st} f'(t) \to 0$ as $t \to \infty$, and $f'(0)$ is the value of $f'(t)$ at $t = 0$. Hence $\{f''(t)\} = -f'(0) + s[s(f(t)) - f(0)]$, from equation (3),

i.e. $$\left.\begin{array}{l} \mathcal{L}\{f''(t)\} \\ = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0) \end{array}\right\} \quad (4)$$

or $$\mathcal{L}\left\{\frac{d^2y}{dx^2}\right\} = s^2\mathcal{L}\{y\} - sy(0) - y'(0)$$

where $y'(0)$ is the value of $\dfrac{dy}{dx}$ at $x = 0$.

Equations (3) and (4) are important and are used in the solution of differential equations (see Chapter 64) and simultaneous differential equations (Chapter 65).

Problem 5. Use the Laplace transform of the first derivative to derive:

(a) $\mathcal{L}\{k\} = \dfrac{k}{s}$ (b) $\mathcal{L}\{2t\} = \dfrac{2}{s^2}$

(c) $\mathcal{L}\{e^{-at}\} = \dfrac{1}{s+a}$

From equation (3), $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$.

(a) Let $f(t) = k$, then $f'(t) = 0$ and $f(0) = k$.

Substituting into equation (3) gives:

$$\mathcal{L}\{0\} = s\mathcal{L}\{k\} - k$$

i.e. $$k = s\mathcal{L}\{k\}$$

Hence $$\boldsymbol{\mathcal{L}\{k\} = \frac{k}{s}}$$

(b) Let $f(t) = 2t$ then $f'(t) = 2$ and $f(0) = 0$.

Substituting into equation (3) gives:

$$\mathcal{L}\{2\} = s\mathcal{L}\{2t\} - 0$$

i.e. $$\frac{2}{s} = s\mathcal{L}\{2t\}$$

Hence $$\boldsymbol{\mathcal{L}\{2t\} = \frac{2}{s^2}}$$

(c) Let $f(t) = e^{-at}$ then $f'(t) = -ae^{-at}$ and $f(0) = 1$.

Substituting into equation (3) gives:

$$\mathcal{L}\{-ae^{-at}\} = s\mathcal{L}\{e^{-at}\} - 1$$

$$-a\mathcal{L}\{e^{-at}\} = s\mathcal{L}\{e^{-at}\} - 1$$

$$1 = s\mathcal{L}\{e^{-at}\} + a\mathcal{L}\{e^{-at}\}$$

$$1 = (s+a)\mathcal{L}\{e^{-at}\}$$

Hence $$\boldsymbol{\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}}$$

Problem 6. Use the Laplace transform of the second derivative to derive

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2 + a^2}$$

From equation (4),

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

Let $f(t) = \cos at$, then $f'(t) = -a\sin at$ and $f''(t) = -a^2\cos at$, $f(0) = 1$ and $f'(0) = 0$

Substituting into equation (4) gives:

$$\mathcal{L}\{-a^2\cos at\} = s^2\{\cos at\} - s(1) - 0$$

i.e. $$-a^2\mathcal{L}\{\cos at\} = s^2\mathcal{L}\{\cos at\} - s$$

Hence $$s = (s^2 + a^2)\mathcal{L}\{\cos at\}$$

from which, $$\mathcal{L}\{\cos at\} = \boldsymbol{\frac{s}{s^2 + a^2}}$$

Now try the following exercise

Exercise 221 Further problems on the Laplace transforms of derivatives

1. Derive the Laplace transform of the first derivative from the definition of a Laplace transform. Hence derive the transform

$$\mathcal{L}\{1\} = \frac{1}{s}$$

2. Use the Laplace transform of the first derivative to derive the transforms:
   (a) $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$ (b) $\mathcal{L}\{3t^2\} = \dfrac{6}{s^3}$
3. Derive the Laplace transform of the second derivative from the definition of a Laplace transform. Hence derive the transform
   $$\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$$
4. Use the Laplace transform of the second derivative to derive the transforms:
   (a) $\mathcal{L}\{\sinh at\} = \dfrac{a}{s^2-a^2}$
   (b) $\mathcal{L}\{\cosh at\} = \dfrac{s}{s^2-a^2}$

62.4 The initial and final value theorems

There are several Laplace transform theorems used to simplify and interpret the solution of certain problems. Two such theorems are the initial value theorem and the final value theorem.

(a) The initial value theorem states:

$$\lim_{t\to 0}[f(t)] = \lim_{s\to\infty}[s\mathcal{L}\{f(t)\}]$$

For example, if $f(t) = 3e^{4t}$ then

$$\mathcal{L}\{3e^{4t}\} = \frac{3}{s-4}$$

from (iii) of Table 61.1, page 584.

By the initial value theorem,

$$\lim_{t\to 0}[3e^{4t}] = \lim_{s\to\infty}\left[s\left(\frac{3}{s-4}\right)\right]$$

i.e. $$3e^0 = \infty\left(\frac{3}{\infty-4}\right)$$

i.e. **3 = 3**, which illustrates the theorem.

Problem 7. Verify the initial value theorem for the voltage function $(5+2\cos 3t)$ volts, and state its initial value.

Let $f(t) = 5+2\cos 3t$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{5+2\cos 3t\} = \frac{5}{s} + \frac{2s}{s^2+9}$$

from (ii) and (v) of Table 61.1, page 584.

By the initial value theorem,

$$\lim_{t\to 0}[f(t)] = \lim_{s\to\infty}[s\mathcal{L}\{f(t)\}]$$

i.e. $$\lim_{t\to 0}[5+2\cos 3t] = \lim_{s\to\infty}\left[s\left(\frac{5}{s}+\frac{2s}{s^2+9}\right)\right]$$

$$= \lim_{s\to\infty}\left[5+\frac{2s^2}{s^2+9}\right]$$

i.e. $$5+2(1) = 5+\frac{2\infty^2}{\infty^2+9} = 5+2$$

i.e. **7 = 7**, which verifies the theorem in this case.

The initial value of the voltage is thus **7 V**.

Problem 8. Verify the initial value theorem for the function $(2t-3)^2$ and state its initial value.

Let $f(t) = (2t-3)^2 = 4t^2 - 12t + 9$

Let $\mathcal{L}\{f(t)\} = \mathcal{L}(4t^2 - 12t + 9)$

$$= 4\left(\frac{2}{s^3}\right) - \frac{12}{s^2} + \frac{9}{s}$$

from (vii), (vi) and (ii) of Table 61.1, page 584.

By the initial value theorem,

$$\lim_{t\to 0}[(2t-3)^2] = \lim_{s\to\infty}\left[s\left(\frac{8}{s^3} - \frac{12}{s^2} + \frac{9}{s}\right)\right]$$

$$= \lim_{s\to\infty}\left[\frac{8}{s^2} - \frac{12}{s} + 9\right]$$

i.e. $$(0-3)^2 = \frac{8}{\infty^2} - \frac{12}{\infty} + 9$$

i.e. **9 = 9**, which verifies the theorem in this case.

The initial value of the given function is thus **9**.

(b) The final value theorem states:

$$\lim_{t\to\infty}[f(t)] = \lim_{s\to 0}[s\mathcal{L}\{f(t)\}]$$

For example, if $f(t) = 3e^{-4t}$ then:

$$\lim_{t\to\infty}[3e^{-4t}] = \lim_{s\to 0}\left[s\left(\frac{3}{s+4}\right)\right]$$

i.e. $3e^{-\infty} = (0)\left(\frac{3}{0+4}\right)$

i.e. $\mathbf{0 = 0}$, which illustrates the theorem.

Problem 9. Verify the final value theorem for the function $(2 + 3e^{-2t}\sin 4t)$ cm, which represents the displacement of a particle. State its final steady value.

Let $f(t) = 2 + 3e^{-2t}\sin 4t$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{2 + 3e^{-2t}\sin 4t\}$$

$$= \frac{2}{s} + 3\left(\frac{4}{(s-(-2))^2 + 4^2}\right)$$

$$= \frac{2}{s} + \frac{12}{(s+2)^2 + 16}$$

from (ii) of Table 61.1, page 584 and (ii) of Table 62.1 on page 587.

By the final value theorem,

$$\lim_{t\to\infty}[f(t)] = \lim_{s\to 0}[s\mathcal{L}\{f(t)\}]$$

i.e. $\lim_{t\to\infty}[2 + 3e^{-2t}\sin 4t]$

$$= \lim_{s\to 0}\left[s\left(\frac{2}{s} + \frac{12}{(s+2)^2 + 16}\right)\right]$$

$$= \lim_{s\to 0}\left[2 + \frac{12s}{(s+2)^2 + 16}\right]$$

i.e. $2 + 0 = 2 + 0$

i.e. $\mathbf{2 = 2}$, which verifies the theorem in this case.

The final value of the displacement is thus 2 cm.

The initial and final value theorems are used in pulse circuit applications where the response of the circuit for small periods of time, or the behaviour immediately after the switch is closed, are of interest. The final value theorem is particularly useful in investigating the stability of systems (such as in automatic aircraft-landing systems) and is concerned with the steady state response for large values of time t, i.e. after all transient effects have died away.

Now try the following exercise

Exercise 222 Further problems on initial and final value theorems

1. State the initial value theorem. Verify the theorem for the functions (a) $3 - 4\sin t$ (b) $(t-4)^2$ and state their initial values. [(a) 3 (b) 16]

2. Verify the initial value theorem for the voltage functions: (a) $4 + 2\cos t$ (b) $t - \cos 3t$ and state their initial values. [(a) 6 (b) -1]

3. State the final value theorem and state a practical application where it is of use. Verify the theorem for the function $4 + e^{-2t}(\sin t + \cos t)$ representing a displacement and state its final value. [4]

4. Verify the final value theorem for the function $3t^2e^{-4t}$ and determine its steady state value. [0]

Chapter 63

Inverse Laplace transforms

63.1 Definition of the inverse Laplace transform

If the Laplace transform of a function $f(t)$ is $F(s)$, i.e. $\mathcal{L}\{f(t)\}=F(s)$, then $f(t)$ is called the **inverse Laplace transform** of $F(s)$ and is written as $f(t)=\mathcal{L}^{-1}\{F(s)\}$.

For example, since $\mathcal{L}\{1\}=\dfrac{1}{s}$ then $\boldsymbol{\mathcal{L}^{-1}\left\{\dfrac{1}{s}\right\}=1}$.

Similarly, since $\mathcal{L}\{\sin at\}=\dfrac{a}{s^2+a^2}$ then

$$\boldsymbol{\mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\}=\sin at}, \text{ and so on.}$$

63.2 Inverse Laplace transforms of simple functions

Tables of Laplace transforms, such as the tables in Chapters 61 and 62 (see pages 584 and 587) may be used to find inverse Laplace transforms.

However, for convenience, a summary of inverse Laplace transforms is shown in Table 63.1.

Problem 1. Find the following inverse Laplace transforms:

(a) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+9}\right\}$ (b) $\mathcal{L}^{-1}\left\{\dfrac{5}{3s-1}\right\}$

(a) From (iv) of Table 63.1,

$$\mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\}=\sin at,$$

Table 63.1 Inverse Laplace transforms

	$F(s)=\mathcal{L}\{f(t)\}$	$\mathcal{L}^{-1}\{F(s)\}=f(t)$
(i)	$\dfrac{1}{s}$	1
(ii)	$\dfrac{k}{s}$	k
(iii)	$\dfrac{1}{s-a}$	e^{at}
(iv)	$\dfrac{a}{s^2+a^2}$	$\sin at$
(v)	$\dfrac{s}{s^2+a^2}$	$\cos at$
(vi)	$\dfrac{1}{s^2}$	t
(vii)	$\dfrac{2!}{s^3}$	t^2
(viii)	$\dfrac{n!}{s^{n+1}}$	t^n
(ix)	$\dfrac{a}{s^2-a^2}$	$\sinh at$
(x)	$\dfrac{s}{s^2-a^2}$	$\cosh at$
(xi)	$\dfrac{n!}{(s-a)^{n+1}}$	$e^{at}t^n$
(xii)	$\dfrac{\omega}{(s-a)^2+\omega^2}$	$e^{at}\sin\omega t$
(xiii)	$\dfrac{s-a}{(s-a)^2+\omega^2}$	$e^{at}\cos\omega t$
(xiv)	$\dfrac{\omega}{(s-a)^2-\omega^2}$	$e^{at}\sinh\omega t$
(xv)	$\dfrac{s-a}{(s-a)^2-\omega^2}$	$e^{at}\cosh\omega t$

Hence $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+9}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s^2+3^2}\right\}$

$$= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}$$

$$= \frac{\mathbf{1}}{\mathbf{3}}\mathbf{sin\,3}t$$

(b) $\mathcal{L}^{-1}\left\{\dfrac{5}{3s-1}\right\} = \mathcal{L}^{-1}\left\{\dfrac{5}{3\left(s-\frac{1}{3}\right)}\right\}$

$$= \frac{5}{3}\mathcal{L}^{-1}\left\{\frac{1}{\left(s-\frac{1}{3}\right)}\right\} = \frac{\mathbf{5}}{\mathbf{3}}\mathbf{e}^{\frac{1}{3}t}$$

from (iii) of Table 63.1

Problem 2. Find the following inverse Laplace transforms:

(a) $\mathcal{L}^{-1}\left\{\dfrac{6}{s^3}\right\}$ (b) $\mathcal{L}^{-1}\left\{\dfrac{3}{s^4}\right\}$

(a) From (vii) of Table 63.1, $\mathcal{L}^{-1}\left\{\dfrac{2}{s^3}\right\} = t^2$

Hence $\mathcal{L}^{-1}\left\{\dfrac{6}{s^3}\right\} = 3\mathcal{L}^{-1}\left\{\dfrac{2}{s^3}\right\} = \mathbf{3}t^2$.

(b) From (viii) of Table 63.1, if s is to have a power of 4 then $n=3$.

Thus $\mathcal{L}^{-1}\left\{\dfrac{3!}{s^4}\right\} = t^3$ i.e. $\mathcal{L}^{-1}\left\{\dfrac{6}{s^4}\right\} = t^3$

Hence $\mathcal{L}^{-1}\left\{\dfrac{3}{s^4}\right\} = \dfrac{1}{2}\mathcal{L}^{-1}\left\{\dfrac{6}{s^4}\right\} = \dfrac{\mathbf{1}}{\mathbf{2}}\boldsymbol{t}^3$.

Problem 3. Determine

(a) $\mathcal{L}^{-1}\left\{\dfrac{7s}{s^2+4}\right\}$ (b) $\mathcal{L}^{-1}\left\{\dfrac{4s}{s^2-16}\right\}$

(a) $\mathcal{L}^{-1}\left\{\dfrac{7s}{s^2+4}\right\} = 7\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+2^2}\right\} = \mathbf{7\,cos\,2}t$,

from (v) of Table 63.1

(b) $\mathcal{L}^{-1}\left\{\dfrac{4s}{s^2-16}\right\} = 4\mathcal{L}^{-1}\left\{\dfrac{s}{s^2-4^2}\right\}$

$$= \mathbf{4\,cosh\,4}t,$$

from (x) of Table 63.1

Problem 4. Find

(a) $\mathcal{L}^{-1}\left\{\dfrac{3}{s^2-7}\right\}$ (b) $\mathcal{L}^{-1}\left\{\dfrac{2}{(s-3)^5}\right\}$

(a) From (ix) of Table 63.1,

$$\mathcal{L}^{-1}\left\{\frac{a}{s^2-a^2}\right\} = \sinh at$$

Thus

$$\mathcal{L}^{-1}\left\{\frac{3}{s^2-7}\right\} = 3\mathcal{L}^{-1}\left\{\frac{1}{s^2-(\sqrt{7})^2}\right\}$$

$$= \frac{3}{\sqrt{7}}\mathcal{L}^{-1}\left\{\frac{\sqrt{7}}{s^2-(\sqrt{7})^2}\right\}$$

$$= \frac{\mathbf{3}}{\sqrt{\mathbf{7}}}\mathbf{sinh}\sqrt{\mathbf{7}}\boldsymbol{t}$$

(b) From (xi) of Table 63.1,

$$\mathcal{L}^{-1}\left\{\frac{n!}{(s-a)^{n+1}}\right\} = \mathrm{e}^{at}t^n$$

Thus $\mathcal{L}^{-1}\left\{\dfrac{1}{(s-a)^{n+1}}\right\} = \dfrac{1}{n!}\mathrm{e}^{at}t^n$

and comparing with $\mathcal{L}^{-1}\left\{\dfrac{2}{(s-3)^5}\right\}$ shows that $n=4$ and $a=3$.

Hence

$$\mathcal{L}^{-1}\left\{\frac{2}{(s-3)^5}\right\} = 2\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^5}\right\}$$

$$= 2\left(\frac{1}{4!}\mathrm{e}^{3t}t^4\right) = \frac{\mathbf{1}}{\mathbf{12}}\mathbf{e}^{\mathbf{3}t}\boldsymbol{t}^{\mathbf{4}}$$

Problem 5. Determine

(a) $\mathcal{L}^{-1}\left\{\dfrac{3}{s^2-4s+13}\right\}$

(b) $\mathcal{L}^{-1}\left\{\dfrac{2(s+1)}{s^2+2s+10}\right\}$

(a) $\mathcal{L}^{-1}\left\{\dfrac{3}{s^2-4s+13}\right\} = \mathcal{L}^{-1}\left\{\dfrac{3}{(s-2)^2+3^2}\right\}$

$= \mathbf{e^{2t}\sin 3t}$,

from (xii) of Table 63.1

(b) $\mathcal{L}^{-1}\left\{\dfrac{2(s+1)}{s^2+2s+10}\right\} = \mathcal{L}^{-1}\left\{\dfrac{2(s+1)}{(s+1)^2+3^2}\right\}$

$= \mathbf{2e^{-t}\cos 3t}$,

from (xiii) of Table 63.1

Problem 6. Determine

(a) $\mathcal{L}^{-1}\left\{\dfrac{5}{s^2+2s-3}\right\}$

(b) $\mathcal{L}^{-1}\left\{\dfrac{4s-3}{s^2-4s-5}\right\}$

(a) $\mathcal{L}^{-1}\left\{\dfrac{5}{s^2+2s-3}\right\} = \mathcal{L}^{-1}\left\{\dfrac{5}{(s+1)^2-2^2}\right\}$

$= \mathcal{L}^{-1}\left\{\dfrac{\frac{5}{2}(2)}{(s+1)^2-2^2}\right\}$

$= \mathbf{\dfrac{5}{2}e^{-t}\sinh 2t}$,

from (xiv) of Table 63.1

(b) $\mathcal{L}^{-1}\left\{\dfrac{4s-3}{s^2-4s-5}\right\} = \mathcal{L}^{-1}\left\{\dfrac{4s-3}{(s-2)^2-3^2}\right\}$

$= \mathcal{L}^{-1}\left\{\dfrac{4(s-2)+5}{(s-2)^2-3^2}\right\}$

$= \mathcal{L}^{-1}\left\{\dfrac{4(s-2)}{(s-2)^2-3^2}\right\}$

$+ \mathcal{L}^{-1}\left\{\dfrac{5}{(s-2)^2-3^2}\right\}$

$= 4e^{2t}\cosh 3t + \mathcal{L}^{-1}\left\{\dfrac{\frac{5}{3}(3)}{(s-2)^2-3^2}\right\}$

from (xv) of Table 63.1

$= \mathbf{4e^{2t}\cosh 3t + \dfrac{5}{3}e^{2t}\sinh 3t}$,

from (xiv) of Table 63.1

Now try the following exercise

Exercise 223 Further problems on inverse Laplace transforms of simple functions

Determine the inverse Laplace transforms of the following:

1. (a) $\dfrac{7}{s}$ (b) $\dfrac{2}{s-5}$ [(a) 7 (b) $2e^{5t}$]

2. (a) $\dfrac{3}{2s+1}$ (b) $\dfrac{2s}{s^2+4}$

$\left[\text{(a) } \dfrac{3}{2}e^{-\frac{1}{2}t} \quad \text{(b) } 2\cos 2t\right]$

3. (a) $\dfrac{1}{s^2+25}$ (b) $\dfrac{4}{s^2+9}$

$\left[\text{(a) } \dfrac{1}{5}\sin 5t \quad \text{(b) } \dfrac{4}{3}\sin 3t\right]$

4. (a) $\dfrac{5s}{2s^2+18}$ (b) $\dfrac{6}{s^2}$

$\left[\text{(a) } \dfrac{5}{2}\cos 3t \quad \text{(b) } 6t\right]$

5. (a) $\dfrac{5}{s^3}$ (b) $\dfrac{8}{s^4}$ $\left[\text{(a) } \dfrac{5}{2}t^2 \quad \text{(b) } \dfrac{4}{3}t^3\right]$

6. (a) $\dfrac{3s}{\frac{1}{2}s^2-8}$ (b) $\dfrac{7}{s^2-16}$

$\left[\text{(a) } 6\cosh 4t \quad \text{(b) } \dfrac{7}{4}\sinh 4t\right]$

7. (a) $\dfrac{15}{3s^2-27}$ (b) $\dfrac{4}{(s-1)^3}$

$\left[\text{(a) } \dfrac{5}{3}\sinh 3t \quad \text{(b) } 2e^t t^2\right]$

8. (a) $\dfrac{1}{(s+2)^4}$ (b) $\dfrac{3}{(s-3)^5}$

$\left[\text{(a) } \dfrac{1}{6}e^{-2t}t^3 \quad \text{(b) } \dfrac{1}{8}e^{3t}t^4\right]$

9. (a) $\dfrac{s+1}{s^2+2s+10}$ (b) $\dfrac{3}{s^2+6s+13}$

$$\left[\text{(a) } \mathrm{e}^{-t}\cos 3t \quad \text{(b) } \frac{3}{2}\mathrm{e}^{-3t}\sin 2t\right]$$

10. (a) $\dfrac{2(s-3)}{s^2-6s+13}$ (b) $\dfrac{7}{s^2-8s+12}$

$$\left[(a)\ 2\mathrm{e}^{3t}\cos 2t \quad \text{(b) } \frac{7}{2}\mathrm{e}^{4t}\sinh 2t\right]$$

11. (a) $\dfrac{2s+5}{s^2+4s-5}$ (b) $\dfrac{3s+2}{s^2-8s+25}$

$$\left[\begin{array}{l}\text{(a) } 2\mathrm{e}^{-2t}\cosh 3t+\frac{1}{3}\mathrm{e}^{-2t}\sinh 3t \\ \text{(b) } 3\mathrm{e}^{4t}\cos 3t+\frac{14}{3}\mathrm{e}^{4t}\sin 3t\end{array}\right]$$

63.3 Inverse Laplace transforms using partial fractions

Sometimes the function whose inverse is required is not recognisable as a standard type, such as those listed in Table 63.1. In such cases it may be possible, by using partial fractions, to resolve the function into simpler fractions which may be inverted on sight. For example, the function,

$$F(s)=\frac{2s-3}{s(s-3)}$$

cannot be inverted on sight from Table 63.1. However, by using partial fractions, $\dfrac{2s-3}{s(s-3)} \equiv \dfrac{1}{s}+\dfrac{1}{s-3}$ which may be inverted as $1+\mathrm{e}^{3t}$ from (i) and (iii) of Table 61.1. Partial fractions are discussed in Chapter 2, and a summary of the forms of partial fractions is given in Table 2.1 on page 13.

Problem 7. Determine $\mathcal{L}^{-1}\left\{\dfrac{4s-5}{s^2-s-2}\right\}$

$$\frac{4s-5}{s^2-s-2} \equiv \frac{4s-5}{(s-2)(s+1)} \equiv \frac{A}{(s-2)}+\frac{B}{(s+1)}$$

$$\equiv \frac{A(s+1)+B(s-2)}{(s-2)(s+1)}$$

Hence $4s-5 \equiv A(s+1)+B(s-2)$.

When $s=2$, $3=3A$, from which, $A=1$.

When $s=-1$, $-9=-3B$, from which, $B=3$.

Hence $\mathcal{L}^{-1}\left\{\dfrac{4s-5}{s^2-s-2}\right\}$

$$\equiv \mathcal{L}^{-1}\left\{\frac{1}{s-2}+\frac{3}{s+1}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}+\mathcal{L}^{-1}\left\{\frac{3}{s+1}\right\}$$

$$=\mathbf{e^{2t}+3e^{-t}},\ \text{from (iii) of Table 63.1}$$

Problem 8. Find $\mathcal{L}^{-1}\left\{\dfrac{3s^3+s^2+12s+2}{(s-3)(s+1)^3}\right\}$

$$\frac{3s^3+s^2+12s+2}{(s-3)(s+1)^3}$$

$$\equiv \frac{A}{s-3}+\frac{B}{s+1}+\frac{C}{(s+1)^2}+\frac{D}{(s+1)^3}$$

$$\equiv \frac{\left(\begin{array}{l}A(s+1)^3+B(s-3)(s+1)^2 \\ +C(s-3)(s+1)+D(s-3)\end{array}\right)}{(s-3)(s+1)^3}$$

Hence

$$3s^3+s^2+12s+2 \equiv A(s+1)^3+B(s-3)(s+1)^2 +C(s-3)(s+1)+D(s-3)$$

When $s=3$, $128=64A$, from which, $A=2$.

When $s=-1$, $-12=-4D$, from which, $D=3$.

Equating s^3 terms gives: $3=A+B$, from which, $B=1$.

Equating constant terms gives:

$$2=A-3B-3C-3D,$$

i.e. $$2=2-3-3C-9,$$

from which, $3C=-12$ and $C=-4$

Hence

$$\mathcal{L}^{-1}\left\{\frac{3s^3+s^2+12s+2}{(s-3)(s+1)^3}\right\}$$

$$\equiv \mathcal{L}^{-1}\left\{\frac{2}{s-3}+\frac{1}{s+1}-\frac{4}{(s+1)^2}+\frac{3}{(s+1)^3}\right\}$$

$$=\mathbf{2e^{3t}+e^{-t}-4e^{-t}t+\frac{3}{2}e^{-t}t^2},$$

from (iii) and (xi) of Table 63.1

Problem 9. Determine $\mathcal{L}^{-1}\left\{\dfrac{5s^2+8s-1}{(s+3)(s^2+1)}\right\}$

$$\frac{5s^2+8s-1}{(s+3)(s^2+1)} \equiv \frac{A}{s+3}+\frac{Bs+C}{(s^2+1)} \equiv \frac{A(s^2+1)+(Bs+C)(s+3)}{(s+3)(s^2+1)}$$

Hence $5s^2+8s-1 \equiv A(s^2+1)+(Bs+C)(s+3)$.

When $s=-3$, $20=10A$, from which, $A=2$.

Equating s^2 terms gives: $5=A+B$, from which, $B=3$, since $A=2$.

Equating s terms gives: $8=3B+C$, from which, $C=-1$, since $B=3$.

Hence $\mathcal{L}^{-1}\left\{\dfrac{5s^2+8s-1}{(s+3)(s^2+1)}\right\}$

$$\equiv \mathcal{L}^{-1}\left\{\frac{2}{s+3}+\frac{3s-1}{s^2+1}\right\}$$

$$\equiv \mathcal{L}^{-1}\left\{\frac{2}{s+3}\right\}+\mathcal{L}^{-1}\left\{\frac{3s}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= \mathbf{2e^{-3t}+3\cos t-\sin t},$$

from (iii), (v) and (iv) of Table 63.1

Problem 10. Find $\mathcal{L}^{-1}\left\{\dfrac{7s+13}{s(s^2+4s+13)}\right\}$

$$\frac{7s+13}{s(s^2+4s+13)} \equiv \frac{A}{s}+\frac{Bs+C}{s^2+4s+13} \equiv \frac{A(s^2+4s+13)+(Bs+C)(s)}{s(s^2+4s+13)}$$

Hence $7s+13 \equiv A(s^2+4s+13)+(Bs+C)(s)$.

When $s=0$, $13=13A$, from which, $A=1$.

Equating s^2 terms gives: $0=A+B$, from which, $B=-1$.

Equating s terms gives: $7=4A+C$, from which, $C=3$.

Hence $\mathcal{L}^{-1}\left\{\dfrac{7s+13}{s(s^2+4s+13)}\right\}$

$$\equiv \mathcal{L}^{-1}\left\{\frac{1}{s}+\frac{-s+3}{s^2+4s+13}\right\}$$

$$\equiv \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}+\mathcal{L}^{-1}\left\{\frac{-s+3}{(s+2)^2+3^2}\right\}$$

$$\equiv \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}+\mathcal{L}^{-1}\left\{\frac{-(s+2)+5}{(s+2)^2+3^2}\right\}$$

$$\equiv \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}-\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+3^2}\right\} + \mathcal{L}^{-1}\left\{\frac{5}{(s+2)^2+3^2}\right\}$$

$$\equiv \mathbf{1-e^{-2t}\cos 3t+\frac{5}{3}e^{-2t}\sin 3t}$$

from (i), (xiii) and (xii) of Table 63.1

Now try the following exercise

Exercise 224 Further problems on inverse Laplace transforms using partial fractions

Use partial fractions to find the inverse Laplace transforms of the following functions:

1. $\dfrac{11-3s}{s^2+2s-3}$ $[2e^t-5e^{-3t}]$
2. $\dfrac{2s^2-9s-35}{(s+1)(s-2)(s+3)}$ $[4e^{-t}-3e^{2t}+e^{-3t}]$
3. $\dfrac{5s^2-2s-19}{(s+3)(s-1)^2}$ $[2e^{-3t}+3e^t-4e^t t]$
4. $\dfrac{3s^2+16s+15}{(s+3)^3}$ $[e^{-3t}(3-2t-3t^2)]$
5. $\dfrac{7s^2+5s+13}{(s^2+2)(s+1)}$ $\left[2\cos\sqrt{2}t+\dfrac{3}{\sqrt{2}}\sin\sqrt{2}t+5e^{-t}\right]$
6. $\dfrac{3+6s+4s^2-2s^3}{s^2(s^2+3)}$ $[2+t+\sqrt{3}\sin\sqrt{3}t-4\cos\sqrt{3}t]$

7. $\dfrac{26-s^2}{s(s^2+4s+13)}$

$$\left[2-3\mathrm{e}^{-2t}\cos 3t-\frac{2}{3}\mathrm{e}^{-2t}\sin 3t\right]$$

63.4 Poles and zeros

It was seen in the previous section that Laplace transforms, in general, have the form $f(s)=\dfrac{\phi(s)}{\theta(s)}$. This is the same form as most transfer functions for engineering systems, a **transfer function** being one that relates the response at a given pair of terminals to a source or stimulus at another pair of terminals.

Let a function in the s domain be given by: $f(s)=\dfrac{\phi(s)}{(s-a)(s-b)(s-c)}$ where $\phi(s)$ is of less degree than the denominator.

Poles: The values $a, b, c, \ldots$ that makes the denominator zero, and hence $f(s)$ infinite, are called the system poles of $f(s)$.
If there are no repeated factors, the poles are **simple poles**.
If there are repeated factors, the poles are **multiple poles**.

Zeros: Values of s that make the numerator $\phi(s)$ zero, and hence $f(s)$ zero, are called the system zeros of $f(s)$.

For example: $\dfrac{s-4}{(s+1)(s-2)}$ has simple poles at $s=-1$ and $s=+2$, and a zero at $s=4$ $\dfrac{s+3}{(s+1)^2(2s+5)}$ has a simple pole at $s=-\dfrac{5}{2}$ and double poles at $s=-1$, and a zero at $s=-3$ and $\dfrac{s+2}{s(s-1)(s+4)(2s+1)}$ has simple poles at $s=0, +1, -4$, and $-\dfrac{1}{2}$ and a zero at $s=-2$

Pole-zero diagram

The poles and zeros of a function are values of complex frequency s and can therefore be plotted on the complex frequency or s-plane. The resulting plot is the **pole-zero diagram** or **pole-zero map**. On the rectangular axes, the real part is labelled the **σ-axis** and the imaginary part the **$j\omega$-axis**.

The location of a pole in the s-plane is denoted by a cross ($\times$) and the location of a zero by a small circle (o). This is demonstrated in the following examples. From the pole-zero diagram it may be determined that the magnitude of the transfer function will be larger when it is closer to the poles and smaller when it is close to the zeros. This is important in understanding what the system does at various frequencies and is crucial in the study of **stability** and **control theory** in general.

Problem 11. Determine for the transfer function:

$$R(s)=\frac{400\,(s+10)}{s\,(s+25)(s^2+10s+125)}$$

(a) the zero and (b) the poles. Show the poles and zero on a pole-zero diagram.

(a) For the numerator to be zero, $(s+10)=0$.

Hence, **$s=-10$ is a zero** of $R(s)$.

(b) For the denominator to be zero, $s=0$ or $s=-25$ or $s^2+10s+125=0$.

Using the quadratic formula.

$$s=\frac{-10\pm\sqrt{10^2-4(1)(125)}}{2}=\frac{-10\pm\sqrt{-400}}{2}$$

$$=\frac{-10\pm j20}{2}$$

$$=(-5\pm j10)$$

Hence, **poles occur at $s=0$, $s=-25$, $(-5+j10)$ and $(-5-j10)$**

The pole-zero diagram is shown in Figure 63.1.

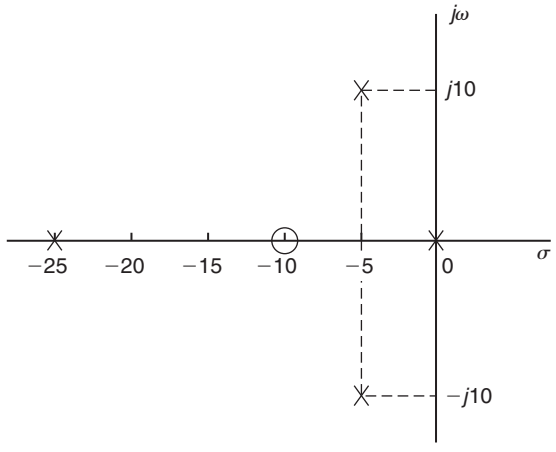


Figure 63.1

Problem 12. Determine the poles and zeros for the function: $F(s)=\dfrac{(s+3)(s-2)}{(s+4)(s^2+2s+2)}$ and plot them on a pole-zero map.

For the numerator to be zero, $(s+3)=0$ and $(s-2)=0$, hence **zeros occur at $s=-3$** and at **$s=+2$** Poles occur when the denominator is zero, i.e. when $(s+4)=0$, i.e. **$s=-4$**, and when $s^2+2s+2=0$,

$$\text{i.e. } s=\frac{-2\pm\sqrt{2^2-4(1)(2)}}{2}=\frac{-2\pm\sqrt{-4}}{2}$$

$$=\frac{-2\pm j2}{2}=(\mathbf{-1+j})\text{ or }(\mathbf{-1-j})$$

The poles and zeros are shown on the pole-zero map of $F(s)$ in Figure 63.2.

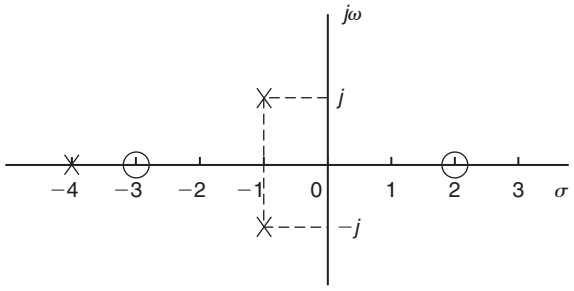


Figure 63.2

It is seen from these problems that poles and zeros are always real or complex conjugate.

Now try the following exercise

Exercise 225 Further problems on poles and zeros

1. Determine for the transfer function: $R(s)=\dfrac{50(s+4)}{s(s+2)(s^2-8s+25)}$ (a) the zero and (b) the poles. Show the poles and zeros on a pole-zero diagram.

   $$\left[\begin{array}{c}\text{(a) } s=-4 \quad \text{(b) } s=0,\ s=-2,\\ s=4+j3,\ s=4-j3\end{array}\right]$$

2. Determine the poles and zeros for the function: $F(s)=\dfrac{(s-1)(s+2)}{(s+3)(s^2-2s+5)}$ and plot them on a pole-zero map.

   $$\left[\begin{array}{l}\text{poles at } s=-3,\ s=1+j2,\ s=1-j2,\\ \text{zeros at } s=+1,\ s=-2\end{array}\right]$$

3. For the function $G(s)=\dfrac{s-1}{(s+2)(s^2+2s+5)}$ determine the poles and zeros and show them on a pole-zero diagram.

   $$\left[\begin{array}{c}\text{poles at } s=-2,\ s=-1+j2,\\ s=-1-j2,\\ \text{zero at } s=1\end{array}\right]$$

4. Find the poles and zeros for the transfer function: $H(s)=\dfrac{s^2-5s-6}{s(s^2+4)}$ and plot the results in the s-plane.

   $$\left[\begin{array}{l}\text{poles at } s=0,\ s=+j2,\ s=-j2,\\ \text{zeros at } s=-1,\ s=6\end{array}\right]$$

Chapter 64

The solution of differential equations using Laplace transforms

64.1 Introduction

An alternative method of solving differential equations to that used in Chapters 46 to 51 is possible by using Laplace transforms.

64.2 Procedure to solve differential equations by using Laplace transforms

(i) Take the Laplace transform of both sides of the differential equation by applying the formulae for the Laplace transforms of derivatives (i.e. equations (3) and (4) of Chapter 62) and, where necessary, using a list of standard Laplace transforms, such as Tables 61.1 and 62.1 on pages 584 and 587.

(ii) Put in the given initial conditions, i.e. $y(0)$ and $y'(0)$.

(iii) Rearrange the equation to make $\mathcal{L}\{y\}$ the subject.

(iv) Determine y by using, where necessary, partial fractions, and taking the inverse of each term by using Table 63.1 on page 593.

64.3 Worked problems on solving differential equations using Laplace transforms

Problem 1. Use Laplace transforms to solve the differential equation $2\dfrac{d^2y}{dx^2}+5\dfrac{dy}{dx}-3y=0$, given that when $x=0$, $y=4$ and $\dfrac{dy}{dx}=9$.

This is the same problem as Problem 1 of Chapter 50, page 478 and a comparison of methods can be made. Using the above procedure:

(i) $$2\mathcal{L}\left\{\frac{d^2y}{dx^2}\right\}+5\mathcal{L}\left\{\frac{dy}{dx}\right\}-3\mathcal{L}\{y\}=\mathcal{L}\{0\}$$

$$2[s^2\mathcal{L}\{y\}-sy(0)-y'(0)]+5[s\mathcal{L}\{y\}-y(0)]-3\mathcal{L}\{y\}=0,$$

from equations (3) and (4) of Chapter 62.

(ii) $y(0)=4$ and $y'(0)=9$

Thus $2[s^2\mathcal{L}\{y\}-4s-9]+5[s\mathcal{L}\{y\}-4]-3\mathcal{L}\{y\}=0$

i.e. $2s^2\mathcal{L}\{y\}-8s-18+5s\mathcal{L}\{y\}-20-3\mathcal{L}\{y\}=0$

(iii) Rearranging gives:

$$(2s^2+5s-3)\mathcal{L}\{y\}=8s+38$$

i.e. $$\mathcal{L}\{y\}=\frac{8s+38}{2s^2+5s-3}$$

(iv) $$y=\mathcal{L}^{-1}\left\{\frac{8s+38}{2s^2+5s-3}\right\}$$

$$\frac{8s+38}{2s^2+5s-3}\equiv\frac{8s+38}{(2s-1)(s+3)}\equiv\frac{A}{2s-1}+\frac{B}{s+3}\equiv\frac{A(s+3)+B(2s-1)}{(2s-1)(s+3)}$$

Hence $8s+38=A(s+3)+B(2s-1)$.

When $s=\frac{1}{2}$, $42=3\frac{1}{2}A$, from which, $A=12$.

When $s=-3$, $14=-7B$, from which, $B=-2$.

Hence $$y=\mathcal{L}^{-1}\left\{\frac{8s+38}{2s^2+5s-3}\right\}=\mathcal{L}^{-1}\left\{\frac{12}{2s-1}-\frac{2}{s+3}\right\}=\mathcal{L}^{-1}\left\{\frac{12}{2\left(s-\frac{1}{2}\right)}\right\}-\mathcal{L}^{-1}\left\{\frac{2}{s+3}\right\}$$

Hence $\boldsymbol{y=6e^{\frac{1}{2}x}-2e^{-3x}}$, from (iii) of Table 63.1.

Problem 2. Use Laplace transforms to solve the differential equation: $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+13y=0$, given that when $x=0$, $y=3$ and $\frac{dy}{dx}=7$.

This is the same as Problem 3 of Chapter 50, page 479. Using the above procedure:

(i) $$\mathcal{L}\left\{\frac{d^2x}{dy^2}\right\}+6\mathcal{L}\left\{\frac{dy}{dx}\right\}+13\mathcal{L}\{y\}=\mathcal{L}\{0\}$$

Hence $[s^2\mathcal{L}\{y\}-sy(0)-y'(0)]+6[s\mathcal{L}\{y\}-y(0)]+13\mathcal{L}\{y\}=0$,

from equations (3) and (4) of Chapter 62.

(ii) $y(0)=3$ and $y'(0)=7$

Thus $s^2\mathcal{L}\{y\}-3s-7+6s\mathcal{L}\{y\}-18+13\mathcal{L}\{y\}=0$

(iii) Rearranging gives:

$$(s^2+6s+13)\mathcal{L}\{y\}=3s+25$$

i.e. $$\mathcal{L}\{y\}=\frac{3s+25}{s^2+6s+13}$$

(iv) $$y=\mathcal{L}^{-1}\left\{\frac{3s+25}{s^2+6s+13}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{3s+25}{(s+3)^2+2^2}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{3(s+3)+16}{(s+3)^2+2^2}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{3(s+3)}{(s+3)^2+2^2}\right\}+\mathcal{L}^{-1}\left\{\frac{8(2)}{(s+3)^2+2^2}\right\}$$

$=3e^{-3t}\cos 2t+8e^{-3t}\sin 2t$, from (xiii) and (xii) of Table 63.1

Hence $\boldsymbol{y=e^{-3t}(3\cos 2t+8\sin 2t)}$

Problem 3. Use Laplace transforms to solve the differential equation: $\frac{d^2y}{dx^2}-3\frac{dy}{dx}=9$, given that when $x=0$, $y=0$ and $\frac{dy}{dx}=0$.

This is the same problem as Problem 2 of Chapter 51, page 485. Using the procedure:

(i) $\mathcal{L}\left\{\frac{d^2y}{dx^2}\right\} - 3\mathcal{L}\left\{\frac{dy}{dx}\right\} = \mathcal{L}\{9\}$

Hence $[s^2\mathcal{L}\{y\} - sy(0) - y'(0)]$
$-3[s\mathcal{L}\{y\} - y(0)] = \frac{9}{s}$

(ii) $y(0) = 0$ and $y'(0) = 0$

Hence $s^2\mathcal{L}\{y\} - 3s\mathcal{L}\{y\} = \frac{9}{s}$

(iii) Rearranging gives:

$$(s^2 - 3s)\mathcal{L}\{y\} = \frac{9}{s}$$

i.e. $\mathcal{L}\{y\} = \frac{9}{s(s^2 - 3s)} = \frac{9}{s^2(s-3)}$

(iv) $y = \mathcal{L}^{-1}\left\{\frac{9}{s^2(s-3)}\right\}$

$$\frac{9}{s^2(s-3)} \equiv \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-3}$$
$$\equiv \frac{A(s)(s-3) + B(s-3) + Cs^2}{s^2(s-3)}$$

Hence $9 \equiv A(s)(s-3) + B(s-3) + Cs^2$.

When $s = 0$, $9 = -3B$, from which, $B = -3$.

When $s = 3$, $9 = 9C$, from which, $C = 1$.

Equating s^2 terms gives: $0 = A + C$, from which, $A = -1$, since $C = 1$. Hence,

$$\mathcal{L}^{-1}\left\{\frac{9}{s^2(s-3)}\right\} = \mathcal{L}^{-1}\left\{-\frac{1}{s} - \frac{3}{s^2} + \frac{1}{s-3}\right\}$$
$$= -1 - 3x + e^{3x}, \text{ from (i), (vi) and (iii) of Table 63.1.}$$

i.e. $\boldsymbol{y = e^{3x} - 3x - 1}$

Problem 4. Use Laplace transforms to solve the differential equation:

$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 10y = e^{2x} + 20$, given that when $x = 0$, $y = 0$ and $\frac{dy}{dx} = -\frac{1}{3}$

Using the procedure:

(i) $\mathcal{L}\left\{\frac{d^2y}{dx^2}\right\} - 7\mathcal{L}\left\{\frac{dy}{dx}\right\} + 10\mathcal{L}\{y\} = \mathcal{L}\{e^{2x} + 20\}$

Hence $[s^2\mathcal{L}\{y\} - sy(0) - y'(0)] - 7[s\mathcal{L}\{y\}$
$- y(0)] + 10\mathcal{L}\{y\} = \frac{1}{s-2} + \frac{20}{s}$

(ii) $y(0) = 0$ and $y'(0) = -\frac{1}{3}$

Hence $s^2\mathcal{L}\{y\} - 0 - \left(-\frac{1}{3}\right) - 7s\mathcal{L}\{y\} + 0$
$+ 10\mathcal{L}\{y\} = \frac{21s - 40}{s(s-2)}$

(iii) $(s^2 - 7s + 10)\mathcal{L}\{y\} = \frac{21s - 40}{s(s-2)} - \frac{1}{3}$
$$= \frac{3(21s - 40) - s(s-2)}{3s(s-2)}$$
$$= \frac{-s^2 + 65s - 120}{3s(s-2)}$$

Hence $\mathcal{L}\{y\} = \frac{-s^2 + 65s - 120}{3s(s-2)(s^2 - 7s + 10)}$
$$= \frac{1}{3}\left[\frac{-s^2 + 65s - 120}{s(s-2)(s-2)(s-5)}\right]$$
$$= \frac{1}{3}\left[\frac{-s^2 + 65s - 120}{s(s-5)(s-2)^2}\right]$$

(iv) $y = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{-s^2 + 65s - 120}{s(s-5)(s-2)^2}\right\}$

$$\frac{-s^2 + 65s - 120}{s(s-5)(s-2)^2}$$
$$\equiv \frac{A}{s} + \frac{B}{s-5} + \frac{C}{s-2} + \frac{D}{(s-2)^2}$$
$$\equiv \frac{\left(\begin{array}{c} A(s-5)(s-2)^2 + B(s)(s-2)^2 \\ +C(s)(s-5)(s-2) + D(s)(s-5) \end{array}\right)}{s(s-5)(s-2)^2}$$

Hence

$-s^2 + 65s - 120$
$\equiv A(s-5)(s-2)^2 + B(s)(s-2)^2$
$+ C(s)(s-5)(s-2) + D(s)(s-5)$

When $s=0, -120=-20A$, from which, $A=6$.

When $s=5, 180=45B$, from which, $B=4$.

When $s=2, 6=-6D$, from which, $D=-1$.

Equating s^3 terms gives: $0=A+B+C$, from which, $C=-10$.

Hence $\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{-s^2+65s-120}{s(s-5)(s-2)^2}\right\}$

$$=\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{6}{s}+\frac{4}{s-5}-\frac{10}{s-2}-\frac{1}{(s-2)^2}\right\}$$

$$=\frac{1}{3}[6+4\mathrm{e}^{5x}-10\mathrm{e}^{2x}-x\,\mathrm{e}^{2x}]$$

Thus $\boldsymbol{y=2+\frac{4}{3}e^{5x}-\frac{10}{3}e^{2x}-\frac{x}{3}e^{2x}}$

Problem 5. The current flowing in an electrical circuit is given by the differential equation $Ri+L(\mathrm{d}i/\mathrm{d}t)=E$, where E, L and R are constants. Use Laplace transforms to solve the equation for current i given that when $t=0$, $i=0$.

Using the procedure:

(i) $\mathcal{L}\{Ri\}+\mathcal{L}\left\{L\frac{\mathrm{d}i}{\mathrm{d}t}\right\}=\mathcal{L}\{E\}$

i.e. $R\mathcal{L}\{i\}+L[s\mathcal{L}\{i\}-i(0)]=\frac{E}{s}$

(ii) $i(0)=0$, hence $R\mathcal{L}\{i\}+Ls\mathcal{L}\{i\}=\frac{E}{s}$

(iii) Rearranging gives:

$$(R+Ls)\mathcal{L}\{i\}=\frac{E}{s}$$

i.e. $\mathcal{L}\{i\}=\frac{E}{s(R+Ls)}$

(iv) $i=\mathcal{L}^{-1}\left\{\frac{E}{s(R+Ls)}\right\}$

$$\frac{E}{s(R+Ls)}\equiv\frac{A}{s}+\frac{B}{R+Ls}$$

$$\equiv\frac{A(R+Ls)+Bs}{s(R+Ls)}$$

Hence $E=A(R+Ls)+Bs$

When $s=0$, $E=AR$,

from which, $A=\frac{E}{R}$

When $s=-\frac{R}{L}$, $E=B\left(-\frac{R}{L}\right)$

from which, $B=-\frac{EL}{R}$

Hence $\mathcal{L}^{-1}\left\{\frac{E}{s(R+Ls)}\right\}$

$$=\mathcal{L}^{-1}\left\{\frac{E/R}{s}+\frac{-EL/R}{R+Ls}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{E}{Rs}-\frac{EL}{R(R+Ls)}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{E}{R}\left(\frac{1}{s}\right)-\frac{E}{R}\left(\frac{1}{\frac{R}{L}+s}\right)\right\}$$

$$=\frac{E}{R}\mathcal{L}^{-1}\left\{\frac{1}{s}-\frac{1}{\left(s+\frac{R}{L}\right)}\right\}$$

Hence **current** $\boldsymbol{i=\frac{E}{R}\left(1-e^{-\frac{Rt}{L}}\right)}$

Now try the following exercise

Exercise 226 Further problems on solving differential equations using Laplace transforms

1. A first order differential equation involving current i in a series $R-L$ circuit is given by: $\frac{\mathrm{d}i}{\mathrm{d}t}+5i=\frac{E}{2}$ and $i=0$ at time $t=0$.

   Use Laplace transforms to solve for i when (a) $E=20$ (b) $E=40\mathrm{e}^{-3t}$ and (c) $E=50\sin 5t$.

   [(a) $i=2(1-\mathrm{e}^{-5t})$
   (b) $i=10(\mathrm{e}^{-3t}-\mathrm{e}^{-5t})$
   (c) $i=\frac{5}{2}(\mathrm{e}^{-5t}-\cos 5t+\sin 5t)$]

In Problems 2 to 9, use Laplace transforms to solve the given differential equations.

2. $9\frac{d^2y}{dt^2} - 24\frac{dy}{dt} + 16y = 0$, given $y(0) = 3$ and $y'(0) = 3$. $\left[y = (3 - t)e^{\frac{4}{3}t}\right]$

3. $\frac{d^2x}{dt^2} + 100x = 0$, given $x(0) = 2$ and $x'(0) = 0$. $[x = 2\cos 10t]$

4. $\frac{d^2i}{dt^2} + 1000\frac{di}{dt} + 250000i = 0$, given $i(0) = 0$ and $i'(0) = 100$. $[i = 100te^{-500t}]$

5. $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 8x = 0$, given $x(0) = 4$ and $x'(0) = 8$. $[x = 4(3e^{-2t} - 2e^{-4t})]$

6. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 3e^{4x}$, given $y(0) = -\frac{2}{3}$ and $y'(0) = 4\frac{1}{3}$ $\left[y = (4x - 1)e^x + \frac{1}{3}e^{4x}\right]$

7. $\frac{d^2y}{dx^2} + 16y = 10\cos 4x$, given $y(0) = 3$ and $y'(0) = 4$. $\left[y = 3\cos 4x + \sin 4x + \frac{5}{4}x\sin 4x\right]$

8. $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 3\cos 3x - 11\sin 3x$, given $y(0) = 0$ and $y'(0) = 6$ $[y = e^x - e^{-2x} + \sin 3x]$

9. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 3e^x\cos 2x$, given $y(0) = 2$ and $y'(0) = 5$ $\left[y = 3e^x(\cos x + \sin x) - e^x\cos 2x\right]$

10. Solve, using Laplace transforms, Problems 4 to 9 of Exercise 187, page 480 and Problems 1 to 5 of Exercise 188, page 482.

11. Solve, using Laplace transforms, Problems 3 to 6 of Exercise 189, page 486, Problems 5 and 6 of Exercise 190, page 488, Problems 4 and 7 of Exercise 191, page 490 and Problems 5 and 6 of Exercise 192, page 492.

Chapter 65

The solution of simultaneous differential equations using Laplace transforms

65.1 Introduction

It is sometimes necessary to solve simultaneous differential equations. An example occurs when two electrical circuits are coupled magnetically where the equations relating the two currents i_1 and i_2 are typically:

$$L_1\frac{\mathrm{d}i_1}{\mathrm{d}t} + M\frac{\mathrm{d}i_2}{\mathrm{d}t} + R_1 i_1 = E_1$$

$$L_2\frac{\mathrm{d}i_2}{\mathrm{d}t} + M\frac{\mathrm{d}i_1}{\mathrm{d}t} + R_2 i_2 = 0$$

where L represents inductance, R resistance, M mutual inductance and E_1 the p.d. applied to one of the circuits.

65.2 Procedure to solve simultaneous differential equations using Laplace transforms

(i) Take the Laplace transform of both sides of each simultaneous equation by applying the formulae for the Laplace transforms of derivatives (i.e. equations (3) and (4) of Chapter 62, page 589) and using a list of standard Laplace transforms, as in Table 61.1, page 584 and Table 62.1, page 587.

(ii) Put in the initial conditions, i.e. $x(0)$, $y(0)$, $x'(0)$, $y'(0)$.

(iii) Solve the simultaneous equations for $\mathcal{L}\{y\}$ and $\mathcal{L}\{x\}$ by the normal algebraic method.

(iv) Determine y and x by using, where necessary, partial fractions, and taking the inverse of each term.

65.3 Worked problems on solving simultaneous differential equations by using Laplace transforms

Problem 1. Solve the following pair of simultaneous differential equations

$$\frac{\mathrm{d}y}{\mathrm{d}t} + x = 1$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} - y + 4\mathrm{e}^t = 0$$

given that at $t = 0$, $x = 0$ and $y = 0$.

Using the above procedure:

(i) $$\mathcal{L}\left\{\frac{\mathrm{d}y}{\mathrm{d}t}\right\} + \mathcal{L}\{x\} = \mathcal{L}\{1\} \qquad (1)$$

$$\mathcal{L}\left\{\frac{dx}{dt}\right\} - \mathcal{L}\{y\} + 4\mathcal{L}\{e^t\} = 0 \qquad (2)$$

Equation (1) becomes:

$$[s\mathcal{L}\{y\} - y(0)] + \mathcal{L}\{x\} = \frac{1}{s} \qquad (1')$$

from equation (3), page 589 and Table 61.1, page 584.

Equation (2) becomes:

$$[s\mathcal{L}\{x\} - x(0)] - \mathcal{L}\{y\} = -\frac{4}{s-1} \qquad (2')$$

(ii) $x(0) = 0$ and $y(0) = 0$ hence

Equation $(1')$ becomes:

$$s\mathcal{L}\{y\} + \mathcal{L}\{x\} = \frac{1}{s} \qquad (1'')$$

and equation $(2')$ becomes:

$$s\mathcal{L}\{x\} - \mathcal{L}\{y\} = -\frac{4}{s-1}$$

$$\text{or } -\mathcal{L}\{y\} + s\mathcal{L}\{x\} = -\frac{4}{s-1} \qquad (2'')$$

(iii) $1 \times$ equation $(1'')$ and $s \times$ equation $(2'')$ gives:

$$s\mathcal{L}\{y\} + \mathcal{L}\{x\} = \frac{1}{s} \qquad (3)$$

$$-s\mathcal{L}\{y\} + s^2\mathcal{L}\{x\} = -\frac{4s}{s-1} \qquad (4)$$

Adding equations (3) and (4) gives:

$$(s^2+1)\mathcal{L}\{x\} = \frac{1}{s} - \frac{4s}{s-1}$$

$$= \frac{(s-1) - s(4s)}{s(s-1)}$$

$$= \frac{-4s^2 + s - 1}{s(s-1)}$$

$$\text{from which, } \quad \mathcal{L}\{x\} = \frac{-4s^2 + s - 1}{s(s-1)(s^2+1)} \qquad (5)$$

Using partial fractions

$$\frac{-4s^2 + s - 1}{s(s-1)(s^2+1)} \equiv \frac{A}{s} + \frac{B}{(s-1)} + \frac{Cs + D}{(s^2+1)}$$

$$= \frac{\left(\begin{array}{c} A(s-1)(s^2+1) + Bs(s^2+1) \\ +(Cs+D)s(s-1) \end{array}\right)}{s(s-1)(s^2+1)}$$

Hence

$$-4s^2 + s - 1 = A(s-1)(s^2+1) + Bs(s^2+1) + (Cs+D)s(s-1)$$

When $s = 0$, $\quad -1 = -A \quad$ hence $\boldsymbol{A = 1}$

When $s = 1$, $\quad -4 = 2B \quad$ hence $\boldsymbol{B = -2}$

Equating s^3 coefficients:

$$0 = A + B + C \quad \text{hence} \quad \boldsymbol{C = 1}$$

(since $A = 1$ and $B = -2$)

Equating s^2 coefficients:

$$-4 = -A + D - C \quad \text{hence} \quad \boldsymbol{D = -2}$$

(since $A = 1$ and $C = 1$)

$$\text{Thus} \quad \mathcal{L}\{x\} = \frac{-4s^2 + s - 1}{s(s-1)(s^2+1)}$$

$$= \frac{1}{s} - \frac{2}{(s-1)} + \frac{s-2}{(s^2+1)}$$

(iv) Hence

$$x = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{2}{(s-1)} + \frac{s-2}{(s^2+1)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{2}{(s-1)} + \frac{s}{(s^2+1)} - \frac{2}{(s^2+1)}\right\}$$

i.e. $\quad \boldsymbol{x = 1 - 2e^t + \cos t - 2\sin t}$,

from Table 63.1, page 593

From the second equation given in the question,

$$\frac{dx}{dt} - y + 4e^t = 0$$

from which,

$$y = \frac{dx}{dt} + 4e^t$$

$$= \frac{d}{dt}(1 - 2e^t + \cos t - 2\sin t) + 4e^t$$

$$= -2e^t - \sin t - 2\cos t + 4e^t$$

i.e. $\boldsymbol{y = 2e^t - \sin t - 2\cos t}$

[Alternatively, to determine y, return to equations $(1'')$ and $(2'')$]

Problem 2. Solve the following pair of simultaneous differential equations

$$3\frac{dx}{dt} - 5\frac{dy}{dt} + 2x = 6$$
$$2\frac{dy}{dt} - \frac{dx}{dt} - y = -1$$

given that at $t=0$, $x=8$ and $y=3$.

Using the above procedure:

(i) $$3\mathcal{L}\left\{\frac{dx}{dt}\right\} - 5\mathcal{L}\left\{\frac{dy}{dt}\right\} + 2\mathcal{L}\{x\} = \mathcal{L}\{6\} \quad (1)$$

$$2\mathcal{L}\left\{\frac{dy}{dt}\right\} - \mathcal{L}\left\{\frac{dx}{dt}\right\} - \mathcal{L}\{y\} = \mathcal{L}\{-1\} \quad (2)$$

Equation (1) becomes:

$$3[s\mathcal{L}\{x\} - x(0)] - 5[s\mathcal{L}\{y\} - y(0)] + 2\mathcal{L}\{x\} = \frac{6}{s}$$

from equation (3), page 589, and Table 61.1, page 584.

i.e. $$3s\mathcal{L}\{x\} - 3x(0) - 5s\mathcal{L}\{y\} + 5y(0) + 2\mathcal{L}\{x\} = \frac{6}{s}$$

i.e. $$(3s+2)\mathcal{L}\{x\} - 3x(0) - 5s\mathcal{L}\{y\} + 5y(0) = \frac{6}{s} \quad (1')$$

Equation (2) becomes:

$$2[s\mathcal{L}\{y\} - y(0)] - [s\mathcal{L}\{x\} - x(0)] - \mathcal{L}\{y\} = -\frac{1}{s}$$

from equation (3), page 589, and Table 61.1, page 584,

i.e. $$2s\mathcal{L}\{y\} - 2y(0) - s\mathcal{L}\{x\} + x(0) - \mathcal{L}\{y\} = -\frac{1}{s}$$

i.e. $$(2s-1)\mathcal{L}\{y\} - 2y(0) - s\mathcal{L}\{x\} + x(0) = -\frac{1}{s} \quad (2')$$

(ii) $x(0)=8$ and $y(0)=3$, hence equation $(1')$ becomes

$$(3s+2)\mathcal{L}\{x\} - 3(8) - 5s\mathcal{L}\{y\} + 5(3) = \frac{6}{s} \quad (1'')$$

and equation $(2')$ becomes

$$(2s-1)\mathcal{L}\{y\} - 2(3) - s\mathcal{L}\{x\} + 8 = -\frac{1}{s} \quad (2'')$$

i.e. $$(3s+2)\mathcal{L}\{x\} - 5s\mathcal{L}\{y\} = \frac{6}{s} + 9 \quad (1'')$$

$$\left.\begin{aligned} &(3s+2)\mathcal{L}\{x\} - 5s\mathcal{L}\{y\} = \frac{6}{s} + 9 \quad (1''') \\ &-s\mathcal{L}\{x\} + (2s-1)\mathcal{L}\{y\} = -\frac{1}{s} - 2 \quad (2''') \end{aligned}\right\} \quad (A)$$

(iii) $s\times$ equation $(1''')$ and $(3s+2)\times$ equation $(2''')$ gives:

$$s(3s+2)\mathcal{L}\{x\} - 5s^2\mathcal{L}\{y\} = s\left(\frac{6}{s} + 9\right) \quad (3)$$

$$-s(3s+2)\mathcal{L}\{x\} + (3s+2)(2s-1)\mathcal{L}\{y\} = (3s+2)\left(-\frac{1}{s} - 2\right) \quad (4)$$

i.e. $$s(3s+2)\mathcal{L}\{x\} - 5s^2\mathcal{L}\{y\} = 6 + 9s \quad (3')$$

$$-s(3s+2)\mathcal{L}\{x\} + (6s^2 + s - 2)\mathcal{L}\{y\} = -6s - \frac{2}{s} - 7 \quad (4')$$

Adding equations $(3')$ and $(4')$ gives:

$$(s^2 + s - 2)\mathcal{L}\{y\} = -1 + 3s - \frac{2}{s} = \frac{-s + 3s^2 - 2}{s}$$

from which, $$\mathcal{L}\{y\} = \frac{3s^2 - s - 2}{s(s^2 + s - 2)}$$

Using partial fractions

$$\frac{3s^2 - s - 2}{s(s^2 + s - 2)} \equiv \frac{A}{s} + \frac{B}{(s+2)} + \frac{C}{(s-1)} = \frac{A(s+2)(s-1) + Bs(s-1) + Cs(s+2)}{s(s+2)(s-1)}$$

i.e. $3s^2 - s - 2 = A(s+2)(s-1) + Bs(s-1) + Cs(s+2)$

When $s = 0, -2 = -2A$, hence $\boldsymbol{A = 1}$

When $s = 1, 0 = 3C$, hence $\boldsymbol{C = 0}$

When $s = -2, 12 = 6B$, hence $\boldsymbol{B = 2}$

Thus $\mathcal{L}\{y\} = \dfrac{3s^2 - s - 2}{s(s^2 + s - 2)} = \dfrac{1}{s} + \dfrac{2}{(s+2)}$

(iv) Hence $\boldsymbol{y = \mathcal{L}^{-1}\left\{\dfrac{1}{s} + \dfrac{2}{s+2}\right\} = 1 + 2e^{-2t}}$

Returning to equations (A) to determine $\mathcal{L}\{x\}$ and hence x:

$(2s-1) \times$ equation $(1''')$ and $5s \times (2''')$ gives:

$$(2s-1)(3s+2)\mathcal{L}\{x\} - 5s(2s-1)\mathcal{L}\{y\} = (2s-1)\left(\frac{6}{s} + 9\right) \quad (5)$$

and $-s(5s)\mathcal{L}\{x\} + 5s(2s-1)\mathcal{L}\{y\} = 5s\left(-\dfrac{1}{s} - 2\right)$ (6)

i.e. $(6s^2 + s - 2)\mathcal{L}\{x\} - 5s(2s-1)\mathcal{L}\{y\} = 12 + 18s - \dfrac{6}{s} - 9$ $(5')$

and $-5s^2\mathcal{L}\{x\} + 5s(2s-1)\mathcal{L}\{y\} = -5 - 10s$ $(6')$

Adding equations $(5')$ and $(6')$ gives:

$$(s^2 + s - 2)\mathcal{L}\{x\} = -2 + 8s - \frac{6}{s} = \frac{-2s + 8s^2 - 6}{s}$$

from which, $\mathcal{L}\{x\} = \dfrac{8s^2 - 2s - 6}{s(s^2 + s - 2)} = \dfrac{8s^2 - 2s - 6}{s(s+2)(s-1)}$

Using partial fractions

$$\frac{8s^2 - 2s - 6}{s(s+2)(s-1)} \equiv \frac{A}{s} + \frac{B}{(s+2)} + \frac{C}{(s-1)} = \frac{A(s+2)(s-1) + Bs(s-1) + Cs(s+2)}{s(s+2)(s-1)}$$

i.e. $8s^2 - 2s - 6 = A(s+2)(s-1) + Bs(s-1) + Cs(s+2)$

When $s = 0, -6 = -2A$, hence $\boldsymbol{A = 3}$

When $s = 1, 0 = 3C$, hence $\boldsymbol{C = 0}$

When $s = -2, 30 = 6B$, hence $\boldsymbol{B = 5}$

Thus $\mathcal{L}\{x\} = \dfrac{8s^2 - 2s - 6}{s(s+2)(s-1)} = \dfrac{3}{s} + \dfrac{5}{(s+2)}$

Hence $\boldsymbol{x = \mathcal{L}^{-1}\left\{\dfrac{3}{s} + \dfrac{5}{s+2}\right\} = 3 + 5e^{-2t}}$

Therefore the solutions of the given simultaneous differential equations are

$$\boldsymbol{y = 1 + 2e^{-2t}} \quad \text{and} \quad \boldsymbol{x = 3 + 5e^{-2t}}$$

(These solutions may be checked by substituting the expressions for x and y into the original equations.)

Problem 3. Solve the following pair of simultaneous differential equations

$$\frac{d^2x}{dt^2} - x = y$$

$$\frac{d^2y}{dt^2} + y = -x$$

given that at $t = 0$, $x = 2$, $y = -1$, $\dfrac{dx}{dt} = 0$ and $\dfrac{dy}{dt} = 0$.

Using the procedure:

(i) $[s^2\mathcal{L}\{x\} - sx(0) - x'(0)] - \mathcal{L}\{x\} = \mathcal{L}\{y\}$ (1)

$[s^2\mathcal{L}\{y\} - sy(0) - y'(0)] + \mathcal{L}\{y\} = -\mathcal{L}\{x\}$ (2)

from equation (4), page 590

(ii) $x(0)=2$, $y(0)=-1$, $x'(0)=0$ and $y'(0)=0$

hence $s^2\mathcal{L}\{x\} - 2s - \mathcal{L}\{x\} = \mathcal{L}\{y\}$ (1′)

$s^2\mathcal{L}\{y\} + s + \mathcal{L}\{y\} = -\mathcal{L}\{x\}$ (2′)

(iii) Rearranging gives:

$(s^2-1)\mathcal{L}\{x\} - \mathcal{L}\{y\} = 2s$ (3)

$\mathcal{L}\{x\} + (s^2+1)\mathcal{L}\{y\} = -s$ (4)

Equation (3) $\times (s^2+1)$ and equation (4) $\times 1$ gives:

$(s^2+1)(s^2-1)\mathcal{L}\{x\} - (s^2+1)\mathcal{L}\{y\} = (s^2+1)2s$ (5)

$\mathcal{L}\{x\} + (s^2+1)\mathcal{L}\{y\} = -s$ (6)

Adding equations (5) and (6) gives:

$[(s^2+1)(s^2-1)+1]\mathcal{L}\{x\} = (s^2+1)2s - s$

i.e. $s^4\mathcal{L}\{x\} = 2s^3 + s = s(2s^2+1)$

from which, $\mathcal{L}\{x\} = \dfrac{s(2s^2+1)}{s^4} = \dfrac{2s^2+1}{s^3}$

$= \dfrac{2s^2}{s^3} + \dfrac{1}{s^3} = \dfrac{2}{s} + \dfrac{1}{s^3}$

(iv) Hence $x = \mathcal{L}^{-1}\left\{\dfrac{2}{s} + \dfrac{1}{s^3}\right\}$

i.e. $\boldsymbol{x = 2 + \frac{1}{2}t^2}$

Returning to equations (3) and (4) to determine y:

$1 \times$ equation (3) and $(s^2-1) \times$ equation (4) gives:

$(s^2-1)\mathcal{L}\{x\} - \mathcal{L}\{y\} = 2s$ (7)

$(s^2-1)\mathcal{L}\{x\} + (s^2-1)(s^2+1)\mathcal{L}\{y\} = -s(s^2-1)$ (8)

Equation (7) − equation (8) gives:

$[-1 - (s^2-1)(s^2+1)]\mathcal{L}\{y\} = 2s + s(s^2-1)$

i.e. $-s^4\mathcal{L}\{y\} = s^3 + s$

and $\mathcal{L}\{y\} = \dfrac{s^3+s}{-s^4} = -\dfrac{1}{s} - \dfrac{1}{s^3}$

from which, $y = \mathcal{L}^{-1}\left\{-\dfrac{1}{s} - \dfrac{1}{s^3}\right\}$

i.e. $\boldsymbol{y = -1 - \frac{1}{2}t^2}$

Now try the following exercise

Exercise 227 Further problems on solving simultaneous differential equations using Laplace transforms

Solve the following pairs of simultaneous differential equations:

1. $2\dfrac{dx}{dt} + \dfrac{dy}{dt} = 5e^t$

   $\dfrac{dy}{dt} - 3\dfrac{dx}{dt} = 5$

   given that when $t=0$, $x=0$ and $y=0$.

   $[x = e^t - t - 1$ and $y = 2t - 3 + 3e^t]$

2. $2\dfrac{dy}{dt} - y + x + \dfrac{dx}{dt} - 5\sin t = 0$

   $3\dfrac{dy}{dt} + x - y + 2\dfrac{dx}{dt} - e^t = 0$

   given that at $t=0$, $x=0$ and $y=0$.

   $\left[\begin{array}{l} x = 5\cos t + 5\sin t - e^{2t} - e^t - 3 \text{ and} \\ y = e^{2t} + 2e^t - 3 - 5\sin t \end{array}\right]$

3. $\dfrac{d^2x}{dt^2} + 2x = y$

   $\dfrac{d^2y}{dt^2} + 2y = x$

   given that at $t=0$, $x=4$, $y=2$, $\dfrac{dx}{dt}=0$ and $\dfrac{dy}{dt}=0$.

   $\left[\begin{array}{l} x = 3\cos t + \cos(\sqrt{3}t) \text{ and} \\ y = 3\cos t - \cos(\sqrt{3}t) \end{array}\right]$

Revision Test 18

This Revision Test covers the material contained in Chapters 61 to 65. *The marks for each question are shown in brackets at the end of each question.*

1. Find the Laplace transforms of the following functions:

   (a) $2t^3 - 4t + 5$ (b) $3e^{-2t} - 4\sin 2t$

   (c) $3\cosh 2t$ (d) $2t^4e^{-3t}$

   (e) $5e^{2t}\cos 3t$ (f) $2e^{3t}\sinh 4t$ (16)

2. Find the inverse Laplace transforms of the following functions:

   (a) $\dfrac{5}{2s+1}$ (b) $\dfrac{12}{s^5}$

   (c) $\dfrac{4s}{s^2+9}$ (d) $\dfrac{5}{s^2-9}$

   (e) $\dfrac{3}{(s+2)^4}$ (f) $\dfrac{s-4}{s^2-8s-20}$

   (g) $\dfrac{8}{s^2-4s+3}$ (17)

3. Use partial fractions to determine the following:

   (a) $\mathcal{L}^{-1}\left\{\dfrac{5s-1}{s^2-s-2}\right\}$

   (b) $\mathcal{L}^{-1}\left\{\dfrac{2s^2+11s-9}{s(s-1)(s+3)}\right\}$

   (c) $\mathcal{L}^{-1}\left\{\dfrac{13-s^2}{s(s^2+4s+13)}\right\}$ (24)

4. In a galvanometer the deflection θ satisfies the differential equation:

$$\frac{d^2\theta}{dt^2} + 2\frac{d\theta}{dt} + \theta = 4$$

   Use Laplace transforms to solve the equation for θ given that when $t = 0, \theta = 0$ and $\dfrac{d\theta}{dt} = 0$. (13)

5. Solve the following pair of simultaneous differential equations:

$$3\frac{dx}{dt} = 3x + 2y$$

$$2\frac{dy}{dt} + 3x = 6y$$

   given that when $t = 0$, $x = 1$ and $y = 3$. (20)

6. Determine the poles and zeros for the transfer function: $F(s) = \dfrac{(s+2)(s-3)}{(s+3)(s^2+2s+5)}$ and plot them on a pole-zero diagram. (10)

Chapter 66

Fourier series for periodic functions of period 2π

66.1 Introduction

Fourier series provides a method of analysing periodic functions into their constituent components. Alternating currents and voltages, displacement, velocity and acceleration of slider-crank mechanisms and acoustic waves are typical practical examples in engineering and science where periodic functions are involved and often requiring analysis.

66.2 Periodic functions

A function $f(x)$ is said to be **periodic** if $f(x+T)=f(x)$ for all values of x, where T is some positive number. T is the interval between two successive repetitions and is called the **period** of the functions $f(x)$. For example, $y=\sin x$ is periodic in x with period 2π since $\sin x=\sin(x+2\pi)=\sin(x+4\pi)$, and so on. In general, if $y=\sin\omega t$ then the period of the waveform is $2\pi/\omega$. The function shown in Fig. 66.1 is also periodic of period 2π and is defined by:

$$f(x)=\begin{cases}-1, & \text{when } -\pi<x<0\\ 1, & \text{when } 0<x<\pi\end{cases}$$

If a graph of a function has no sudden jumps or breaks it is called a **continuous function**, examples being the graphs of sine and cosine functions. However, other graphs make finite jumps at a point or points in the interval. The square wave shown in Fig. 66.1 has finite discontinuities at $x=\pi$, 2π, 3π, and so on. A great advantage of Fourier series over other series is that it can be applied to functions which are discontinuous as well as those which are continuous.

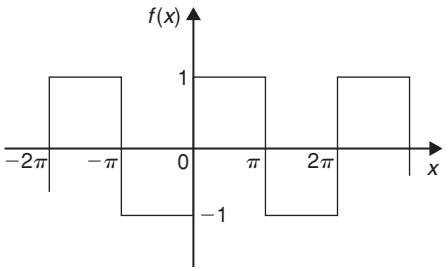


Figure 66.1

66.3 Fourier series

(i) The basis of a Fourier series is that all functions of practical significance which are defined in the interval $-\pi\le x\le\pi$ can be expressed in terms of a convergent trigonometric series of the form:

$$f(x)=a_0+a_1\cos x+a_2\cos 2x+a_3\cos 3x+\cdots+b_1\sin x+b_2\sin 2x+b_3\sin 3x+\cdots$$

when $a_0, a_1, a_2, \ldots b_1, b_2, \ldots$ are real constants, i.e.

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad (1)$$

where for the range $-\pi$ to π:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$(n = 1, 2, 3, \ldots)$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$(n = 1, 2, 3, \ldots)$$

(ii) a_0, a_n and b_n are called the **Fourier coefficients** of the series and if these can be determined, the series of equation (1) is called the **Fourier series** corresponding to $f(x)$.

(iii) An alternative way of writing the series is by using the $a\cos x + b\sin x = c\sin(x+\alpha)$ relationship introduced in Chapter 17, i.e.

$$f(x) = a_0 + c_1\sin(x+\alpha_1) + c_2\sin(2x+\alpha_2) + \cdots + c_n\sin(nx+\alpha_n),$$

where a_0 is a constant,

$$c_1 = \sqrt{(a_1^2 + b_1^2)}, \ldots c_n = \sqrt{(a_n^2 + b_n^2)}$$

are the amplitudes of the various components, and phase angle

$$\alpha_n = \tan^{-1}\frac{a_n}{b_n}$$

(iv) For the series of equation (1): the term $(a_1\cos x + b_1\sin x)$ or $c_1\sin(x+\alpha_1)$ is called the **first harmonic** or the **fundamental**, the term $(a_2\cos 2x + b_2\sin 2x)$ or $c_2\sin(2x+\alpha_2)$ is called the **second harmonic**, and so on.

For an exact representation of a complex wave, an infinite number of terms are, in general, required. In many practical cases, however, it is sufficient to take the first few terms only (see Problem 2).

The sum of a Fourier series at a point of discontinuity is given by the arithmetic mean of the two limiting values of $f(x)$ as x approaches the point of discontinuity from the two sides. For example, for the waveform shown in Fig. 66.2, the sum of the Fourier series at the points of discontinuity (i.e. at $\frac{\pi}{2}, \pi, \ldots$ is given by:

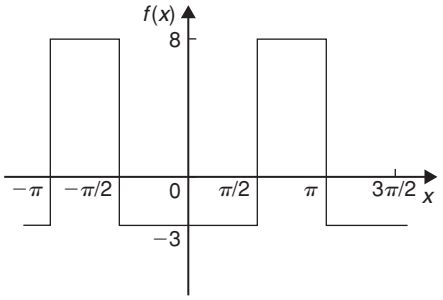


Figure 66.2

$$\frac{8 + (-3)}{2} = \frac{5}{2} \text{ or } 2\frac{1}{2}$$

66.4 Worked problems on Fourier series of periodic functions of period 2π

Problem 1. Obtain a Fourier series for the periodic function $f(x)$ defined as:

$$f(x) = \begin{cases} -k, & \text{when } -\pi < x < 0 \\ +k, & \text{when } 0 < x < \pi \end{cases}$$

The function is periodic outside of this range with period 2π.

The square wave function defined is shown in Fig. 66.3. Since $f(x)$ is given by two different expressions in the two halves of the range the integration is performed in two parts, one from $-\pi$ to 0 and the other from 0 to π.

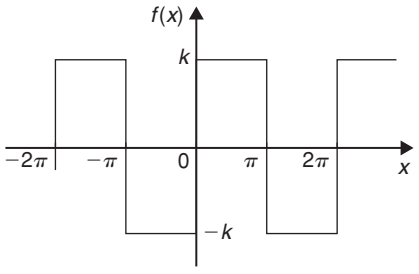


Figure 66.3

From Section 66.3(i):

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x$$
$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} -k\,\mathrm{d}x + \int_{0}^{\pi} k\,\mathrm{d}x\right]$$
$$= \frac{1}{2\pi}\{[-kx]_{-\pi}^{0} + [kx]_{0}^{\pi}\} = 0$$

[a_0 is in fact the **mean value** of the waveform over a complete period of 2π and this could have been deduced on sight from Fig. 66.3.]

From Section 66.3(i):

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x$$
$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} -k\cos nx\,\mathrm{d}x + \int_{0}^{\pi} k\cos nx\,\mathrm{d}x\right\}$$
$$= \frac{1}{\pi}\left\{\left[\frac{-k\sin nx}{n}\right]_{-\pi}^{0} + \left[\frac{k\sin nx}{n}\right]_{0}^{\pi}\right\} = 0$$

Hence a_1, a_2, a_3, ... are all zero (since $\sin 0 = \sin(-n\pi) = \sin n\pi = 0$), and therefore no cosine terms will appear in the Fourier series.

From Section 66.3(i):

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x$$
$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} -k\sin nx\,\mathrm{d}x + \int_{0}^{\pi} k\sin nx\,\mathrm{d}x\right\}$$
$$= \frac{1}{\pi}\left\{\left[\frac{k\cos nx}{n}\right]_{-\pi}^{0} + \left[\frac{-k\cos nx}{n}\right]_{0}^{\pi}\right\}$$

When n is odd:

$$b_n = \frac{k}{\pi}\left\{\left[\left(\frac{1}{n}\right) - \left(-\frac{1}{n}\right)\right] + \left[-\left(-\frac{1}{n}\right) - \left(-\frac{1}{n}\right)\right]\right\}$$
$$= \frac{k}{\pi}\left\{\frac{2}{n} + \frac{2}{n}\right\} = \frac{4k}{n\pi}$$

Hence $b_1 = \frac{4k}{\pi}$, $b_3 = \frac{4k}{3\pi}$, $b_5 = \frac{4k}{5\pi}$, and so on.

When n is even:

$$b_n = \frac{k}{\pi}\left\{\left[\frac{1}{n} - \frac{1}{n}\right] + \left[-\frac{1}{n} - \left(-\frac{1}{n}\right)\right]\right\} = 0$$

Hence, from equation (1), the Fourier series for the function shown in Fig. 66.3 is given by:

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$
$$= 0 + \sum_{n=1}^{\infty}(0 + b_n\sin nx)$$

i.e. $$f(x) = \frac{4k}{\pi}\sin x + \frac{4k}{3\pi}\sin 3x + \frac{4k}{5\pi}\sin 5x + \cdots$$

i.e. $$\boldsymbol{f(x) = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)}$$

Problem 2. For the Fourier series of Problem 1 let $k = \pi$. Show by plotting the first three partial sums of this Fourier series that as the series is added together term by term the result approximates more and more closely to the function it represents.

If $k = \pi$ in the Fourier series of Problem 1 then:

$$f(x) = 4(\sin x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x + \cdots)$$

$4\sin x$ is termed the first partial sum of the Fourier series of $f(x)$, $(4\sin x + \frac{4}{3}\sin 3x)$ is termed the second partial sum of the Fourier series, and $(4\sin x + \frac{4}{3}\sin 3x + \frac{4}{5}\sin 5x)$ is termed the third partial sum, and so on.

Let $P_1 = 4\sin x$,

$P_2 = \left(4\sin x + \frac{4}{3}\sin 3x\right)$

and $P_3 = \left(4\sin x + \frac{4}{3}\sin 3x + \frac{4}{5}\sin 5x\right)$.

Graphs of P_1, P_2 and P_3, obtained by drawing up tables of values, and adding waveforms, are shown in Figs. 66.4(a) to (c) and they show that the series is convergent, i.e. continually approximating towards a definite limit as more and more partial sums are taken, and in the limit will have the sum $f(x) = \pi$.

Even with just three partial sums, the waveform is starting to approach the rectangular wave the Fourier series is representing.

Problem 3. If in the Fourier series of Problem 1, $k = 1$, deduce a series for $\frac{\pi}{4}$ at the point $x = \frac{\pi}{2}$

If $k = 1$ in the Fourier series of Problem 1:

$$f(x) = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$$

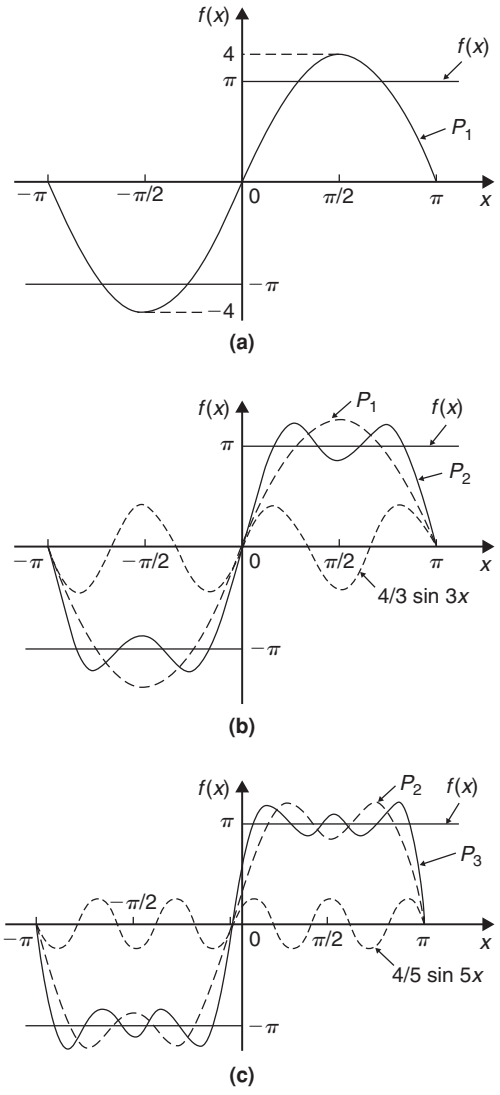


Figure 66.4

When $x = \frac{\pi}{2}$, $f(x) = 1$,

$$\sin x = \sin\frac{\pi}{2} = 1,$$

$$\sin 3x = \sin\frac{3\pi}{2} = -1,$$

$$\sin 5x = \sin\frac{5\pi}{2} = 1, \text{ and so on.}$$

Hence $1 = \frac{4}{\pi}\left[1 + \frac{1}{3}(-1) + \frac{1}{5}(1) + \frac{1}{7}(-1) + \cdots\right]$

i.e. $$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Problem 4. Determine the Fourier series for the full wave rectified sine wave $i = 5\sin\frac{\theta}{2}$ shown in Fig. 66.5.

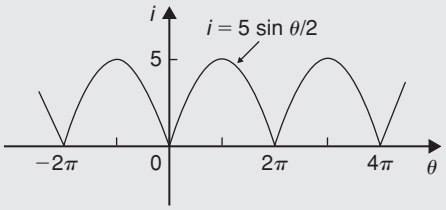


Figure 66.5

$i = 5\sin\frac{\theta}{2}$ is a periodic function of period 2π. Thus

$$i = f(\theta) = a_0 + \sum_{n=1}^{\infty}(a_n\cos n\theta + b_n\sin n\theta)$$

In this case it is better to take the range 0 to 2π instead of $-\pi$ to $+\pi$ since the waveform is continuous between 0 and 2π.

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} 5\sin\frac{\theta}{2}\,\mathrm{d}\theta$$

$$= \frac{5}{2\pi}\left[-2\cos\frac{\theta}{2}\right]_0^{2\pi}$$

$$= \frac{5}{\pi}\left[\left(-\cos\frac{2\pi}{2}\right) - (-\cos 0)\right]$$

$$= \frac{5}{\pi}[(1) - (-1)] = \frac{10}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} 5\sin\frac{\theta}{2}\cos n\theta\,\mathrm{d}\theta$$

$$= \frac{5}{\pi}\int_0^{2\pi}\frac{1}{2}\left\{\sin\left(\frac{\theta}{2} + n\theta\right) + \sin\left(\frac{\theta}{2} - n\theta\right)\right\}\mathrm{d}\theta$$

(see Chapter 40, page 401)

$$= \frac{5}{2\pi}\left[\frac{-\cos\left[\theta\left(\frac{1}{2}+n\right)\right]}{\left(\frac{1}{2}+n\right)} - \frac{\cos\left[\theta\left(\frac{1}{2}-n\right)\right]}{\left(\frac{1}{2}-n\right)}\right]_0^{2\pi}$$

$$= \frac{5}{2\pi}\left\{\left[\frac{-\cos\left[2\pi\left(\frac{1}{2}+n\right)\right]}{\left(\frac{1}{2}+n\right)} - \frac{\cos\left[2\pi\left(\frac{1}{2}-n\right)\right]}{\left(\frac{1}{2}-n\right)}\right] - \left[\frac{-\cos 0}{\left(\frac{1}{2}+n\right)} - \frac{\cos 0}{\left(\frac{1}{2}-n\right)}\right]\right\}$$

When n is both odd and even,

$$a_n = \frac{5}{2\pi}\left\{\left[\frac{1}{\left(\frac{1}{2}+n\right)} + \frac{1}{\left(\frac{1}{2}-n\right)}\right] - \left[\frac{-1}{\left(\frac{1}{2}+n\right)} - \frac{1}{\left(\frac{1}{2}-n\right)}\right]\right\}$$

$$= \frac{5}{2\pi}\left\{\frac{2}{\left(\frac{1}{2}+n\right)} + \frac{2}{\left(\frac{1}{2}-n\right)}\right\}$$

$$= \frac{5}{\pi}\left\{\frac{1}{\left(\frac{1}{2}+n\right)} + \frac{1}{\left(\frac{1}{2}-n\right)}\right\}$$

Hence

$$a_1 = \frac{5}{\pi}\left[\frac{1}{\frac{3}{2}} + \frac{1}{-\frac{1}{2}}\right] = \frac{5}{\pi}\left[\frac{2}{3} - \frac{2}{1}\right] = \frac{-20}{3\pi}$$

$$a_2 = \frac{5}{\pi}\left[\frac{1}{\frac{5}{2}} + \frac{1}{-\frac{3}{2}}\right] = \frac{5}{\pi}\left[\frac{2}{5} - \frac{2}{3}\right] = \frac{-20}{(3)(5)\pi}$$

$$a_3 = \frac{5}{\pi}\left[\frac{1}{\frac{7}{2}} + \frac{1}{-\frac{5}{2}}\right] = \frac{5}{\pi}\left[\frac{2}{7} - \frac{2}{5}\right] = \frac{-20}{(5)(7)\pi}$$

and so on

$$b_n = \frac{1}{\pi}\int_0^{2\pi} 5\sin\frac{\theta}{2}\sin n\theta\,\mathrm{d}\theta$$

$$= \frac{5}{\pi}\int_0^{2\pi} -\frac{1}{2}\left\{\cos\left[\theta\left(\frac{1}{2}+n\right)\right] - \cos\left[\theta\left(\frac{1}{2}-n\right)\right]\right\}\mathrm{d}\theta$$

from Chapter 40

$$= \frac{5}{2\pi}\left[\frac{\sin\left[\theta\left(\frac{1}{2}-n\right)\right]}{\left(\frac{1}{2}-n\right)} - \frac{\sin\left[\theta\left(\frac{1}{2}+n\right)\right]}{\left(\frac{1}{2}+n\right)}\right]_0^{2\pi}$$

$$= \frac{5}{2\pi}\left\{\left[\frac{\sin 2\pi\left(\frac{1}{2}-n\right)}{\left(\frac{1}{2}-n\right)} - \frac{\sin 2\pi\left(\frac{1}{2}+n\right)}{\left(\frac{1}{2}+n\right)}\right] - \left[\frac{\sin 0}{\left(\frac{1}{2}-n\right)} - \frac{\sin 0}{\left(\frac{1}{2}+n\right)}\right]\right\}$$

When n is both odd and even, $b_n = 0$ since $\sin(-\pi)$, $\sin 0$, $\sin\pi$, $\sin 3\pi$, ... are all zero. Hence the Fourier series for the rectified sine wave, $i = 5\sin\frac{\theta}{2}$ is given by:

$$f(\theta) = a_0 + \sum_{n=1}^{\infty}(a_n\cos n\theta + b_n\sin n\theta)$$

$$\text{i.e.}\quad i = f(\theta) = \frac{10}{\pi} - \frac{20}{3\pi}\cos\theta - \frac{20}{(3)(5)\pi}\cos 2\theta - \frac{20}{(5)(7)\pi}\cos 3\theta - \cdots$$

$$\text{i.e.}\quad \boldsymbol{i = \frac{20}{\pi}\left(\frac{1}{2} - \frac{\cos\theta}{(3)} - \frac{\cos 2\theta}{(3)(5)} - \frac{\cos 3\theta}{(5)(7)} - \cdots\right)}$$

Now try the following exercise

Exercises 228 Further problems on Fourier series of periodic functions of period 2π

1. Determine the Fourier series for the periodic function:

$$f(x) = \begin{cases} -2, & \text{when } -\pi < x < 0 \\ +2, & \text{when } 0 < x < \pi \end{cases}$$

which is periodic outside this range of period 2π.

$$\left[f(x) = \frac{8}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)\right]$$

2. For the Fourier series in Problem 1, deduce a series for $\frac{\pi}{4}$ at the point where $x = \frac{\pi}{2}$

$$\left[\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right]$$

3. For the waveform shown in Fig. 66.6 determine (a) the Fourier series for the function and (b) the sum of the Fourier series at the points of discontinuity.

$$\left[\begin{array}{l}\text{(a) } f(x)=\dfrac{1}{2}+\dfrac{2}{\pi}\left(\cos x-\dfrac{1}{3}\cos 3x\right.\\ \qquad\qquad\left.+\dfrac{1}{5}\cos 5x-\cdots\right)\\ \text{(b) } \dfrac{1}{2}\end{array}\right]$$

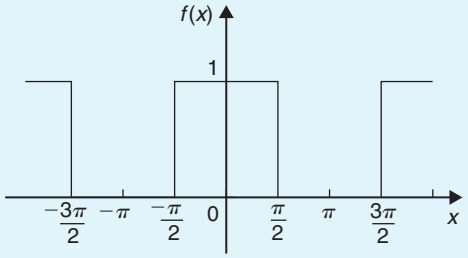


Figure 66.6

4. For Problem 3, draw graphs of the first three partial sums of the Fourier series and show that as the series is added together term by term the result approximates more and more closely to the function it represents.

5. Find the term representing the third harmonic for the periodic function of period 2π given by:

$$f(x)=\begin{cases}0, & \text{when } -\pi<x<0\\ 1, & \text{when } 0<x<\pi\end{cases}$$

$$\left[\frac{2}{3\pi}\sin 3x\right]$$

6. Determine the Fourier series for the periodic function of period 2π defined by:

$$f(t)=\begin{cases}0, & \text{when } -\pi<t<0\\ 1, & \text{when } 0<t<\dfrac{\pi}{2}\\ -1, & \text{when } \dfrac{\pi}{2}<t<\pi\end{cases}$$

The function has a period of 2π.

$$\left[f(t)=\frac{2}{\pi}\left(\begin{array}{l}\cos t-\dfrac{1}{3}\cos 3t\\ \quad+\dfrac{1}{5}\cos 5t-\cdots\\ \quad+\sin 2t+\dfrac{1}{3}\sin 6t\\ \quad+\dfrac{1}{5}\sin 10t+\cdots\end{array}\right)\right]$$

7. Show that the Fourier series for the periodic function of period 2π defined by

$$f(\theta)=\begin{cases}0, & \text{when } -\pi<\theta<0\\ \sin\theta, & \text{when } 0<\theta<\pi\end{cases}$$

is given by:

$$f(\theta)=\frac{2}{\pi}\left(\frac{1}{2}-\frac{\cos 2\theta}{(3)}-\frac{\cos 4\theta}{(3)(5)}-\frac{\cos 6\theta}{(5)(7)}-\cdots\right)$$

Chapter 67

Fourier series for a non-periodic function over range 2π

67.1 Expansion of non-periodic functions

If a function $f(x)$ is not periodic then it cannot be expanded in a Fourier series for **all** values of x. However, it is possible to determine a Fourier series to represent the function over any range of width 2π.

Given a non-periodic function, a new function may be constructed by taking the values of $f(x)$ in the given range and then repeating them outside of the given range at intervals of 2π. Since this new function is, by construction, periodic with period 2π, it may then be expanded in a Fourier series for all values of x. For example, the function $f(x)=x$ is not a periodic function. However, if a Fourier series for $f(x)=x$ is required then the function is constructed outside of this range so that it is periodic with period 2π as shown by the broken lines in Fig. 67.1.

For non-periodic functions, such as $f(x)=x$, the sum of the Fourier series is equal to $f(x)$ at all points in the given range but it is not equal to $f(x)$ at points outside of the range.

For determining a Fourier series of a non-periodic function over a range 2π, exactly the same formulae for the Fourier coefficients are used as in Section 66.3(i).

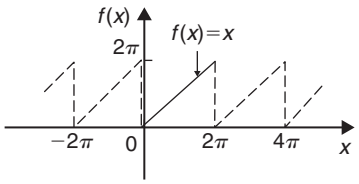


Figure 67.1

67.2 Worked problems on Fourier series of non-periodic functions over a range of 2π

Problem 1. Determine the Fourier series to represent the function $f(x)=2x$ in the range $-\pi$ to $+\pi$.

The function $f(x)=2x$ is not periodic. The function is shown in the range $-\pi$ to π in Fig. 67.2 and is then constructed outside of that range so that it is periodic of period 2π (see broken lines) with the resulting saw-tooth waveform.

For a Fourier series:

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

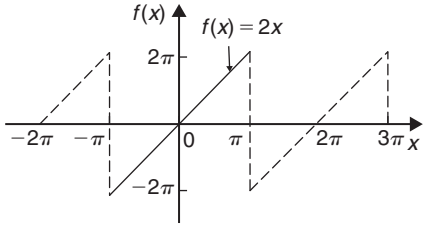


Figure 67.2

From Section 66.3(i),

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} 2x\,\mathrm{d}x = \frac{2}{2\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{\pi} = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x = \frac{1}{\pi}\int_{-\pi}^{\pi} 2x\cos nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\left[\frac{x\sin nx}{n} - \int \frac{\sin nx}{n}\,\mathrm{d}x\right]_{-\pi}^{\pi}$$

by parts (see Chapter 43)

$$= \frac{2}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_{-\pi}^{\pi}$$

$$= \frac{2}{\pi}\left[\left(0 + \frac{\cos n\pi}{n^2}\right) - \left(0 + \frac{\cos n(-\pi)}{n^2}\right)\right] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x = \frac{1}{\pi}\int_{-\pi}^{\pi} 2x\sin nx\,\mathrm{d}x$$

$$= \frac{2}{\pi}\left[\frac{-x\cos nx}{n} - \int\left(\frac{-\cos nx}{n}\right)\mathrm{d}x\right]_{-\pi}^{\pi}$$

by parts

$$= \frac{2}{\pi}\left[\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_{-\pi}^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\frac{-\pi\cos n\pi}{n} + \frac{\sin n\pi}{n^2}\right)\right.$$

$$\left. - \left(\frac{-(-\pi)\cos n(-\pi)}{n} + \frac{\sin n(-\pi)}{n^2}\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{-\pi\cos n\pi}{n} - \frac{\pi\cos(-n\pi)}{n}\right] = \frac{-4}{n}\cos n\pi$$

When n is odd, $b_n = \frac{4}{n}$. Thus $b_1 = 4$, $b_3 = \frac{4}{3}$, $b_5 = \frac{4}{5}$, and so on.

When n is even, $b_n = \frac{-4}{n}$. Thus $b_2 = -\frac{4}{2}$, $b_4 = -\frac{4}{4}$, $b_6 = -\frac{4}{6}$, and so on.

Thus $f(x) = 2x = 4\sin x - \frac{4}{2}\sin 2x + \frac{4}{3}\sin 3x - \frac{4}{4}\sin 4x + \frac{4}{5}\sin 5x - \frac{4}{6}\sin 6x + \cdots$

i.e. $$\mathbf{2x = 4\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \frac{1}{5}\sin 5x - \frac{1}{6}\sin 6x + \cdots\right)} \qquad (1)$$

for values of $f(x)$ between $-\pi$ and π. For values of $f(x)$ outside the range $-\pi$ to $+\pi$ the sum of the series is not equal to $f(x)$.

Problem 2. In the Fourier series of Problem 1, by letting $x = \pi/2$, deduce a series for $\pi/4$.

When $x = \pi/2$, $f(x) = \pi$ from Fig. 67.2.
Thus, from the Fourier series of equation (1):

$$2\left(\frac{\pi}{2}\right) = 4\left(\sin\frac{\pi}{2} - \frac{1}{2}\sin\frac{2\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{2} - \frac{1}{4}\sin\frac{4\pi}{2} + \frac{1}{5}\sin\frac{5\pi}{2} - \frac{1}{6}\sin\frac{6\pi}{2} + \cdots\right)$$

$$\pi = 4\left(1 - 0 - \frac{1}{3} - 0 + \frac{1}{5} - 0 - \frac{1}{7} - \cdots\right)$$

i.e. $$\mathbf{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots}$$

Problem 3. Obtain a Fourier series for the function defined by:

$$f(x) = \begin{cases} x, & \text{when } 0 < x < \pi \\ 0, & \text{when } \pi < x < 2\pi. \end{cases}$$

The defined function is shown in Fig. 67.3 between 0 and 2π. The function is constructed outside of this range so that it is periodic of period 2π, as shown by the broken line in Fig. 67.3.
For a Fourier series:

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

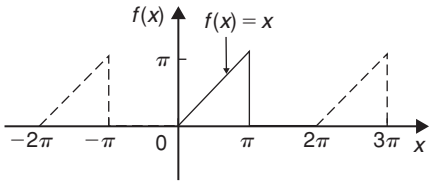


Figure 67.3

It is more convenient in this case to take the limits from 0 to 2π instead of from $-\pi$ to $+\pi$. The value of the Fourier coefficients are unaltered by this change of limits. Hence

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,\mathrm{d}x = \frac{1}{2\pi}\left[\int_0^{\pi} x\,\mathrm{d}x + \int_{\pi}^{2\pi} 0\,\mathrm{d}x\right]$$

$$= \frac{1}{2\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{1}{2\pi}\left(\frac{\pi^2}{2}\right) = \frac{\pi}{4}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\cos nx\,\mathrm{d}x + \int_{\pi}^{2\pi} 0\,\mathrm{d}x\right]$$

$$= \frac{1}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

(from Problem 1, by parts)

$$= \frac{1}{\pi}\left\{\left[\frac{\pi\sin n\pi}{n} + \frac{\cos n\pi}{n^2}\right] - \left[0 + \frac{\cos 0}{n^2}\right]\right\}$$

$$= \frac{1}{\pi n^2}(\cos n\pi - 1)$$

When n is even, $a_n = 0$.

When n is odd, $a_n = \dfrac{-2}{\pi n^2}$

Hence $a_1 = \dfrac{-2}{\pi}$, $a_3 = \dfrac{-2}{3^2\pi}$, $a_5 = \dfrac{-2}{5^2\pi}$, and so on

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\sin nx\,\mathrm{d}x - \int_{\pi}^{2\pi} 0\,\mathrm{d}x\right]$$

$$= \frac{1}{\pi}\left[\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

(from Problem 1, by parts)

$$= \frac{1}{\pi}\left\{\left[\frac{-\pi\cos n\pi}{n} + \frac{\sin n\pi}{n^2}\right] - \left[0 + \frac{\sin 0}{n^2}\right]\right\}$$

$$= \frac{1}{\pi}\left[\frac{-\pi\cos n\pi}{n}\right] = \frac{-\cos n\pi}{n}$$

Hence $b_1 = -\cos\pi = 1$, $b_2 = -\dfrac{1}{2}$, $b_3 = \dfrac{1}{3}$, and so on.

Thus the Fourier series is:

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

i.e. $$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\cos x - \frac{2}{3^2\pi}\cos 3x - \frac{2}{5^2\pi}\cos 5x - \cdots + \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots$$

i.e. $\boldsymbol{f(x)}$

$$\boldsymbol{= \frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right) + \left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right)}$$

Problem 4. For the Fourier series of Problem 3: (a) what is the sum of the series at the point of discontinuity (i.e. at $x = \pi$)? (b) what is the amplitude and phase angle of the third harmonic? and (c) let $x = 0$, and deduce a series for $\pi^2/8$.

(a) The sum of the Fourier series at the point of discontinuity is given by the arithmetic mean of the two limiting values of $f(x)$ as x approaches the point of discontinuity from the two sides.

Hence sum of the series at $x = \pi$ is

$$\frac{\pi - 0}{2} = \boldsymbol{\frac{\pi}{2}}$$

(b) The third harmonic term of the Fourier series is

$$\left(-\frac{2}{3^2\pi}\cos 3x + \frac{1}{3}\sin 3x\right)$$

This may also be written in the form $c\sin(3x + \alpha)$,

where amplitude, $c = \sqrt{\left[\left(\dfrac{-2}{3^2\pi}\right)^2 + \left(\dfrac{1}{3}\right)^2\right]}$

$$= \mathbf{0.341}$$

and phase angle,

$$\alpha = \tan^{-1}\left(\frac{\frac{-2}{3^2\pi}}{\frac{1}{3}}\right)$$

$$= \mathbf{-11.98^\circ} \quad \text{or} \quad \mathbf{-0.209\ radians}$$

Hence the third harmonic is given by

$$\mathbf{0.341\sin(3x - 0.209)}$$

(c) When $x = 0$, $f(x) = 0$ (see Fig. 67.3).

Hence, from the Fourier series:

$$0 = \frac{\pi}{4} - \frac{2}{\pi}\left(\cos 0 + \frac{1}{3^2}\cos 0 + \frac{1}{5^2}\cos 0 + \cdots\right) + (0)$$

$$\text{i.e.} \quad -\frac{\pi}{4} = -\frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right)$$

$$\textbf{Hence} \quad \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

Problem 5. Deduce the Fourier series for the function $f(\theta) = \theta^2$ in the range 0 to 2π.

$f(\theta) = \theta^2$ is shown in Fig. 67.4 in the range 0 to 2π. The function is not periodic but is constructed outside of this range so that it is periodic of period 2π, as shown by the broken lines.

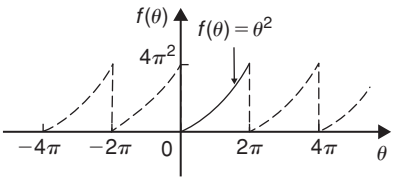


Figure 67.4

For a Fourier series:

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} \theta^2\,\mathrm{d}\theta$$

$$= \frac{1}{2\pi}\left[\frac{\theta^3}{3}\right]_0^{2\pi} = \frac{1}{2\pi}\left[\frac{8\pi^3}{3} - 0\right] = \frac{4\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\,\mathrm{d}\theta$$

$$= \frac{1}{\pi}\int_0^{2\pi} \theta^2 \cos n\theta\,\mathrm{d}\theta$$

$$= \frac{1}{\pi}\left[\frac{\theta^2 \sin n\theta}{n} + \frac{2\theta\cos n\theta}{n^2} - \frac{2\sin n\theta}{n^3}\right]_0^{2\pi}$$

by parts

$$= \frac{1}{\pi}\left[\left(0 + \frac{4\pi\cos 2\pi n}{n^2} - 0\right) - (0)\right]$$

$$= \frac{4}{n^2}\cos 2\pi n = \frac{4}{n^2} \text{ when } n = 1, 2, 3, \ldots$$

Hence $a_1 = \frac{4}{1^2}$, $a_2 = \frac{4}{2^2}$, $a_3 = \frac{4}{3^2}$ and so on

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\,\mathrm{d}\theta = \frac{1}{\pi}\int_0^{2\pi} \theta^2 \sin n\theta\,\mathrm{d}\theta$$

$$= \frac{1}{\pi}\left[\frac{-\theta^2\cos n\theta}{n} + \frac{2\theta\sin n\theta}{n^2} + \frac{2\cos n\theta}{n^3}\right]_0^{2\pi}$$

by parts

$$= \frac{1}{\pi}\left[\left(\frac{-4\pi^2\cos 2\pi n}{n} + 0 + \frac{2\cos 2\pi n}{n^3}\right) - \left(0 + 0 + \frac{2\cos 0}{n^3}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{-4\pi^2}{n} + \frac{2}{n^3} - \frac{2}{n^3}\right] = \frac{-4\pi}{n}$$

Hence $b_1 = \frac{-4\pi}{1}$, $b_2 = \frac{-4\pi}{2}$, $b_3 = \frac{-4\pi}{3}$, and so on.

Thus $f(\theta) = \theta^2$

$$= \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos n\theta - \frac{4\pi}{n}\sin n\theta\right)$$

i.e. $\boldsymbol{\theta^2 =}$

$$\mathbf{\frac{4\pi^2}{3} + 4\left(\cos\theta + \frac{1}{2^2}\cos 2\theta + \frac{1}{3^2}\cos 3\theta + \cdots\right) - 4\pi\left(\sin\theta + \frac{1}{2}\sin 2\theta + \frac{1}{3}\sin 3\theta + \cdots\right)}$$

for values of θ between 0 and 2π.

Problem 6. In the Fourier series of Problem 5, let $\theta = \pi$ and determine a series for $\frac{\pi^2}{12}$

When $\theta=\pi$, $f(\theta)=\pi^2$

Hence $$\pi^2=\frac{4\pi^2}{3}+4\left(\cos\pi+\frac{1}{4}\cos 2\pi+\frac{1}{9}\cos 3\pi+\frac{1}{16}\cos 4\pi+\cdots\right)-4\pi\left(\sin\pi+\frac{1}{2}\sin 2\pi+\frac{1}{3}\sin 3\pi+\cdots\right)$$

i.e. $$\pi^2-\frac{4\pi^2}{3}=4\left(-1+\frac{1}{4}-\frac{1}{9}+\frac{1}{16}-\cdots\right)-4\pi(0)$$

$$-\frac{\pi^2}{3}=4\left(-1+\frac{1}{4}-\frac{1}{9}+\frac{1}{16}-\cdots\right)$$

$$\frac{\pi^2}{3}=4\left(1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\cdots\right)$$

Hence $$\boldsymbol{\frac{\pi^2}{12}=1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\cdots}$$

or $$\boldsymbol{\frac{\pi^2}{12}=1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots}$$

Now try the following exercise

Exercise 229 Further problems on Fourier series of non-periodic functions over a range of 2π

1. Show that the Fourier series for the function $f(x)=x$ over the range $x=0$ to $x=2\pi$ is given by:
$$f(x)=\pi-2\left(\sin x+\tfrac{1}{2}\sin 2x+\tfrac{1}{3}\sin 3x+\tfrac{1}{4}\sin 4x+\cdots\right)$$

2. Determine the Fourier series for the function defined by:
$$f(t)=\begin{cases}1-t, & \text{when } -\pi<t<0\\ 1+t, & \text{when } 0<t<\pi\end{cases}$$
Draw a graph of the function within and outside of the given range.
$$\left[f(t)=\frac{\pi}{2}+1-\frac{4}{\pi}\left(\cos t+\frac{\cos 3t}{3^2}+\frac{\cos 5t}{5^2}+\cdots\right)\right]$$

3. Find the Fourier series for the function $f(x)=x+\pi$ within the range $-\pi<x<\pi$.
$$\left[f(x)=\pi+2\left(\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-\cdots\right)\right]$$

4. Determine the Fourier series up to and including the third harmonic for the function defined by:
$$f(x)=\begin{cases}x, & \text{when } 0<x<\pi\\ 2\pi-x, & \text{when } \pi<x<2\pi\end{cases}$$
Sketch a graph of the function within and outside of the given range, assuming the period is 2π.
$$\left[f(x)=\frac{\pi}{2}-\frac{4}{\pi}\left(\cos x+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\cdots\right)\right]$$

5. Expand the function $f(\theta)=\theta^2$ in a Fourier series in the range $-\pi<\theta<\pi$.
Sketch the function within and outside of the given range.
$$\left[f(\theta)=\frac{\pi^2}{3}-4\left(\cos\theta-\frac{1}{2^2}\cos 2\theta+\frac{1}{3^2}\cos 3\theta-\cdots\right)\right]$$

6. For the Fourier series obtained in Problem 5, let $\theta=\pi$ and deduce the series for $\sum_{n=1}^{\infty}\frac{1}{n^2}$
$$\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{6}\right]$$

7. Show that the Fourier series for the triangular waveform shown in Fig. 67.5 is given by:
$$y=\frac{8}{\pi^2}\left(\sin\theta-\frac{1}{3^2}\sin 3\theta+\frac{1}{5^2}\sin 5\theta-\frac{1}{7^2}\sin 7\theta+\cdots\right)$$

in the range 0 to 2π.

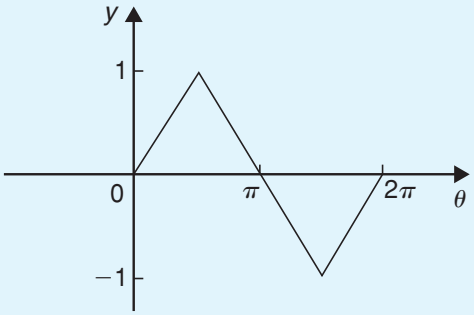


Figure 67.5

8. Sketch the waveform defined by:

$$f(x) = \begin{cases} 1 + \dfrac{2x}{\pi}, & \text{when } -\pi < x < 0 \\ 1 - \dfrac{2x}{\pi}, & \text{when } 0 < x < \pi \end{cases}$$

Determine the Fourier series in this range.

$$\left[f(x) = \frac{8}{\pi^2}\left(\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \frac{1}{7^2}\cos 7x + \cdots\right)\right]$$

9. For the Fourier series of Problem 8, deduce a series for $\dfrac{\pi^2}{8}$

$$\left[\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots\right]$$

Chapter 68

Even and odd functions and half-range Fourier series

68.1 Even and odd functions

Even functions

A function $y = f(x)$ is said to be **even** if $f(-x) = f(x)$ for all values of x. Graphs of even functions are always **symmetrical about the *y*-axis** (i.e. is a mirror image). Two examples of even functions are $y = x^2$ and $y = \cos x$ as shown in Fig. 18.25, page 186.

Odd functions

A function $y = f(x)$ is said to be **odd** if $f(-x) = -f(x)$ for all values of x. Graphs of odd functions are always **symmetrical about the origin**. Two examples of odd functions are $y = x^3$ and $y = \sin x$ as shown in Fig. 18.26, page 187.
Many functions are neither even nor odd, two such examples being shown in Fig. 18.27, page 187.
See also Problems 3 and 4, page 187.

68.2 Fourier cosine and Fourier sine series

(a) Fourier cosine series

The Fourier series of an **even periodic** function $f(x)$ having period 2π contains **cosine terms only** (i.e. contains no sine terms) and may contain a constant term.

Hence $\boldsymbol{f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx}$

where $\boldsymbol{a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx}$

$\boldsymbol{= \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx}$

(due to symmetry)

and $\boldsymbol{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx}$

$\boldsymbol{= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx}$

(b) Fourier sine series

The Fourier series of an **odd** periodic function $f(x)$ having period 2π contains sine terms only (i.e. contains no constant term and no cosine terms).

Hence $\boldsymbol{f(x) = \sum_{n=1}^{\infty} b_n \sin nx}$

where $\boldsymbol{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx}$

$\boldsymbol{= \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx}$

Problem 1. Determine the Fourier series for the periodic function defined by:

$$f(x)=\begin{cases}-2, & \text{when } -\pi<x<-\dfrac{\pi}{2}\\ 2, & \text{when } -\dfrac{\pi}{2}<x<\dfrac{\pi}{2}\\ -2, & \text{when } \dfrac{\pi}{2}<x<\pi.\end{cases}$$

and has a period of 2π.

The square wave shown in Fig. 68.1 is an even function since it is symmetrical about the $f(x)$ axis.

Hence from para. (a) the Fourier series is given by:

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos nx$$

(i.e. the series contains no sine terms.)

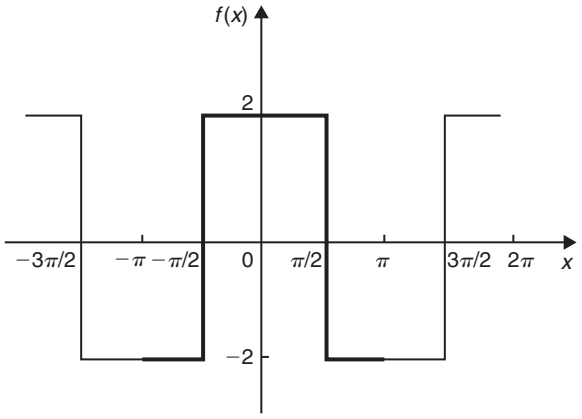


Figure 68.1

From para. (a),

$$a_0=\frac{1}{\pi}\int_0^{\pi}f(x)\,dx$$

$$=\frac{1}{\pi}\left\{\int_0^{\pi/2}2\,dx+\int_{\pi/2}^{\pi}-2\,dx\right\}$$

$$=\frac{1}{\pi}\left\{[2x]_0^{\pi/2}+[-2x]_{\pi/2}^{\pi}\right\}$$

$$=\frac{1}{\pi}[(\pi)+[(-2\pi)-(-\pi)]=0$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nx\,dx$$

$$=\frac{2}{\pi}\left\{\int_0^{\pi/2}2\cos nx\,dx+\int_{\pi/2}^{\pi}-2\cos nx\,dx\right\}$$

$$=\frac{4}{\pi}\left\{\left[\frac{\sin nx}{n}\right]_0^{\pi/2}+\left[\frac{-\sin nx}{n}\right]_{\pi/2}^{\pi}\right\}$$

$$=\frac{4}{\pi}\left\{\left(\frac{\sin(\pi/2)n}{n}-0\right)+\left(0-\frac{-\sin(\pi/2)n}{n}\right)\right\}$$

$$=\frac{4}{\pi}\left(\frac{2\sin(\pi/2)n}{n}\right)=\frac{8}{\pi n}\left(\sin\frac{n\pi}{2}\right)$$

When n is even, $a_n=0$

When n is odd, $a_n=\dfrac{8}{\pi n}$ for $n=1,5,9,\ldots$

and $a_n=\dfrac{-8}{\pi n}$ for $n=3,7,11,\ldots$

Hence $a_1=\dfrac{8}{\pi}$, $a_3=\dfrac{-8}{3\pi}$, $a_5=\dfrac{8}{5\pi}$, and so on.

Hence the Fourier series for the waveform of Fig. 68.1 is given by:

$$\boldsymbol{f(x)=\frac{8}{\pi}\left(\cos x-\frac{1}{3}\cos 3x+\frac{1}{5}\cos 5x-\frac{1}{7}\cos 7x+\cdots\right)}$$

Problem 2. In the Fourier series of Problem 1 let $x=0$ and deduce a series for $\pi/4$.

When $x=0$, $f(x)=2$ (from Fig. 68.1).

Thus, from the Fourier series,

$$2=\frac{8}{\pi}\left(\cos 0-\frac{1}{3}\cos 0+\frac{1}{5}\cos 0-\frac{1}{7}\cos 0+\cdots\right)$$

Hence $\dfrac{2\pi}{8}=1-\dfrac{1}{3}+\dfrac{1}{5}-\dfrac{1}{7}+\cdots$

i.e. $\boldsymbol{\dfrac{\pi}{4}=1-\dfrac{1}{3}+\dfrac{1}{5}-\dfrac{1}{7}+\cdots}$

Problem 3. Obtain the Fourier series for the square wave shown in Fig. 68.2.

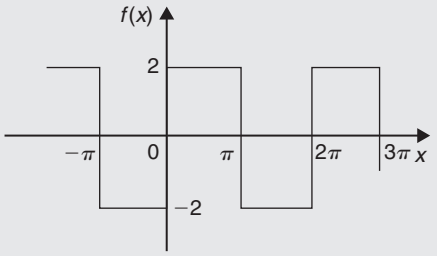


Figure 68.2

The square wave shown in Fig. 68.2 is an odd function since it is symmetrical about the origin.

Hence, from para. (b), the Fourier series is given by:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

The function is defined by:

$$f(x) = \begin{cases} -2, & \text{when } -\pi < x < 0 \\ 2, & \text{when } 0 < x < \pi \end{cases}$$

From para. (b), $b_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x)\sin nx\,\mathrm{d}x$

$$= \frac{2}{\pi}\int_0^{\pi} 2\sin nx\,\mathrm{d}x$$

$$= \frac{4}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi}$$

$$= \frac{4}{\pi}\left[\left(\frac{-\cos n\pi}{n}\right) - \left(-\frac{1}{n}\right)\right]$$

$$= \frac{4}{\pi n}(1 - \cos n\pi)$$

When n is even, $b_n = 0$.

When n is odd, $b_n = \dfrac{4}{\pi n}(1-(-1)) = \dfrac{8}{\pi n}$

Hence $b_1 = \dfrac{8}{\pi}$, $b_3 = \dfrac{8}{3\pi}$, $b_5 = \dfrac{8}{5\pi}$,

and so on

Hence the Fourier series is:

$$f(x) = \frac{8}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \cdots\right)$$

Problem 4. Determine the Fourier series for the function $f(\theta) = \theta^2$ in the range $-\pi < \theta < \pi$. The function has a period of 2π.

A graph of $f(\theta) = \theta^2$ is shown in Fig. 68.3 in the range $-\pi$ to π with period 2π. The function is symmetrical about the $f(\theta)$ axis and is thus an even function. Thus a Fourier cosine series will result of the form:

$$f(\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta$$

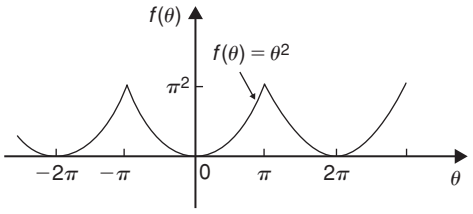


Figure 68.3

From para. (a),

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(\theta)\mathrm{d}\theta = \frac{1}{\pi}\int_0^{\pi} \theta^2\,\mathrm{d}\theta$$

$$= \frac{1}{\pi}\left[\frac{\theta^3}{3}\right]_0^{\pi} = \frac{\pi^2}{3}$$

and $a_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(\theta)\cos n\theta\,\mathrm{d}\theta$

$$= \frac{2}{\pi}\int_0^{\pi} \theta^2 \cos n\theta\,\mathrm{d}\theta$$

$$= \frac{2}{\pi}\left[\frac{\theta^2\sin n\theta}{n} + \frac{2\theta\cos n\theta}{n^2} - \frac{2\sin n\theta}{n^3}\right]_0^{\pi}$$

by parts

$$= \frac{2}{\pi}\left[\left(0 + \frac{2\pi\cos n\pi}{n^2} - 0\right) - (0)\right]$$

$$= \frac{4}{n^2}\cos n\pi$$

When n is odd, $a_n = \dfrac{-4}{n^2}$. Hence $a_1 = \dfrac{-4}{1^2}$, $a_3 = \dfrac{-4}{3^2}$, $a_5 = \dfrac{-4}{5^2}$, and so on.

When n is even, $a_n = \dfrac{4}{n^2}$. Hence $a_2 = \dfrac{4}{2^2}$, $a_4 = \dfrac{4}{4^2}$, and so on.

Hence the Fourier series is:

$$f(\theta)=\theta^2=\frac{\pi^2}{3}-4\left(\cos\theta-\frac{1}{2^2}\cos 2\theta+\frac{1}{3^2}\cos 3\theta-\frac{1}{4^2}\cos 4\theta+\frac{1}{5^2}\cos 5\theta-\cdots\right)$$

Problem 5. For the Fourier series of Problem 4, let $\theta=\pi$ and show that $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$

When $\theta=\pi$, $f(\theta)=\pi^2$ (see Fig. 68.3). Hence from the Fourier series:

$$\pi^2=\frac{\pi^2}{3}-4\left(\cos\pi-\frac{1}{2^2}\cos 2\pi+\frac{1}{3^2}\cos 3\pi-\frac{1}{4^2}\cos 4\pi+\frac{1}{5^2}\cos 5\pi-\cdots\right)$$

i.e.

$$\pi^2-\frac{\pi^2}{3}=-4\left(-1-\frac{1}{2^2}-\frac{1}{3^2}-\frac{1}{4^2}-\frac{1}{5^2}-\cdots\right)$$

$$\frac{2\pi^2}{3}=4\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots\right)$$

i.e. $$\frac{2\pi^2}{(3)(4)}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots$$

i.e. $$\frac{\pi^2}{6}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots$$

Hence

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

Now try the following exercise

Exercise 230 Further problems on Fourier cosine and Fourier sine series

1. Determine the Fourier series for the function defined by:

$$f(x)=\begin{cases}-1, & -\pi<x<-\frac{\pi}{2}\\ 1, & -\frac{\pi}{2}<x<\frac{\pi}{2}\\ -1, & \frac{\pi}{2}<x<\pi\end{cases}$$

which is periodic outside of this range of period 2π.

$$\left[f(x)=\frac{4}{\pi}\left(\cos x-\frac{1}{3}\cos 3x+\frac{1}{5}\cos 5x-\frac{1}{7}\cos 7x+\cdots\right)\right]$$

2. Obtain the Fourier series of the function defined by:

$$f(t)=\begin{cases}t+\pi, & -\pi<t<0\\ t-\pi, & 0<t<\pi\end{cases}$$

which is periodic of period 2π. Sketch the given function.

$$\left[f(t)=-2\left(\sin t+\tfrac{1}{2}\sin 2t+\tfrac{1}{3}\sin 3t+\tfrac{1}{4}\sin 4t+\cdots\right)\right]$$

3. Determine the Fourier series defined by

$$f(x)=\begin{cases}1-x, & -\pi<x<0\\ 1+x, & 0<x<\pi\end{cases}$$

which is periodic of period 2π.

$$\left[f(x)=\frac{\pi}{2}+1-\frac{4}{\pi}\left(\cos x+\frac{1}{3^2}\cos 3x+\frac{1}{5^2}\cos 5x+\cdots\right)\right]$$

4. In the Fourier series of Problem 3, let $x=0$ and deduce a series for $\pi^2/8$.

$$\left[\frac{\pi^2}{8}=1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots\right]$$

68.3 Half-range Fourier series

(a) When a function is defined over the range say 0 to π instead of from 0 to 2π it may be expanded in a series of sine terms only or of cosine terms only. The series produced is called a **half-range Fourier series**.

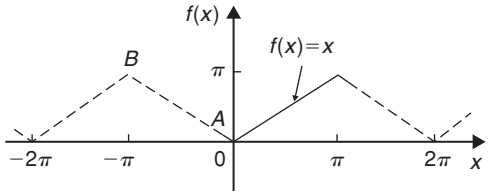


Figure 68.4

(b) If a **half-range cosine series** is required for the function $f(x)=x$ in the range 0 to π then an **even** periodic function is required. In Figure 68.4, $f(x)=x$ is shown plotted from $x=0$ to $x=\pi$. Since an even function is symmetrical about the $f(x)$ axis the line AB is constructed as shown. If the triangular waveform produced is assumed to be periodic of period 2π outside of this range then the waveform is as shown in Fig. 68.4. When a half-range cosine series is required then the Fourier coefficients a_0 and a_n are calculated as in Section 68.2(a), i.e.

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

where $$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$$

and $$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

(c) If a **half-range sine series** is required for the function $f(x)=x$ in the range 0 to π then an odd periodic function is required. In Figure 68.5, $f(x)=x$ is shown plotted from $x=0$ to $x=\pi$. Since an odd function is symmetrical about the origin the line CD is constructed as shown. If the sawtooth waveform produced is assumed to be periodic of period 2π outside of this range, then the waveform is as shown in Fig. 68.5. When a half-range sine series is required then the Fourier coefficient b_n is

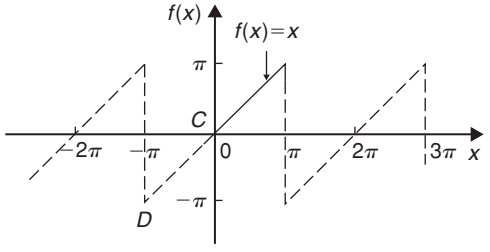


Figure 68.5

calculated as in Section 68.2(b), i.e.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where $$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Problem 6. Determine the half-range Fourier cosine series to represent the function $f(x)=3x$ in the range $0 \le x \le \pi$.

From para. (b), for a half-range cosine series:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

When $f(x)=3x$,

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} 3x\,dx$$

$$= \frac{3}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{3\pi}{2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} 3x\cos nx\,dx$$

$$= \frac{6}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi} \quad \text{by parts}$$

$$= \frac{6}{\pi}\left[\left(\frac{\pi\sin n\pi}{n} + \frac{\cos n\pi}{n^2}\right) - \left(0 + \frac{\cos 0}{n^2}\right)\right]$$

$$= \frac{6}{\pi}\left(0 + \frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right)$$

$$= \frac{6}{\pi n^2}(\cos n\pi - 1)$$

When n is even, $a_n = 0$

When n is odd, $a_n = \frac{6}{\pi n^2}(-1-1) = \frac{-12}{\pi n^2}$

Hence $a_1 = \frac{-12}{\pi}$, $a_3 = \frac{-12}{\pi 3^2}$, $a_5 = \frac{-12}{\pi 5^2}$, and so on.

Hence the half-range Fourier cosine series is given by:

$$f(x) = 3x = \frac{3\pi}{2} - \frac{12}{\pi}\left(\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right)$$

Problem 7. Find the half-range Fourier sine series to represent the function $f(x)=3x$ in the range $0 \le x \le \pi$.

From para. (c), for a half-range sine series:

$$f(x)=\sum_{n=1}^{\infty} b_n \sin nx$$

When $f(x)=3x$,

$$b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx=\frac{2}{\pi}\int_0^{\pi} 3x\sin nx\,dx$$

$$=\frac{6}{\pi}\left[\frac{-x\cos nx}{n}+\frac{\sin nx}{n^2}\right]_0^{\pi} \quad \text{by parts}$$

$$=\frac{6}{\pi}\left[\left(\frac{-\pi\cos n\pi}{n}+\frac{\sin n\pi}{n^2}\right)-(0+0)\right]$$

$$=-\frac{6}{n}\cos n\pi$$

When n is odd, $b_n=\dfrac{6}{n}$

Hence $b_1=\dfrac{6}{1}$, $b_3=\dfrac{6}{3}$, $b_5=\dfrac{6}{5}$ and so on.

When n is even, $b_n=-\dfrac{6}{n}$

Hence $b_2=-\dfrac{6}{2}$, $b_4=-\dfrac{6}{4}$, $b_6=-\dfrac{6}{6}$ and so on.

Hence the half-range Fourier sine series is given by:

$$\boldsymbol{f(x)=3x=6\left(\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-\frac{1}{4}\sin 4x+\frac{1}{5}\sin 5x-\cdots\right)}$$

Problem 8. Expand $f(x)=\cos x$ as a half-range Fourier sine series in the range $0 \le x \le \pi$, and sketch the function within and outside of the given range.

When a half-range sine series is required then an odd function is implied, i.e. a function symmetrical about the origin. A graph of $y=\cos x$ is shown in Fig. 68.6 in the range 0 to π. For $\cos x$ to be symmetrical about the origin the function is as shown by the broken lines in Fig. 68.6 outside of the given range.

From para. (c), for a half-range Fourier sine series:

$$f(x)=\sum_{n=1}^{\infty} b_n \sin nx\,dx$$

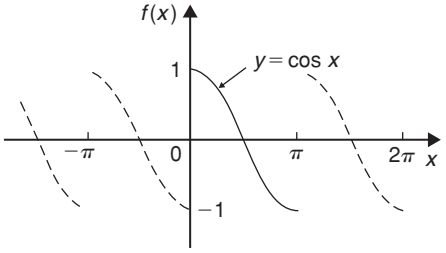


Figure 68.6

$$b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi}\cos x\sin nx\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi}\frac{1}{2}[\sin(x+nx)-\sin(x-nx)]\,dx$$

$$=\frac{1}{\pi}\left[\frac{-\cos[x(1+n)]}{(1+n)}+\frac{\cos[x(1-n)]}{(1-n)}\right]_0^{\pi}$$

$$=\frac{1}{\pi}\left[\left(\frac{-\cos[\pi(1+n)]}{(1+n)}+\frac{\cos[\pi(1-n)]}{(1-n)}\right)-\left(\frac{-\cos 0}{(1+n)}+\frac{\cos 0}{(1-n)}\right)\right]$$

When n is odd,

$$b_n=\frac{1}{\pi}\left[\left(\frac{-1}{(1+n)}+\frac{1}{(1-n)}\right)-\left(\frac{-1}{(1+n)}+\frac{1}{(1-n)}\right)\right]=0$$

When n is even,

$$b_n=\frac{1}{\pi}\left[\left(\frac{1}{(1+n)}-\frac{1}{(1-n)}\right)-\left(\frac{-1}{(1+n)}+\frac{1}{(1-n)}\right)\right]$$

$$=\frac{1}{\pi}\left(\frac{2}{(1+n)}-\frac{2}{(1-n)}\right)$$

$$=\frac{1}{\pi}\left(\frac{2(1-n)-2(1+n)}{1-n^2}\right)$$

$$=\frac{1}{\pi}\left(\frac{-4n}{1-n^2}\right)=\frac{4n}{\pi(n^2-1)}$$

Hence $b_2=\dfrac{8}{3\pi}$, $b_4=\dfrac{16}{15\pi}$, $b_6=\dfrac{24}{35\pi}$ and so on.

Hence the half-range Fourier sine series for $f(x)$ in the range 0 to π is given by:

$$f(x) = \frac{8}{3\pi}\sin 2x + \frac{16}{15\pi}\sin 4x + \frac{24}{35\pi}\sin 6x + \cdots$$

or $$\boldsymbol{f(x) = \frac{8}{\pi}\left(\frac{1}{3}\sin 2x + \frac{2}{(3)(5)}\sin 4x + \frac{3}{(5)(7)}\sin 6x + \cdots\right)}$$

Now try the following exercise

Exercise 231 Further problems on half-range Fourier series

1. Determine the half-range sine series for the function defined by:

$$f(x) = \begin{cases} x, & 0 < x < \dfrac{\pi}{2} \\ 0, & \dfrac{\pi}{2} < x < \pi \end{cases}$$

$$\left[f(x) = \frac{2}{\pi}\left(\sin x + \frac{\pi}{4}\sin 2x - \frac{1}{9}\sin 3x - \frac{\pi}{8}\sin 4x + \cdots\right) \right]$$

2. Obtain (a) the half-range cosine series and (b) the half-range sine series for the function

$$f(t) = \begin{cases} 0, & 0 < t < \dfrac{\pi}{2} \\ 1, & \dfrac{\pi}{2} < t < \pi \end{cases}$$

$$\left[\text{(a)} \quad f(t) = \frac{1}{2} - \frac{2}{\pi}\left(\cos t - \frac{1}{3}\cos 3t + \frac{1}{5}\cos 5t - \cdots\right) \right.$$

$$\left. \text{(b)} \quad f(t) = \frac{2}{\pi}\left(\sin t - \sin 2t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t - \frac{1}{3}\sin 6t + \cdots\right) \right]$$

3. Find (a) the half-range Fourier sine series and (b) the half-range Fourier cosine series for the function $f(x) = \sin^2 x$ in the range $0 \leq x \leq \pi$. Sketch the function within and outside of the given range.

$$\left[\text{(a)} \quad f(x) = \frac{8}{\pi}\left(\frac{\sin x}{(1)(3)} - \frac{\sin 3x}{(1)(3)(5)} - \frac{\sin 5x}{(3)(5)(7)} - \frac{\sin 7x}{(5)(7)(9)} - \cdots\right) \right.$$

$$\left. \text{(b)} \quad f(x) = \frac{1}{2}(1 - \cos 2x) \right]$$

4. Determine the half-range Fourier cosine series in the range $x = 0$ to $x = \pi$ for the function defined by:

$$f(x) = \begin{cases} x, & 0 < x < \dfrac{\pi}{2} \\ (\pi - x), & \dfrac{\pi}{2} < x < \pi \end{cases}$$

$$\left[f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\cos 2x + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \cdots\right) \right]$$

Chapter 69

Fourier series over any range

69.1 Expansion of a periodic function of period L

(a) A periodic function $f(x)$ of period L repeats itself when x increases by L, i.e. $f(x+L)=f(x)$. The change from functions dealt with previously having period 2π to functions having period L is not difficult since it may be achieved by a change of variable.

(b) To find a Fourier series for a function $f(x)$ in the range $-\frac{L}{2}\leq x\leq\frac{L}{2}$ a new variable u is introduced such that $f(x)$, as a function of u, has period 2π. If $u=\frac{2\pi x}{L}$ then, when $x=-\frac{L}{2}$, $u=-\pi$ and when $x=\frac{L}{2}$, $u=+\pi$. Also, let $f(x)=f\left(\frac{Lu}{2\pi}\right)=F(u)$. The Fourier series for $F(u)$ is given by:

$$F(u)=a_0+\sum_{n=1}^{\infty}(a_n\cos nu+b_n\sin nu),$$

where $a_0=\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}F(u)\,\mathrm{d}u,$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}F(u)\cos nu\,\mathrm{d}u$$

and $\quad b_n=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}F(u)\sin nu\,\mathrm{d}u$

(c) It is however more usual to change the formula of para. (b) to terms of x. Since $u=\frac{2\pi x}{L}$, then

$$\mathrm{d}u=\frac{2\pi}{L}\mathrm{d}x,$$

and the limits of integration are $-\frac{L}{2}$ to $+\frac{L}{2}$ instead of from $-\pi$ to $+\pi$. Hence the Fourier series expressed in terms of x is given by:

$$f(x)=a_0+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{2\pi nx}{L}\right)+b_n\sin\left(\frac{2\pi nx}{L}\right)\right]$$

where, in the range $-\frac{L}{2}$ to $+\frac{L}{2}$:

$$a_0=\frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}}f(x)\,\mathrm{d}x,$$

$$a_n=\frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}}f(x)\cos\left(\frac{2\pi nx}{L}\right)\mathrm{d}x$$

and $\quad b_n=\dfrac{2}{L}\displaystyle\int_{-\frac{L}{2}}^{\frac{L}{2}}f(x)\sin\left(\frac{2\pi nx}{L}\right)\mathrm{d}x$

The limits of integration may be replaced by any interval of length L, such as from 0 to L.

Problem 1. The voltage from a square wave generator is of the form:

$$v(t)=\begin{cases}0, & -4<t<0\\ 10, & 0<t<4\end{cases}$$

and has a period of 8 ms.

Find the Fourier series for this periodic function.

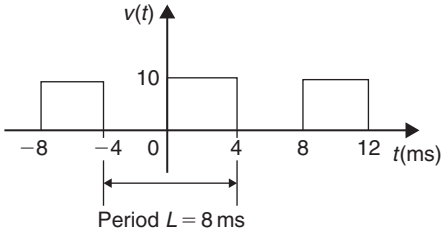


Figure 69.1

The square wave is shown in Fig. 69.1. From para. (c), the Fourier series is of the form:

$$v(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi nt}{L}\right) + b_n \sin\left(\frac{2\pi nt}{L}\right)\right]$$

$$a_0 = \frac{1}{L}\int_{\frac{-L}{2}}^{\frac{L}{2}} v(t)\,\mathrm{d}t = \frac{1}{8}\int_{-4}^{4} v(t)\,\mathrm{d}t$$

$$= \frac{1}{8}\left\{\int_{-4}^{0} 0\,\mathrm{d}t + \int_{0}^{4} 10\,\mathrm{d}t\right\} = \frac{1}{8}[10t]_0^4 = 5$$

$$a_n = \frac{2}{L}\int_{\frac{-L}{2}}^{\frac{L}{2}} v(t)\cos\left(\frac{2\pi nt}{L}\right)\mathrm{d}t$$

$$= \frac{2}{8}\int_{-4}^{4} v(t)\cos\left(\frac{2\pi nt}{8}\right)\mathrm{d}t$$

$$= \frac{1}{4}\left\{\int_{-4}^{0} 0\cos\left(\frac{\pi nt}{4}\right)\mathrm{d}t + \int_{0}^{4} 10\cos\left(\frac{\pi nt}{4}\right)\mathrm{d}t\right\}$$

$$= \frac{1}{4}\left[\frac{10\sin\left(\frac{\pi nt}{4}\right)}{\left(\frac{\pi n}{4}\right)}\right]_0^4 = \frac{10}{\pi n}[\sin\pi n - \sin 0]$$

$= 0$ for $n = 1, 2, 3, \ldots$

$$b_n = \frac{2}{L}\int_{\frac{-L}{2}}^{\frac{L}{2}} v(t)\sin\left(\frac{2\pi nt}{L}\right)\mathrm{d}t$$

$$= \frac{2}{8}\int_{-4}^{4} v(t)\sin\left(\frac{2\pi nt}{8}\right)\mathrm{d}t$$

$$= \frac{1}{4}\left\{\int_{-4}^{0} 0\sin\left(\frac{\pi nt}{4}\right)\mathrm{d}t + \int_{0}^{4} 10\sin\left(\frac{\pi nt}{4}\right)\mathrm{d}t\right\}$$

$$= \frac{1}{4}\left[\frac{-10\cos\left(\frac{\pi nt}{4}\right)}{\left(\frac{\pi n}{4}\right)}\right]_0^4$$

$$= \frac{-10}{\pi n}[\cos\pi n - \cos 0]$$

When n is even, $b_n = 0$

When n is odd, $b_1 = \frac{-10}{\pi}(-1-1) = \frac{20}{\pi}$,

$$b_3 = \frac{-10}{3\pi}(-1-1) = \frac{20}{3\pi},$$

$$b_5 = \frac{20}{5\pi}, \text{ and so on.}$$

Thus the Fourier series for the function $v(t)$ is given by:

$$\boldsymbol{v(t) = 5 + \frac{20}{\pi}\left[\sin\left(\frac{\pi t}{4}\right) + \frac{1}{3}\sin\left(\frac{3\pi t}{4}\right) + \frac{1}{5}\sin\left(\frac{5\pi t}{4}\right) + \cdots\right]}$$

Problem 2. Obtain the Fourier series for the function defined by:

$$f(x) = \begin{cases} 0, & \text{when } -2 < x < -1 \\ 5, & \text{when } -1 < x < 1 \\ 0, & \text{when } 1 < x < 2 \end{cases}$$

The function is periodic outside of this range of period 4.

The function $f(x)$ is shown in Fig. 69.2 where period, $L = 4$. Since the function is symmetrical about the $f(x)$ axis it is an even function and the Fourier series contains no sine terms (i.e. $b_n = 0$).

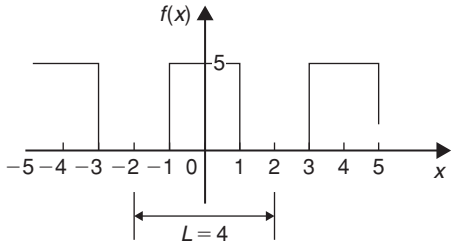


Figure 69.2

Thus, from para. (c),

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nx}{L}\right)$$

$$a_0 = \frac{1}{L}\int_{\frac{-L}{2}}^{\frac{L}{2}} f(x)\,dx = \frac{1}{4}\int_{-2}^{2} f(x)\,dx$$

$$= \frac{1}{4}\left\{\int_{-2}^{-1} 0\,dx + \int_{-1}^{1} 5\,dx + \int_{1}^{2} 0\,dx\right\}$$

$$= \frac{1}{4}[5x]_{-1}^{1} = \frac{1}{4}[(5)-(-5)] = \frac{10}{4} = \frac{5}{2}$$

$$a_n = \frac{2}{L}\int_{\frac{-L}{2}}^{\frac{L}{2}} f(x)\cos\left(\frac{2\pi nx}{L}\right)dx$$

$$= \frac{2}{4}\int_{-2}^{2} f(x)\cos\left(\frac{2\pi nx}{4}\right)dx$$

$$= \frac{1}{2}\left\{\int_{-2}^{-1} 0\cos\left(\frac{\pi nx}{2}\right)dx + \int_{-1}^{1} 5\cos\left(\frac{\pi nx}{2}\right)dx + \int_{1}^{2} 0\cos\left(\frac{\pi nx}{2}\right)dx\right\}$$

$$= \frac{5}{2}\left[\frac{\sin\frac{\pi nx}{2}}{\frac{\pi n}{2}}\right]_{-1}^{1}$$

$$= \frac{5}{\pi n}\left[\sin\left(\frac{\pi n}{2}\right) - \sin\left(\frac{-\pi n}{2}\right)\right]$$

When n is even, $a_n = 0$
When n is odd,

$$a_1 = \frac{5}{\pi}(1-(-1)) = \frac{10}{\pi}$$

$$a_3 = \frac{5}{3\pi}(-1-1) = \frac{-10}{3\pi}$$

$$a_5 = \frac{5}{5\pi}(1-(-1)) = \frac{10}{5\pi} \text{ and so on.}$$

Hence the Fourier series for the function $f(x)$ is given by:

$$\boldsymbol{f(x) = \frac{5}{2} + \frac{10}{\pi}\left[\cos\left(\frac{\pi x}{2}\right) - \frac{1}{3}\cos\left(\frac{3\pi x}{2}\right) + \frac{1}{5}\cos\left(\frac{5\pi x}{2}\right) - \frac{1}{7}\cos\left(\frac{7\pi x}{2}\right) + \cdots\right]}$$

Problem 3. Determine the Fourier series for the function $f(t) = t$ in the range $t = 0$ to $t = 3$.

The function $f(t) = t$ in the interval 0 to 3 is shown in Fig. 69.3. Although the function is not periodic it may be constructed outside of this range so that it is periodic of period 3, as shown by the broken lines in Fig. 69.3.

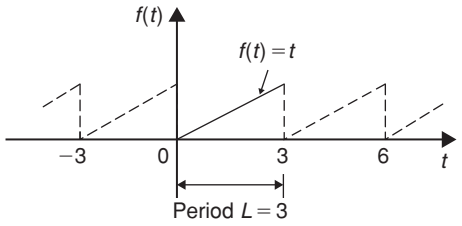


Figure 69.3

From para.(c), the Fourier series is given by:

$$f(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi nt}{L}\right) + b_n \sin\left(\frac{2\pi nt}{L}\right)\right]$$

$$a_0 = \frac{1}{L}\int_{\frac{-L}{2}}^{\frac{L}{2}} f(t)\,dx = \frac{1}{L}\int_{0}^{L} f(t)\,dx$$

$$= \frac{1}{3}\int_{0}^{3} t\,dt = \frac{1}{3}\left[\frac{t^2}{2}\right]_{0}^{3} = \frac{3}{2}$$

$$a_n = \frac{2}{L}\int_{\frac{-L}{2}}^{\frac{L}{2}} f(t)\cos\left(\frac{2\pi nt}{L}\right)dt$$

$$= \frac{2}{L}\int_{0}^{L} t\cos\left(\frac{2\pi nt}{L}\right)dt$$

$$= \frac{2}{3}\int_{0}^{3} t\cos\left(\frac{2\pi nt}{3}\right)dt$$

$$= \frac{2}{3}\left[\frac{t\sin\left(\frac{2\pi nt}{3}\right)}{\left(\frac{2\pi n}{3}\right)} + \frac{\cos\left(\frac{2\pi nt}{3}\right)}{\left(\frac{2\pi n}{3}\right)^2}\right]_0^3$$

by parts

$$= \frac{2}{3}\left[\left\{\frac{3\sin 2\pi n}{\left(\frac{2\pi n}{3}\right)} + \frac{\cos 2\pi n}{\left(\frac{2\pi n}{3}\right)^2}\right\} - \left\{0 + \frac{\cos 0}{\left(\frac{2\pi n}{3}\right)^2}\right\}\right] = 0$$

$$b_n = \frac{2}{L}\int_{\frac{-L}{2}}^{\frac{L}{2}} f(t)\sin\left(\frac{2\pi nt}{L}\right)\mathrm{d}t$$

$$= \frac{2}{L}\int_0^L t\sin\left(\frac{2\pi nt}{L}\right)\mathrm{d}t$$

$$= \frac{2}{3}\int_0^3 t\sin\left(\frac{2\pi nt}{3}\right)\mathrm{d}t$$

$$= \frac{2}{3}\left[\frac{-t\cos\left(\frac{2\pi nt}{3}\right)}{\left(\frac{2\pi n}{3}\right)} + \frac{\sin\left(\frac{2\pi nt}{3}\right)}{\left(\frac{2\pi n}{3}\right)^2}\right]_0^3$$

by parts

$$= \frac{2}{3}\left[\left\{\frac{-3\cos 2\pi n}{\left(\frac{2\pi n}{3}\right)} + \frac{\sin 2\pi n}{\left(\frac{2\pi n}{3}\right)^2}\right\} - \left\{0 + \frac{\sin 0}{\left(\frac{2\pi n}{3}\right)^2}\right\}\right]$$

$$= \frac{2}{3}\left[\frac{-3\cos 2\pi n}{\left(\frac{2\pi n}{3}\right)}\right] = \frac{-3}{\pi n}\cos 2\pi n = \frac{-3}{\pi n}$$

Hence $b_1 = \frac{-3}{\pi}$, $b_2 = \frac{-3}{2\pi}$, $b_3 = \frac{-3}{3\pi}$ and so on.

Thus the Fourier series for the function $f(t)$ in the range 0 to 3 is given by:

$$f(t) = \frac{3}{2} - \frac{3}{\pi}\left[\sin\left(\frac{2\pi t}{3}\right) + \frac{1}{2}\sin\left(\frac{4\pi t}{3}\right) + \frac{1}{3}\sin\left(\frac{6\pi t}{3}\right) + \cdots\right]$$

Now try the following exercise

Exercise 232 Further problems on Fourier series over any range L

1. The voltage from a square wave generator is of the form:

$$v(t) = \begin{cases} 0, & -10 < t < 0 \\ 5, & 0 < t < 10 \end{cases}$$

and is periodic of period 20. Show that the Fourier series for the function is given by:

$$v(t) = \frac{5}{2} + \frac{10}{\pi}\left[\sin\left(\frac{\pi t}{10}\right) + \frac{1}{3}\sin\left(\frac{3\pi t}{10}\right) + \frac{1}{5}\sin\left(\frac{5\pi t}{10}\right) + \cdots\right]$$

2. Find the Fourier series for $f(x) = x$ in the range $x = 0$ to $x = 5$.

$$\left[f(x) = \frac{5}{2} - \frac{5}{\pi}\left[\sin\left(\frac{2\pi x}{5}\right) + \frac{1}{2}\sin\left(\frac{4\pi x}{5}\right) + \frac{1}{3}\sin\left(\frac{6\pi x}{5}\right) + \cdots\right]\right]$$

3. A periodic function of period 4 is defined by:

$$f(x) = \begin{cases} -3, & -2 < x < 0 \\ +3, & 0 < x < 2 \end{cases}$$

Sketch the function and obtain the Fourier series for the function.

$$\left[f(x) = \frac{12}{\pi}\left(\sin\left(\frac{\pi x}{2}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{2}\right) + \frac{1}{5}\sin\left(\frac{5\pi x}{2}\right) + \cdots\right)\right]$$

4. Determine the Fourier series for the half wave rectified sinusoidal voltage $V\sin\omega t$ defined by:

$$f(t)=\begin{cases} V\sin\omega t, & 0<t<\dfrac{\pi}{\omega} \\ 0, & \dfrac{\pi}{\omega}<t<\dfrac{2\pi}{\omega} \end{cases}$$

which is periodic of period $\dfrac{2\pi}{\omega}$

$$\left[\begin{array}{l} f(t)=\dfrac{V}{\pi}+\dfrac{V}{2}\sin\omega t \\ \qquad -\dfrac{2V}{\pi}\left(\dfrac{\cos 2\omega t}{(1)(3)}\right. \\ \qquad \left.+\dfrac{\cos 4\omega t}{(3)(5)}+\dfrac{\cos 6\omega t}{(5)(7)}+\cdots\right) \end{array}\right]$$

69.2 Half-range Fourier series for functions defined over range L

(a) By making the substitution $u=\dfrac{\pi x}{L}$ (see Section 69.1), the range $x=0$ to $x=L$ corresponds to the range $u=0$ to $u=\pi$. Hence a function may be expanded in a series of either cosine terms or sine terms only, i.e. a **half-range Fourier series**.

(b) A **half-range cosine series** in the range 0 to L can be expanded as:

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{L}\right)$$

where $$a_0=\frac{1}{L}\int_0^L f(x)\,\mathrm{d}x \quad \text{and}$$

$$a_n=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x$$

(c) A **half-range sine series** in the range 0 to L can be expanded as:

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{L}\right)$$

where $$b_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x$$

Problem 4. Determine the half-range Fourier cosine series for the function $f(x)=x$ in the range $0\leq x\leq 2$. Sketch the function within and outside of the given range.

A half-range Fourier cosine series indicates an even function. Thus the graph of $f(x)=x$ in the range 0 to 2 is shown in Fig. 69.4 and is extended outside of this range so as to be symmetrical about the $f(x)$ axis as shown by the broken lines.

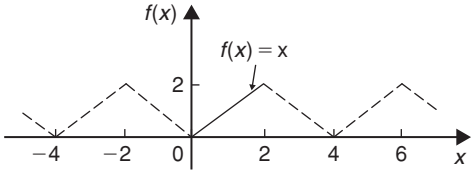


Figure 69.4

From para. (b), for a half-range cosine series:

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{L}\right)$$

$$a_0=\frac{1}{L}\int_0^L f(x)\,\mathrm{d}x=\frac{1}{2}\int_0^2 x\,\mathrm{d}x$$

$$=\frac{1}{2}\left[\frac{x^2}{2}\right]_0^2=1$$

$$a_n=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x$$

$$=\frac{2}{2}\int_0^2 x\cos\left(\frac{n\pi x}{2}\right)\mathrm{d}x$$

$$=\left[\frac{x\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)}+\frac{\cos\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^2}\right]_0^2$$

$$=\left[\left(\frac{2\sin n\pi}{\left(\frac{n\pi}{2}\right)}+\frac{\cos n\pi}{\left(\frac{n\pi}{2}\right)^2}\right)-\left(0+\frac{\cos 0}{\left(\frac{n\pi}{2}\right)^2}\right)\right]$$

$$=\left[\frac{\cos n\pi}{\left(\frac{n\pi}{2}\right)^2}-\frac{1}{\left(\frac{n\pi}{2}\right)^2}\right]$$

$$=\left(\frac{2}{\pi n}\right)^2(\cos n\pi-1)$$

When n is even, $a_n = 0$

$a_1 = \dfrac{-8}{\pi^2}$, $a_3 = \dfrac{-8}{\pi^2 3^2}$, $a_5 = \dfrac{-8}{\pi^2 5^2}$ and so on.

Hence the half-range Fourier cosine series for $f(x)$ in the range 0 to 2 is given by:

$$f(x) = 1 - \frac{8}{\pi^2}\left[\cos\left(\frac{\pi x}{2}\right) + \frac{1}{3^2}\cos\left(\frac{3\pi x}{2}\right) + \frac{1}{5^2}\cos\left(\frac{5\pi x}{2}\right) + \cdots\right]$$

Problem 5. Find the half-range Fourier sine series for the function $f(x) = x$ in the range $0 \leq x \leq 2$. Sketch the function within and outside of the given range.

A half-range Fourier sine series indicates an odd function. Thus the graph of $f(x) = x$ in the range 0 to 2 is shown in Fig. 69.5 and is extended outside of this range so as to be symmetrical about the origin, as shown by the broken lines.

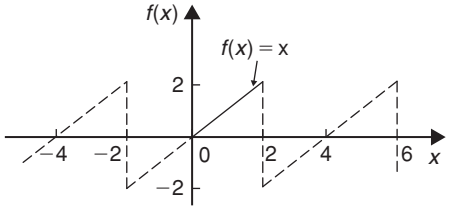


Figure 69.5

From para. (c), for a half-range sine series:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x$$

$$= \frac{2}{2}\int_0^2 x\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x$$

$$= \left[\frac{-x\cos\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)} + \frac{\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^2}\right]_0^2$$

$$= \left[\left(\frac{-2\cos n\pi}{\left(\frac{n\pi}{2}\right)} + \frac{\sin n\pi}{\left(\frac{n\pi}{2}\right)^2}\right) - \left(0 + \frac{\sin 0}{\left(\frac{n\pi}{2}\right)^2}\right)\right]$$

$$= \frac{-2\cos n\pi}{\frac{n\pi}{2}} = \frac{-4}{n\pi}\cos n\pi$$

Hence $b_1 = \dfrac{-4}{\pi}(-1) = \dfrac{4}{\pi}$

$b_2 = \dfrac{-4}{2\pi}(1) = \dfrac{-4}{2\pi}$

$b_3 = \dfrac{-4}{3\pi}(-1) = \dfrac{4}{3\pi}$ and so on.

Thus the half-range Fourier sine series in the range 0 to 2 is given by:

$$f(x) = \frac{4}{\pi}\left[\sin\left(\frac{\pi x}{2}\right) - \frac{1}{2}\sin\left(\frac{2\pi x}{2}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{2}\right) - \frac{1}{4}\sin\left(\frac{4\pi x}{2}\right) + \cdots\right]$$

Now try the following exercise

Exercise 233 Further problems on half-range Fourier series over range L

1. Determine the half-range Fourier cosine series for the function $f(x) = x$ in the range $0 \leq x \leq 3$. Sketch the function within and outside of the given range.

$$\left[f(x) = \frac{3}{2} - \frac{12}{\pi^2}\left\{\cos\left(\frac{\pi x}{3}\right) + \frac{1}{3^2}\cos\left(\frac{3\pi x}{3}\right) + \frac{1}{5^2}\cos\left(\frac{5\pi x}{3}\right) + \cdots\right\}\right]$$

2. Find the half-range Fourier sine series for the function $f(x) = x$ in the range $0 \leq x \leq 3$. Sketch the function within and outside of the given range.

$$\left[f(x) = \frac{6}{\pi}\left(\sin\left(\frac{\pi x}{3}\right) - \frac{1}{2}\sin\left(\frac{2\pi x}{3}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{3}\right) - \frac{1}{4}\sin\left(\frac{4\pi x}{3}\right) + \cdots\right)\right]$$

3. Determine the half-range Fourier sine series for the function defined by:

$$f(t) = \begin{cases} t, & 0 < t < 1 \\ (2-t), & 1 < t < 2 \end{cases}$$

$$\left[f(t) = \frac{8}{\pi^2}\left(\sin\left(\frac{\pi t}{2}\right) - \frac{1}{3^2}\sin\left(\frac{3\pi t}{2}\right) + \frac{1}{5^2}\sin\left(\frac{5\pi t}{2}\right) - \cdots\right)\right]$$

4. Show that the half-range Fourier cosine series for the function $f(\theta) = \theta^2$ in the range 0 to 4 is given by:

$$f(\theta) = \frac{16}{3} - \frac{64}{\pi^2}\left(\cos\left(\frac{\pi\theta}{4}\right) - \frac{1}{2^2}\cos\left(\frac{2\pi\theta}{4}\right) + \frac{1}{3^2}\cos\left(\frac{3\pi\theta}{4}\right) - \cdots\right)$$

Sketch the function within and outside of the given range.

Chapter 70

A numerical method of harmonic analysis

70.1 Introduction

Many practical waveforms can be represented by simple mathematical expressions, and, by using Fourier series, the magnitude of their harmonic components determined, as shown in Chapters 66 to 69. For waveforms not in this category, analysis may be achieved by numerical methods. **Harmonic analysis** is the process of resolving a periodic, non-sinusoidal quantity into a series of sinusoidal components of ascending order of frequency.

70.2 Harmonic analysis on data given in tabular or graphical form

The Fourier coefficients a_0, a_n and b_n used in Chapters 66 to 69 all require functions to be integrated, i.e.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x = \frac{1}{2\pi}\int_{0}^{2\pi} f(x)\,\mathrm{d}x$$

$$= \text{mean value of } f(x) \text{ in the range } -\pi \text{ to } \pi \text{ or } 0 \text{ to } 2\pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_{0}^{2\pi} f(x)\cos nx\,\mathrm{d}x$$

$$= \text{twice the mean value of } f(x)\cos nx \text{ in the range 0 to } 2\pi$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_{0}^{2\pi} f(x)\sin nx\,\mathrm{d}x$$

$$= \text{twice the mean value of } f(x)\sin nx \text{ in the range 0 to } 2\pi$$

However, irregular waveforms are not usually defined by mathematical expressions and thus the Fourier coefficients cannot be determined by using calculus. In these cases, approximate methods, such as the **trapezoidal rule**, can be used to evaluate the Fourier coefficients.

Most practical waveforms to be analysed are periodic. Let the period of a waveform be 2π and be divided into p equal parts as shown in Fig. 70.1. The width of each

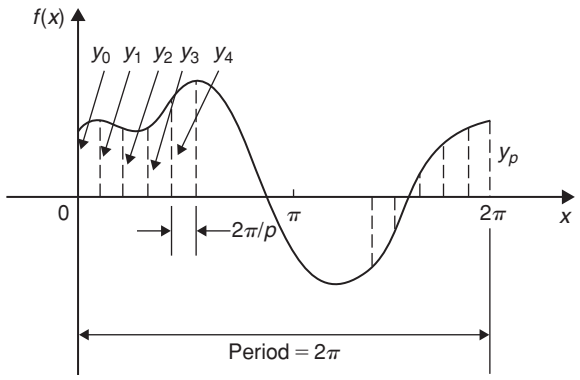


Figure 70.1

interval is thus $\frac{2\pi}{p}$. Let the ordinates be labelled y_0, y_1, $y_2, \ldots y_p$ (note that $y_0 = y_p$). The trapezoidal rule states:

$$\text{Area} = (\text{width of interval})\left[\frac{1}{2}(\text{first} + \text{last ordinate}) + \text{sum of remaining ordinates}\right]$$

$$\approx \frac{2\pi}{p}\left[\frac{1}{2}(y_0 + y_p) + y_1 + y_2 + y_3 + \cdots\right]$$

Since $y_0 = y_p$, then $\frac{1}{2}(y_0 + y_p) = y_0 = y_p$

Hence area $\approx \frac{2\pi}{p}\sum_{k=1}^{p} y_k$

$$\text{Mean value} = \frac{\text{area}}{\text{length of base}}$$

$$\approx \frac{1}{2\pi}\left(\frac{2\pi}{p}\right)\sum_{k=1}^{p} y_k \approx \frac{1}{p}\sum_{k=1}^{p} y_k$$

However, a_0 = mean value of $f(x)$ in the range 0 to 2π.

Thus $$\boldsymbol{a_0 \approx \frac{1}{p}\sum_{k=1}^{p} y_k} \qquad (1)$$

Similarly, a_n = twice the mean value of $f(x)\cos nx$ in the range 0 to 2π,

thus $$\boldsymbol{a_n \approx \frac{2}{p}\sum_{k=1}^{p} y_k \cos nx_k} \qquad (2)$$

and b_n = twice the mean value of $f(x)\sin nx$ in the range 0 to 2π,

thus $$\boldsymbol{b_n \approx \frac{2}{p}\sum_{k=1}^{p} y_k \sin nx_k} \qquad (3)$$

Problem 1. The values of the voltage v volts at different moments in a cycle are given by:

$\theta°$ (degrees)	V (volts)	$\theta°$ (degrees)	V (volts)
30	62	210	−28
60	35	240	24
90	−38	270	80
120	−64	300	96
150	−63	330	90
180	−52	360	70

Draw the graph of voltage V against angle θ and analyse the voltage into its first three constituent harmonics, each coefficient correct to 2 decimal places.

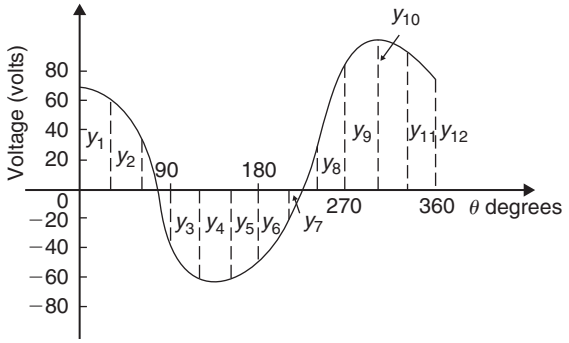


Figure 70.2

The graph of voltage V against angle θ is shown in Fig. 70.2. The range 0 to 2π is divided into 12 equal intervals giving an interval width of $\frac{2\pi}{12}$, i.e. $\frac{\pi}{6}$ rad or 30°. The values of the ordinates y_1, y_2, $y_3, \ldots$ are 62, 35, −38, … from the given table of values. If a larger number of intervals are used, results having a greater accuracy are achieved. The data is tabulated in the proforma shown in Table 70.1, on page 639.

From equation (1), $a_0 \approx \frac{1}{p}\sum_{k=1}^{p} y_k = \frac{1}{12}(212)$

$= 17.67$ (since $p = 12$)

From equation (2), $a_n \approx \frac{2}{p}\sum_{k=1}^{p} y_k \cos nx_k$

hence $$a_1 \approx \frac{2}{12}(417.94) = 69.66$$

$$a_2 \approx \frac{2}{12}(-39) = -6.50$$

and $$a_3 \approx \frac{2}{12}(-49) = -8.17$$

From equation (3), $b_n \approx \frac{2}{p}\sum_{k=1}^{p} y_k \sin nx_k$

hence $$b_1 \approx \frac{2}{12}(-278.53) = -46.42$$

Table 70.1

Ordinates	$\theta°$	V	$\cos\theta$	$V\cos\theta$	$\sin\theta$	$V\sin\theta$	$\cos 2\theta$	$V\cos 2\theta$	$\sin 2\theta$	$V\sin 2\theta$	$\cos 3\theta$	$V\cos 3\theta$	$\sin 3\theta$	$V\sin 3\theta$
y_1	30	62	0.866	53.69	0.5	31	0.5	31	0.866	53.69	0	0	1	62
y_2	60	35	0.5	17.5	0.866	30.31	−0.5	−17.5	0.866	30.31	−1	−35	0	0
y_3	90	−38	0	0	1	−38	−1	38	0	0	0	0	−1	38
y_4	120	−64	−0.5	32	0.866	−55.42	−0.5	32	−0.866	55.42	1	−64	0	0
y_5	150	−63	−0.866	54.56	0.5	−31.5	0.5	−31.5	−0.866	54.56	0	0	1	−63
y_6	180	−52	−1	52	0	0	1	−52	0	0	−1	52	0	0
y_7	210	−28	−0.866	24.25	−0.5	14	0.5	−14	0.866	−24.25	0	0	−1	28
y_8	240	24	−0.5	−12	−0.866	−20.78	−0.5	−12	0.866	20.78	1	24	0	0
y_9	270	80	0	0	−1	−80	−1	−80	0	0	0	0	1	80
y_{10}	300	96	0.5	48	−0.866	−83.14	−0.5	−48	−0.866	−83.14	−1	−96	0	0
y_{11}	330	90	0.866	77.94	−0.5	−45	0.5	45	−0.866	−77.94	0	0	−1	−90
y_{12}	360	70	1	70	0	0	1	70	0	0	1	70	0	0
$\sum_{k=1}^{12} y_k = (212)$			$\sum_{k=1}^{12} y_k\cos\theta_k = 417.94$		$\sum_{k=1}^{12} y_k\sin\theta_k = -278.53$		$\sum_{k=1}^{12} y_k\cos 2\theta_k = -39$		$\sum_{k=1}^{12} y_k\sin 2\theta_k = 29.43$		$\sum_{k=1}^{12} y_k\cos 3\theta_k = -49$		$\sum_{k=1}^{12} y_k\sin 3\theta_k = 55$	

$$b_2 \approx \frac{2}{12}(29.43) = 4.91$$

and $$b_3 \approx \frac{2}{12}(55) = 9.17$$

Substituting these values into the Fourier series:

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

gives: $$\boldsymbol{v = 17.67 + 69.66\cos\theta - 6.50\cos 2\theta - 8.17\cos 3\theta + \cdots - 46.42\sin\theta + 4.91\sin 2\theta + 9.17\sin 3\theta + \cdots} \quad (4)$$

Note that in equation (4), $(-46.42\sin\theta + 69.66\cos\theta)$ comprises the fundamental, $(4.91\sin 2\theta - 6.50\cos 2\theta)$ comprises the second harmonic and $(9.17\sin 3\theta - 8.17\cos 3\theta)$ comprises the third harmonic.
It is shown in Chapter 17 that:

$$a\sin\omega t + b\cos\omega t = R\sin(\omega t + \alpha)$$

where $a = R\cos\alpha$, $b = R\sin\alpha$, $R = \sqrt{a^2 + b^2}$ and $\alpha = \tan^{-1}\frac{b}{a}$

For the fundamental, $R = \sqrt{(-46.42)^2 + (69.66)^2}$

$$= 83.71$$

If $a = R\cos\alpha$, then $\cos\alpha = \frac{a}{R} = \frac{-46.42}{83.71}$ which is negative,

and if $b = R\sin\alpha$, then $\sin\alpha = \frac{b}{R} = \frac{69.66}{83.71}$ which is positive.

The only quadrant where $\cos\alpha$ is negative *and* $\sin\alpha$ is positive is the second quadrant.

Hence $\alpha = \tan^{-1}\frac{b}{a} = \tan^{-1}\frac{69.66}{-46.42}$

$$= 123.68^\circ \text{ or } 2.16\,\text{rad}$$

Thus $(-46.42\sin\theta + 69.66\cos\theta)$

$$= 83.71\sin(\theta + 2.16)$$

By a similar method it may be shown that the second harmonic

$$(4.91\sin 2\theta - 6.50\cos 2\theta) = 8.15\sin(2\theta - 0.92)$$

and the third harmonic

$$(9.17\sin 3\theta - 8.17\cos 3\theta) = 12.28\sin(3\theta - 0.73)$$

Hence equation (4) may be re-written as:

$$\boldsymbol{v = 17.67 + 83.71\sin(\theta + 2.16) + 8.15\sin(2\theta - 0.92) + 12.28\sin(3\theta - 0.73)\ \text{volts}}$$

which is the form used in Chapter 25 with complex waveforms.

Now try the following exercise

Exercise 234 Further problems on numerical harmonic analysis

Determine the Fourier series to represent the periodic functions given by the tables of values in Problems 1 to 3, up to and including the third harmonic and each coefficient correct to 2 decimal places. Use 12 ordinates in each case.

1.

Angle θ°	30	60	90	120	150	180
Displacement y	40	43	38	30	23	17

Angle θ°	210	240	270	300	330	360
Displacement y	11	9	10	13	21	32

$$y = 23.92 + 7.81\cos\theta + 14.61\sin\theta + 0.17\cos 2\theta + 2.31\sin 2\theta - 0.33\cos 3\theta + 0.50\sin 3\theta$$

2.

Angle θ°	0	30	60	90	120	150
Voltage v	−5.0	−1.5	6.0	12.5	16.0	16.5

Angle θ°	180	210	240	270	300	330
Voltage v	15.0	12.5	6.5	−4.0	−7.0	−7.5

$$v = 5.00 - 10.78\cos\theta + 6.83\sin\theta - 1.96\cos 2\theta + 0.80\sin 2\theta + 0.58\cos 3\theta - 1.08\sin 3\theta$$

3.

Angle θ°	30	60	90	120	150	180
Current i	0	−1.4	−1.8	−1.9	−1.8	−1.3

Angle θ°	210	240	270	300	330	360
Current i	0	2.2	3.8	3.9	3.5	2.5

$$i = 0.64 + 1.58\cos\theta - 2.73\sin\theta - 0.23\cos 2\theta - 0.42\sin 2\theta + 0.27\cos 3\theta + 0.05\sin 3\theta$$

70.3 Complex waveform considerations

It is sometimes possible to predict the harmonic content of a waveform on inspection of particular waveform characteristics.

(i) If a periodic waveform is such that the area above the horizontal axis is equal to the area below then the mean value is zero. Hence $a_0 = 0$ (see Fig. 70.3(a)).

(ii) An **even function** is symmetrical about the vertical axis and contains **no sine terms** (see Fig. 70.3(b)).

(iii) An **odd function** is symmetrical about the origin and contains **no cosine terms** (see Fig. 70.3(c)).

(iv) $f(x) = f(x+\pi)$ represents a waveform which repeats after half a cycle and **only even harmonics** are present (see Fig. 70.3(d)).

(v) $f(x) = -f(x+\pi)$ represents a waveform for which the positive and negative cycles are identical in shape and **only odd harmonics** are present (see Fig. 70.3(e)).

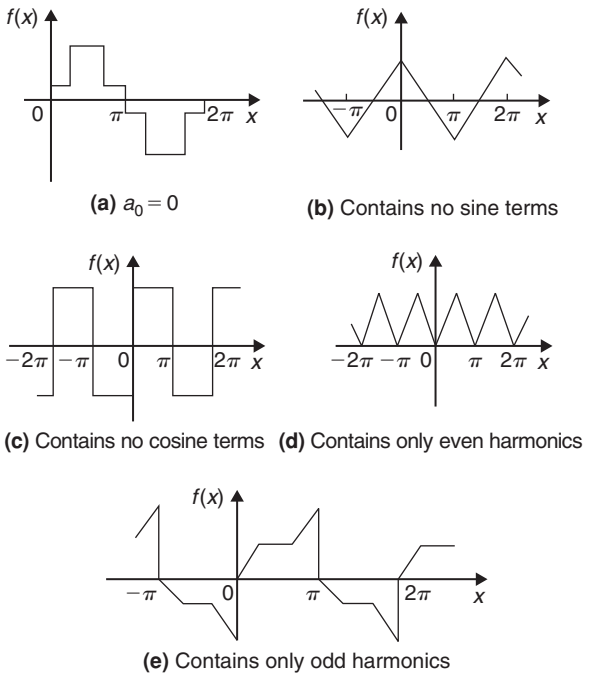


Figure 70.3

Problem 2. Without calculating Fourier coefficients state which harmonics will be present in the waveforms shown in Fig. 70.4.

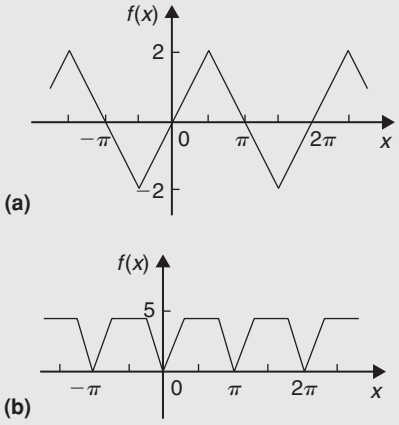


Figure 70.4

(a) The waveform shown in Fig. 70.4(a) is symmetrical about the origin and is thus an odd function. An odd function contains no cosine terms. Also, the waveform has the characteristic $f(x) = -f(x+\pi)$, i.e. the positive and negative half cycles are identical in shape. Only odd harmonics can be present in such a waveform. Thus the waveform shown in Fig. 70.4(a) contains **only odd sine terms**. Since the area above the x-axis is equal to the area below, $a_0 = 0$.

(b) The waveform shown in Fig. 70.4(b) is symmetrical about the $f(x)$ axis and is thus an even function. An even function contains no sine terms. Also, the waveform has the characteristic $f(x) = f(x+\pi)$, i.e. the waveform repeats itself after half a cycle. Only even harmonics can be present in such a waveform. Thus the waveform shown in Fig. 70.4(b) contains **only even cosine terms** (together with a constant term, a_0).

Problem 3. An alternating current i amperes is shown in Fig. 70.5. Analyse the waveform into its constituent harmonics as far as and including the fifth harmonic, correct to 2 decimal places, by taking 30° intervals.

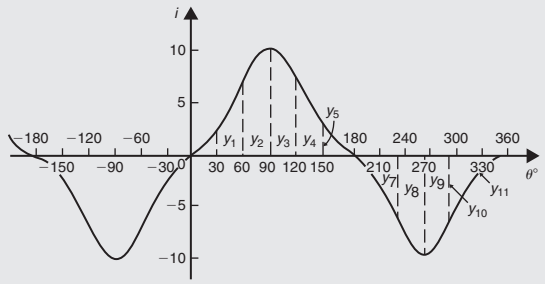


Figure 70.5

With reference to Fig. 70.5, the following characteristics are noted:

(i) The mean value is zero since the area above the θ axis is equal to the area below it. Thus the constant term, or d.c. component, $a_0 = 0$.

(ii) Since the waveform is symmetrical about the origin the function i is odd, which means that there are no cosine terms present in the Fourier series.

(iii) The waveform is of the form $f(\theta) = -f(\theta + \pi)$ which means that only odd harmonics are present.

Investigating waveform characteristics has thus saved unnecessary calculations and in this case the Fourier series has only odd sine terms present, i.e.

$$i = b_1 \sin\theta + b_3 \sin 3\theta + b_5 \sin 5\theta + \cdots$$

A proforma, similar to Table 70.1, but without the 'cosine terms' columns and without the 'even sine terms' columns is shown in Table 70.2 up to, and including, the fifth harmonic, from which the Fourier coefficients b_1, b_3 and b_5 can be determined. Twelve co-ordinates are chosen and labelled $y_1, y_2, y_3, \ldots y_{12}$ as shown in Fig. 70.5

From equation (3), Section 70.2,

$$b_n = \frac{2}{p}\sum_{k=1}^{p} i_k \sin n\theta_k, \text{ where } p = 12$$

Hence $b_1 \approx \frac{2}{12}(48.24) = 8.04,$

$$b_3 \approx \frac{2}{12}(-12) = -2.00,$$

and $b_5 \approx \frac{2}{12}(-0.24) = -0.04$

Table 70.2

Ordinate	θ	i	$\sin\theta$	$i\sin\theta$	$\sin 3\theta$	$i\sin 3\theta$	$\sin 5\theta$	$i\sin 5\theta$
y_1	30	2	0.5	1	1	2	0.5	1
y_2	60	7	0.866	6.06	0	0	−0.866	−6.06
y_3	90	10	1	10	−1	−10	1	10
y_4	120	7	0.866	6.06	0	0	−0.866	−6.06
y_5	150	2	0.5	1	1	2	0.5	1
y_6	180	0	0	0	0	0	0	0
y_7	210	−2	−0.5	1	−1	2	−0.5	1
y_8	240	−7	−0.866	6.06	0	0	0.866	−6.06
y_9	270	−10	−1	10	1	−10	−1	10
y_{10}	300	−7	−0.866	6.06	0	0	0.866	−6.06
y_{11}	330	−2	−0.5	1	−1	2	−0.5	1
y_{12}	360	0	0	0	0	0	0	0
			$\sum_{k=1}^{12} y_k \sin\theta_k = 48.24$		$\sum_{k=1}^{12} y_k \sin 3\theta_k = -12$		$\sum_{k=1}^{12} y_k \sin 5\theta_k = -0.24$	

Thus the Fourier series for current i is given by:

$$i = 8.04\sin\theta - 2.00\sin 3\theta - 0.04\sin 5\theta$$

Now try the following exercise

Exercise 235 Further problems on a numerical method of harmonic analysis

1. Without performing calculations, state which harmonics will be present in the waveforms shown in Fig. 70.6

[(a) only odd cosine terms present
(b) only even sine terms present]

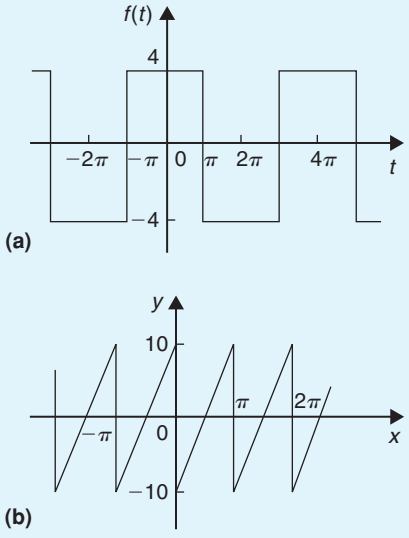


Figure 70.6

2. Analyse the periodic waveform of displacement y against angle θ in Fig. 70.7(a) into its constituent harmonics as far as and including the third harmonic, by taking 30° intervals.

$$\left[\begin{array}{r} y = 9.4 + 13.2\cos\theta - 24.1\sin\theta \\ +0.92\cos 2\theta - 0.14\sin 2\theta \\ +0.83\cos 3\theta + 0.67\sin 3\theta \end{array}\right]$$

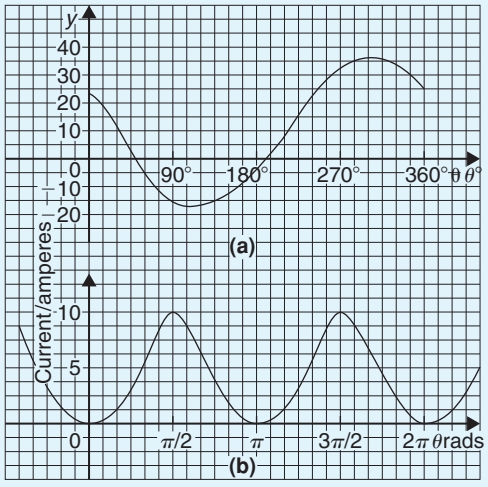


Figure 70.7

3. For the waveform of current shown in Fig. 70.7(b) state why only a d.c. component and even cosine terms will appear in the Fourier series and determine the series, using $\pi/6$ rad intervals, up to and including the sixth harmonic.

$$\left[\begin{array}{r} I = 4.00 - 4.67\cos 2\theta + 1.00\cos 4\theta \\ -0.66\cos 6\theta \end{array}\right]$$

4. Determine the Fourier series as far as the third harmonic to represent the periodic function y given by the waveform in Fig. 70.8. Take 12 intervals when analysing the waveform.

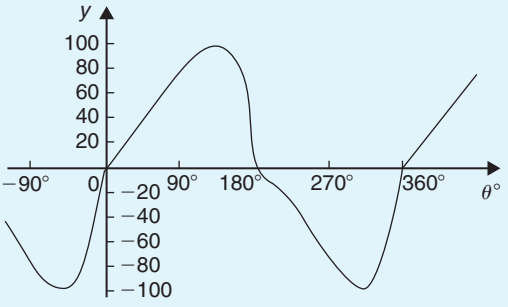


Figure 70.8

$$\left[\begin{array}{r} y = 1.83 - 27.77\cos\theta + 83.74\sin\theta \\ -0.75\cos 2\theta - 1.59\sin 2\theta \\ +16.00\cos 3\theta + 11.00\sin 3\theta \end{array}\right]$$

Chapter 71

The complex or exponential form of a Fourier series

71.1 Introduction

The form used for the Fourier series in Chapters 66 to 70 consisted of cosine and sine terms. However, there is another form that is commonly used—one that directly gives the amplitude terms in the frequency spectrum and relates to phasor notation. This form involves the use of complex numbers (see Chapters 20 and 21). It is called the **exponential** or **complex form** of a Fourier series.

71.2 Exponential or complex notation

It was shown on page 226, equations (4) and (5) that:

$$e^{j\theta} = \cos\theta + j\sin\theta \qquad (1)$$

and $$e^{-j\theta} = \cos\theta - j\sin\theta \qquad (2)$$

Adding equations (1) and (2) gives:

$$e^{j\theta} + e^{-j\theta} = 2\cos\theta$$

from which, $$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad (3)$$

Similarly, equation (1) – equation (2) gives:

$$e^{j\theta} - e^{-j\theta} = 2j\sin\theta$$

from which, $$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \qquad (4)$$

Thus, from page 630, the Fourier series $f(x)$ over any range L,

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{2\pi nx}{L}\right) + b_n\sin\left(\frac{2\pi nx}{L}\right)\right]$$

may be written as:

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n\left(\frac{e^{j\frac{2\pi nx}{L}} + e^{-j\frac{2\pi nx}{L}}}{2}\right) + b_n\left(\frac{e^{j\frac{2\pi nx}{L}} - e^{-j\frac{2\pi nx}{L}}}{2j}\right)\right]$$

Multiplying top and bottom of the b_n term by $-j$ (and remembering that $j^2 = -1$) gives:

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n\left(\frac{e^{j\frac{2\pi nx}{L}} + e^{-j\frac{2\pi nx}{L}}}{2}\right) - jb_n\left(\frac{e^{j\frac{2\pi nx}{L}} - e^{-j\frac{2\pi nx}{L}}}{2}\right)\right]$$

Rearranging gives:

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[\left(\frac{a_n - jb_n}{2}\right)e^{j\frac{2\pi nx}{L}} + \left(\frac{a_n + jb_n}{2}\right)e^{-j\frac{2\pi nx}{L}}\right] \quad (5)$$

The Fourier coefficients a_0, a_n and b_n may be replaced by complex coefficients c_0, c_n and c_{-n} such that

$$c_0 = a_0 \quad (6)$$

$$c_n = \frac{a_n - jb_n}{2} \quad (7)$$

and $$c_{-n} = \frac{a_n + jb_n}{2} \quad (8)$$

where c_{-n} represents the complex conjugate of c_n (see page 216).
Thus, equation (5) may be rewritten as:

$$f(x) = c_0 + \sum_{n=1}^{\infty} c_n e^{j\frac{2\pi nx}{L}} + \sum_{n=1}^{\infty} c_{-n} e^{-j\frac{2\pi nx}{L}} \quad (9)$$

Since $e^0 = 1$, the c_0 term can be absorbed into the summation since it is just another term to be added to the summation of the c_n term when $n = 0$. Thus,

$$f(x) = \sum_{n=0}^{\infty} c_n e^{j\frac{2\pi nx}{L}} + \sum_{n=1}^{\infty} c_{-n} e^{-j\frac{2\pi nx}{L}} \quad (10)$$

The c_{-n} term may be rewritten by changing the limits $n = 1$ to $n = \infty$ to $n = -1$ to $n = -\infty$. Since n has been made negative, the exponential term becomes $e^{j\frac{2\pi nx}{L}}$ and c_{-n} becomes c_n. Thus,

$$f(x) = \sum_{n=0}^{\infty} c_n e^{j\frac{2\pi nx}{L}} + \sum_{n=-1}^{-\infty} c_n e^{j\frac{2\pi nx}{L}}$$

Since the summations now extend from $-\infty$ to -1 and from 0 to $+\infty$, equation (10) may be written as:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi nx}{L}} \quad (11)$$

Equation (11) is the **complex** or **exponential form** of the Fourier series.

71.3 The complex coefficients

From equation (7), the complex coefficient c_n was defined as: $c_n = \frac{a_n - jb_n}{2}$
However, a_n and b_n are defined (from page 630) by:

$$a_n = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\cos\left(\frac{2\pi nx}{L}\right)dx \quad \text{and}$$

$$b_n = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\sin\left(\frac{2\pi nx}{L}\right)dx$$

$$\text{Thus,} \quad c_n = \frac{\left(\begin{array}{c}\frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\cos\left(\frac{2\pi nx}{L}\right)dx \\ -j\frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\sin\left(\frac{2\pi nx}{L}\right)dx\end{array}\right)}{2}$$

$$= \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\cos\left(\frac{2\pi nx}{L}\right)dx - j\frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\sin\left(\frac{2\pi nx}{L}\right)dx$$

From equations (3) and (4),

$$c_n = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\left(\frac{e^{j\frac{2\pi nx}{L}} + e^{-j\frac{2\pi nx}{L}}}{2}\right)dx - j\frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\left(\frac{e^{j\frac{2\pi nx}{L}} - e^{-j\frac{2\pi nx}{L}}}{2j}\right)dx$$

from which,

$$c_n = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\left(\frac{e^{j\frac{2\pi nx}{L}} + e^{-j\frac{2\pi nx}{L}}}{2}\right)dx - \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\left(\frac{e^{j\frac{2\pi nx}{L}} - e^{-j\frac{2\pi nx}{L}}}{2}\right)dx$$

i.e. $$\boldsymbol{c_n = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)e^{-j\frac{2\pi nx}{L}}dx} \quad (12)$$

Care needs to be taken when determining c_0. If n appears in the denominator of an expression the expansion can be invalid when $n = 0$. In such circumstances it is usually simpler to evaluate c_0 by using the relationship:

$$c_0 = a_0 = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)dx \quad \text{(from page 630).} \quad (13)$$

Problem 1. Determine the complex Fourier series for the function defined by:

$$f(x)=\begin{cases}0, & \text{when } -2\le x\le -1\\ 5, & \text{when } -1\le x\le 1\\ 0, & \text{when } \ 1\le x\le 2\end{cases}$$

The function is periodic outside this range of period 4.

This is the same Problem as Problem 2 on page 631 and we can use this to demonstrate that the two forms of Fourier series are equivalent.

The function $f(x)$ is shown in Figure 71.1, where the period, $L=4$.

From equation (11), the complex Fourier series is given by:

$$f(x)=\sum_{n=-\infty}^{\infty} c_n\, e^{j\frac{2\pi nx}{L}}$$

where c_n is given by:

$$c_n=\frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\, e^{-j\frac{2\pi nx}{L}}\,dx \quad \text{(from equation 12).}$$

With reference to Figure 71.1, when $L=4$,

$$c_n=\frac{1}{4}\left\{\int_{-2}^{-1}0\,dx+\int_{-1}^{1}5e^{-j\frac{2\pi nx}{4}}dx+\int_{1}^{2}0\,dx\right\}$$

$$=\frac{1}{4}\int_{-1}^{1}5e^{-\frac{j\pi nx}{2}}dx=\frac{5}{4}\left[\frac{e^{-\frac{j\pi nx}{2}}}{-\frac{j\pi n}{2}}\right]_{-1}^{1}$$

$$=\frac{-5}{j2\pi n}\left[e^{-\frac{j\pi nx}{2}}\right]_{-1}^{1}=\frac{-5}{j2\pi n}\left(e^{-\frac{j\pi n}{2}}-e^{\frac{j\pi n}{2}}\right)$$

$$=\frac{5}{\pi n}\left(\frac{e^{j\frac{\pi n}{2}}-e^{-j\frac{\pi n}{2}}}{2j}\right)$$

$$=\frac{5}{\pi n}\sin\frac{\pi n}{2} \quad \text{(from equation (4)).}$$

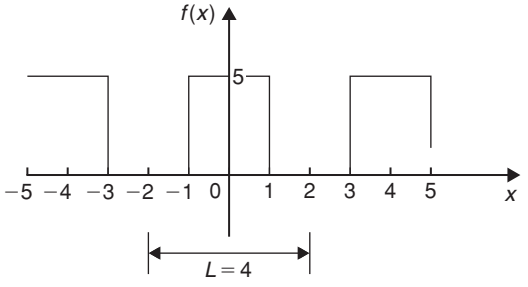


Figure 71.1

Hence, from equation (11), **the complex form of the Fourier series** is given by:

$$\boldsymbol{f(x)=\sum_{n=-\infty}^{\infty} c_n\, e^{j\frac{2\pi nx}{L}}=\sum_{n=-\infty}^{\infty}\frac{5}{\pi n}\sin\frac{\pi n}{2}\,e^{j\frac{\pi nx}{2}}} \tag{14}$$

Let us show how this result is equivalent to the result involving sine and cosine terms determined on page 632.

From equation (13),

$$\boldsymbol{c_0}=a_0=\frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}}f(x)dx=\frac{1}{4}\int_{-1}^{1}5\,dx$$

$$=\frac{5}{4}[x]_{-1}^{1}=\frac{5}{4}[1-(-1)]=\boldsymbol{\frac{5}{2}}$$

Since $c_n=\frac{5}{\pi n}\sin\frac{\pi n}{2}$, then

$$\boldsymbol{c_1}=\frac{5}{\pi}\sin\frac{\pi}{2}=\boldsymbol{\frac{5}{\pi}}$$

$$\boldsymbol{c_2}=\frac{5}{2\pi}\sin\pi=\boldsymbol{0}$$

(in fact, **all even terms will be zero** since $\sin n\pi=0$)

$$\boldsymbol{c_3}=\frac{5}{\pi n}\sin\frac{\pi n}{2}=\frac{5}{3\pi}\sin\frac{3\pi}{2}=\boldsymbol{-\frac{5}{3\pi}}$$

By similar substitution,

$$\boldsymbol{c_5}=\boldsymbol{\frac{5}{5\pi}} \qquad \boldsymbol{c_7}=\boldsymbol{-\frac{5}{7\pi}}, \text{ and so on.}$$

Similarly,

$$\boldsymbol{c_{-1}}=\frac{5}{-\pi}\sin\frac{-\pi}{2}=\boldsymbol{\frac{5}{\pi}}$$

$$\boldsymbol{c_{-2}}=-\frac{5}{2\pi}\sin\frac{-2\pi}{2}=\boldsymbol{0}=\boldsymbol{c_{-4}}=\boldsymbol{c_{-6}}, \text{ and so on.}$$

$$\boldsymbol{c_{-3}}=-\frac{5}{3\pi}\sin\frac{-3\pi}{2}=\boldsymbol{-\frac{5}{3\pi}}$$

$$\boldsymbol{c_{-5}}=-\frac{5}{5\pi}\sin\frac{-5\pi}{2}=\boldsymbol{\frac{5}{5\pi}}, \text{ and so on.}$$

Hence, the extended complex form of the Fourier series shown in equation (14) becomes:

$$\begin{aligned}
f(x) &= \frac{5}{2} + \frac{5}{\pi}e^{j\frac{\pi x}{2}} - \frac{5}{3\pi}e^{j\frac{3\pi x}{2}} + \frac{5}{5\pi}e^{j\frac{5\pi x}{2}} \\
&\quad - \frac{5}{7\pi}e^{j\frac{7\pi x}{2}} + \cdots + \frac{5}{\pi}e^{-j\frac{\pi x}{2}} \\
&\quad - \frac{5}{3\pi}e^{-j\frac{3\pi x}{2}} + \frac{5}{5\pi}e^{-j\frac{5\pi x}{2}} \\
&\quad - \frac{5}{7\pi}e^{-j\frac{7\pi x}{2}} + \cdots \\
&= \frac{5}{2} + \frac{5}{\pi}\left(e^{j\frac{\pi x}{2}} + e^{-j\frac{\pi x}{2}}\right) \\
&\qquad - \frac{5}{3\pi}\left(e^{j\frac{3\pi x}{2}} + e^{-j\frac{3\pi x}{2}}\right) \\
&\qquad + \frac{5}{5\pi}\left(e^{\frac{5\pi x}{2}} + e^{-j\frac{5\pi x}{2}}\right) - \cdots \\
&= \frac{5}{2} + \frac{5}{\pi}(2)\left(\frac{e^{j\frac{\pi x}{2}} + e^{-j\frac{\pi x}{2}}}{2}\right) \\
&\qquad - \frac{5}{3\pi}(2)\left(\frac{e^{j\frac{3\pi x}{2}} + e^{-j\frac{3\pi x}{2}}}{2}\right) \\
&\qquad + \frac{5}{5\pi}(2)\left(\frac{e^{j\frac{5\pi x}{2}} + e^{-j\frac{5\pi x}{2}}}{2}\right) - \cdots \\
&= \frac{5}{2} + \frac{10}{\pi}\cos\left(\frac{\pi x}{2}\right) - \frac{10}{3\pi}\cos\left(\frac{3\pi x}{2}\right) \\
&\qquad + \frac{10}{5\pi}\cos\left(\frac{5\pi x}{2}\right) - \cdots \\
&\qquad \text{(from equation (3))}
\end{aligned}$$

i.e. $$\boldsymbol{f(x) = \frac{5}{2} + \frac{10}{\pi}\left[\cos\left(\frac{\pi x}{2}\right) - \frac{1}{3}\cos\left(\frac{3\pi x}{2}\right) + \frac{1}{5}\cos\left(\frac{5\pi x}{2}\right) - \cdots\right]}$$

which is the same as obtained on page 632.

Hence, $\displaystyle\sum_{n=-\infty}^{\infty} \frac{5}{\pi n}\sin\frac{n\pi}{2}\, e^{j\frac{\pi n x}{2}}$ is equivalent to

$$\frac{5}{2} + \frac{10}{\pi}\left[\cos\left(\frac{\pi x}{2}\right) - \frac{1}{3}\cos\left(\frac{3\pi x}{2}\right) + \frac{1}{5}\cos\left(\frac{5\pi x}{2}\right) - \cdots\right]$$

Problem 2. Show that the complex Fourier series for the function $f(t) = t$ in the range $t = 0$ to $t = 1$, and of period 1, may be expressed as:

$$f(t) = \frac{1}{2} + \frac{j}{2\pi}\sum_{n=-\infty}^{\infty}\frac{e^{j2\pi nt}}{n}$$

The saw tooth waveform is shown in Figure 71.2.

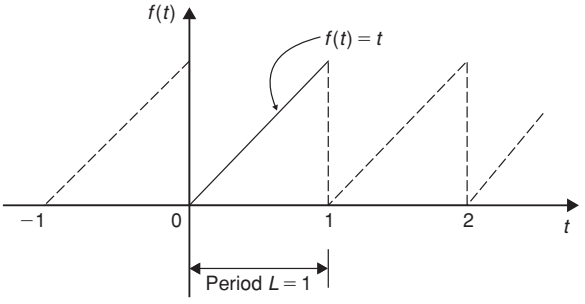


Figure 71.2

From equation (11), the complex Fourier series is given by:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi nt}{L}}$$

and when the period, $L = 1$, then:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi nt}$$

where, from equation (12),

$$c_n = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(t)e^{-j\frac{2\pi nt}{L}}\,dt = \frac{1}{L}\int_0^L f(t)e^{-j\frac{2\pi nt}{L}}\,dt$$

and when $L = 1$ and $f(t) = t$, then:

$$c_n = \frac{1}{1}\int_0^1 t\,e^{-j\frac{2\pi nt}{1}}\,dt = \int_0^1 t\,e^{-j2\pi nt}\,dt$$

Using integration by parts (see Chapter 43), let $u = t$, from which, $\frac{du}{dt} = 1$, and $dt = du$, and

let $dv = e^{-j2\pi nt}$, from which,

$$v = \int e^{-j2\pi nt}\,dt = \frac{e^{-j2\pi nt}}{-j2\pi n}$$

Hence, $c_n = \int_0^1 t\,e^{-j2\pi nt} = uv - \int v\,du$

$$= \left[t\frac{e^{-j2\pi nt}}{-j2\pi n}\right]_0^1 - \int_0^1 \frac{e^{-j2\pi nt}}{-j2\pi n}\,dt$$

$$= \left[t\frac{e^{-j2\pi nt}}{-j2\pi n} - \frac{e^{-j2\pi nt}}{(-j2\pi n)^2}\right]_0^1$$

$$= \left(\frac{e^{-j2\pi n}}{-j2\pi n} - \frac{e^{-j2\pi n}}{(-j2\pi n)^2}\right) - \left(0 - \frac{e^0}{(-j2\pi n)^2}\right)$$

From equation (2),

$$c_n = \left(\frac{\cos 2\pi n - j\sin 2\pi n}{-j2\pi n} - \frac{\cos 2\pi n - j\sin 2\pi n}{(-j2\pi n)^2}\right) + \frac{1}{(-j2\pi n)^2}$$

However, $\cos 2\pi n = 1$ and $\sin 2\pi n = 0$ for all positive and negative integer values of n.

Thus, $c_n = \dfrac{1}{-j2\pi n} - \dfrac{1}{(-j2\pi n)^2} + \dfrac{1}{(-j2\pi n)^2}$

$$= \frac{1}{-j2\pi n} = \frac{1(j)}{-j2\pi n(j)}$$

i.e. $\boldsymbol{c_n = \dfrac{j}{2\pi n}}$

From equation (13),

$$\boldsymbol{c_0} = a_0 = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(t)\,dt$$

$$= \frac{1}{L}\int_0^L f(t)\,dt = \frac{1}{1}\int_0^1 t\,dt$$

$$= \left[\frac{t^2}{2}\right]_0^1 = \left[\frac{1}{2} - 0\right] = \mathbf{\frac{1}{2}}$$

Hence, the complex Fourier series is given by:

$$\boldsymbol{f(t)} = \sum_{n=-\infty}^{\infty} c_n\,e^{j\frac{2\pi nt}{L}} \text{ from equation (11)}$$

i.e. $\boldsymbol{f(t) = \frac{1}{2} + \sum_{n=-\infty}^{\infty} \frac{j}{2\pi n}\,e^{j2\pi nt}}$

$$= \mathbf{\frac{1}{2} + \frac{j}{2\pi}\sum_{n=-\infty}^{\infty}\frac{e^{j2\pi nt}}{n}}$$

Problem 3. Show that the exponential form of the Fourier series for the waveform described by:

$$f(x) = \begin{cases} 0 \text{ when } -4 \le x \le 0 \\ 10 \text{ when } \ 0 \le x \le 4 \end{cases}$$

and has a period of 8, is given by:

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{5j}{n\pi}(\cos n\pi - 1)\,e^{j\frac{n\pi x}{4}}.$$

From equation (12),

$$c_n = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\,e^{-j\frac{2\pi nx}{L}}\,dx$$

$$= \frac{1}{8}\left[\int_{-4}^{0} 0\,e^{-j\frac{\pi nx}{4}}\,dx + \int_0^4 10\,e^{-j\frac{\pi nx}{4}}\,dx\right]$$

$$= \frac{10}{8}\left[\frac{e^{-j\frac{\pi nt}{4}}}{-j\frac{\pi n}{4}}\right]_0^4 = \frac{10}{8}\left(\frac{4}{-j\pi n}\right)\left[e^{-j\pi n} - 1\right]$$

$$= \frac{5j}{-j^2\pi n}\left(e^{-j\pi n} - 1\right) = \frac{5j}{\pi n}\left(e^{-j\pi n} - 1\right)$$

From equation (2), $e^{-j\theta} = \cos\theta - j\sin\theta$, thus $e^{-j\pi n} = \cos\pi n - j\sin\pi n = \cos\pi n$ for all integer values of n. Hence,

$$c_n = \frac{5j}{\pi n}\left(e^{-j\pi n} - 1\right) = \frac{5j}{\pi n}(\cos n\pi - 1)$$

From equation (11), the exponential Fourier series is given by:

$$\boldsymbol{f(x)} = \sum_{n=-\infty}^{\infty} c_n\,e^{j\frac{2\pi nx}{L}}$$

$$= \sum_{n=-\infty}^{\infty} \frac{5j}{n\pi}(\cos n\pi - 1)\,e^{j\frac{n\pi x}{4}}$$

Now try the following exercise

Exercise 236 Further problems on the complex form of a Fourier series

1. Determine the complex Fourier series for the function defined by:

$$f(t)=\begin{cases}0, & \text{when } -\pi \le t \le 0\\ 2, & \text{when } 0 \le t \le \pi\end{cases}$$

The function is periodic outside of this range of period 2π.

$$\left[f(t)=\sum_{n=-\infty}^{\infty}\frac{j}{n\pi}(\cos n\pi-1)\,e^{jnt}\right.$$
$$=1-j\frac{2}{\pi}\left(e^{jt}+\frac{1}{3}e^{j3t}+\frac{1}{5}e^{j5t}+\cdots\right)$$
$$\left.+j\frac{2}{\pi}\left(e^{-jt}+\frac{1}{3}e^{-j3t}+\frac{1}{5}e^{-j5t}+\cdots\right)\right]$$

2. Show that the complex Fourier series for the waveform shown in Figure 71.3, that has period 2, may be represented by:

$$f(t)=2+\sum_{\substack{n=-\infty\\(n\neq 0)}}^{\infty}\frac{j2}{\pi n}(\cos n\pi-1)\,e^{j\pi nt}$$

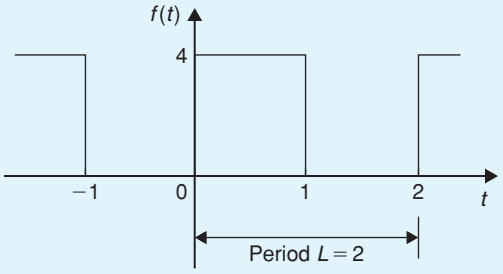


Figure 71.3

3. Show that the complex Fourier series of Problem 2 is equivalent to:

$$f(t)=2+\frac{8}{\pi}\left(\sin\pi t+\frac{1}{3}\sin 3\pi t+\frac{1}{5}\sin 5\pi t+\cdots\right)$$

4. Determine the exponential form of the Fourier series for the function defined by: $f(t)=e^{2t}$ when $-1<t<1$ and has period 2.

$$\left[f(t)=\frac{1}{2}\sum_{n=-\infty}^{\infty}\left(\frac{e^{(2-j\pi n)}-e^{-(2-j\pi n)}}{2-j\pi n}\right)e^{j\pi nt}\right]$$

71.4 Symmetry relationships

If even or odd symmetry is noted in a function, then time can be saved in determining coefficients.

The Fourier coefficients present in the complex Fourier series form are affected by symmetry. Summarising from previous chapters:

An **even function** is symmetrical about the vertical axis and contains no sine terms, i.e. $b_n=0$.

For even symmetry,

$$a_0=\frac{1}{L}\int_0^L f(x)\,dx \quad \text{and}$$

$$a_n=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{2\pi nx}{L}\right)dx$$
$$=\frac{4}{L}\int_0^{\frac{L}{2}} f(x)\cos\left(\frac{2\pi nx}{L}\right)dx$$

An **odd function** is symmetrical about the origin and contains no cosine terms, $a_0=a_n=0$.

For odd symmetry,

$$b_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{2\pi nx}{L}\right)dx$$
$$=\frac{4}{L}\int_0^{\frac{L}{2}} f(x)\sin\left(\frac{2\pi nx}{L}\right)dx$$

From equation (7), page 645, $c_n=\dfrac{a_n-jb_n}{2}$

Thus, for **even symmetry**, $b_n=0$ and

$$\boldsymbol{c_n=\frac{a_n}{2}=\frac{2}{L}\int_0^{\frac{L}{2}} f(x)\cos\left(\frac{2\pi nx}{L}\right)dx} \qquad (15)$$

For **odd symmetry**, $a_n=0$ and

$$\boldsymbol{c_n=\frac{-jb_n}{2}=-j\frac{2}{L}\int_0^{\frac{L}{2}} f(x)\sin\left(\frac{2\pi nx}{L}\right)dx} \qquad (16)$$

For example, in Problem 1 on page 646, the function $f(x)$ is even, since the waveform is symmetrical about

the $f(x)$ axis. Thus equation (15) could have been used, giving:

$$c_n = \frac{2}{L}\int_0^{\frac{L}{2}} f(x)\cos\left(\frac{2\pi nx}{L}\right)\mathrm{d}x$$

$$= \frac{2}{4}\int_0^{2} f(x)\cos\left(\frac{2\pi nx}{4}\right)\mathrm{d}x$$

$$= \frac{1}{2}\left\{\int_0^1 5\cos\left(\frac{\pi nx}{2}\right)\mathrm{d}x + \int_1^2 0\,\mathrm{d}x\right\}$$

$$= \frac{5}{2}\left[\frac{\sin\left(\frac{\pi nx}{2}\right)}{\frac{\pi n}{2}}\right]_0^1 = \frac{5}{2}\left(\frac{2}{\pi n}\right)\left(\sin\frac{n\pi}{2} - 0\right)$$

$$= \boldsymbol{\frac{5}{\pi n}\sin\frac{n\pi}{2}}$$

which is the same answer as in Problem 1; however, a knowledge of even functions has produced the coefficient more quickly.

Problem 4. Obtain the Fourier series, in complex form, for the square wave shown in Figure 71.4.

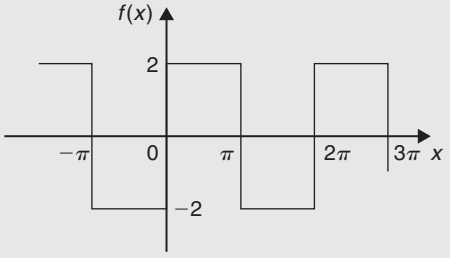


Figure 71.4

Method A

The square wave shown in Figure 71.4 is an **odd function** since it is symmetrical about the origin.
The period of the waveform, $L = 2\pi$.

Thus, using equation (16):

$$c_n = -j\frac{2}{L}\int_0^{\frac{L}{2}} f(x)\sin\left(\frac{2\pi nx}{L}\right)\mathrm{d}x$$

$$= -j\frac{2}{2\pi}\int_0^{\pi} 2\sin\left(\frac{2\pi nx}{2\pi}\right)\mathrm{d}x$$

$$= -j\frac{2}{\pi}\int_0^{\pi}\sin nx\,\mathrm{d}x = -j\frac{2}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi}$$

$$= -j\frac{2}{\pi n}\Big((-\cos\pi n) - (-\cos 0)\Big)$$

i.e. $$\boldsymbol{c_n = -j\frac{2}{\pi n}[1 - \cos\pi n]} \qquad (17)$$

Method B

If it had **not** been noted that the function was odd, equation (12) would have been used, i.e.

$$c_n = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\,\mathrm{e}^{-j\frac{2\pi nx}{L}}\,\mathrm{d}x$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{e}^{-j\frac{2\pi nx}{2\pi}}\,\mathrm{d}x$$

$$= \frac{1}{2\pi}\left\{\int_{-\pi}^{0} -2\mathrm{e}^{-jnx}\,\mathrm{d}x + \int_0^{\pi} 2\mathrm{e}^{-jnx}\,\mathrm{d}x\right\}$$

$$= \frac{1}{2\pi}\left\{\left[\frac{-2\mathrm{e}^{-jnx}}{-jn}\right]_{-\pi}^{0} + \left[\frac{2\mathrm{e}^{-jnx}}{-jn}\right]_0^{\pi}\right\}$$

$$= \frac{1}{2\pi}\left(\frac{2}{jn}\right)\left\{\left[\mathrm{e}^{-jnx}\right]_{-\pi}^{0} - \left[\mathrm{e}^{-jnx}\right]_0^{\pi}\right\}$$

$$= \frac{1}{2\pi}\left(\frac{2}{jn}\right)\left\{\left[\mathrm{e}^{0} - \mathrm{e}^{+jn\pi}\right] - \left[\mathrm{e}^{-jn\pi} - \mathrm{e}^{0}\right]\right\}$$

$$= \frac{1}{jn\pi}\left\{1 - \mathrm{e}^{jn\pi} - \mathrm{e}^{-jn\pi} + 1\right\}$$

$$= \frac{1}{jn\pi}\left\{2 - 2\left(\frac{\mathrm{e}^{jn\pi} + \mathrm{e}^{-jn\pi}}{2}\right)\right\}$$

by rearranging

$$= \frac{2}{jn\pi}\left\{1 - \left(\frac{\mathrm{e}^{jn\pi} + \mathrm{e}^{-jn\pi}}{2}\right)\right\}$$

$$= \frac{2}{jn\pi}\{1 - \cos n\pi\} \qquad \text{from equation (3)}$$

$$= \frac{-j2}{-j(jn\pi)}\{1 - \cos n\pi\}$$

by multiplying top and bottom by $-j$

i.e. $$\boldsymbol{c_n = -j\frac{2}{n\pi}(1 - \cos n\pi)} \qquad (17)$$

It is clear that method A is by far the shorter of the two methods.

From equation (11), the complex Fourier series is given by:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi nx}{L}}$$

$$= \sum_{n=-\infty}^{\infty} -j\frac{2}{n\pi}(1-\cos n\pi)e^{jnx} \qquad (18)$$

Problem 5. Show that the complex Fourier series obtained in problem 4 above is equivalent to

$$f(x) = \frac{8}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \cdots\right)$$

(which was the Fourier series obtained in terms of sines and cosines in Problem 3 on page 625).

From equation (17) above, $c_n = -j\frac{2}{n\pi}(1-\cos n\pi)$

When $n=1$,

$$c_1 = -j\frac{2}{(1)\pi}(1-\cos\pi)$$

$$= -j\frac{2}{\pi}\left(1-(-1)\right) = -\frac{j4}{\pi}$$

When $n=2$,

$$c_2 = -j\frac{2}{2\pi}(1-\cos 2\pi) = 0;$$

in fact, all even values of c_n will be zero.

When $n=3$,

$$c_3 = -j\frac{2}{3\pi}(1-\cos 3\pi)$$

$$= -j\frac{2}{3\pi}(1-(-1)) = -\frac{j4}{3\pi}$$

By similar reasoning,

$$c_5 = -\frac{j4}{5\pi},\ c_7 = -\frac{j4}{7\pi},\ \text{and so on.}$$

When $n=-1$,

$$c_{-1} = -j\frac{2}{(-1)\pi}(1-\cos(-\pi))$$

$$= +j\frac{2}{\pi}(1-(-1)) = +\frac{j4}{\pi}$$

When $n=-3$,

$$c_{-3} = -j\frac{2}{(-3)\pi}(1-\cos(-3\pi))$$

$$= +j\frac{2}{3\pi}(1-(-1)) = +\frac{j4}{3\pi}$$

By similar reasoning,

$$c_{-5} = +\frac{j4}{5\pi},\ c_{-7} = +\frac{j4}{7\pi},\ \text{and so on.}$$

Since the waveform is odd, $c_0 = a_0 = 0$.

From equation (18) above,

$$f(x) = \sum_{n=-\infty}^{\infty} -j\frac{2}{n\pi}(1-\cos n\pi)e^{jnx}$$

Hence,

$$f(x) = -\frac{j4}{\pi}e^{jx} - \frac{j4}{3\pi}e^{j3x} - \frac{j4}{5\pi}e^{j5x} - \frac{j4}{7\pi}e^{j7x} - \cdots + \frac{j4}{\pi}e^{-jx} + \frac{j4}{3\pi}e^{-j3x} + \frac{j4}{5\pi}e^{-j5x} + \frac{j4}{7\pi}e^{-j7x} + \cdots$$

$$= \left(-\frac{j4}{\pi}e^{jx} + \frac{j4}{\pi}e^{-jx}\right) + \left(-\frac{j4}{3\pi}e^{3x} + \frac{j4}{3\pi}e^{-3x}\right) + \left(-\frac{j4}{5\pi}e^{5x} + \frac{j4}{5\pi}e^{-5x}\right) + \cdots$$

$$= -\frac{j4}{\pi}\left(e^{jx} - e^{-jx}\right) - \frac{j4}{3\pi}\left(e^{3x} - e^{-3x}\right) - \frac{j4}{5\pi}\left(e^{5x} - e^{-5x}\right) + \cdots$$

$$= \frac{4}{j\pi}\left(e^{jx} - e^{-jx}\right) + \frac{4}{j3\pi}\left(e^{3x} - e^{-3x}\right) + \frac{4}{j5\pi}\left(e^{5x} - e^{-5x}\right) + \cdots$$

by multiplying top and bottom by j

$$= \frac{8}{\pi}\left(\frac{e^{jx} - e^{-jx}}{2j}\right) + \frac{8}{3\pi}\left(\frac{e^{j3x} - e^{-j3}}{2j}\right) + \frac{8}{5\pi}\left(\frac{e^{j5x} - e^{-j5x}}{2j}\right) + \cdots$$

by rearranging

$$= \frac{8}{\pi}\sin x + \frac{8}{3\pi}\sin 3x + \frac{8}{3x}\sin 5x + \cdots$$

from equation (4), page 644

i.e.

$$f(x)=\frac{8}{\pi}\left(\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+\frac{1}{7}\sin 7x+\cdots\right)$$

Hence,

$$f(x)=\sum_{n=-\infty}^{\infty}-j\frac{2}{n\pi}(1-\cos n\pi)\,e^{jnx}$$
$$\equiv\frac{8}{\pi}\left(\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+\frac{1}{7}\sin 7x+\cdots\right)$$

Now try the following exercise

Exercise 237 Further problems on symmetry relationships

1. Determine the exponential form of the Fourier series for the periodic function defined by:

$$f(x)=\begin{cases}-2, & \text{when } -\pi\le x\le -\frac{\pi}{2}\\ 2, & \text{when } -\frac{\pi}{2}\le x\le +\frac{\pi}{2}\\ -2, & \text{when } +\frac{\pi}{2}\le x\le +\pi\end{cases}$$

and has a period of 2π.

$$\left[f(x)=\sum_{n=-\infty}^{\infty}\left(\frac{4}{n\pi}\sin\frac{n\pi}{2}\right)e^{jnx}\right]$$

2 Show that the exponential form of the Fourier series in problem 1 above is equivalent to:

$$f(x)=\frac{8}{\pi}\left(\cos x-\frac{1}{3}\cos 3x+\frac{1}{5}\cos 5x-\frac{1}{7}\cos 7x+\cdots\right)$$

3. Determine the complex Fourier series to represent the function $f(t)=2t$ in the range $-\pi$ to $+\pi$. $\left[f(t)=\sum_{n=-\infty}^{\infty}\left(\frac{j2}{n}\cos n\pi\right)e^{jnt}\right]$

4. Show that the complex Fourier series in problem 3 above is equivalent to:

$$f(t)=4\left(\sin t-\frac{1}{2}\sin 2t+\frac{1}{3}\sin 3t-\frac{1}{4}\sin 4t+\cdots\right)$$

71.5 The frequency spectrum

In the Fourier analysis of periodic waveforms seen in previous chapters, although waveforms physically exist in the time domain, they can be regarded as comprising components with a variety of frequencies. The amplitude and phase of these components are obtained from the Fourier coefficients a_n and b_n; this is known as a **frequency domain**. Plots of amplitude/frequency and phase/frequency are together known as the **spectrum** of a waveform. A simple example is demonstrated in Problem 6 following.

Problem 6. A pulse of height 20 and width 2 has a period of 10. Sketch the spectrum of the waveform.

The pulse is shown in Figure 71.5.

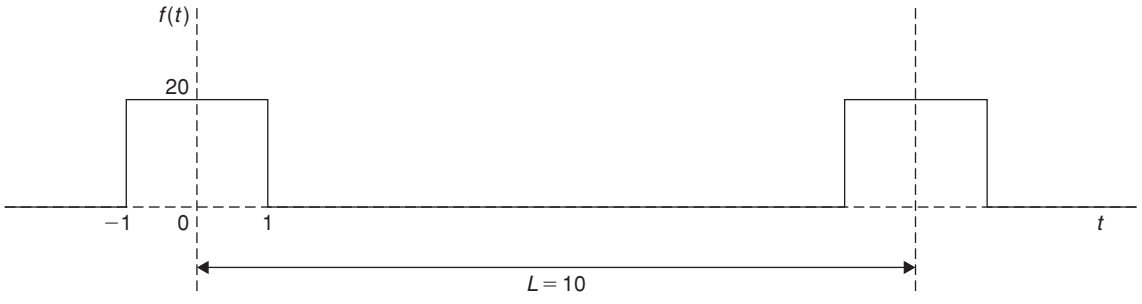


Figure 71.5

The complex coefficient is given by equation (12):

$$c_n = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(t)\mathrm{e}^{-j\frac{2\pi nt}{L}}\,\mathrm{d}t$$

$$= \frac{1}{10}\int_{-1}^{1} 20\mathrm{e}^{-j\frac{2\pi nt}{10}}\,\mathrm{d}t = \frac{20}{10}\left[\frac{\mathrm{e}^{-j\frac{\pi nt}{5}}}{\frac{-j\pi n}{5}}\right]_{-1}^{1}$$

$$= \frac{20}{10}\left(\frac{5}{-j\pi n}\right)\left[\mathrm{e}^{-j\frac{\pi n}{5}} - \mathrm{e}^{j\frac{\pi n}{5}}\right]$$

$$= \frac{20}{\pi n}\left[\frac{\mathrm{e}^{j\frac{\pi n}{5}} - \mathrm{e}^{-j\frac{\pi n}{5}}}{2j}\right]$$

i.e. $\boldsymbol{c_n = \frac{20}{\pi n}\sin\frac{n\pi}{5}}$

from equation (4), page 644.

From equation (13),

$$c_0 = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\,\mathrm{d}x = \frac{1}{10}\int_{-1}^{1} 20\,\mathrm{d}t$$

$$= \frac{1}{10}[20t]_{-1}^{1} = \frac{1}{10}[20-(-20)] = \mathbf{4}$$

$$c_1 = \frac{20}{\pi}\sin\frac{\pi}{5} = \mathbf{3.74} \text{ and}$$

$$c_{-1} = -\frac{20}{\pi}\sin\left(-\frac{\pi}{5}\right) = \mathbf{3.74}$$

Further values of c_n and c_{-n}, up to $n = 10$, are calculated and are shown in the following table.

n	c_n	c_{-n}
0	4	4
1	3.74	3.74
2	3.03	3.03
3	2.02	2.02
4	0.94	0.94
5	0	0
6	−0.62	−0.62
7	−0.86	−0.86
8	−0.76	−0.76
9	−0.42	−0.42
10	0	0

A graph of $|c_n|$ plotted against the number of the harmonic, n, is shown in Figure 71.6.

Figure 71.7 shows the corresponding plot of c_n against n.

Since c_n is real (i.e. no j terms) then the phase must be either 0° or ±180°, depending on the sign of the sine, as shown in Figure 71.8.

When c_n is positive, i.e. between $n=-4$ and $n=+4$, angle $\alpha_n = 0°$.

When c_n is negative, then $\alpha_n = \pm 180°$; between $n=+6$ and $n=+9$, α_n is taken as $+180°$, and between $n=-6$ and $n=-9$, α_n is taken as $-180°$.

Figures 71.6 to 71.8 together form the spectrum of the waveform shown in Figure 71.5.

71.6 Phasors

Electrical engineers in particular often need to analyse alternating current circuits, i.e. circuits containing a sinusoidal input and resulting sinusoidal currents and voltages within the circuit.

It was shown in chapter 14, page 143, that a general sinusoidal voltage function can be represented by:

$$v = V_m \sin(\omega t + \alpha) \text{ volts} \qquad (19)$$

where V_m is the maximum voltage or amplitude of the voltage v, ω is the angular velocity ($=2\pi f$, where f is the frequency), and α is the phase angle compared with $v = V_m \sin\omega t$.

Similarly, a sinusoidal expression may also be expressed in terms of cosine as:

$$v = V_m \cos(\omega t + \alpha) \text{ volts} \qquad (20)$$

It is quite complicated to add, subtract, multiply and divide quantities in the time domain form of equations (19) and (20). As an alternative method of analysis a waveform representation called a **phasor** is used. A phasor has two distinct parts—a magnitude and an angle; for example, the polar form of a complex number, say $5\angle\pi/6$, can represent a phasor, where 5 is the magnitude or modulus, and $\pi/6$ radians is the angle or argument. Also, it was shown on page 228 that $5\angle\pi/6$ may be written as $5\mathrm{e}^{j\pi/6}$ in exponential form.

In chapter 21, equation (4), page 228, it is shown that:

$$\mathrm{e}^{j\theta} = \cos\theta + j\sin\theta \qquad (21)$$

which is known as **Euler's formula**.

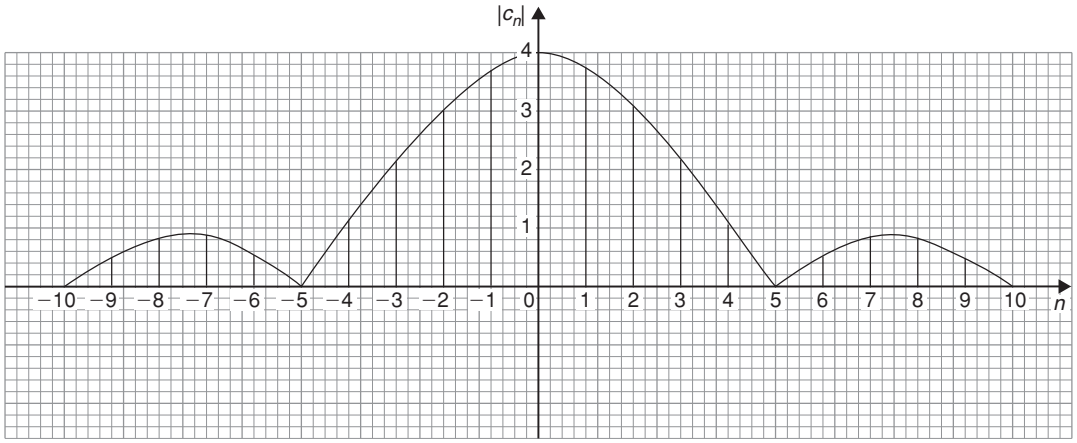


Figure 71.6

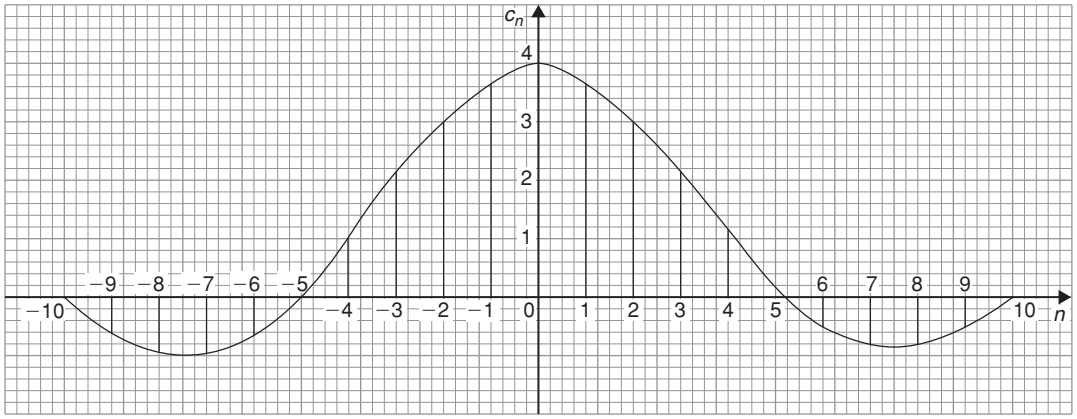


Figure 71.7

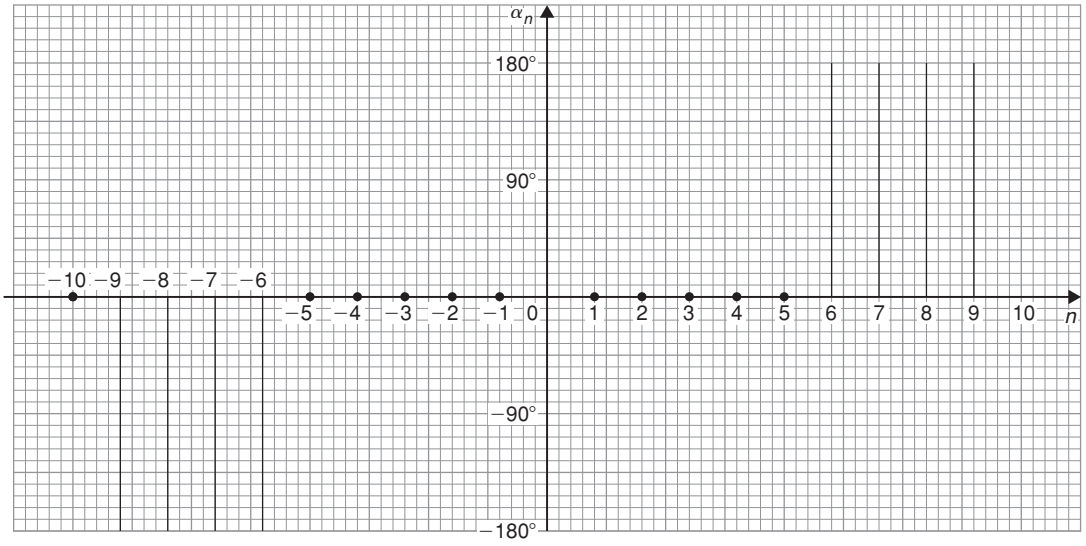


Figure 71.8

From equation (21),

$$e^{j(\omega t+\alpha)} = \cos(\omega t+\alpha) + j\sin(\omega t+\alpha)$$

and $$V_m e^{j(\omega t+\alpha)} = V_m\cos(\omega t+\alpha) + jV_m\sin(\omega t+\alpha)$$

Thus a sinusoidal varying voltage such as in equation (19) or equation (20) can be considered to be either the real or the imaginary part of $V_m e^{j(\omega t+\alpha)}$, depending on whether the cosine or sine function is being considered.

$V_m e^{j(\omega t+\alpha)}$ may be rewritten as $V_m e^{j\omega t}e^{j\alpha}$ since $a^{m+n} = a^m \times a^n$ from the laws of indices, page 1.

The $e^{j\omega t}$ term can be considered to arise from the fact that a radius is rotated with an angular velocity ω, and α is the angle at which the radius starts to rotate at time $t = 0$ (see Chapter 14, page 143).

Thus, $V_m e^{j\omega t}e^{j\alpha}$ defines a **phasor**. In a particular circuit the angular velocity ω is the same for all the elements thus the phasor can be adequately described by $V_m\angle\alpha$, as suggested above.

Alternatively, if

$$v = V_m\cos(\omega t+\alpha) \text{ volts}$$

and $$\cos\theta = \frac{1}{2}\left(e^{j\vartheta} + e^{-j\theta}\right)$$

from equation (3), page 644

then $$v = V_m\left[\frac{1}{2}\left(e^{j(\omega t+\alpha)} + e^{-j(\omega t+\alpha)}\right)\right]$$

i.e. $$v = \frac{1}{2}V_m e^{j\omega t}e^{j\alpha} + \frac{1}{2}V_m e^{-j\omega t}e^{-j\alpha}$$

Thus, v is the sum of two phasors, each with half the amplitude, with one having a positive value of angular velocity (i.e. rotating anticlockwise) and a positive value of α, and the other having a negative value of angular velocity (i.e. rotating clockwise) and a negative value of α, as shown in Figure 71.9.

The two phasors are $\frac{1}{2}V_m\angle\alpha$ and $\frac{1}{2}V_m\angle-\alpha$.

From equation (11), page 645, the Fourier representation of a waveform in complex form is:

$c_n e^{j\frac{2\pi nt}{L}} = c_n e^{j\omega nt}$ for positive values of n $\left(\text{since } \omega = \frac{2\pi}{L}\right)$

and $c_n e^{-j\omega nt}$ for negative values of n.

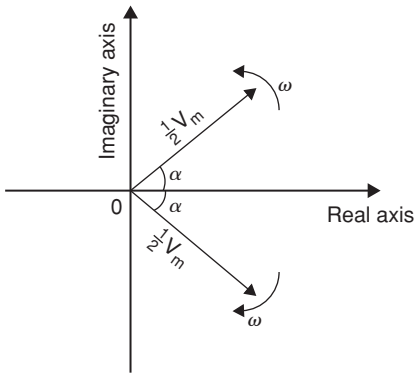


Figure 71.9

It can thus be considered that these terms represent phasors, those with positives powers being phasors rotating with a positive angular velocity (i.e. anticlockwise), and those with negative powers being phasors rotating with a negative angular velocity (i.e. clockwise).

In the above equations,

$n = 0$ represents a non-rotating component, since $e^0 = 1$,
$n = 1$ represents a rotating component with angular velocity of 1ω,
$n = 2$ represents a rotating component with angular velocity of 2ω, and so on.

Thus we have a set of phasors, the algebraic sum of which at some instant of time gives the magnitude of the waveform at that time.

Problem 7. Determine the pair of phasors that can be used to represent the following voltages:
(a) $v = 8\cos 2t$ (b) $v = 8\cos(2t - 1.5)$

(a) From equation (3), page 644,

$$\cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$$

Hence,

$$v = 8\cos 2t = 8\left[\frac{1}{2}\left(e^{j2t} + e^{-j2t}\right)\right]$$

$$= 4e^{j2t} + 4e^{-j2t}$$

This represents a phasor of length 4 rotating anticlockwise (i.e. in the positive direction) with an angular velocity of 2 rad/s, and another phasor of length 4 and rotating clockwise (i.e. in the

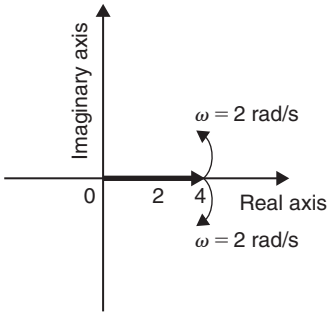


Figure 71.10

negative direction) with an angular velocity of 2 rad/s. Both phasors have zero phase angle. Figure 71.10 shows the two phasors.

(b) From equation (3), page 644,

$$\cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)$$

Hence, $v = 8\cos(2t - 1.5)$

$$= 8\left[\frac{1}{2}\left(e^{j(2t-1.5)} + e^{-j(2t-1.5)}\right)\right]$$

$$= 4e^{j(2t-1.5)} + 4e^{-j(2t-1.5)}$$

i.e. $\boldsymbol{v = 4e^{2t}\,e^{-j1.5} + 4e^{-j2t}\,e^{j1.5}}$

This represents a phasor of length 4 and phase angle -1.5 radians rotating anticlockwise (i.e. in the positive direction) with an angular velocity of 2 rad/s, and another phasor of length 4 and phase angle $+1.5$ radians and rotating clockwise (i.e. in the negative direction) with an angular velocity of 2 rad/s. Figure 71.11 shows the two phasors.

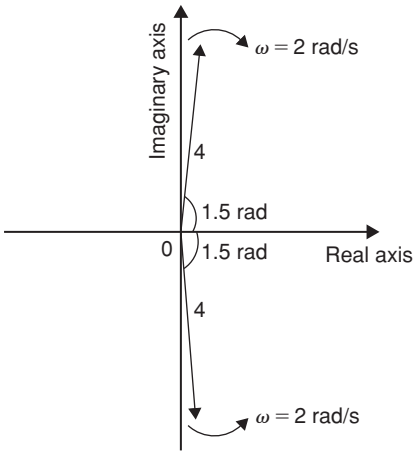


Figure 71.11

Problem 8. Determine – the pair of phasors that can be used to represent the third harmonic

$$v = 8\cos 3t - 20\sin 3t.$$

Using $\cos t = \frac{1}{2}\left(e^{jt} + e^{-jt}\right)$

and $\sin t = \frac{1}{2j}\left(e^{jt} - e^{-jt}\right)$ from page 644

gives: $v = 8\cos 3t - 20\sin 3t$

$$= 8\left[\frac{1}{2}\left(e^{j3t} + e^{-j3t}\right)\right] - 20\left[\frac{1}{2j}\left(e^{j3t} - e^{-j3t}\right)\right]$$

$$= 4e^{j3t} + 4e^{-j3t} - \frac{10}{j}e^{j3t} + \frac{10}{j}e^{-j3t}$$

$$= 4e^{j3t} + 4e^{-j3t} - \frac{10(j)}{j(j)}e^{j3t} + \frac{10(j)}{j(j)}e^{-j3t}$$

$$= 4e^{j3t} + 4e^{-j3t} + 10j\,e^{j3t} - 10j\,e^{-j3t}$$

since $j^2 = -1$

$$= (4 + j10)e^{j3t} + (4 - j10)e^{-j3t}$$

$$(4 + j10) = \sqrt{4^2 + 10^2}\angle\tan^{-1}\left(\frac{10}{4}\right)$$

$$= 10.77\angle 1.19$$

and $(4 - j10)$

$$= 10.77\angle -1.19$$

Hence, $\boldsymbol{v = 10.77\angle 1.19 + 10.77\angle -1.19}$

Thus v comprises a phasor $10.77\angle 1.19$ rotating anticlockwise with an angular velocity if 3 rad/s, and a phasor $10.77\angle -1.19$ rotating clockwise with an angular velocity of 3 rad/s.

Now try the following exercise

Exercise 238 Further problems on phasors

1. Determine the pair of phasors that can be used to represent the following voltages:

   (a) $v = 4\cos 4t$ (b) $v = 4\cos(4t + \pi/2)$.

   [(a) $2e^{j4t} + 2e^{-j4t}$, $2\angle 0°$ anticlockwise, $2\angle 0°$ clockwise, each with $\omega = 4$ rad/s

   (b) $2e^{j4t}e^{j\pi/2} + 2e^{-j4t}e^{-j\pi/2}$, $2\angle \pi/2$ anticlockwise, $2\angle -\pi/2$ clockwise, each with $\omega = 4$ rad/s]

2. Determine the pair of phasors that can represent the harmonic given by: $v = 10\cos 2t - 12\sin 2t$.

   [$(5 + j6)e^{j2t} + (5 - j6)e^{-j2t}$, $7.81\angle 0.88$ rotating anticlockwise, $7.81\angle -0.88$ rotating clockwise, each with $\omega = 2$ rad/s]

3. Find the pair of phasors that can represent the fundamental current: $i = 6\sin t + 4\cos t$.

   [$(2 - j3)e^{jt} + (2 + j3)e^{-jt}$, $3.61\angle -0.98$ rotating anticlockwise, $3.61\angle 0.98$ rotating clockwise, each with $\omega = 1$ rad/s]

Revision Test 19

This Revision Test covers the material contained in Chapters 66 to 71. *The marks for each question are shown in brackets at the end of each question.*

1. Obtain a Fourier series for the periodic function $f(x)$ defined as follows:

$$f(x) = \begin{cases} -1, & \text{when } -\pi \le x \le 0 \\ 1, & \text{when } 0 \le x \le \pi \end{cases}$$

The function is periodic outside of this range with period 2π. (13)

2. Obtain a Fourier series to represent $f(t) = t$ in the range $-\pi$ to $+\pi$. (13)

3. Expand the function $f(\theta) = \theta$ in the range $0 \le \theta \le \pi$ into (a) a half range cosine series, and (b) a half range sine series. (18)

4. (a) Sketch the waveform defined by:

$$f(x) = \begin{cases} 0, & \text{when } -4 \le x \le -2 \\ 3, & \text{when } -2 \le x \le 2 \\ 0, & \text{when } 2 \le x \le 4 \end{cases}$$

and is periodic outside of this range of period 8.

(b) State whether the waveform in (a) is odd, even or neither odd nor even.

(c) Deduce the Fourier series for the function defined in (a). (15)

5. Displacement y on a point on a pulley when turned through an angle of θ degrees is given by:

θ	y
30	3.99
60	4.01
90	3.60
120	2.84
150	1.84
180	0.88
210	0.27
240	0.13
270	0.45
300	1.25
330	2.37
360	3.41

Sketch the waveform and construct a Fourier series for the first three harmonics (23)

6. A rectangular waveform is shown in Fig. RT19.1.

(a) State whether the waveform is an odd or even function.

(b) Obtain the Fourier series for the waveform in complex form.

(c) Show that the complex Fourier series in (b) is equivalent to:

$$f(x) = \frac{20}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \cdots\right)$$

(18)

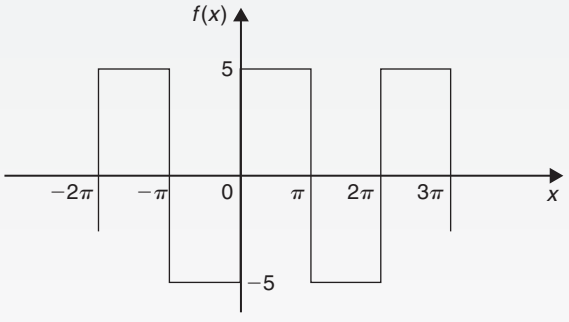


Figure RT19.1

Essential formulae

Number and Algebra

Laws of indices:

$$a^m \times a^n = a^{m+n} \qquad \frac{a^m}{a^n} = a^{m-n} \qquad (a^m)^n = a^{mn}$$

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} \qquad a^{-n} = \frac{1}{a^n} \qquad a^0 = 1$$

Quadratic formula:

$$\text{If} \quad ax^2 + bx + c = 0 \text{ then } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Factor theorem:

If $x = a$ is a root of the equation $f(x) = 0$, then $(x - a)$ is a factor of $f(x)$.

Remainder theorem:

If $(ax^2 + bx + c)$ is divided by $(x - p)$, the remainder will be: $ap^2 + bp + c$.

or if $(ax^3 + bx^2 + cx + d)$ is divided by $(x - p)$, the remainder will be: $ap^3 + bp^2 + cp + d$.

Partial fractions:

Provided that the numerator $f(x)$ is of less degree than the relevant denominator, the following identities are typical examples of the form of partial fractions used:

$$\frac{f(x)}{(x+a)(x+b)(x+c)} \equiv \frac{A}{(x+a)} + \frac{B}{(x+b)} + \frac{C}{(x+c)}$$

$$\frac{f(x)}{(x+a)^3(x+b)} \equiv \frac{A}{(x+a)} + \frac{B}{(x+a)^2} + \frac{C}{(x+a)^3} + \frac{D}{(x+b)}$$

$$\frac{f(x)}{(ax^2+bx+c)(x+d)} \equiv \frac{Ax+B}{(ax^2+bx+c)} + \frac{C}{(x+d)}$$

Definition of a logarithm:

If $y = a^x$ then $x = \log_a y$

Laws of logarithms:

$$\log(A \times B) = \log A + \log B$$

$$\log\left(\frac{A}{B}\right) = \log A - \log B$$

$$\log A^n = n \times \log A$$

Exponential series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

(valid for all values of x)

Hyperbolic functions:

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\cosh^2 x - \sinh^2 = 1 \quad 1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\coth^2 x - 1 = \operatorname{cosech}^2 x$$

Arithmetic progression:

If a = first term and d = common difference, then the arithmetic progression is: $a, a+d, a+2d, \ldots$

The n'th term is: $a+(n-1)d$

Sum of n terms, $S_n = \frac{n}{2}[2a+(n-1)d]$

Geometric progression:

If a = first term and r = common ratio, then the geometric progression is: $a, ar, ar^2, \ldots$

The n'th term is: ar^{n-1}

Sum of n terms, $S_n = \frac{a(1-r^n)}{(1-r)}$ or $\frac{a(r^n-1)}{(r-1)}$

If $-1 < r < 1$, $S_\infty = \frac{a}{(1-r)}$

Binomial series:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

Maclaurin's series:

$$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$$

Newton Raphson iterative method:

If r_1 is the approximate value for a real root of the equation $f(x)=0$, then a closer approximation to the root, r_2, is given by:

$$r_2 = r_1 - \frac{f(r_1)}{f'(r_1)}$$

Boolean algebra:

Laws and rules of Boolean algebra

Commutative Laws:	$A+B=B+A$
	$A\cdot B=B\cdot A$
Associative Laws:	$A+B+C=(A+B)+C$
	$A\cdot B\cdot C=(A\cdot B)\cdot C$
Distributive Laws:	$A\cdot(B+C)=A\cdot B+A\cdot C$
	$A+(B\cdot C)=(A+B)\cdot(A+C)$
Sum rules:	$A+\overline{A}=1$
	$A+1=1$
	$A+0=A$
	$A+A=A$
Product rules:	$A\cdot\overline{A}=0$
	$A\cdot 0=0$
	$A\cdot 1=A$
	$A\cdot A=A$
Absorption rules:	$A+A\cdot B=A$
	$A\cdot(A+B)=A$
	$A+\overline{A}\cdot B=A+B$
De Morgan's Laws:	$\overline{A+B}=\overline{A}\cdot\overline{B}$
	$\overline{A\cdot B}=\overline{A}+\overline{B}$

Geometry and Trigonometry

Theorem of Pythagoras:

$$b^2 = a^2 + c^2$$

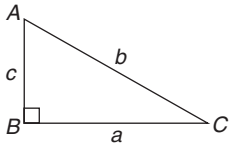


Figure FA1

Identities:

$\sec\theta = \frac{1}{\cos\theta}$ $\quad$ $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$

$\cot\theta = \frac{1}{\tan\theta}$ $\quad$ $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\cos^2\theta + \sin^2\theta = 1$ $\quad$ $1 + \tan^2\theta = \sec^2\theta$

$\cot^2\theta + 1 = \operatorname{cosec}^2\theta$

Triangle formulae:

With reference to Fig. FA2:

Sine rule $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Cosine rule $a^2 = b^2 + c^2 - 2bc\cos A$

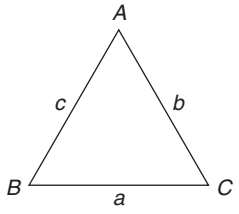


Figure FA2

Area of any triangle

(i) $\frac{1}{2} \times$ base $\times$ perpendicular height

(ii) $\frac{1}{2}ab\sin C$ or $\frac{1}{2}ac\sin B$ or $\frac{1}{2}bc\sin A$

(iii) $\sqrt{[s(s-a)(s-b)(s-c)]}$ where $s = \dfrac{a+b+c}{2}$

Compound angle formulae:

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$$

$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$$

If $\boldsymbol{R\sin(\omega t + \alpha) = a\sin\omega t + b\cos\omega t}$,

then $a = R\cos\alpha$, $b = R\sin\alpha$,

$$R = \sqrt{(a^2 + b^2)} \text{ and } \alpha = \tan^{-1}\frac{b}{a}$$

Double angles:

$$\sin 2A = 2\sin A\cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1$$

$$= 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Products of sines and cosines into sums or differences:

$$\sin A\cos B = \tfrac{1}{2}[\sin(A+B) + \sin(A-B)]$$

$$\cos A\sin B = \tfrac{1}{2}[\sin(A+B) - \sin(A-B)]$$

$$\cos A\cos B = \tfrac{1}{2}[\cos(A+B) + \cos(A-B)]$$

$$\sin A\sin B = -\tfrac{1}{2}[\cos(A+B) - \cos(A-B)]$$

Sums or differences of sines and cosines into products:

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

For a **general sinusoidal function** $\boldsymbol{y = A\sin(\omega t \pm \alpha)}$, then:

A = amplitude

ω = angular velocity $= 2\pi f$ rad/s

$\dfrac{2\pi}{\omega}$ = periodic time T seconds

$\dfrac{\omega}{2\pi}$ = frequency, f hertz

α = angle of lead or lag (compared with $y = A\sin\omega t$)

Cartesian and polar co-ordinates:

If co-ordinate $(x, y) = (r, \theta)$ then $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\dfrac{y}{x}$

If co-ordinate $(r, \theta) = (x, y)$ then $x = r\cos\theta$ and $y = r\sin\theta$.

The circle:

With reference to Fig. FA3.

$$\text{Area} = \pi r^2 \quad \text{Circumference} = 2\pi r$$

π radians $= 180°$

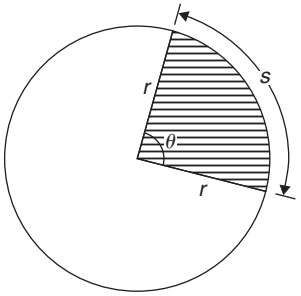


Figure FA3

For sector of circle:

$$s = r\theta \quad (\theta \text{ in rad})$$

$$\text{shaded area} = \tfrac{1}{2}r^2\theta \quad (\theta \text{ in rad})$$

Equation of a circle, centre at (a, b), radius r:

$$(x-a)^2 + (y-b)^2 = r^2$$

Linear and angular velocity:

If v = linear velocity (m/s), s = displacement (m), t = time (s), n = speed of revolution (rev/s), θ = angle (rad), ω = angular velocity (rad/s), r = radius of circle (m) then:

$$v = \frac{s}{t} \quad \omega = \frac{\theta}{t} = 2\pi n \quad v = \omega r$$

$$\textbf{centripetal force} = \frac{mv^2}{r}$$

where m = mass of rotating object.

Graphs

Equations of functions:

Equation of a straight line: $y = mx + c$

Equation of a parabola: $y = ax^2 + bx + c$

Circle, centre (a, b), radius r:

$$(x-a)^2 + (y-b)^2 = r^2$$

Equation of an ellipse, centre at origin, semi-axes a and b:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Equation of a hyperbola:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Equation of a rectangular hyperbola: $xy = c^2$

Irregular areas:

Trapezoidal rule

$$\text{Area} \approx \begin{pmatrix}\text{width of}\\ \text{interval}\end{pmatrix}\left[\frac{1}{2}\begin{pmatrix}\text{first + last}\\ \text{ordinates}\end{pmatrix} + \begin{pmatrix}\text{sum of remaining}\\ \text{ordinates}\end{pmatrix}\right]$$

Mid-ordinate rule

$$\text{Area} \approx \begin{pmatrix}\text{width of}\\ \text{interval}\end{pmatrix}\begin{pmatrix}\text{sum of}\\ \text{mid-ordinates}\end{pmatrix}$$

Simpson's rule

$$\text{Area} \approx \frac{1}{3}\begin{pmatrix}\text{width of}\\ \text{interval}\end{pmatrix}\left[\begin{pmatrix}\text{first + last}\\ \text{ordinate}\end{pmatrix} + 4\begin{pmatrix}\text{sum of even}\\ \text{ordinates}\end{pmatrix} + 2\begin{pmatrix}\text{sum of remaining}\\ \text{odd ordinates}\end{pmatrix}\right]$$

Vector Geometry

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $b = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

$$|a| = \sqrt{a_1^2 + a_2^2 + a_3^2} \quad \cos\theta = \frac{a\cdot b}{|a|\,|b|}$$

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$|a\times b| = \sqrt{[(a\cdot a)(b\cdot b) - (a\cdot b)^2]}$$

Complex Numbers

$z = a + jb = r(\cos\theta + j\sin\theta) = r\angle\theta = re^{j\theta}$ where $j^2 = -1$

Modulus $r = |z| = \sqrt{(a^2 + b^2)}$

Argument $\theta = \arg z = \tan^{-1}\dfrac{b}{a}$

Addition: $(a + jb) + (c + jd) = (a + c) + j(b + d)$

Subtraction: $(a + jb) - (c + jd) = (a - c) + j(b - d)$

Complex equations: If $m + jn = p + jq$ then $m = p$ and $n = q$

Multiplication: $z_1\ z_2 = r_1\ r_2\angle(\theta_1 + \theta_2)$

Division: $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\angle(\theta_1 - \theta_2)$

De Moivre's theorem:

$[r\angle\theta]^n = r^n\angle n\theta = r^n(\cos n\theta + j\sin n\theta) = re^{j\theta}$

Matrices and Determinants

Matrices:

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ then

$$A + B = \begin{pmatrix} a + e & b + f \\ c + g & d + h \end{pmatrix}$$

$$A - B = \begin{pmatrix} a - e & b - f \\ c - g & d - h \end{pmatrix}$$

$$A \times B = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$$

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

If $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ then $A^{-1} = \dfrac{B^T}{|A|}$ where

B^T = transpose of cofactors of matrix A

Determinants:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Differential Calculus

Standard derivatives:

y or $f(x)$	$\dfrac{dy}{dx}$ or $f'(x)$
ax^n	anx^{n-1}
$\sin ax$	$a\cos ax$
$\cos ax$	$-a\sin ax$
$\tan ax$	$a\sec^2 ax$
$\sec ax$	$a\sec ax\tan ax$
$\operatorname{cosec} ax$	$-a\operatorname{cosec} ax\cot ax$
$\cot ax$	$-a\operatorname{cosec}^2 ax$
e^{ax}	ae^{ax}
$\ln ax$	$\dfrac{1}{x}$
$\sinh ax$	$a\cosh ax$
$\cosh ax$	$a\sinh ax$
$\tanh ax$	$a\operatorname{sech}^2 ax$
$\operatorname{sech} ax$	$-a\operatorname{sech} ax\tanh ax$
$\operatorname{cosech} ax$	$-a\operatorname{cosech} ax\coth ax$
$\coth ax$	$-a\operatorname{cosech}^2 ax$
$\sin^{-1}\dfrac{x}{a}$	$\dfrac{1}{\sqrt{a^2 - x^2}}$
$\sin^{-1} f(x)$	$\dfrac{f'(x)}{\sqrt{1 - [f(x)]^2}}$
$\cos^{-1}\dfrac{x}{a}$	$\dfrac{-1}{\sqrt{a^2 - x^2}}$

y or $f(x)$	$\frac{dy}{dx}$ or $f'(x)$
$\cos^{-1} f(x)$	$\frac{-f'(x)}{\sqrt{1-[f(x)]^2}}$
$\tan^{-1}\frac{x}{a}$	$\frac{a}{a^2+x^2}$
$\tan^{-1} f(x)$	$\frac{f'(x)}{1+[f(x)]^2}$
$\sec^{-1}\frac{x}{a}$	$\frac{a}{x\sqrt{x^2-a^2}}$
$\sec^{-1} f(x)$	$\frac{f'(x)}{f(x)\sqrt{[f(x)]^2-1}}$
$\operatorname{cosec}^{-1}\frac{x}{a}$	$\frac{-a}{x\sqrt{x^2-a^2}}$
$\operatorname{cosec}^{-1} f(x)$	$\frac{-f'(x)}{f(x)\sqrt{[f(x)]^2-1}}$
$\cot^{-1}\frac{x}{a}$	$\frac{-a}{a^2+x^2}$
$\cot^{-1} f(x)$	$\frac{-f'(x)}{1+[f(x)]^2}$
$\sinh^{-1}\frac{x}{a}$	$\frac{1}{\sqrt{x^2+a^2}}$
$\sinh^{-1} f(x)$	$\frac{f'(x)}{\sqrt{[f(x)]^2+1}}$
$\cosh^{-1}\frac{x}{a}$	$\frac{1}{\sqrt{x^2-a^2}}$
$\cosh^{-1} f(x)$	$\frac{f'(x)}{\sqrt{[f(x)]^2-1}}$
$\tanh^{-1}\frac{x}{a}$	$\frac{a}{a^2-x^2}$
$\tanh^{-1} f(x)$	$\frac{f'(x)}{1-[f(x)]^2}$
$\operatorname{sech}^{-1}\frac{x}{a}$	$\frac{-a}{x\sqrt{a^2-x^2}}$
$\operatorname{sech}^{-1} f(x)$	$\frac{-f'(x)}{f(x)\sqrt{1-[f(x)]^2}}$
$\operatorname{cosech}^{-1}\frac{x}{a}$	$\frac{-a}{x\sqrt{x^2+a^2}}$
$\operatorname{cosech}^{-1} f(x)$	$\frac{-f'(x)}{f(x)\sqrt{[f(x)]^2+1}}$
$\coth^{-1}\frac{x}{a}$	$\frac{a}{a^2-x^2}$
$\coth^{-1} f(x)$	$\frac{f'(x)}{1-[f(x)]^2}$

Product rule:

When $y = uv$ and u and v are functions of x then:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Quotient rule:

When $y = \frac{u}{v}$ and u and v are functions of x then:

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Function of a function:

If u is a function of x then:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Parametric differentiation:

If x and y are both functions of θ, then:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left(\frac{dy}{dx}\right)}{\frac{dx}{d\theta}}$$

Implicit function:

$$\frac{d}{dx}[f(y)] = \frac{d}{dy}[f(y)] \times \frac{dy}{dx}$$

Maximum and minimum values:

If $y = f(x)$ then $\mathbf{\frac{dy}{dx} = 0}$ for stationary points.

Let a solution of $\frac{dy}{dx} = 0$ be $x = a$; if the value of $\frac{d^2y}{dx^2}$ when $x = a$ is: *positive*, the point is a *minimum*, *negative*, the point is a *maximum*.

Velocity and acceleration:

If distance $x = f(t)$, then

velocity $v = f'(t)$ or $\frac{dx}{dt}$ and

acceleration $a = f''(t)$ or $\frac{d^2x}{dt^2}$

Tangents and normals:

Equation of tangent to curve $y = f(x)$ at the point (x_1, y_1) is:

$$y - y_1 = m(x - x_1)$$

where m = gradient of curve at (x_1, y_1).

Equation of normal to curve $y = f(x)$ at the point (x_1, y_1) is:

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

Partial differentiation:

Total differential

If $z = f(u, v, \ldots)$, then the total differential,

$$dz = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv + \ldots.$$

Rate of change

If $z = f(u, v, \ldots)$ and $\frac{du}{dt}, \frac{dv}{dt}, \ldots$ denote the rate of change of $u, v, \ldots$ respectively, then the rate of change of z,

$$\frac{dz}{dt} = \frac{\partial z}{\partial u} \cdot \frac{du}{dt} + \frac{\partial z}{\partial v} \cdot \frac{dv}{dt} + \ldots$$

Small changes

If $z = f(u, v, \ldots)$ and $\delta x, \delta y, \ldots$ denote small changes in $x, y, \ldots$ respectively, then the corresponding change,

$$\delta z \approx \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y + \ldots.$$

To determine maxima, minima and saddle points for functions of two variables: Given $z = f(x, y)$,

(i) determine $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

(ii) for stationary points, $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$,

(iii) solve the simultaneous equations $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$ for x and y, which gives the co-ordinates of the stationary points,

(iv) determine $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$ and $\frac{\partial^2 z}{\partial x \partial y}$

(v) for each of the co-ordinates of the stationary points, substitute values of x and y into $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$ and $\frac{\partial^2 z}{\partial x \partial y}$ and evaluate each,

(vi) evaluate $\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$ for each stationary point,

(vii) substitute the values of $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$ and $\frac{\partial^2 z}{\partial x \partial y}$ into the equation $\Delta = \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right)$ and evaluate,

(viii) (a) if $\mathbf{\Delta > 0}$ then the stationary point is a **saddle point**

(b) if $\mathbf{\Delta < 0}$ and $\mathbf{\frac{\partial^2 z}{\partial x^2} < 0}$, then the stationary point is a **maximum point**, and

(c) if $\mathbf{\Delta < 0}$ and $\mathbf{\frac{\partial^2 z}{\partial x^2} > 0}$, then the stationary point is a **minimum point**

Integral Calculus

Standard integrals:

y	$\int y\,dx$
ax^n	$a\dfrac{x^{n+1}}{n+1}+c$ (except where $n=-1$)
$\cos ax$	$\dfrac{1}{a}\sin ax+c$
$\sin ax$	$-\dfrac{1}{a}\cos ax+c$
$\sec^2 ax$	$\dfrac{1}{a}\tan ax+c$
$\operatorname{cosec}^2 ax$	$-\dfrac{1}{a}\cot ax+c$
$\operatorname{cosec} ax\cot ax$	$-\dfrac{1}{a}\operatorname{cosec} ax+c$
$\sec ax\tan ax$	$\dfrac{1}{a}\sec ax+c$
e^{ax}	$\dfrac{1}{a}e^{ax}+c$
$\dfrac{1}{x}$	$\ln x+c$
$\tan ax$	$\dfrac{1}{a}\ln(\sec ax)+c$
$\cos^2 x$	$\dfrac{1}{2}\left(x+\dfrac{\sin 2x}{2}\right)+c$
$\sin^2 x$	$\dfrac{1}{2}\left(x-\dfrac{\sin 2x}{2}\right)+c$
$\tan^2 x$	$\tan x-x+c$
$\cot^2 x$	$-\cot x-x+c$
$\dfrac{1}{\sqrt{(a^2-x^2)}}$	$\sin^{-1}\dfrac{x}{a}+c$
$\sqrt{(a^2-x^2)}$	$\dfrac{a^2}{2}\sin^{-1}\dfrac{x}{a}+\dfrac{x}{2}\sqrt{(a^2-x^2)}+c$
$\dfrac{1}{(a^2+x^2)}$	$\dfrac{1}{a}\tan^{-1}\dfrac{x}{a}+c$
$\dfrac{1}{\sqrt{(x^2+a^2)}}$	$\sinh^{-1}\dfrac{x}{a}+c$ or $\ln\left[\dfrac{x+\sqrt{(x^2+a^2)}}{a}\right]+c$
$\sqrt{(x^2+a^2)}$	$\dfrac{a^2}{2}\sinh^{-1}\dfrac{x}{a}+\dfrac{x}{2}\sqrt{(x^2+a^2)}+c$
$\dfrac{1}{\sqrt{(x^2-a^2)}}$	$\cosh^{-1}\dfrac{x}{a}+c$ or $\ln\left[\dfrac{x+\sqrt{(x^2-a^2)}}{a}\right]+c$
$\sqrt{(x^2-a^2)}$	$\dfrac{x}{2}\sqrt{(x^2-a^2)}-\dfrac{a^2}{1}\cosh^{-1}\dfrac{x}{a}+c$

$t=\tan\dfrac{\theta}{2}$ substitution

To determine $\displaystyle\int\frac{1}{a\cos\theta+b\sin\theta+c}d\theta$ let

$$\sin\theta=\frac{2t}{(1+t^2)}\quad\cos\theta=\frac{1-t^2}{1+t^2}\quad\text{and}$$

$$d\theta=\frac{2\,dt}{(1+t^2)}$$

Integration by parts:

If u and v are both functions of x then:

$$\int u\frac{dv}{dx}dx=uv-\int v\frac{du}{dx}dx$$

Reduction formulae:

$$\int x^n e^x\,dx=I_n=x^n e^x-nI_{n-1}$$

$$\int x^n\cos x\,dx=I_n=x^n\sin x+nx^{n-1}\cos x-n(n-1)I_{n-2}$$

$$\int_0^{\pi} x^n \cos x \, dx = I_n = -n\pi^{n-1} - n(n-1)I_{n-2}$$

$$\int x^n \sin x \, dx = I_n = -x^n \cos x + nx^{n-1} \sin x - n(n-1)I_{n-2}$$

$$\int \sin^n x \, dx = I_n = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} I_{n-2}$$

$$\int \cos^n x \, dx = I_n = \frac{1}{n} \cos^{n-1} \sin x + \frac{n-1}{n} I_{n-2}$$

$$\int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \cos^n x \, dx = I_n = \frac{n-1}{n} I_{n-2}$$

$$\int \tan^n x \, dx = I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$

$$\int (\ln x)^n \, dx = I_n = x(\ln x)^n - nI_{n-1}$$

With reference to Fig. FA4.

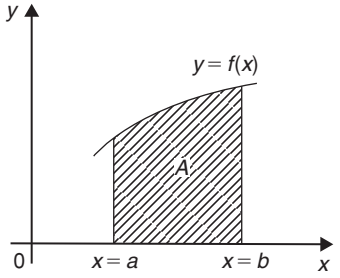


Figure FA4

Area under a curve:

$$\text{area } A = \int_a^b y \, dx$$

Mean value:

$$\text{mean value} = \frac{1}{b-a} \int_a^b y \, dx$$

R.m.s. value:

$$\text{r.m.s. value} = \sqrt{\left\{ \frac{1}{b-a} \int_a^b y^2 \, dx \right\}}$$

Volume of solid of revolution:

$$\text{volume} = \int_a^b \pi y^2 \, dx \text{ about the } x\text{-axis}$$

Centroids:

With reference to Fig. FA5:

$$\bar{x} = \frac{\int_a^b xy \, dx}{\int_a^b y \, dx} \quad \text{and} \quad \bar{y} = \frac{\frac{1}{2}\int_a^b y^2 \, dx}{\int_a^b y \, dx}$$

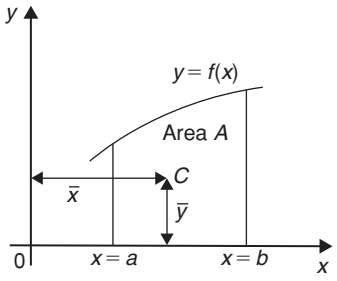


Figure FA5

Theorem of Pappus:

With reference to Fig. FA5, when the curve is rotated one revolution about the x-axis between the limits $x = a$ and $x = b$, the volume V generated is given by: $\boldsymbol{V = 2\pi A\bar{y}}$.

Parallel axis theorem:

If C is the centroid of area A in Fig. FA6 then

$$Ak_{BB}^2 = Ak_{GG}^2 + Ad^2 \text{ or } k_{BB}^2 = k_{GG}^2 + d^2$$

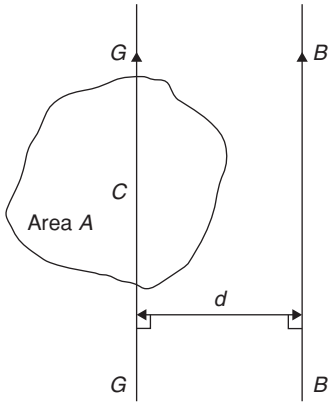


Figure FA6

Second moment of area and radius of gyration:

Shape	Position of axis	Second moment of area, I	Radius of gyration, k
Rectangle length l breadth b	(1) Coinciding with b	$\frac{bl^3}{3}$	$\frac{1}{\sqrt{3}}$
	(2) Coinciding with l	$\frac{lb^3}{3}$	$\frac{b}{\sqrt{3}}$
	(3) Through centroid, parallel to b	$\frac{bl^3}{12}$	$\frac{1}{\sqrt{12}}$
	(4) Through centroid, parallel to l	$\frac{lb^3}{12}$	$\frac{b}{\sqrt{12}}$
Triangle Perpendicular height h base b	(1) Coinciding with b	$\frac{bh^3}{12}$	$\frac{h}{\sqrt{6}}$
	(2) Through centroid, parallel to base	$\frac{bh^3}{36}$	$\frac{h}{\sqrt{18}}$
	(3) Through vertex, parallel to base	$\frac{bh^3}{4}$	$\frac{h}{\sqrt{2}}$
Circle radius r	(1) Through centre, perpendicular to plane (i.e. polar axis)	$\frac{\pi r^4}{2}$	$\frac{r}{\sqrt{2}}$
	(2) Coinciding with diameter	$\frac{\pi r^4}{4}$	$\frac{r}{2}$
	(3) About a tangent	$\frac{5\pi r^4}{4}$	$\frac{\sqrt{5}}{2}r$
Semicircle radius r	Coinciding with diameter	$\frac{\pi r^4}{8}$	$\frac{r}{2}$

Perpendicular axis theorem:

If OX and OY lie in the plane of area A in Fig. FA7, then $Ak_{OZ}^2 = Ak_{OX}^2 + Ak_{OY}^2$ or $k_{OZ}^2 = k_{OX}^2 + k_{OY}^2$

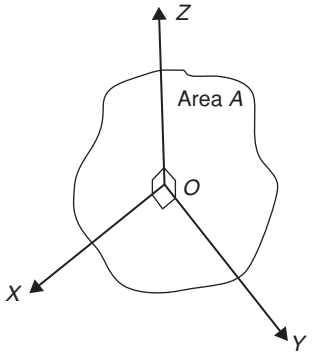


Figure FA7

Numerical integration:

Trapezoidal rule

$$\int y\,dx \approx \begin{pmatrix}\text{width of}\\ \text{interval}\end{pmatrix}\left[\frac{1}{2}\begin{pmatrix}\text{first} + \text{last}\\ \text{ordinates}\end{pmatrix} + \begin{pmatrix}\text{sum of remaining}\\ \text{ordinates}\end{pmatrix}\right]$$

Mid-ordinate rule

$$\int y\,dx \approx \begin{pmatrix}\text{width of}\\ \text{interval}\end{pmatrix}\begin{pmatrix}\text{sum of}\\ \text{mid-ordinates}\end{pmatrix}$$

Simpson's rule

$$\int y\,dx \approx \frac{1}{3}\begin{pmatrix}\text{width of}\\ \text{interval}\end{pmatrix}\left[\begin{pmatrix}\text{first + last}\\ \text{ordinate}\end{pmatrix} + 4\begin{pmatrix}\text{sum of even}\\ \text{ordinates}\end{pmatrix} + 2\begin{pmatrix}\text{sum of remaining}\\ \text{odd ordinates}\end{pmatrix}\right]$$

Differential Equations

First order differential equations:

Separation of variables

If $\frac{dy}{dx} = f(x)$ then $y = \int f(x)\,dx$

If $\frac{dy}{dx} = f(y)$ then $\int dx = \int \frac{dy}{f(y)}$

If $\frac{dy}{dx} = f(x)\cdot f(y)$ then $\int \frac{dy}{f(y)} = \int f(x)\,dx$

Homogeneous equations:

If $P\frac{dy}{dx} = Q$, where P and Q are functions of both x and y of the same degree throughout (i.e. a homogeneous first order differential equation) then:

(i) Rearrange $P\frac{dy}{dx} = Q$ into the form $\frac{dy}{dx} = \frac{Q}{P}$

(ii) Make the substitution $y = vx$ (where v is a function of x), from which, by the product rule,

$$\frac{dy}{dx} = v(1) + x\frac{dv}{dx}$$

(iii) Substitute for both y and $\frac{dy}{dx}$ in the equation $\frac{dy}{dx} = \frac{Q}{P}$

(iv) Simplify, by cancelling, and then separate the variables and solve using the $\frac{dy}{dx} = f(x)\cdot f(y)$ method

(v) Substitute $v = \frac{y}{x}$ to solve in terms of the original variables.

Linear first order:

If $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x only (i.e. a linear first order differential equation), then

(i) determine the integrating factor, $e^{\int P\,dx}$

(ii) substitute the integrating factor (I.F.) into the equation

$$y\ (\text{I.F.}) = \int (\text{I.F.})\ Q\,dx$$

(iii) determine the integral $\int (\text{I.F.})\,Q\,dx$

Numerical solutions of first order differential equations:

Euler's method: $y_1 = y_0 + h(y')_0$

Euler-Cauchy method: $y_{P_1} = y_0 + h(y')_0$

and $y_{C_1} = y_0 + \frac{1}{2}h[(y')_0 + f(x_1, y_{p_1})]$

Runge-Kutta method:

To solve the differential equation $\frac{dy}{dx} = f(x, y)$ given the initial condition $y = y_0$ at $x = x_0$ for a range of values of $x = x_0(h)x_n$:

1. Identify x_0, y_0 and h, and values of $x_1, x_2, x_3, \ldots$
2. Evaluate $k_1 = f(x_n, y_n)$ starting with $n = 0$
3. Evaluate $k_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right)$
4. Evaluate $k_3 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_2\right)$
5. Evaluate $k_4 = f(x_n + h, y_n + hk_3)$
6. Use the values determined from steps 2 to 5 to evaluate:

   $y_{n+1} = y_n + \frac{h}{6}\{k_1 + 2k_2 + 2k_3 + k_4\}$
7. Repeat steps 2 to 6 for $n = 1, 2, 3, \ldots$

Second order differential equations:

If $\boldsymbol{a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0}$ (where a, b and c are constants) then:

(i) rewrite the differential equation as $(aD^2 + bD + c)y = 0$

(ii) substitute m for D and solve the auxiliary equation $am^2 + bm + c = 0$

(iii) if the roots of the auxiliary equation are:

(a) **real and different**, say $m=\alpha$ and $m=\beta$ then the general solution is

$$y = Ae^{\alpha x} + Be^{\beta x}$$

(b) **real and equal**, say $m=\alpha$ twice, then the general solution is

$$y = (Ax + B)e^{\alpha x}$$

(c) **complex**, say $m=\alpha \pm j\beta$, then the general solution is

$$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$$

(iv) given boundary conditions, constants A and B can be determined and the particular solution obtained.

If $a\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = f(x)$ then:

(i) rewrite the differential equation as $(aD^2 + bD + c)y = 0$.

(ii) substitute m for D and solve the auxiliary equation $am^2 + bm + c = 0$.

(iii) obtain the complimentary function (C.F.), u, as per (iii) above.

(iv) to find the particular integral, v, first assume a particular integral which is suggested by $f(x)$, but which contains undetermined coefficients (See Table 51.1, page 484 for guidance).

(v) substitute the suggested particular integral into the original differential equation and equate relevant coefficients to find the constants introduced.

(vi) the general solution is given by $y = u + v$.

(vii) given boundary conditions, arbitrary constants in the C.F. can be determined and the particular solution obtained.

Higher derivatives:

y	$y^{(n)}$
e^{ax}	$a^n e^{ax}$
$\sin ax$	$a^n \sin\left(ax + \dfrac{n\pi}{2}\right)$
$\cos ax$	$a^n \cos\left(ax + \dfrac{n\pi}{2}\right)$
x^a	$\dfrac{a!}{(a-n)!}x^{a-n}$
$\sinh ax$	$\dfrac{a^n}{2}\{[1+(-1)^n]\sinh ax + [1-(-1)^n]\cosh ax\}$
$\cosh ax$	$\dfrac{a^n}{2}\{[1-(-1)^n]\sinh ax + [1+(-1)^n]\cosh ax\}$
$\ln ax$	$(-1)^{n-1}\dfrac{(n-1)!}{x^n}$

Leibniz's theorem:

To find the n'th derivative of a product $y = uv$:

$$y^{(n)} = (uv)^{(n)} = u^{(n)}v + nu^{(n-1)}v^{(1)} + \frac{n(n-1)}{2!}u^{(n-2)}v^{(2)} + \frac{n(n-1)(n-2)}{3!}u^{(n-3)}v^{(3)} + \cdots$$

Power series solutions of second order differential equations:

(a) **Leibniz-Maclaurin method**

(i) Differentiate the given equation n times, using the Leibniz theorem,

(ii) rearrange the result to obtain the recurrence relation at $x = 0$,

(iii) determine the values of the derivatives at $x = 0$, i.e. find $(y)_0$ and $(y')_0$,

(iv) substitute in the Maclaurin expansion for $y = f(x)$,

(v) simplify the result where possible and apply boundary condition (if given).

(b) **Frobenius method**

(i) Assume a trial solution of the form: $y = x^c\{a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_rx^r + \cdots\}$ $a_0 \neq 0$,

(ii) differentiate the trial series to find y' and y'',

(iii) substitute the results in the given differential equation,

(iv) equate coefficients of corresponding powers of the variable on each side of the equation: this enables index c and coefficients $a_1, a_2, a_3, \ldots$ from the trial solution, to be determined.

Bessel's equation:

The solution of $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + (x^2 - v^2)y = 0$ is:

$$y = Ax^v\left\{1 - \frac{x^2}{2^2(v+1)} + \frac{x^4}{2^4 \times 2!\,(v+1)(v+2)} - \frac{x^6}{2^6 \times 3!\,(v+1)(v+2)(v+3)} + \cdots\right\}$$
$$+ Bx^{-v}\left\{1 + \frac{x^2}{2^2(v-1)} + \frac{x^4}{2^4 \times 2!\,(v-1)(v-2)} + \frac{x^6}{2^6 \times 3!\,(v-1)(v-2)(v-3)} + \cdots\right\}$$

or, in terms of **Bessel functions** and **gamma functions**:

$$y = AJ_v(x) + BJ_{-v}(x)$$
$$= A\left(\frac{x}{2}\right)^v\left\{\frac{1}{\Gamma(v+1)} - \frac{x^2}{2^2(1!)\Gamma(v+2)} + \frac{x^4}{2^4(2!)\Gamma(v+4)} - \cdots\right\}$$
$$+ B\left(\frac{x}{2}\right)^{-v}\left\{\frac{1}{\Gamma(1-v)} - \frac{x^2}{2^2(1!)\Gamma(2-v)} + \frac{x^4}{2^4(2!)\Gamma(3-v)} - \cdots\right\}$$

In general terms:

$$J_v(x) = \left(\frac{x}{2}\right)^v\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{2^{2k}(k!)\Gamma(v+k+1)}$$

and $$J_{-v}(x) = \left(\frac{x}{2}\right)^{-v}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{2^{2k}(k!)\Gamma(k-v+1)}$$

and in particular:

$$J_n(x) = \left(\frac{x}{2}\right)^n\left\{\frac{1}{n!} - \frac{1}{(n+1)!}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)(n+2)!}\left(\frac{x}{2}\right)^4 - \cdots\right\}$$

$$J_0(x) = 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + \cdots$$

and $$J_1(x) = \frac{x}{2} - \frac{x^3}{2^3(1!)(2!)} + \frac{x^5}{2^5(2!)(3!)} - \frac{x^7}{2^7(3!)(4!)} + \cdots$$

Legendre's equation:

The solution of $(1-x^2)\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + k(k+1)y = 0$ is:

$$y = a_0\left\{1 - \frac{k(k+1)}{2!}x^2 + \frac{k(k+1)(k-2)(k+3)}{4!}x^4 - \cdots\right\}$$
$$+ a_1\left\{x - \frac{(k-1)(k+2)}{3!}x^3 + \frac{(k-1)(k-3)(k+2)(k+4)}{5!}x^5 - \cdots\right\}$$

Rodrigue's formula:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n(x^2-1)^n}{dx^n}$$

Statistics and Probability

Mean, median, mode and standard deviation:

If x = variate and f = frequency then:

$$\textbf{mean } \bar{x} = \frac{\sum fx}{\sum f}$$

The **median** is the middle term of a ranked set of data.

The **mode** is the most commonly occurring value in a set of data.

Standard deviation:

$$\sigma = \sqrt{\left[\frac{\sum\{f(x-\bar{x})^2\}}{\sum f}\right]} \text{ for a population}$$

Binomial probability distribution:

If n = number in sample, p = probability of the occurrence of an event and $q = 1 - p$, then the probability of $0, 1, 2, 3, \ldots$ occurrences is given by:

$$q^n, \quad nq^{n-1}p, \quad \frac{n(n-1)}{2!}q^{n-2}p^2,$$
$$\frac{n(n-1)(n-2)}{3!}q^{n-3}p^3, \ldots$$

(i.e. successive terms of the $(q+p)^n$ expansion).

Normal approximation to a binomial distribution:
Mean $= np$ Standard deviation $\sigma = \sqrt{(npq)}$

Poisson distribution:

If λ is the expectation of the occurrence of an event then the probability of $0, 1, 2, 3, \ldots$ occurrences is given by:

$$e^{-\lambda}, \quad \lambda e^{-\lambda}, \quad \lambda^2\frac{e^{-\lambda}}{2!}, \quad \lambda^3\frac{e^{-\lambda}}{3!}, \ldots$$

Product-moment formula for the linear correlation coefficient:

Coefficient of correlation $r = \dfrac{\sum xy}{\sqrt{[(\sum x^2)(\sum y^2)]}}$

where $x = X - \overline{X}$ and $y = Y - \overline{Y}$ and (X_1, Y_1), $(X_2, Y_2), \ldots$ denote a random sample from a bivariate normal distribution and $\overline{X}$ and $\overline{Y}$ are the means of the X and Y values respectively.

Normal probability distribution:

Partial areas under the standardized normal curve — see Table 58.1 on page 564.

Student's *t* distribution:

Percentile values (t_p) for Student's t distribution with ν degrees of freedom — see Table 74.2, page 38, on the website.

Chi-square distribution:

Percentile values (χ^2_p) for the Chi-square distribution with ν degrees of freedom—see Table 76.1, page 60, on the website.
$\chi^2 = \sum\left\{\dfrac{(o-e)^2}{e}\right\}$ where o and e are the observed and expected frequencies.

Symbols:

Population
number of members N_p, mean μ, standard deviation σ.

Sample
number of members N, mean $\overline{x}$, standard deviation s.

Sampling distributions
mean of sampling distribution of means $\mu_{\overline{x}}$
standard error of means $\sigma_{\overline{x}}$
standard error of the standard deviations σ_s.

Standard error of the means:

Standard error of the means of a sample distribution, i.e. the standard deviation of the means of samples, is:

$$\sigma_{\overline{x}} = \frac{\sigma}{\sqrt{N}}\sqrt{\left(\frac{N_p - N}{N_p - 1}\right)}$$

for a finite population and/or for sampling without replacement, and

$$\sigma_{\overline{x}} = \frac{\sigma}{\sqrt{N}}$$

for an infinite population and/or for sampling with replacement.

The relationship between sample mean and population mean:

$\mu_{\overline{x}} = \mu$ for all possible samples of size N are drawn from a population of size N_p.

Estimating the mean of a population (σ known):

The confidence coefficient for a large sample size, ($N \geq 30$) is z_c where:

Confidence level %	Confidence coefficient z_c
99	2.58
98	2.33
96	2.05
95	1.96
90	1.645
80	1.28
50	0.6745

The confidence limits of a population mean based on sample data are given by:

$$\overline{x} \pm \frac{z_c \sigma}{\sqrt{N}} \sqrt{\left(\frac{N_p - N}{N_p - 1}\right)}$$

for a finite population of size N_p, and by

$\overline{x} \pm \dfrac{z_c \sigma}{\sqrt{N}}$ for an infinite population

---

Estimating the mean of a population (σ unknown):

The confidence limits of a population mean based on sample data are given by: $\mu_{\overline{x}} \pm z_c \sigma_{\overline{x}}$.

Estimating the standard deviation of a population:

The confidence limits of the standard deviation of a population based on sample data are given by:
$s \pm z_c \sigma_s$.

Estimating the mean of a population based on a small sample size:

The confidence coefficient for a small sample size ($N < 30$) is t_c which can be determined using Table 74.1, page 33, on the website. The confidence limits of a population mean based on sample data is given by:

$$\overline{x} \pm \frac{t_c s}{\sqrt{(N-1)}}$$

Laplace Transforms

Function $f(t)$	Laplace transforms $\mathcal{L}\{f(t)\} = \int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t$
1	$\frac{1}{s}$
k	$\frac{k}{s}$
e^{at}	$\frac{1}{s-a}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
t	$\frac{1}{s^2}$
$t^n (n = \text{positve integer})$	$\frac{n!}{s^{n+1}}$
$\cosh at$	$\frac{s}{s^2-a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\mathrm{e}^{-at} t^n$	$\frac{n!}{(s+a)^{n+1}}$
$\mathrm{e}^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$\mathrm{e}^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
$\mathrm{e}^{-at} \cosh \omega t$	$\frac{s+a}{(s+a)^2-\omega^2}$
$\mathrm{e}^{-at} \sinh \omega t$	$\frac{\omega}{(s+a)^2-\omega^2}$

The Laplace transforms of derivatives:

First derivative

$$\mathcal{L}\left\{\frac{\mathrm{d}y}{\mathrm{d}x}\right\} = s\mathcal{L}\{y\} - y(0)$$

where $y(0)$ is the value of y at $x = 0$.

Second derivative

$$\mathcal{L}\left\{\frac{\mathrm{d}y}{\mathrm{d}x}\right\} = s^2\mathcal{L}\{y\} - sy(0) - y'(0)$$

where $y'(0)$ is the value of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at $x = 0$.

Fourier Series

If $f(x)$ is a periodic function of period 2π then its Fourier series is given by:

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where, for the range $-\pi$ to $+\pi$:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x \quad (n = 1, 2, 3, \ldots)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x \quad (n = 1, 2, 3, \ldots)$$

If $f(x)$ is a periodic function of period L then its Fourier series is given by:

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left\{a_n \cos\left(\frac{2\pi nx}{L}\right) + b_n \sin\left(\frac{2\pi nx}{L}\right)\right\}$$

where for the range $-\dfrac{L}{2}$ to $+\dfrac{L}{2}$:

$$a_0 = \frac{1}{L}\int_{-L/2}^{L/2} f(x)\,\mathrm{d}x$$

$$a_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\cos\left(\frac{2\pi nx}{L}\right)\mathrm{d}x \quad (n = 1, 2, 3, \ldots)$$

$$b_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\sin\left(\frac{2\pi nx}{L}\right)\mathrm{d}x \quad (n = 1, 2, 3, \ldots)$$

Complex or exponential Fourier series:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{j\frac{2\pi nx}{L}}$$

$$\text{where} \quad c_n = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\mathrm{e}^{-j\frac{2\pi nx}{L}}\mathrm{d}x$$

For even symmetry,

$$c_n = \frac{2}{L}\int_{0}^{\frac{L}{2}} f(x)\cos\left(\frac{2\pi nx}{L}\right)\mathrm{d}x$$

For odd symmetry,

$$c_n = -j\frac{2}{L}\int_{0}^{\frac{L}{2}} f(x)\sin\left(\frac{2\pi nx}{L}\right)\mathrm{d}x$$

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