Q.6.(20 marks). Determine the first four terms of the trigonometric Fourier series for the waveform shown in figure 3.

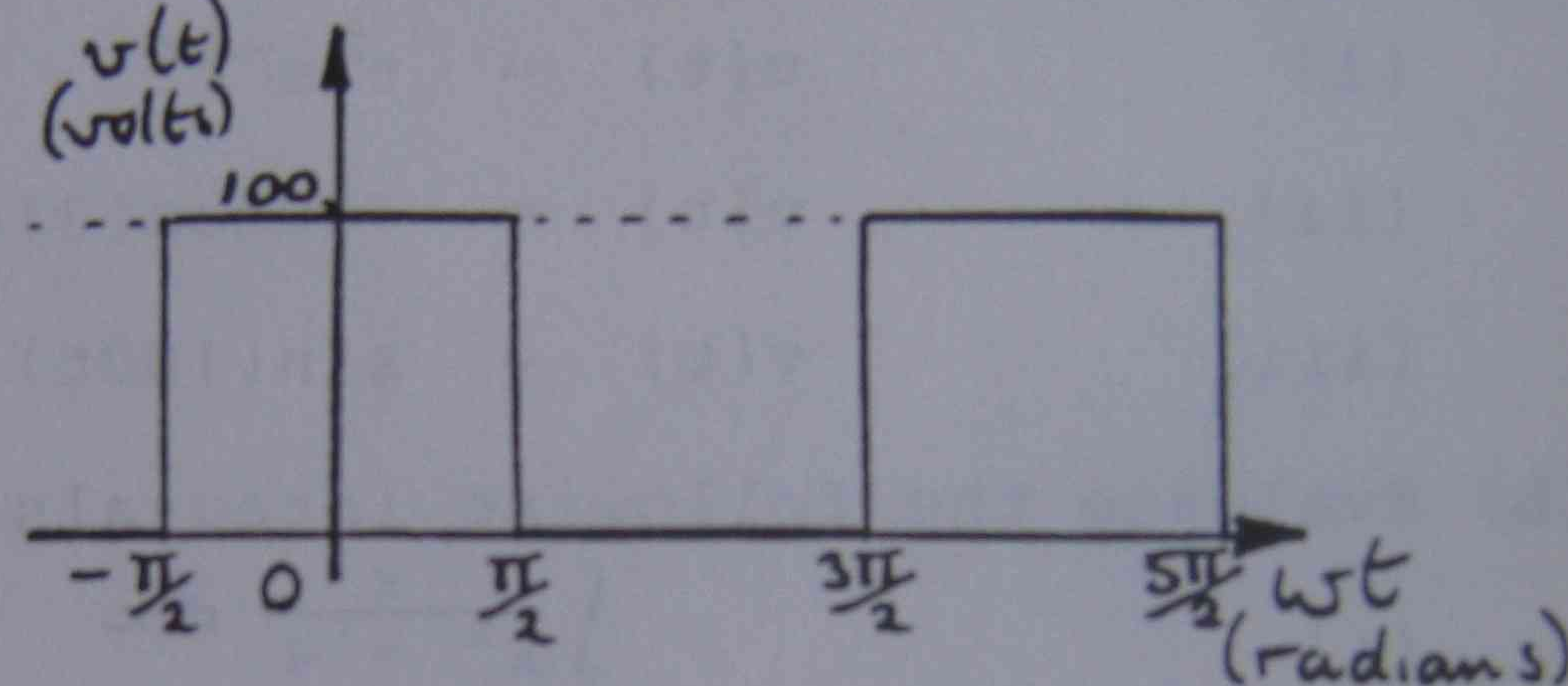
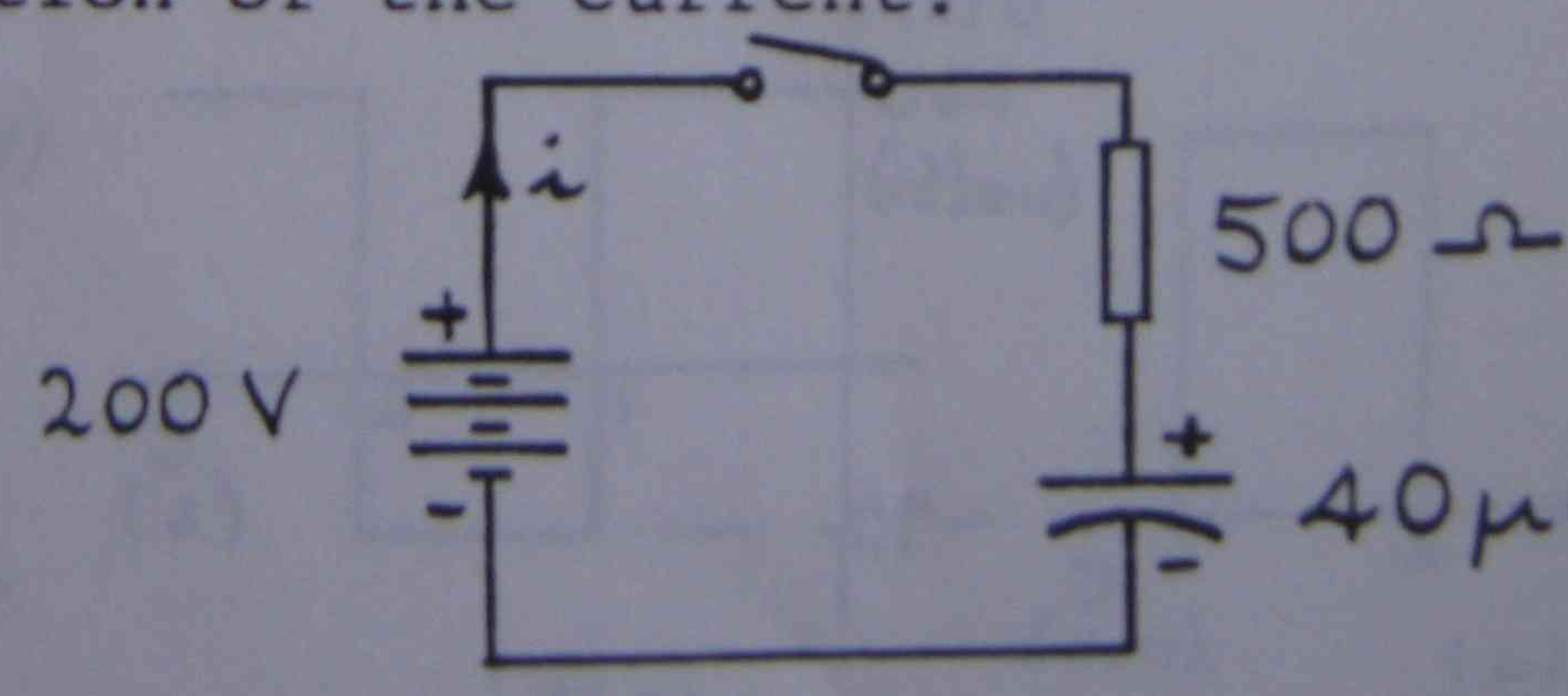


Figure 3

Q.7.(10 marks). After closing the switch in the circuit shown in figure 4 a current will flow. Determine the following:-

(a) the final value of current;
(b) the initial value of current;
(c) the time constant of the circuit;
(d) the equation of the current.



wk 1 + 2

Power Circuit Principles

2840BP

UNIT 3
Harmonics and Fourier analysis

OTEN

Department of School Education

TAFE NSW

Contents

Introduction

All the circuit analysis work you have experienced in studying ac principles up to this point has involved only pure sine waves. Because we have used only pure sine waves, the calculations of impedances involving inductive and capacitive reactances and complex powers have been reasonably simple.

You may not agree with the previous statement, but consider how you would analyse the effect of a square wave on a circuit containing inductance and capacitance. In this unit you will be shown a method called Fourier analysis to help you work out problems similar to that of a square wave in complex circuits.

Much electrical and electronic equipment involve waveforms which are non-sinusoidal. Some of these are deliberately created, others are the results of distortions produced by equipment. In the analysis of these non-sinusoidal waveforms you will be shown that all repetitive waveforms can be made up of combinations of many sinusoidal waves. Also any waveform can be analysed to determine the component quantities. These sine wave components may be used when investigating the response of a complex impedance circuit to a non-sinusoidal supply voltage. The Fourier method provides the means for solving this type of problem.

When you are studying this unit, the reference from the textbook, Edminister, is Chapter 12. The textbook, in covering this topic uses a complete mathematical approach using integral calculus. For the work in this unit, you are not expected to use calculus. Where calculus is required (to obtain the values of constants etc which you need to use in your analysis of waveforms), such values will be given to you. I would advise that the textbook only be used as a supplement for the material presented in this unit. The self-assessment exercises will be your guide to the depth to which each topic has to be understood.

Objectives

The following list of objectives is for your guidance when learning the work in this unit. Do not be concerned if there are terms or concepts stated here which you do not understand. These will become clear as you progress through the unit. At the completion of this unit, review these objectives—you should be able to understand how to:

- determine the trigonometric Fourier series for repetitive waveforms;
- recognise waveform symmetry and hence simplify the Fourier analysis;
- synthesise the Fourier series and show how the series represents the original waveform;
- apply the concepts of Fourier analysis to explain the harmonic content of waveforms;
- state the sources of the production of harmonics;
- explain the problems of non-sinusoidal waveforms in the supply network;
- calculate the effective value of the waveform and the power consumed in the load;
- analyse the effects of harmonics on the value of the rms current in a load.

WK1

Harmonics in waveforms

Reference: Edminister, page 190

Most of the circuit analysis in ac work you have done up to this stage has been possible because you have always considered the supply voltage and current to be sinusoidal. The formulae required the use of sinusoidal waves.

The waveshapes in Figure 1 are common in both electronic and electrical circuits and the calculation of circuit responses is possible using the theory already applied to sine waves.

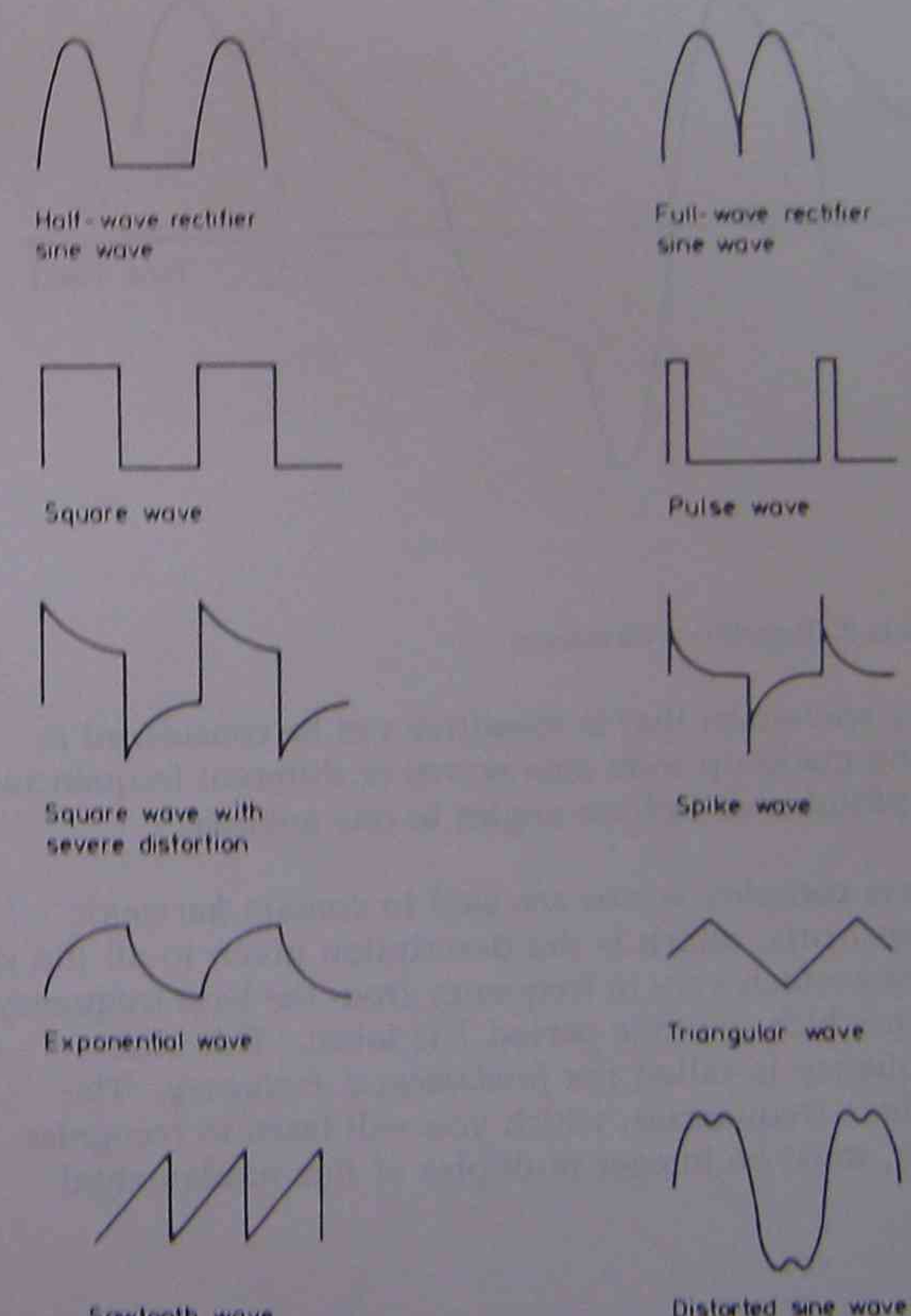


Figure 1

The first important detail to be known about any waveshape is whether it is *repetitive*. The term 'periodic' is used sometimes instead of 'repetitive'.

Consider the waveshape in Figure 2. If we take the value of the wave at any time t_1 then the wave must have exactly the same value at $t_2 = t_1 + T$, where T is the period of the wave and again at $t_3 = t_1 + 2T$. In other words the wave must reproduce itself exactly during each time period T(hence the term 'periodic'), during the time that the supply is considered stable.

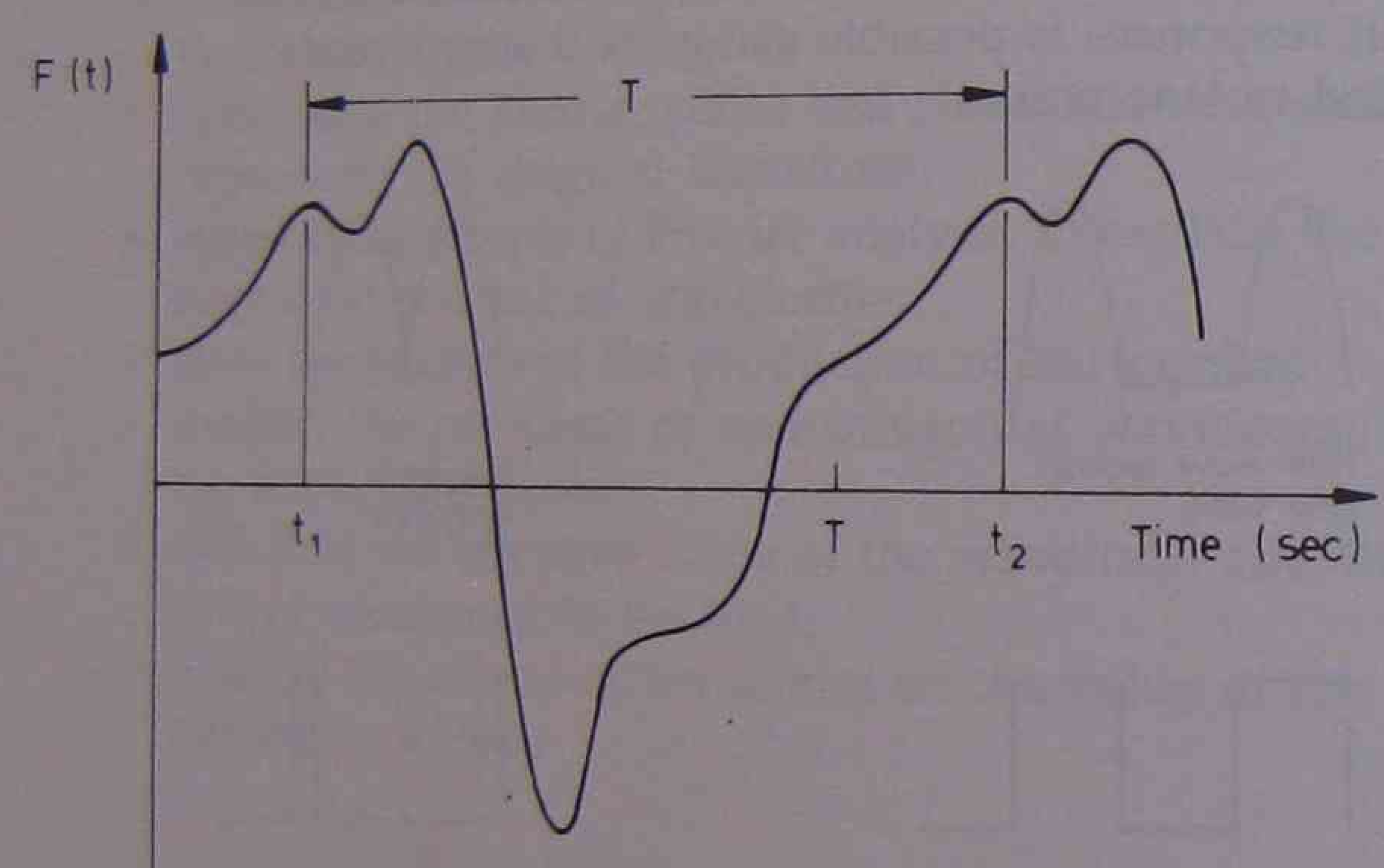


Figure 2: Repetitive waveform

Any waveform that is repetitive can be considered as being made up from sine waves of different frequencies, amplitudes and phase angles to one another.

These complex waves are said to contain *harmonic components*, which is the description given to all the sine waves which vary in frequency from the base frequency from which the time period T is taken. This base frequency is called the *fundamental frequency*. The various frequencies, which you will learn to recognise later, must be integer multiples of this fundamental.

These harmonics take the form shown below:

fundamental = base frequency (sin ωt)
2nd harmonic = 2 x base frequency (sin $2\omega t$)
3rd harmonic = 3 x base frequency (sin $3\omega t$)
to the nth harmonic = n x base frequency (sin $n\omega t$)

Waveform synthesis using Fourier series

Reference: Edminister, page 196

To develop the concept of complex waves being made up of components of sine waves, let us consider an example and develop it through two stages.

If two generators are connected in series, as in Figure 3, with generator 2 having a frequency three times the frequency of generator 1, the output of the system is the sum of all instantaneous amplitudes to give a combined wave shape.

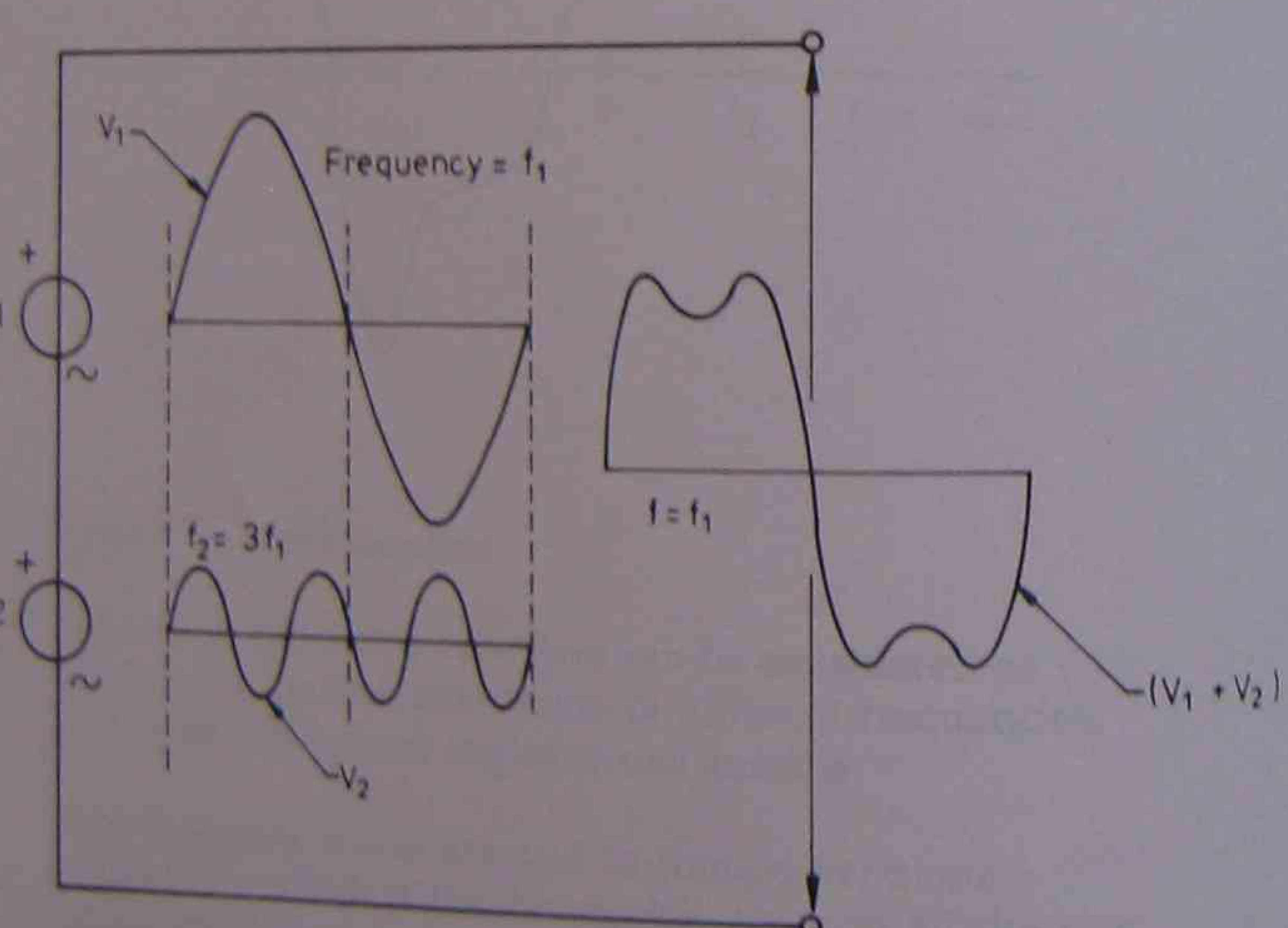


Figure 3

Have you noted that the amplitude of the wave from generator 2 is much less than that of generator 1? Both waves also start at the same point and both rise in a positive direction. If wave 2 were to start in the negative direction, the output wave would be quite different.

Let us now add a third generator to the system with a frequency of five times that of generator 1. This system is shown in Figure 4.

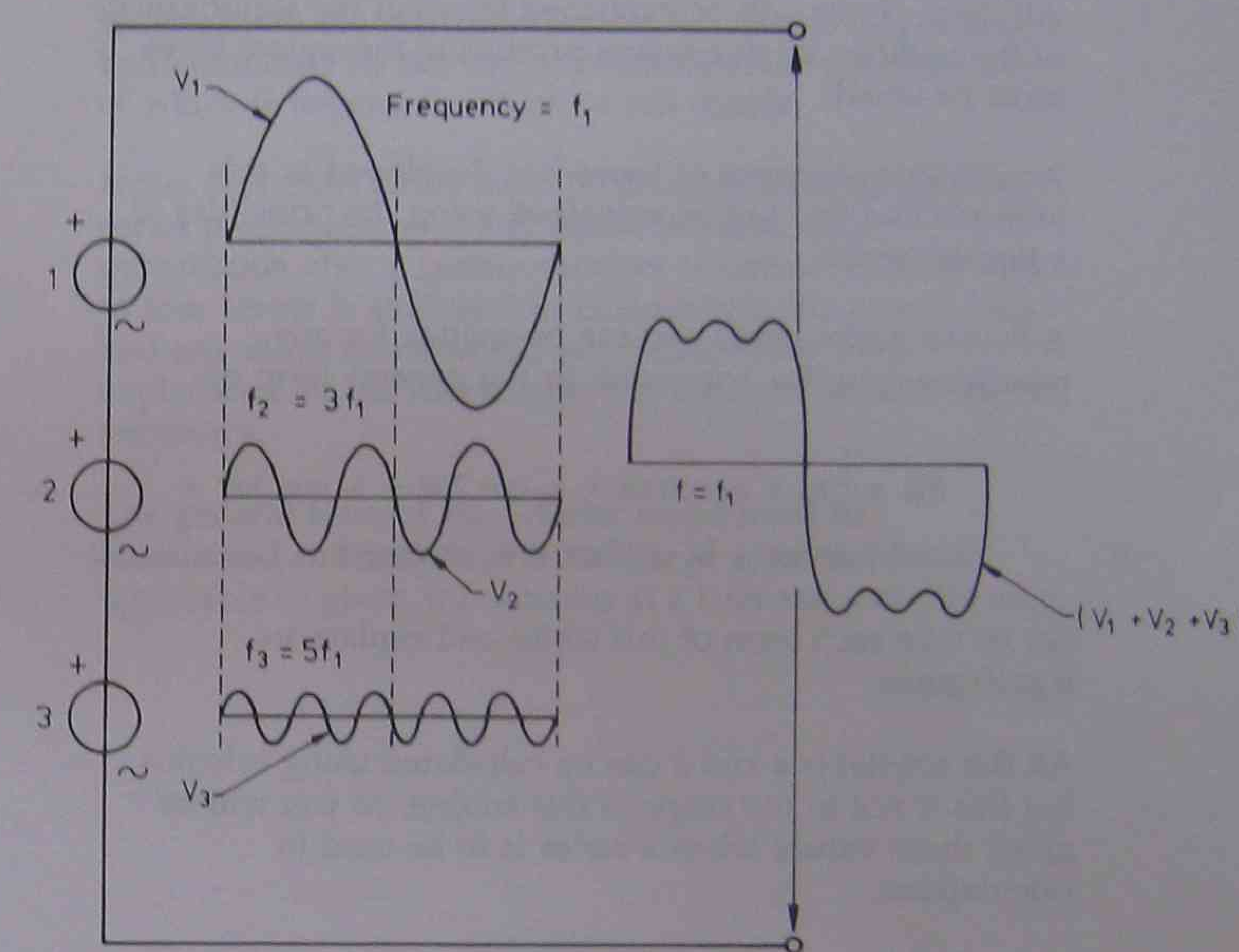


Figure 4

With the addition of the third generator the output wave is now beginning to resemble a poor quality square wave. If time and space permitted, we would add further generators to this system containing the frequencies of seven times, nine times, 11 times the fundamental frequency, and so on. Each additional wave would produce a squarer square wave with less ripple on the top, until, with sufficient generators, a practical pure square wave would be produced.

In this example a number of points must be noted:

- Each successive generator produces a frequency which is an odd number multiple of the fundamental (base) frequency.

- These frequencies are called the third, fifth, seventh (etc) harmonic frequencies.
- Each successive harmonic was at a reduced magnitude to its predecessor.
- All started at the same point and in a positive direction.

All these points will be explained later but the significance of the addition of sine waves producing the square wave must be noted.

An obvious sequence of terms has developed in this example and this can be explained using the principles of a Fourier series.

A Fourier series exists and can be written for any repetitive complex waveform in the general form of:

$$f(t) = \frac{1}{2}a_0 + a_1 \cos \omega t + a_2 \cos 2\omega t + a_3 \cos 3\omega t + \ldots\ldots\ldots\ldots$$

$$+ b_1 \sin \omega t + b_2 \sin 2\omega t + b_3 \sin 3\omega t + \ldots\ldots\ldots\ldots$$

Let us take each term of this series and explain its significance.

All the constants *a* and *b* can be calculated using calculus but this is not in the scope of this subject, so you will be given these values when a series is to be used in calculations.

The first term $\frac{1}{2}a_0$ is an average value, hence it is the value of any dc component which may be present. In the previous example there were only ac generators, but there could have been a dc generator or battery in the system, which would have offset the wave up or down depending on its polarity. You will see this effect in later examples.

Because some complex waves do not start with zero amplitude at time zero, they can be expressed in the terms of cosine functions. So the Fourier series either contains both cosine and sine terms or is made up exclusively of sine, or of cosine, terms. Later in this unit the difference between the sine and cosine terms will be explained and also when either, or both, are used.

The reason we use *a* and *b* constants is to be able to know which constants go with sine terms and which go with cosine terms. In an example where all the *b* terms are zero, then there are no sine terms in the series. Where

the *a* terms are zero, then there are no cosine terms in the series.

At the end of each section of the general series, after the $a_3 \cos 3\omega t$ and the $b_3 \sin 3\omega t$ there is a row of dots. These dots represent the continuation of the terms in the series to infinity in the mathematical sense. In reality it would be determined by the circuit conditions, the consideration of which is beyond the scope of this course.

Note: Sufficient terms must be given to indicate how the series progresses, as it will repeat the arithmetic progression after a certain number of terms. Usually three to four terms is sufficient to demonstrate the progression and any more terms would not be necessary for an explanation of the waveshape, but they can be calculated if necessary.

The general form of the Fourier series must be memorised so that you will be able to construct series applicable to given waveshapes at a later stage of this unit.

Fourier series simplification using waveform symmetry

Reference: Edminister, page 193

There are two basic conditions which are used to simplify the Fourier series.

Condition 1: Half-wave symmetry

When you look at the two examples in Figure 5, you will notice that the horizontal axis is through the centre of the waves. The shape of the positive section of the wave is the same as that of the negative section of the wave.

The test for half-wave symmetry is that both positive and negative half cycles of the wave are indentical. This can be checked by superimposing the negative half cycle between the two positive half cycles and if each wave pulse is identical to the next, then the wave is half-wave symmetric. Compare Figure 5 to Figure 6.

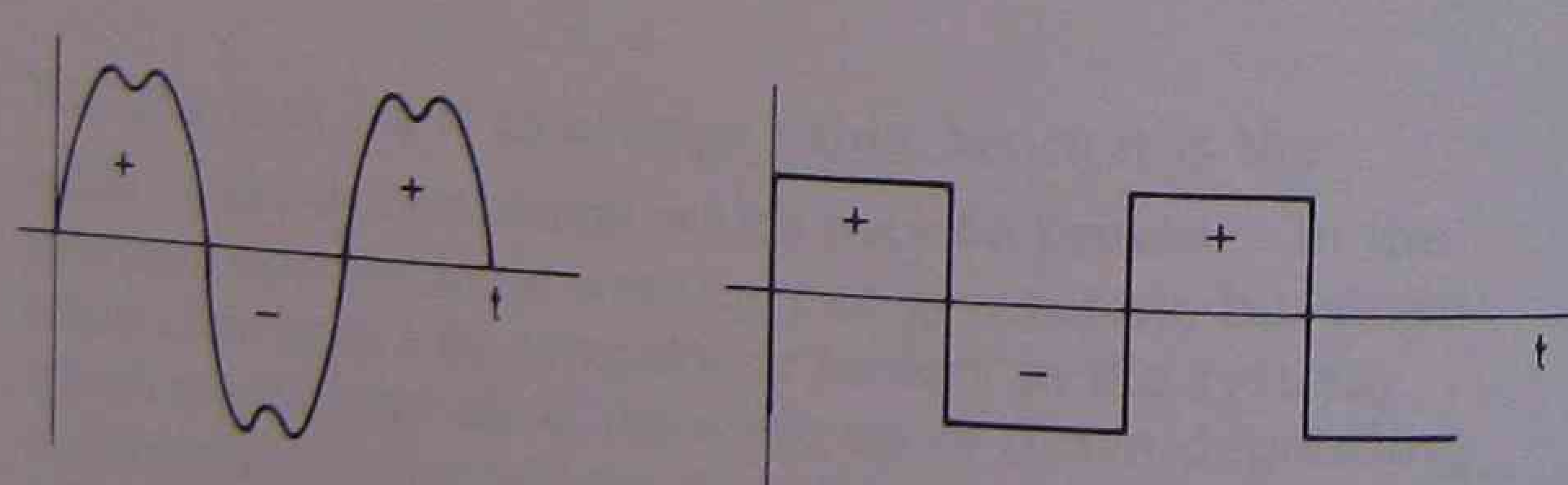


Figure 5: Waveforms with half-wave symmetry

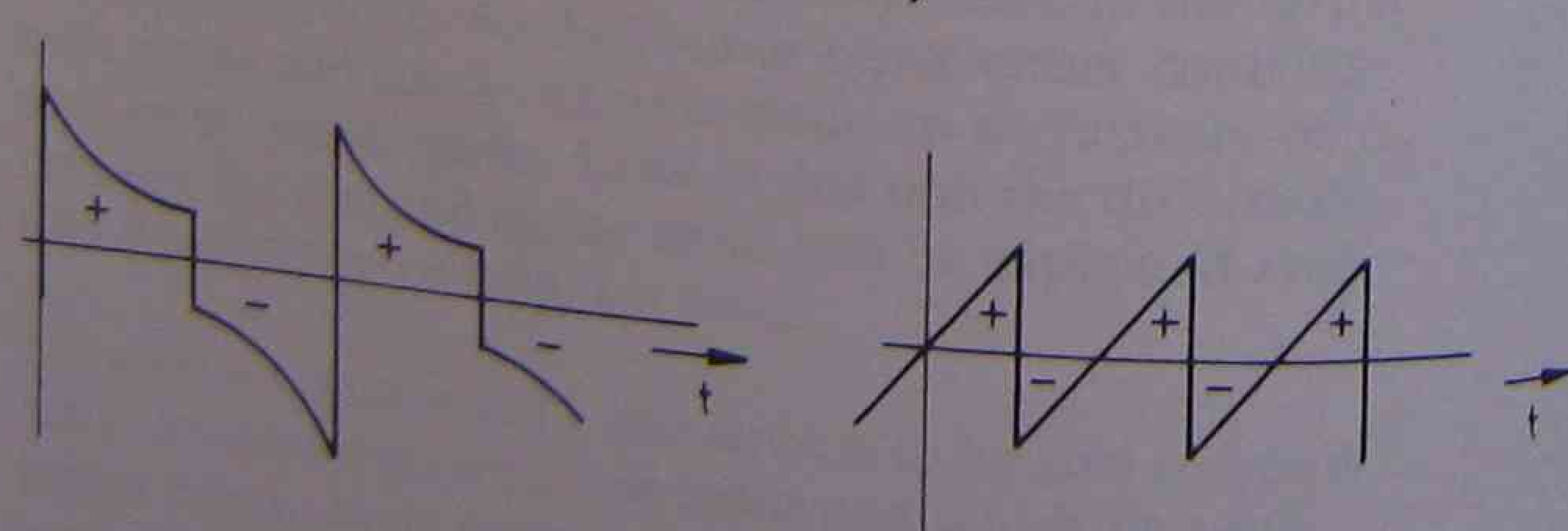


Figure 6: Waveforms not having half-wave symmetry

A function is said to be half-wave symmetric if

$$f(t) = -f(t + \frac{T}{2})$$

where T is the period of the wave.

To explain this expression for the function of *t* look at Figure 7 and note the value of the wave at t_1. If you now move along the horizontal axis, half the period T, to the negative half cycle and to the position $(t_1 + \frac{T}{2})$ then the value must be the negative of the value at t_1.

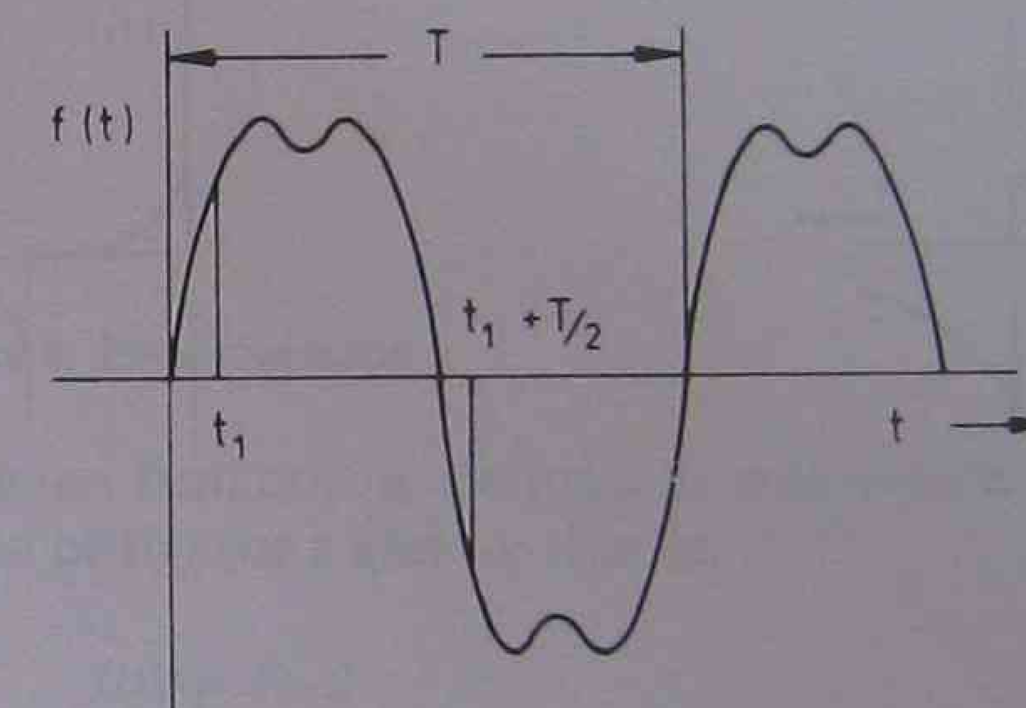


Figure 7

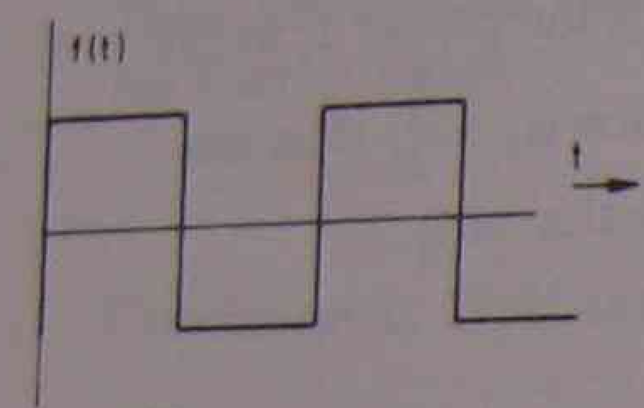
This condition of half-wave symmetry can only exist if **odd harmonics** only are present. Odd harmonics are the 3rd, 5th, 7th, 9th etc. When even harmonics are present, the positive and negative half cycles are different.

Condition 1 must be memorised so you can interpret wave shapes for the purpose of constructing Fourier series.

Now try the exercise in Self-assessment 1. This question is for your self-testing only. Do *not* send your answer to OTEN. Check your answers with those given at the end of the unit.

Self-assessment 1

Indicate whether the following waves are half-wave symmetric.



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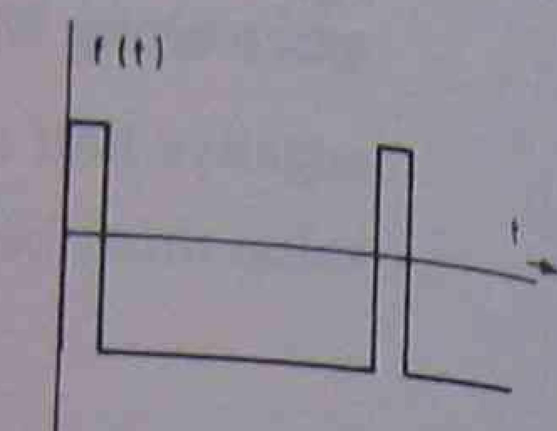

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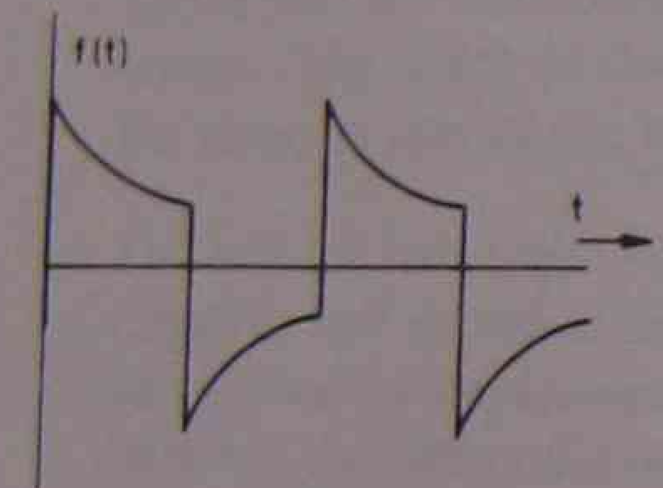

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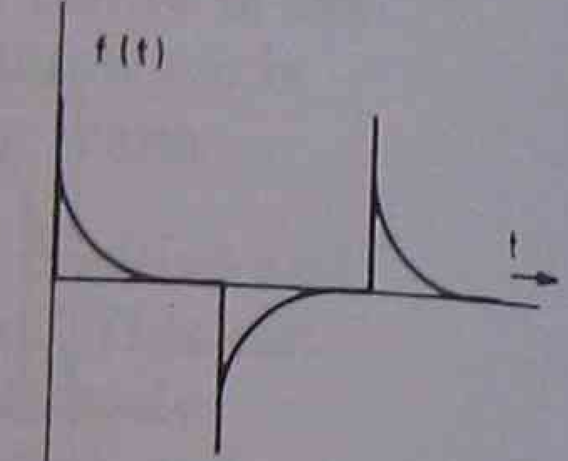

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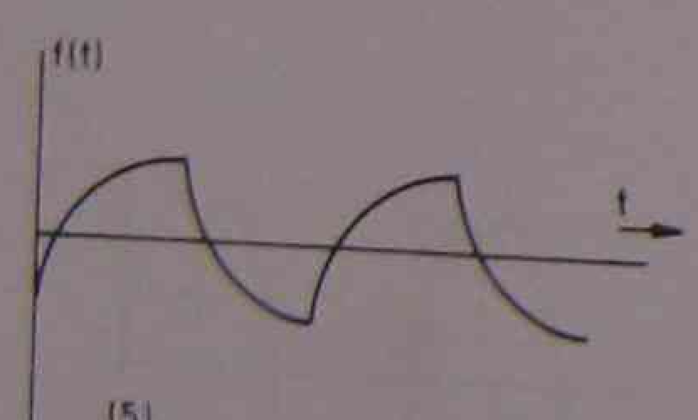

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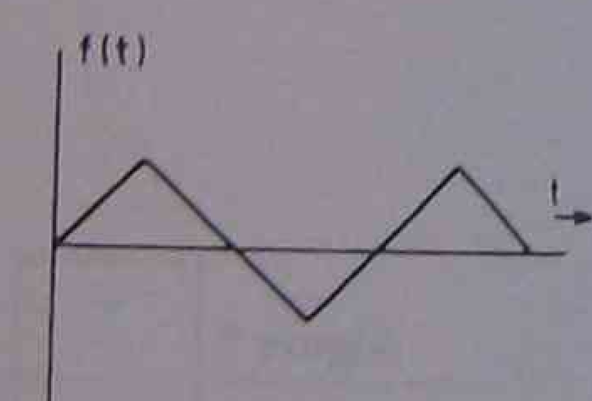

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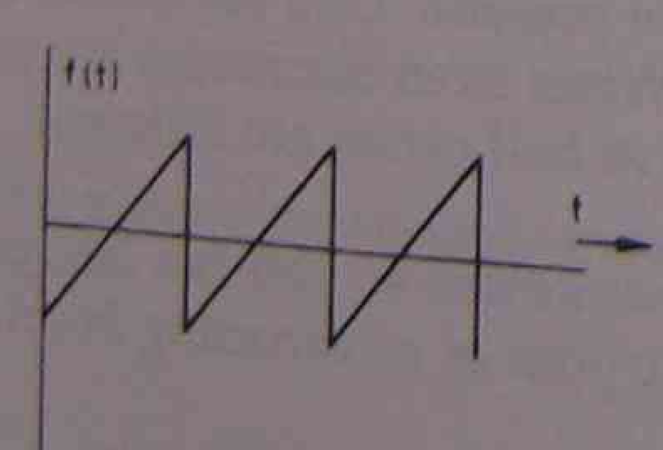


(7)

Condition 2: Odd and even functions

When you look at Figure 8, the two wave shapes have a feature in common which classifies them as **even** functions.

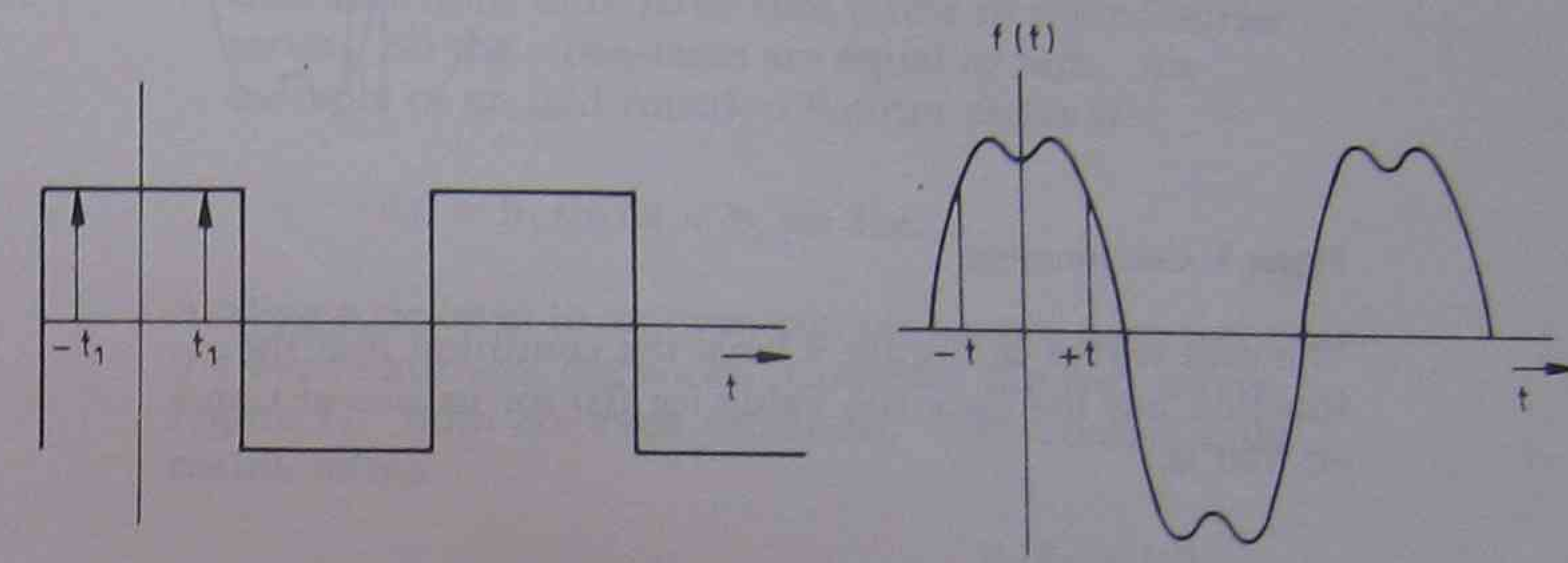


Figure 8: Even functions

An even function is defined as one which has the same value of f(t) for t and −t; that is,

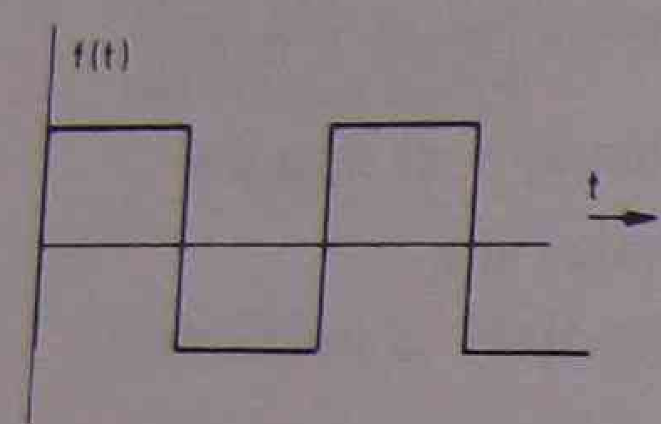
$$f(t) = f(-t)$$

This means the wave must be the same both sides of the vertical axis. One way physically to determine this is to imagine a mirror placed along the vertical axis and the reflection in the positive direction on the time axis is identical to the wave in the negative direction.

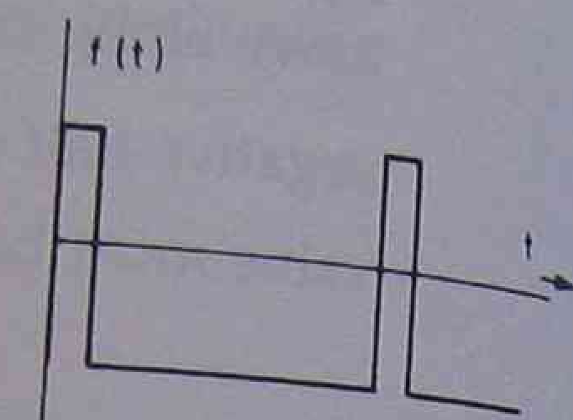
Compare the even waves in Figure 8 with the odd waves of Figure 9.

Self-assessment 1

Indicate whether the following waves are half-wave symmetric.



(1)



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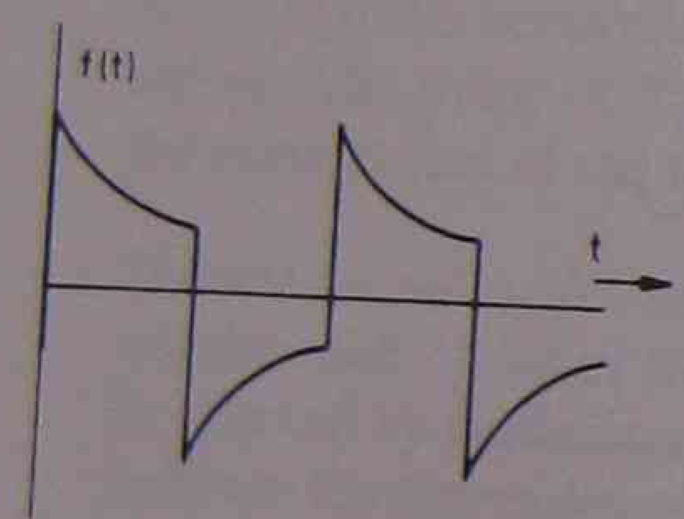

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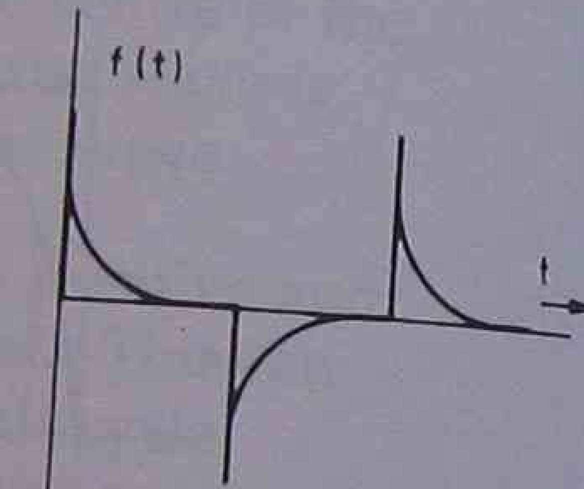

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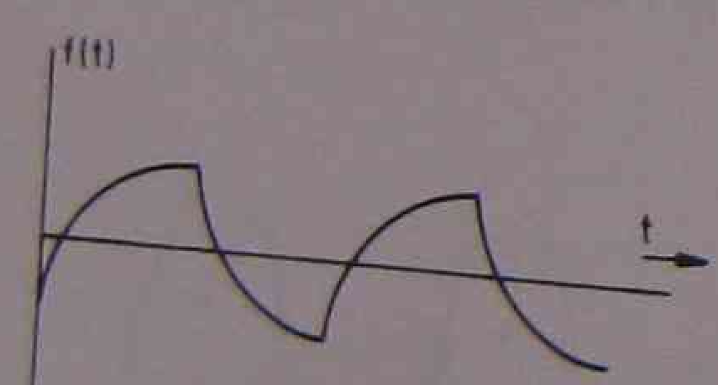

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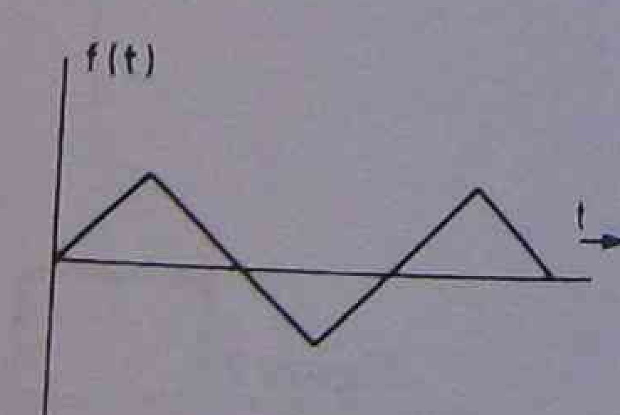

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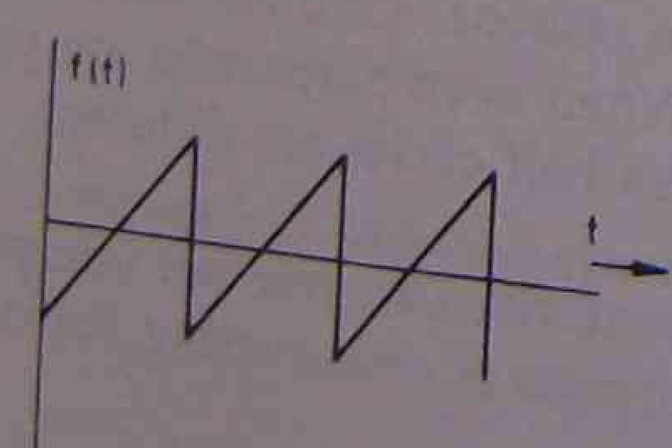


(7)

Condition 2: Odd and even functions

When you look at Figure 8, the two wave shapes have a feature in common which classifies them as **even** functions.

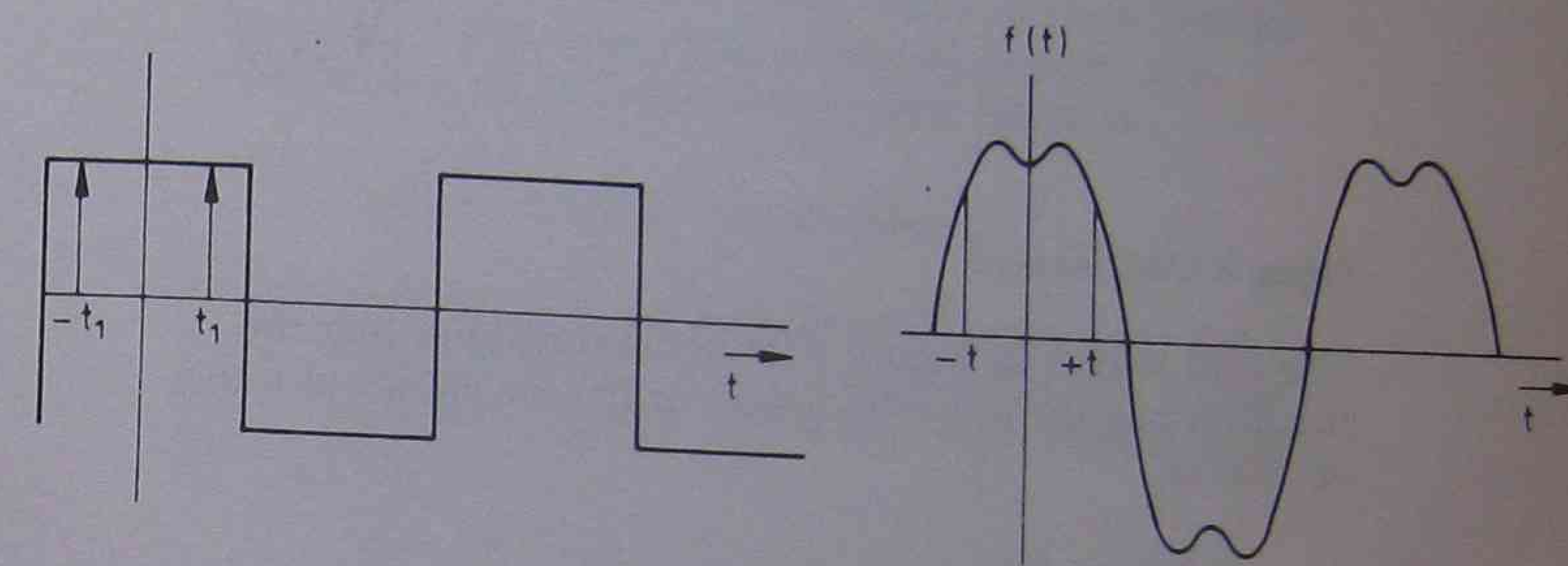


Figure 8: Even functions

An even function is defined as one which has the same value of f(t) for t and –t; that is,

$$f(t) = f(-t)$$

This means the wave must be the same both sides of the vertical axis. One way physically to determine this is to imagine a mirror placed along the vertical axis and the reflection in the positive direction on the time axis is identical to the wave in the negative direction.

Compare the even waves in Figure 8 with the odd waves of Figure 9.

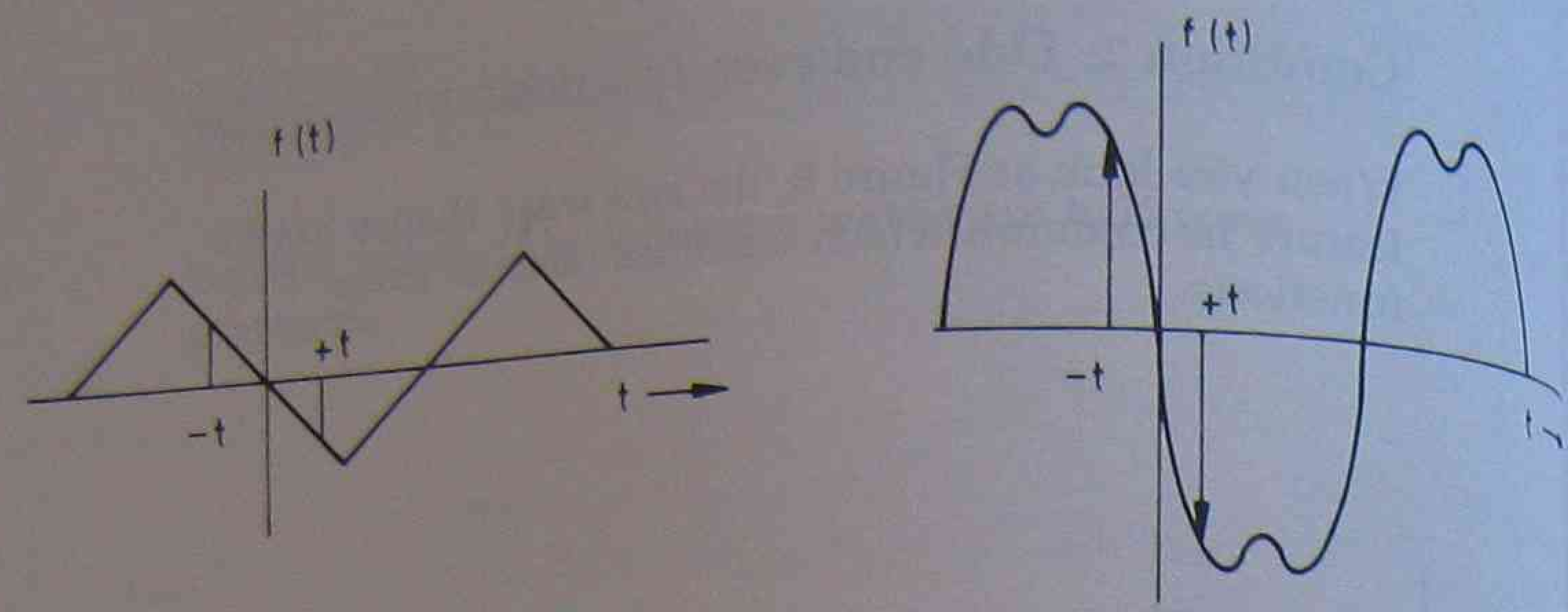
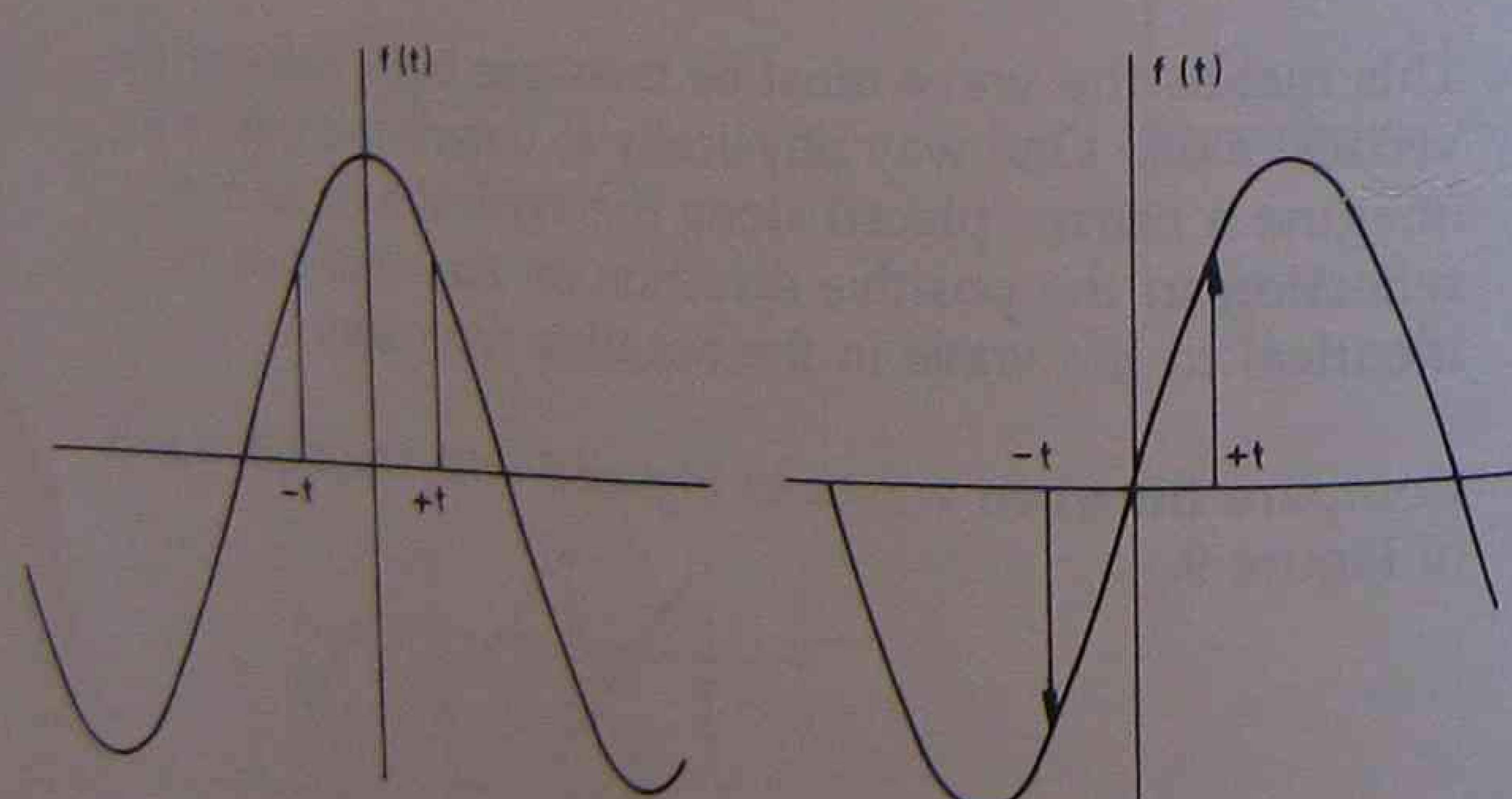


Figure 9: Odd functions

The odd waves in Figure 9 have the condition that the function has the opposite value for f(t) for values of t and –t; that is,

$$f(t) = -f(-t)$$

The most common even and odd functions are the cosine and sine waves respectively. These are shown in Figure 10.



Cosine—even function

Sine—odd function

Figure 10

Figure 10 leads us to the condition that:

- Even functions only have cosine terms in their Fourier series. All the *b* constants are equal to zero. An example of an even function Fourier series is

  $$f(t) = a_0 + a_1 \cos \omega t + a_2 \cos 2\omega t........$$

- Odd functions only have sine terms in their Fourier series. All the a constants are equal to zero. An example of an odd function Fourier series is

  $$f(t) = b_1 \sin \omega t + b_2 \sin 2\omega t........$$

Adding a dc level to an even complex wave does not alter the even nature of the wave. Consider the two waves in Figure 11. Both are even functions. Both contain only cosine terms.

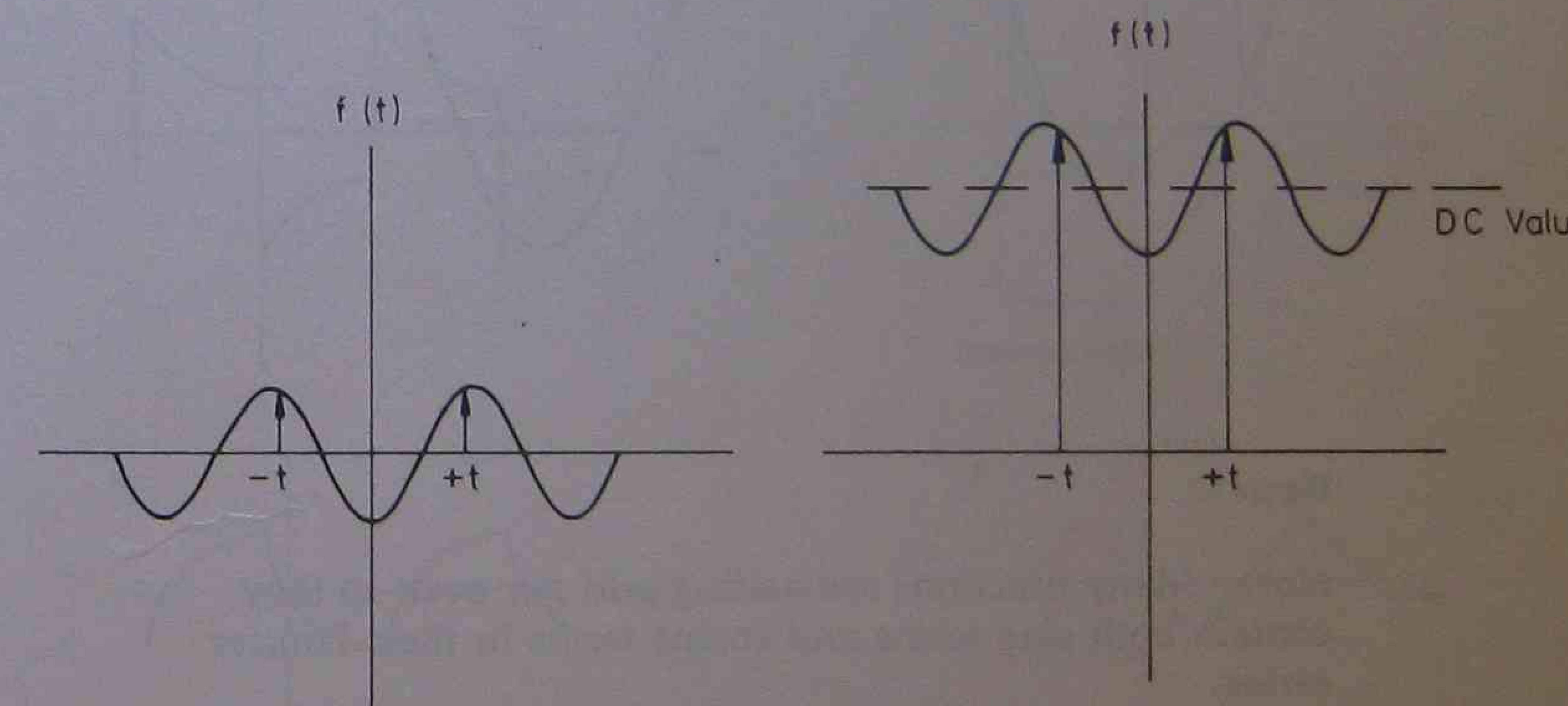


Figure 11

Note: The addition of a constant to an **odd** function removes the **odd** nature of the function and the function will no longer be odd, nor is it even. The function could then contain both sine and cosine terms in the Fourier series.

Moving the wave on the time axis

If we move the wave along the time axis you may make an odd function even or an even function odd.

Consider the even function in Figure 12. The displacement of the vertical axis one quarter wavelength to the right will, in this case, make the function odd. This has made a cosine wave into a sine wave.

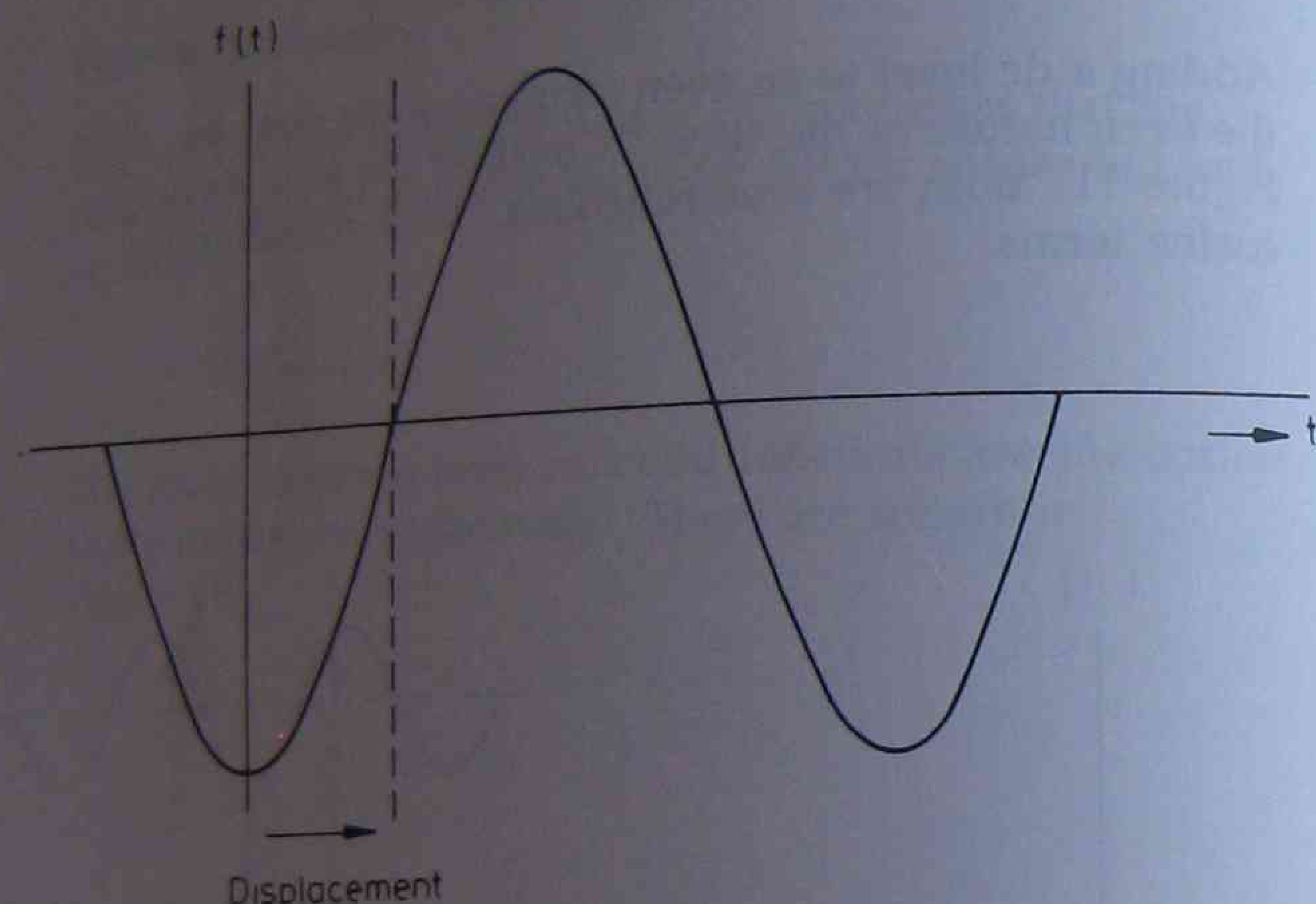


Figure 12

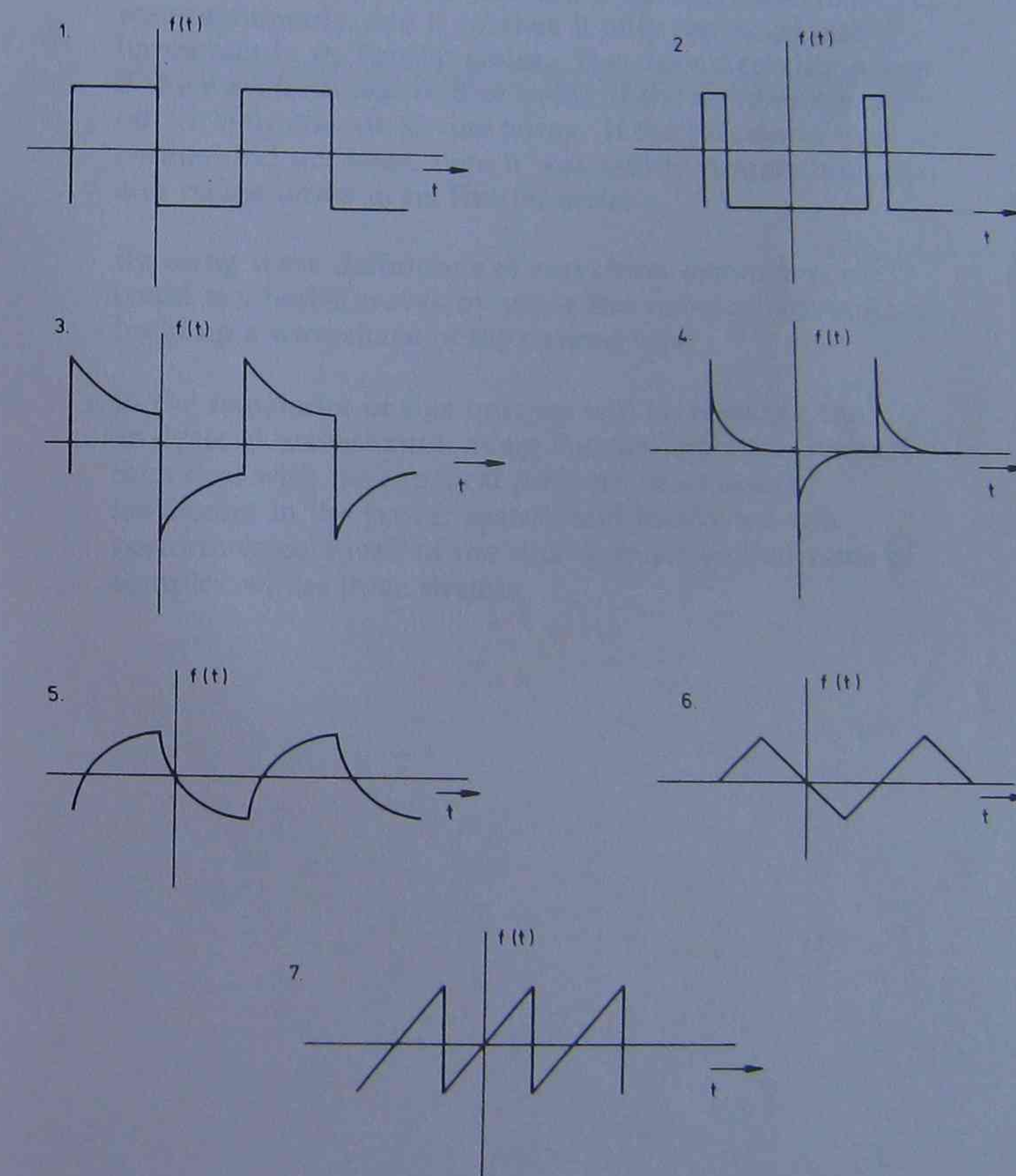
Note: Many functions are neither odd nor even so they contain both sine terms and cosine terms in their Fourier series.

Condition 2 and the variations listed in this section must be memorised so you can interpret wave shapes for the purpose of constructing Fourier series.

Now try the exercise in Self-assessment 2. These questions are for your self testing only. Do *not* send your answers to OTEN. Check your answers with those given at the end of the unit.

Self-assessment 2

1 Indicate whether the following waves are odd, even or neither odd nor even.



2 Indicate whether the following series are even or odd:

(a) $f(t) = \frac{2}{\pi}\sin \omega t + \frac{3}{2\pi}\sin 3\omega t$........

(b) $f(t) = 70 \sin \omega t + 20 \sin 2\omega t + 10 \sin 3\omega t$......
$+ 40 \cos \omega t + 10 \cos 2\omega t$........

(c) $f(t) = 35 - \frac{10}{\pi}\sin \omega t - \frac{10}{2\pi}\sin 2\omega t$........

Review

So far in this unit you have covered the topics of waveform symmetry and waveform synthesis using the Fourier series.

The topic of waveform symmetry was related to two conditions. Condition 1 occurred if the wave was half-wave symmetric, and if so, then it only contained odd harmonics in its Fourier series. The second condition was if the waveform was odd or even. If the waveform was odd, it only contained sine terms. If the waveform was neither odd nor even, then it was said to contain both sine and cosine terms in its Fourier series.

By using these definitions of waveform symmetry, we could synthesise waves by using the various harmonics to build up a waveshape of the desired type.

In the remainder of this unit we will be covering the analysis of waveshapes, using Fourier analysis. We will then deal with the practical problem of sources of harmonics in the power system and finally we will perform calculations of the rms voltages and currents of complex waves in ac circuits.

Fourier analysis of repetitive waveforms

In the previous sections you were shown how to make up (synthesise) complex waves from a series of sine waves using the Fourier series principle. Using the same principle, waveforms can be analysed to determine their harmonic content.

As an example a perfect square wave as shown in Figure 13 can be shown by Fourier analysis to be represented by

$$e = \frac{4Em}{\pi}\left\{ \sin \omega t + \frac{\sin 3\omega t}{3} + \frac{\sin 5\omega t}{5} + \frac{\sin 7\omega t}{7} + \ldots\ldots \right\}$$

where e is the instantaneous voltage at time t, E_m is the peak value of the square wave, and ω is 2π x (fundamental frequency). The sin ωt component is the fundamental, the sin $3\omega t$ quantity is the third harmonic, sin $5\omega t$ represents the fifth harmonic etc. The symmetrical square wave can be said to be made up of a fundamental, odd-numbered harmonics, no even harmonics, and no dc component.

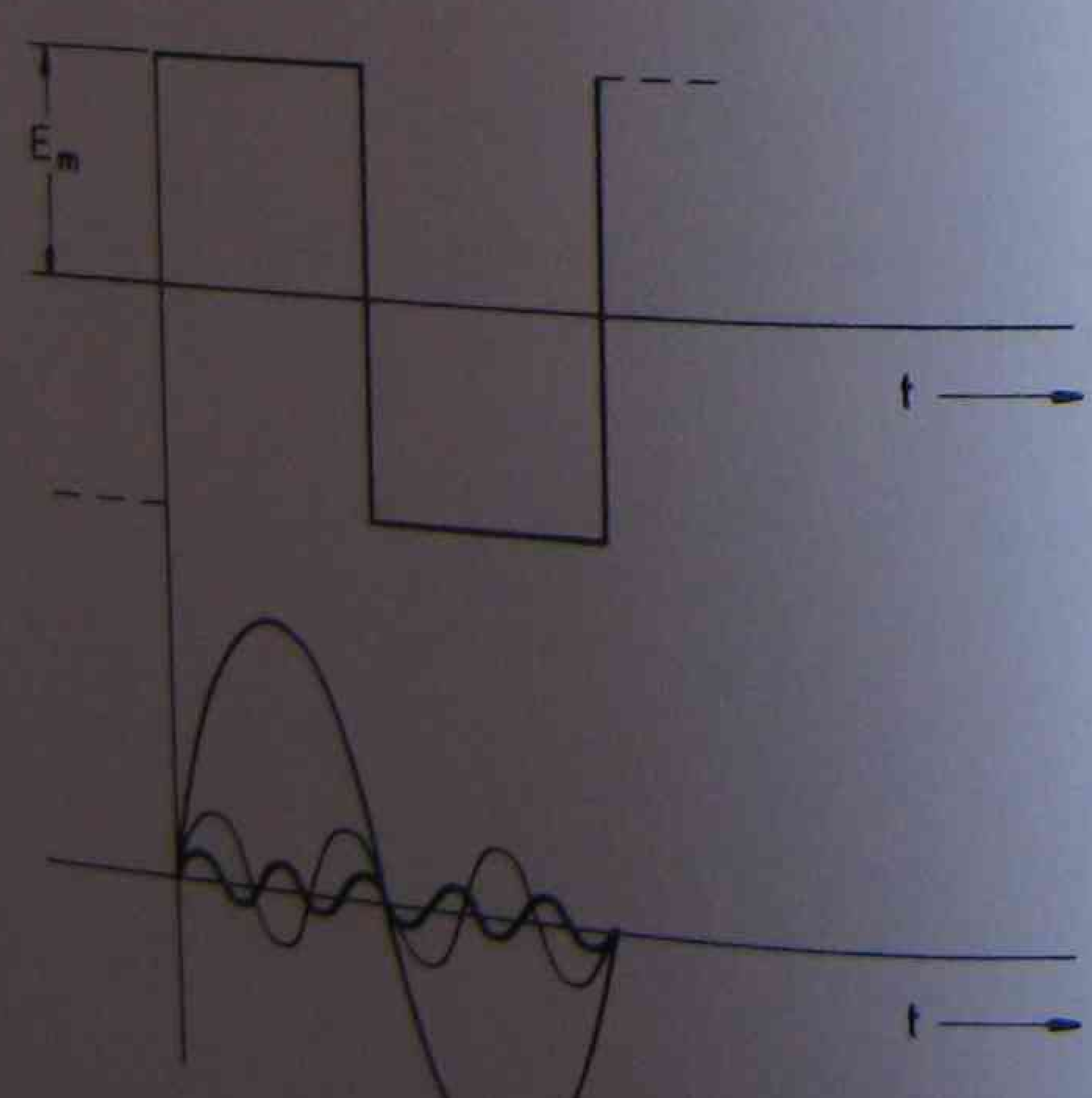


Figure 13

For the level of understanding required by this subject you should be able to analyse the wave using the two conditions in the previous section of this unit.

Let us consider the square wave of Figure 13.

Condition 1: *Is it half-wave symmetric?*
Yes. Then the waveform contains only odd harmonics.
Condition 2: *Is it an odd or even function?*
Odd. Then the waveform contains only sine functions.
Is the wave offset by a dc component?
No, because the positive and negative half cycles have the same maximum value.

We can, from these details, construct a Fourier series as follows:

$$e(t) = b_1 \sin \omega t + b_3 \sin 3\omega t + b_5 \sin 5\omega t + b_7 \sin 7\omega t\ldots\ldots$$

which has the same terms as the equation given except that the values of the constants b_1, b_3 etc are not evaluated as we do not have to perform that level of maths in this subject.

A second example is the sawtooth wave shown in Figure 14, which, by the method of Fourier analysis produces the following equation:

$$e = \frac{2\,Em}{\pi}\left(\sin \omega t - \frac{\sin 2\omega t}{2} + \frac{\sin 3\omega t}{3} - \frac{\sin 4\omega t}{4} + \frac{\sin 5\omega t}{5} \ldots\ldots\right)$$

In this case all the harmonics are present and once again there is no dc component.

Note: In general, a waveform has no dc component when it is symmetrical above and below the horizontal time axis.

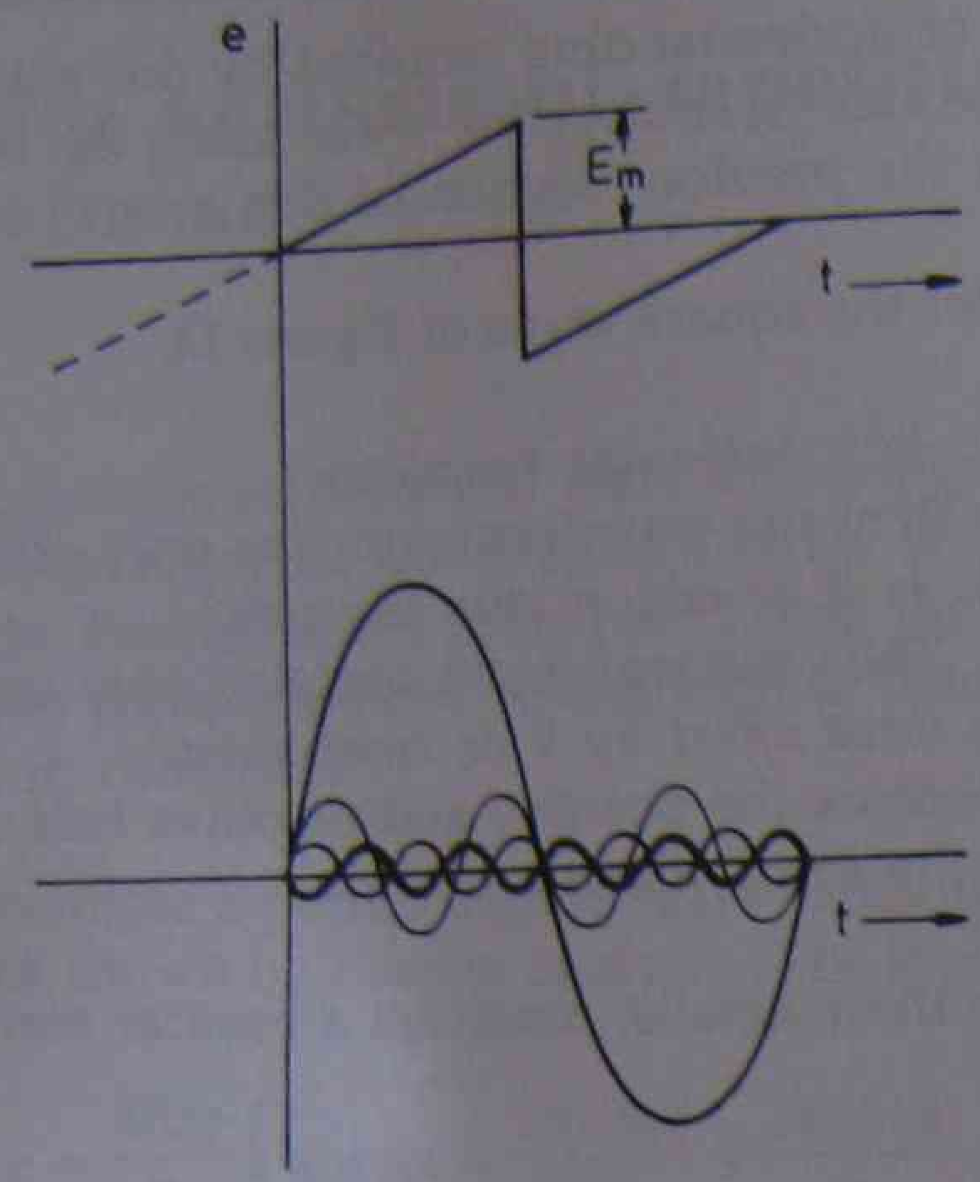


Figure 14

Let us again consider the sawtooth wave of Figure 14 within the scope of this subject.

Condition 1: *Is it half-wave symmetric?*
No. Then the waveform contains both odd and even harmonics.
That means that all harmonics are present.

Condition 2: *Is it an odd or even function?*
Odd. Then the waveform contains only sine functions.
Is the wave offset by a dc component?
No.

We can, from these details, construct a Fourier series as follows:

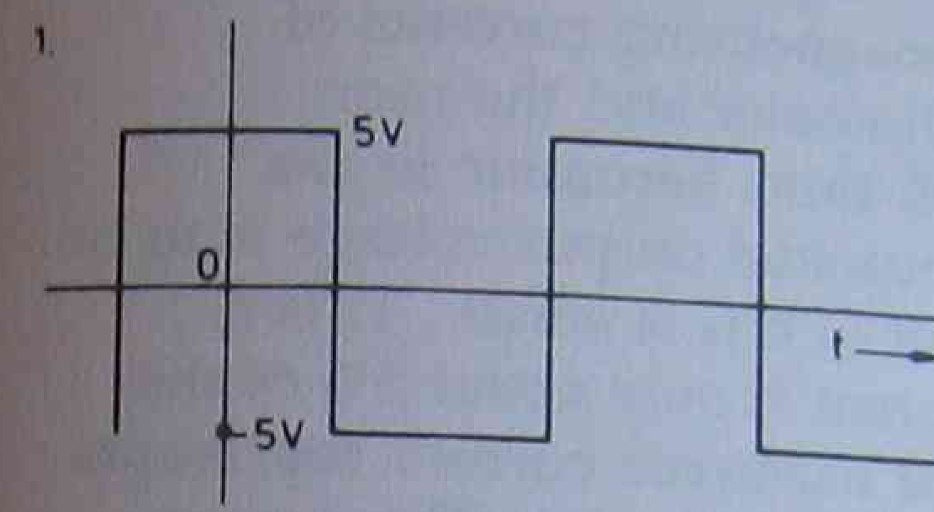
$$e(t) = b_1 \sin \omega t + b_2 \sin 2\omega t + b_3 \sin 3\omega t + b_4 \sin 4\omega t \ldots\ldots$$

Now try the exercises in Self-assessment 3. These questions are for your self-testing only. Do *not* send your answers to OTEN. Check your answers with those given at the end of the unit.

Self-assessment 3

Construct basic Fourier series for the following waveshapes:

1.



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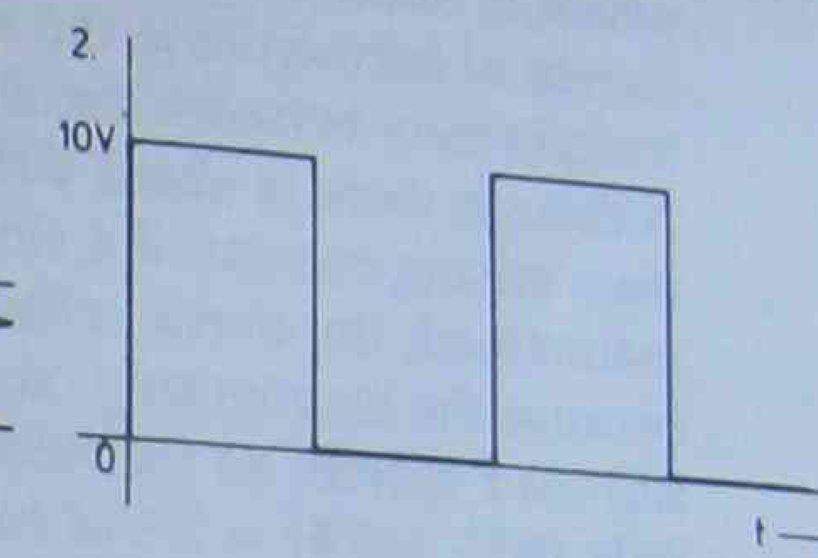

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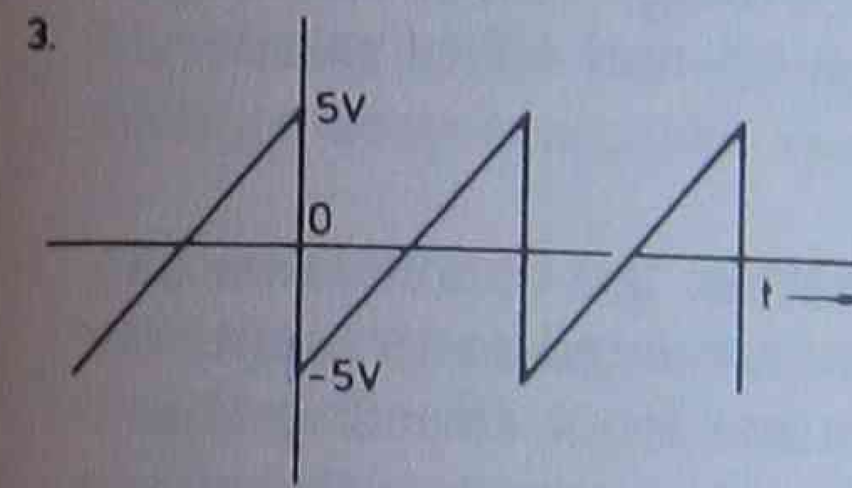

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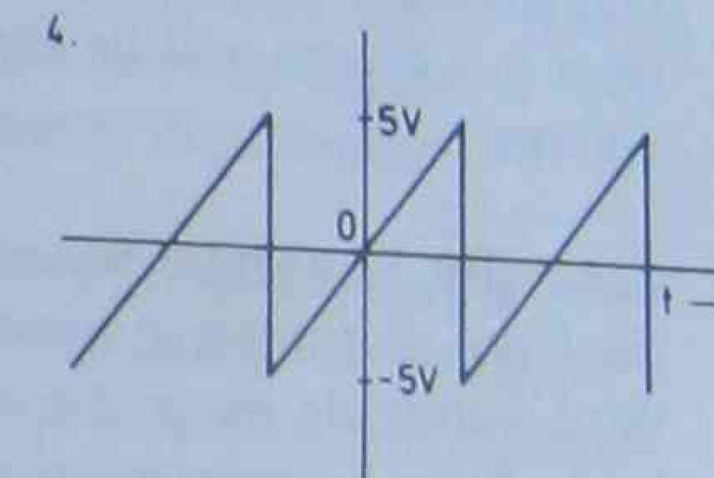

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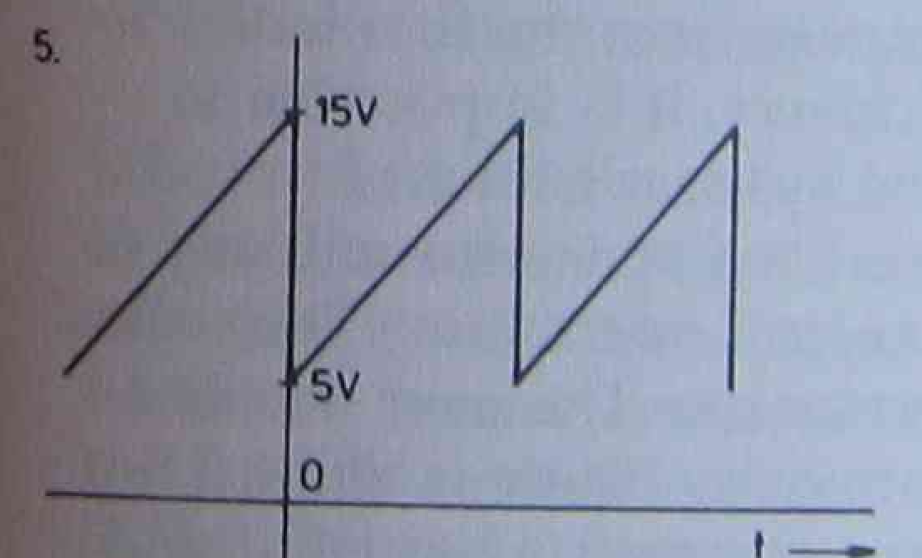

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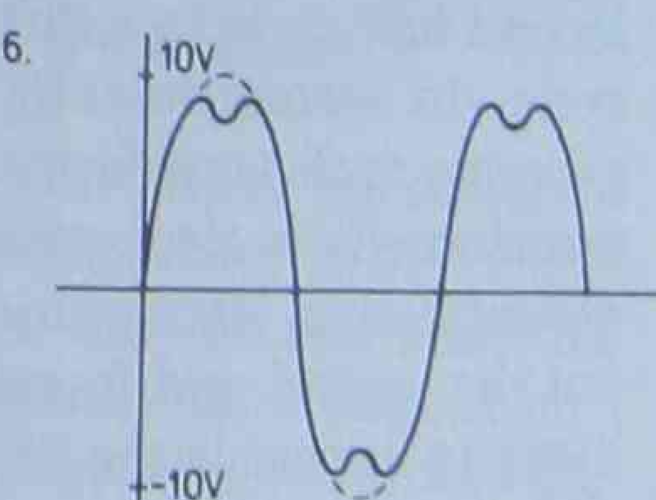

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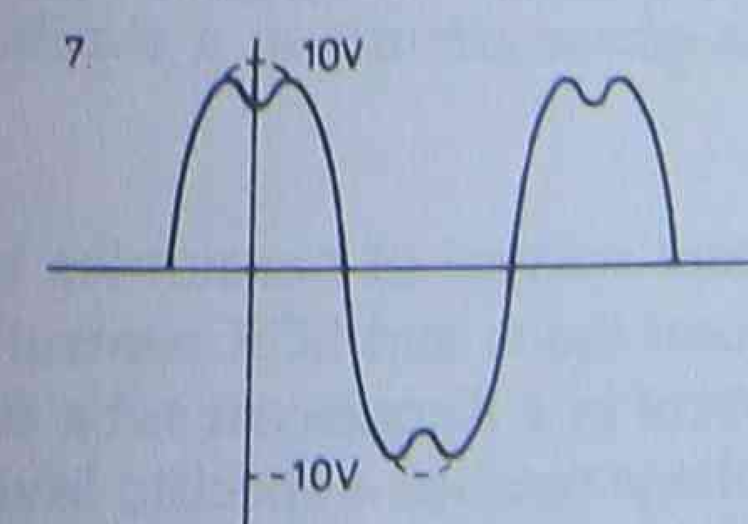


Sources of harmonics

The source of harmonics in a power system, at the most basic level, is the shape of the magnetic field from the alternator rotor and the layout of the stator coils in their slots. In practice, these two combine to produce an emf which is remarkably close to sinusoidal. The next possible source of harmonics is the magnetising currents of transformers between the alternator and the user. Although there is about 40% third harmonic in the magnetising current, if a sinusoidal output voltage is to be maintained, the overall effect of this is small. This is because the magnetising current is only about 5% of the full load current so the third harmonic current represents only (40% x 5%) = 2% of full load current. This means that the level of third harmonic voltage which can occur from this source will lie between 1% and 4% of rated voltage.

Generally, in normal power systems, the connections of the three-phase transformers are arranged to prevent third harmonic magnetising current from affecting the load. Even harmonics are very rare in power systems as they require the generation of waveforms which start the second half cycle in a different way from the first half cycle. In normal rotating machinery, it is impossible to generate such waveforms. The hysteresis loop of transformers is also symmetrical and so is also unlikely to generate even harmonics in normal operation. This rules out the second and fourth harmonics. The next that is likely to appear is the fifth harmonic. There is about 1% fifth harmonic in a transformer magnetising current and this behaves in three-phase circuits in a similar manner to the fundamental.

Increasingly important sources of harmonics in power systems are fluorescent lights and SCR control systems for dc motors. The current in a fluorescent tube does not start to flow until the voltage reaches a specific level in the cycle and it turns off before the voltage returns to zero. Thus the wave shape of current taken is as shown in Figure 15.

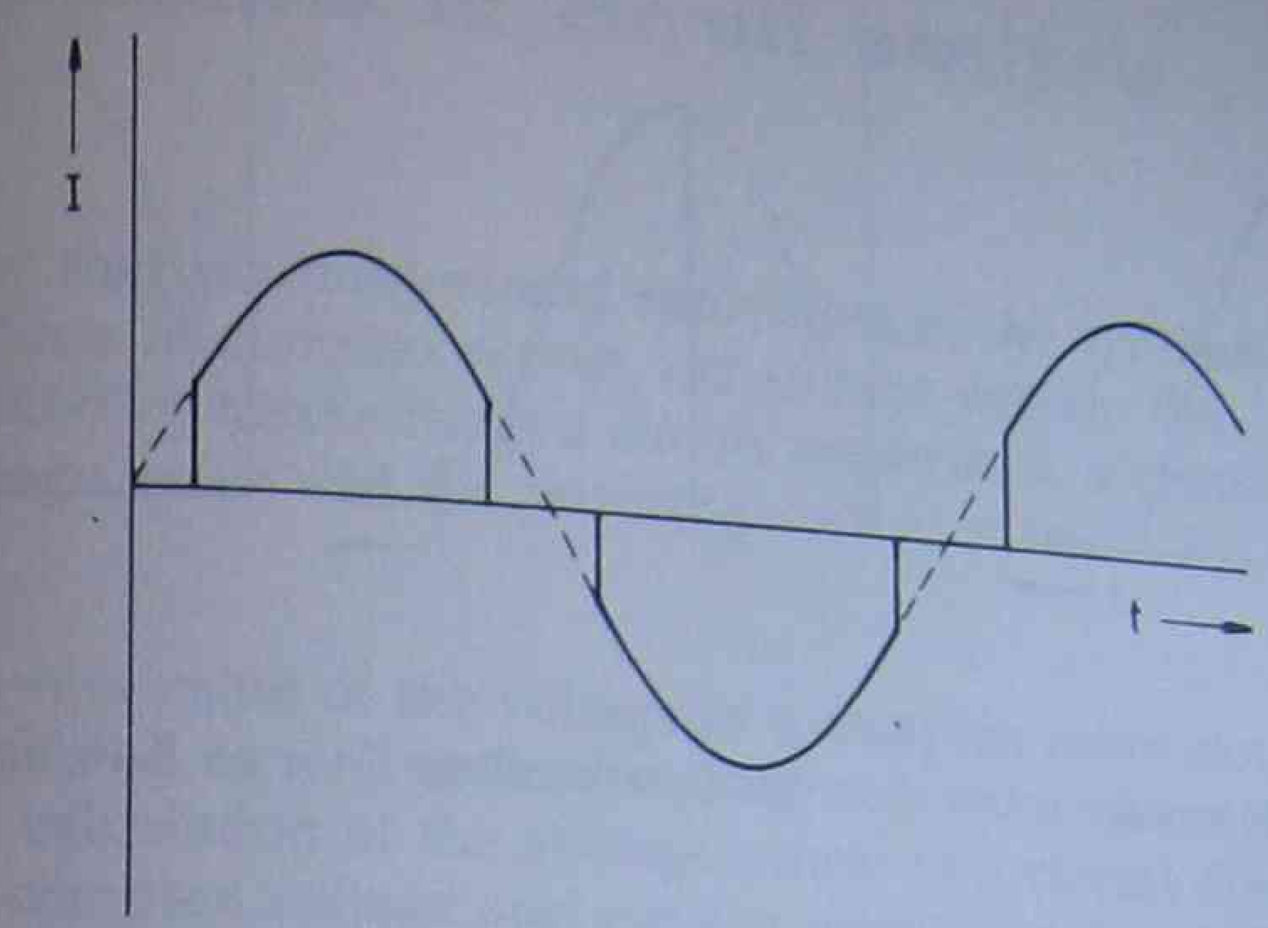
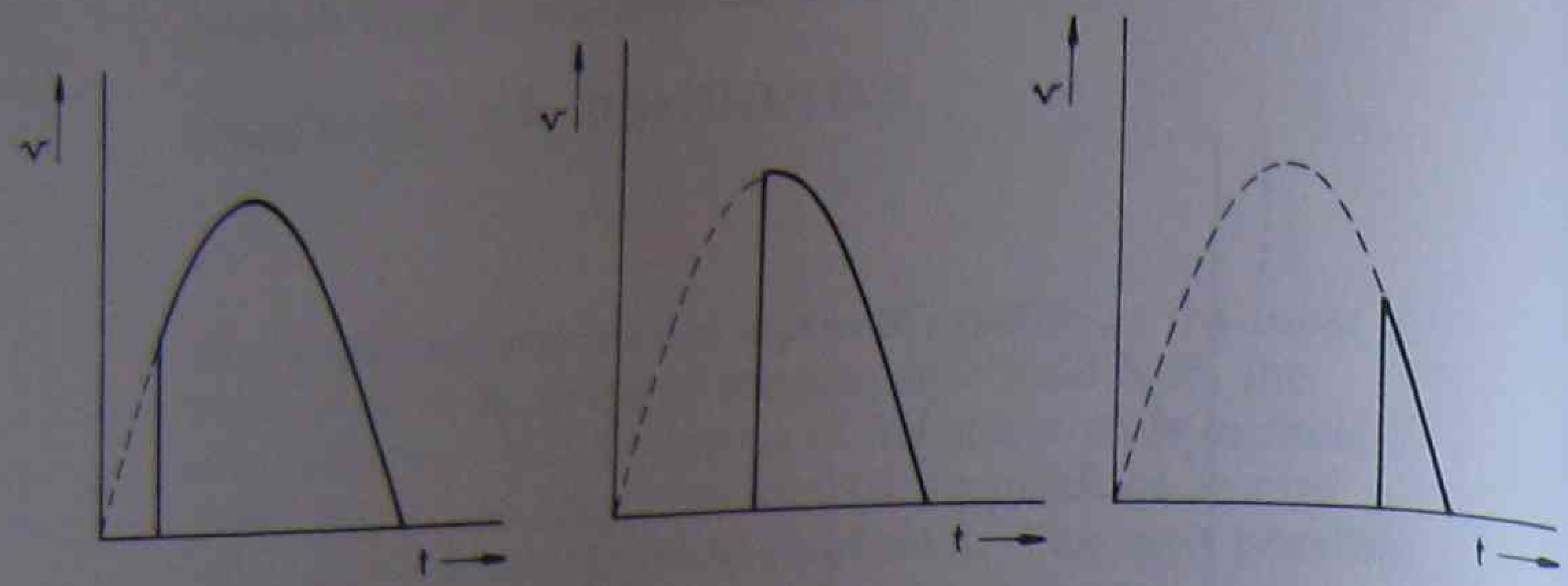


Figure 15

Such a waveform must contain harmonics. These current harmonics affect the voltage waveform by causing a voltage drop across the transformer impedance.

The current flowing in fluorescent tubes contains a variety of harmonics and the third and ninth (*triple-n harmonics*) and so on will add up in the neutral if banks of fluorescent lamps are wired phase-to-neutral and spread over the three phases. This can cause overheating of the neutral connection in large installations.

Silicon controlled rectifier (SCR) units affect the current drawn from the supply in a similar manner to fluorescent tubes, except the wave shape of the current, and hence its harmonic content, varies as the average dc voltage from the rectifier system is varied by phase angle control of the SCRs. Typical voltage waveforms are shown in Figure 16.



High dc voltage Medium dc voltage Low dc voltage

Figure 16

If the SCR unit is driving a motor and the motor must produce constant torque at all speeds, the harmonic content of the current waveform will increase at lower speeds. This is so because T∝ I and the flux under low speed conditions will be constant—therefore, for constant torque, the average value of I must be constant. For I_a average to be constant when the SCR is conducting for a small part of each cycle, it follows that the peak value of the current will be high. As this high current occurs over a small part of a cycle it must contain a large harmonic content. As with fluorescent tubes these current harmonics will show up in the system voltage because of the voltage drop across the transformer impedance.

In the electronics area of engineering non-sinusoidal waveshapes are more common than sinusoidal, so that Fourier analysis is essential to analyse and synthesise waves. The operation of most electronic musical instruments depends on these principles.

wn2

Harmonics in circuit analysis

Now that you understand something of the synthesis and analysis of complex waves, I shall now explain the method of determining a circuit response to a complex voltage wave and the resulting current wave.

The rms value of the voltage of a complex wave can be calculated as well as the resulting rms value of current. The calculation of the average power in a circuit due to the complex voltage and current waves can then be performed.

Consider the circuit shown in Figure 17:

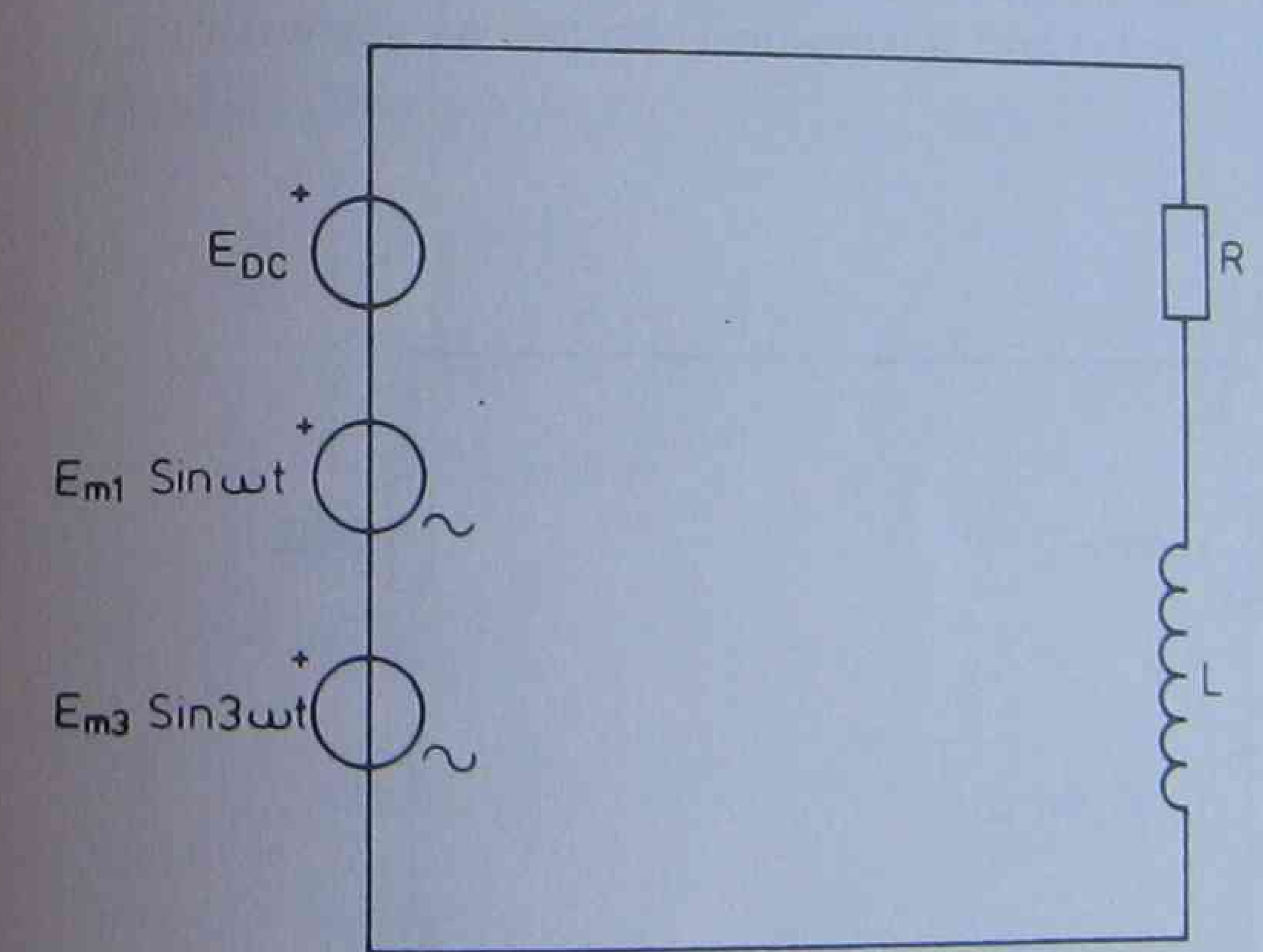


Figure 17

To calculate the total voltage, current and power in the circuit, it is necessary to consider each voltage source separately, calculate its respective current and then to combine these values to obtain the resultant overall values.

This is the *superposition method* which is similar to the method used in multi-sourced dc circuits.

To demonstrate this method you can work through the following example.

The superposition method

A Series R–L circuit has a resistance of 20 ohms and an inductance of 0.2 henry. The applied voltage is

$$V(t) = 25 + 80 \sin \omega t + 20 \sin 3\omega t$$
$$\text{where } \omega = 250 \text{ rad/sec}$$

Find:

1 the instantaneous current
2 rms voltage and current
3 average power supplied to the circuit

Solution

Draw up the circuit with separate voltage sources for the dc, fundamental and harmonic voltages, as shown in Figure 18.

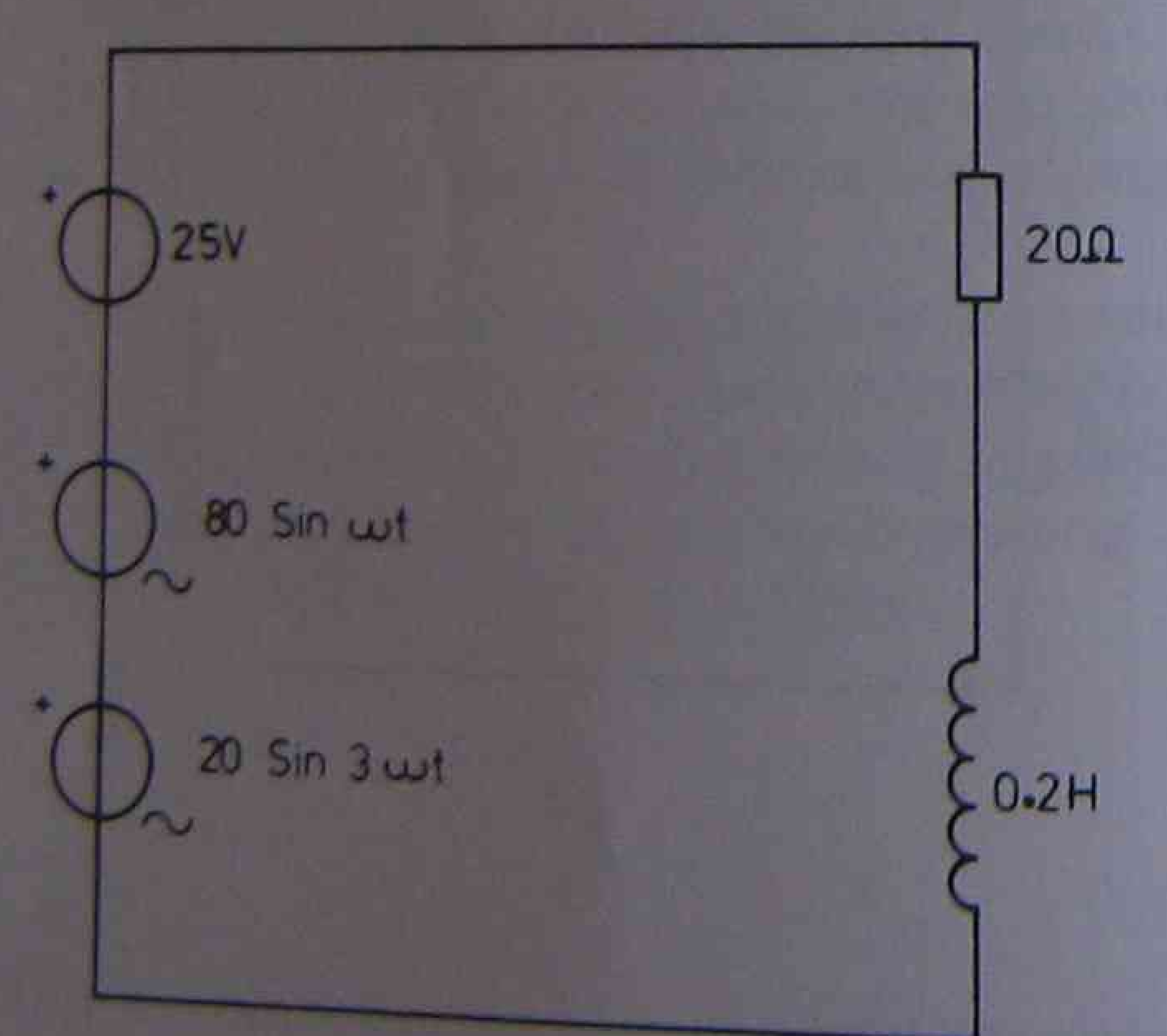


Figure 18

1 The instantaneous current

Now consider each voltage source separately acting on the circuit.

(a) ***dc = 25 volts***

Note: The inductance has no effect on dc.

Then $I = \dfrac{E}{R}$

$= \dfrac{25}{20}$

$= 1.25$ amperes

(b) ***ac fundamental = 80 sin ωt***

Calculate the inductive reactance at $\omega = 250$ rad/s

$$X_L = \omega L$$
$$= 250 \times 0.2$$
$$= 50 \text{ ohms}$$

Calculate impedance of circuit at $\omega = 250$ rad/s

$$Z = \sqrt{R^2 + X_L^2}$$
$$= \sqrt{(20)^2 + (50)^2}$$
$$= 53.85 \text{ ohms}$$

$$\theta = \tan^{-1}\frac{X}{R}$$
$$= \tan^{-1}\frac{50}{20}$$
$$= 68.2°$$

$$Z = 53.85 \angle 68.2° \text{ ohms}$$

Now calculate the current

$$i = \frac{e}{Z}$$
$$= \frac{80 \sin 250t}{53.85 \angle 68.2°}$$

Then $i = 1.49 \sin(250t - 68.2°)$

For the fundamental wave

$$V_{rms} = \frac{80 \angle 0^\circ}{\sqrt{2}}$$

$$= 56.6 \angle 0^\circ \text{ volts}$$

$$I_{rms} = \frac{1.49 \angle -68.2^\circ}{\sqrt{2}}$$

$$= 1.05 \angle -68.2^\circ \text{ amperes}$$

(c) *ac, third harmonic = 20 sin 3ωt*

Calculate the inductive reactance at ω = 750 rad/s

$$X_L = \omega L$$
$$= 750 \times 0.2$$
$$= 150 \text{ ohms}$$

Calculate impedance of circuit at ω = 750 rad/s

$$Z = \sqrt{R^2 + X_L^2}$$
$$= \sqrt{(20)^2 + (150)^2}$$
$$= 151.3 \text{ ohms}$$

$$\theta = \tan^{-1}\frac{X}{R}$$
$$= \tan^{-1}\frac{150}{20}$$
$$= 82.4^\circ$$

$$\bar{Z} = 151.3 \angle 82.4^\circ$$

Now calculate the current for the third harmonic

$$i = \frac{e}{Z}$$
$$= \frac{20 \sin 3wt}{151.3 \angle 82.4^\circ}$$
$$= 0.132 \sin (750t - 82.4^\circ)$$

For the third harmonic wave

$$V_{rms} = \frac{20 \angle 0^\circ}{\sqrt{2}}$$
$$= 14.14 \angle 0^\circ \text{ volts}$$

$$I_{rms} = \frac{0.132 \angle 82.4^\circ}{\sqrt{2}}$$
$$= 0.093 \angle 82.4^\circ \text{ amperes}$$

Instantaneous current i = 1.25 + 1.49 sin (250t − 68.2°) + 0.132 sin (750t − 82.4°)

2 Voltage and current (rms)

To calculate the total rms voltage in the circuit we *cannot* use simple arithmetic with phasor values.

$$P_{dc} = \frac{E_{dc}^2}{R}, \quad P_1 = \frac{E_1^2}{R}, \quad P_3 = \frac{E_3^2}{R}$$

$$P_{total} = P_{dc} + P_1 + P_2$$

$$\frac{E_{rms}^2}{R} = \frac{E_{dc}^2}{R} + \frac{E_1^2}{R} + \frac{E_3^2}{R}$$

Then
$$\boxed{E_{rms} = \sqrt{E_{dc}^2 + E_1^2 + E_3^2}}$$

If we consider the power as

$$P = I^2R$$

Then using the same principle as above

$$\boxed{I_{rms} = \sqrt{I_{dc}^2 + I_1^2 + I_3^2}}$$

Note: These formulae for E_{rms} and I_{rms} can be used for any number of harmonics and *must be memorised.*

For the example above,

$$E_{rms} = \sqrt{(25)^2 + (56.6)^2 + (14.14)^2}$$
$$= 63.5 \text{ volts}$$

and $$I_{rms} = \sqrt{(1.25)^2 + (1.05)^2 + (0.093)^2}$$
$$= 1.64 \text{ amperes}$$

Average power supplied to the circuit

To calculate the average power supplied to the circuit use the total rms current and the resistance of the circuit and then check by calculating each component of power.

$$\text{total power} = I^2R = (1.64)^2 \times 20 = 53.8 \text{ watts}$$

Check

$$\text{dc power} = I^2R = (1.25)^2 \times 20 = 31.25 \text{ watts}$$

$$\text{fundamental wave power} = I^2R = (1.05)^2 \times 20 = 22.05 \text{ watts}$$

$$\text{third harmonic power} = I^2R = (0.093)^2 \times 20 = 0.17 \text{ watts}$$

$$\text{total power} = 31.25 + 22.05 + 0.17 = 53.5 \text{ watts}$$

Notes on the example:

- When calculating dc values, only the resistance of the circuit is taken into consideration.
- The inductive reactance increased proportionally to the order (value) of the harmonic.
- The current lags at an increasing angle for higher value harmonics.
- The harmonic currents contribute to the power dissipation in a circuit.

Now try the exercise in Self-assessment 4. This question is for your self-testing only. Do *not* send your answer to OTEN. Check your answers with those given at the end of the unit.

Self-assessment 4

A circuit containing a coil with resistance of 25 ohms and inductance of 0.25 henrys is in series with a 45 ohm resistor. The supply voltage is given by the expression

$$e(t) = 13 + 120 \sin \omega t + 35 \sin 3\omega t$$

The fundamental frequency is 50 Hz.

Determine:

1 the expression for the instantaneous current

2 the rms value of current

3 the total power dissipated in the coil.

Review

In the second part of this unit you have covered the analysis of a simple repetitive waveform to determine its basic Fourier series components. This was achieved by using waveform symmetry. The next topic contained an explanation of the sources of harmonics in power systems and concentrated on the three main areas of magnetic circuits, transformers, fluorescent lights and SCR-controlled motors. Finally, the effect of non-sinusoidal waveforms on complex impedances in circuits was detailed.

You have to be able to calculate the total rms voltages and current in these circuits, remembering that to add individual rms values, the formula

$$I_{rms} = \sqrt{(I_{1rms})^2 + (I_{2rms})^2 + (I_{3rms})^2}$$

must be used for current and the similar one for voltage. The calculation of the individual rms currents for each harmonic was shown by calculating each harmonic current from the harmonic voltage in the complex impedance. The individual currents we then added in the formula shown above.

Answers to self-assessment questions

Self-assessment 1

1 half-wave symmetric
2 not half-wave symmetric
3 half-wave symmetric
4 half-wave symmetric
5 half-wave symmetric
6 half-wave symmetric
7 not half-wave symmetric

Self-assessment 2

1 1 odd
2 even
3 neither odd nor even
4 neither odd nor even
5 neither odd nor even
6 odd
7 odd

2 (a) odd
(b) neither odd nor even
(c) neither odd nor even

Self-assessment 3

1 $e(t) = a_1 \cos \omega t + a_3 \cos 3\omega t + a_5 \cos 5\omega t$.......

2 $e(t) = 5 + b_1 \sin \omega t + b_2 \sin 2\omega t + b_3 \sin 3\omega t$........
$+ a_1 \cos \omega t + a_2 \cos 2\omega t + a_3 \cos 3\omega t$........

3 $e(t) = b_1 \sin \omega t + b_2 \sin 2\omega t + b_3 \sin 3\omega t$......
$+ a_1 \cos \omega t + a_2 \cos 2\omega t + a_3 \cos 3\omega t$......

4 $e(t) = b_1 \sin \omega t + b_2 \sin 2\omega t + b_3 \sin 3\omega t$......

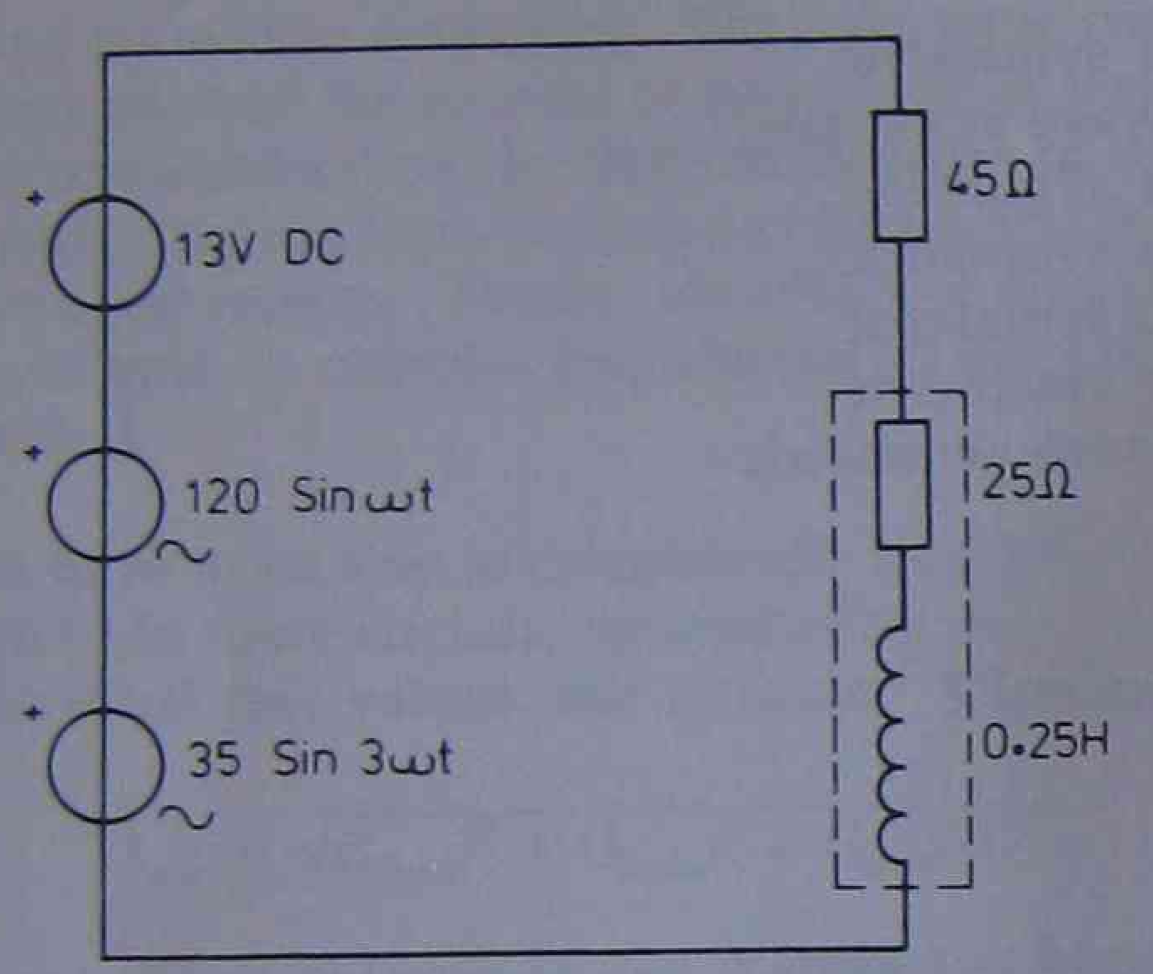
5 $e(t) = 10 + b_1 \sin \omega t + b_2 \sin 2\omega t + b_3 \sin 3\omega t$......
$+ a_1 \cos \omega t + a_2 \cos 2\omega t + a_3 \cos 3\omega t$........

6 $e(t) = b_1 \sin \omega t + b_3 \sin 3\omega t + b_5 \sin 5\omega t$........

7 $e(t) = a_1 \cos \omega t + a_3 \cos 3\omega t + a_5 \sin 5\omega t$........

Self-assessment 4

Circuit diagram



Frequency = 50 Hz = 2π x 50 rad/s = 314 rad/s

1 ***Expression for instantaneous current***

dc: $i = \frac{E}{R_T}$

$= \frac{13}{70}$

$= 0.186$ amperes

$R_T = 45 + 25$
$= 70$ ohms

Fundamental:

$X_L = \omega L$
$= 314 \times 0.25$
$= 78.5$ ohms

$$\bar{Z} = \sqrt{R^2 + X^2} \;\angle \tan^{-1}\frac{X}{R}$$

$$= \sqrt{(70)^2 + (78.5)^2} \;\angle \tan^{-1}\frac{78.5}{70}$$

$$= 105.2 \angle 48.3^\circ \text{ ohms}$$

$$i_1 = \frac{120 \sin 314t}{105.2 \angle 48.3^\circ}$$

$$= 1.14 \sin (314t - 48.3^\circ) \text{ amperes}$$

Third harmonic:

$X_L = \omega L$
$= 942 \times 0.25$
$= 235.5$ ohms

$$Z = \sqrt{R^2 + X_L^2} \;\angle \tan^{-1}\frac{X}{R}$$

$$= \sqrt{(70)^2 + (235.5)^2} \;\angle \tan^{-1}\frac{235.5}{70}$$

$$= 245.7 \angle 73.4^\circ \text{ ohms}$$

$$i_3 = \frac{35 \sin 942t}{245.7 \angle 73.4^\circ}$$

$$= 0.14 \sin (942t - 73.4^\circ)$$

Then expression is

$$i(t) = 0.186 + 1.14 \sin (314t - 48.3^\circ) + 0.14 \sin (942t - 73.4^\circ) \text{ amperes}$$

2 ***rms value of current***

$$I = \sqrt{(0.186)^2 + \left(\frac{1.14^2}{\sqrt{2}}\right) + \left(\frac{0.14^2}{\sqrt{2}}\right)}$$

$$= \sqrt{0.035 + 0.65 + 0.0098}$$

$$= \sqrt{0.695}$$

$$= 0.83 \text{ amperes}$$

3 Power in coil

$$
\begin{aligned}
R &= 25 \text{ ohms} \\
&= I^2R \\
&= (0.83)^2 \times 25 \\
&= 17.4 \text{ watts}
\end{aligned}
$$

Glossary of terms

analysis	breaking down a waveshape into its component parts (harmonics).
distortion	any variation from the original waveshape.
even function	a function which has the same value of f(t) for t and –t. It contains only cosine terms in its Fourier series.
Fourier series	a mathematical series which can be used to represent a complex repetitive wave form.
fundamental frequency	the basic frequency of a complex wave. It is the reciprocal of the period of the complete complex wave.
half-wave symmetry	the shapes of the positive half cycle and negative half cycle of the waveform are identical. The Fourier series of such waves has only odd harmonics.
harmonic	a sine wave of some frequency which is a multiple of the base or fundamental frequency.
integral calculus	a mathematic method which can be used to find areas under waveshapes.
non-sinusoidal	a waveshape which has a form different to a sine wave.
odd function	a function which has the opposite value of f(t) for t and –t. It contains only sine terms in its Fourier series.

wk 3+4+5

Ref 29

Electronic Signals and Systems

7761L

Student Workbook

17908

UEENEEH014B - Troubleshoot frequency dependent circuits

National Module No. EA190
Electrical Engineering
St George TAFE
Sydney Institute

3. State which of the following electrical/electronic systems would use closed loop control.

- The speed control on a kitchen food processor.
- The laser tracking system in a compact disk player.
- A remote controller for a model aeroplane.
- An AGC system in a radio receiver.
- An oven for the crystal in a high stability oscillator.
- The pressure controller in a steel rolling mill.

4. Briefly describe the action of the feedback in a voltage amplifier.

5. Two motor speed controllers are controlled by potentiometers. One uses open loop control and the other uses closed loop control. Briefly compare the likely control characteristics.

Section 2: Signals, spectra and non-linearity

WK3

SUGGESTED DURATION	PREAMBLE
7 hrs	To introduce you to Fourier concepts and the spectra of some common signals, and to extend these concepts in explaining the effect of non-linearity.
This section covers learning outcomes 3 and 4 of the Module Descriptor.	

Objectives

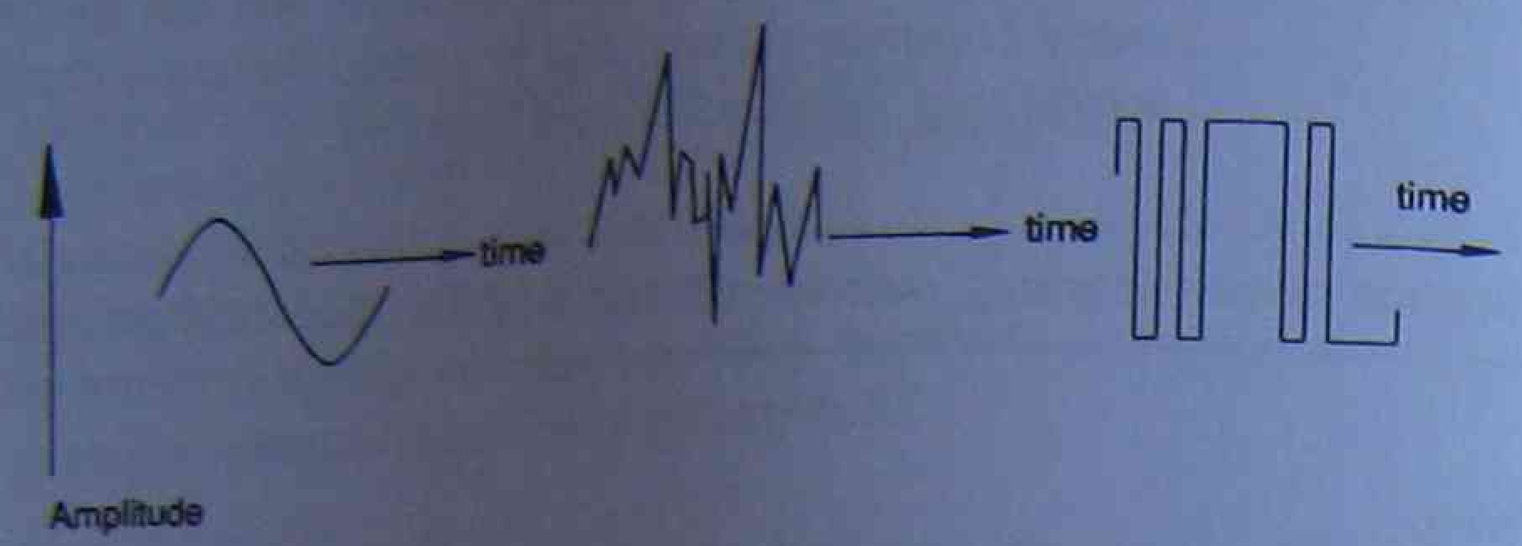
At the end of this section you should be able to:

- ☐ predict the spectral frequencies of a given periodic wave
- ☐ recognise that a non-periodic signal such as speech contains a continuous band of frequencies
- ☐ sketch typical time and frequency domain diagrams for white noise, speech, music, video and random binary data
- ☐ relate one line of video to a grey scale
- ☐ define non-linearity
- ☐ calculate harmonic and intermodulation distortion frequencies.

Time and frequency domains

Time domain

The traditional method of observing electrical signals is to view them in the time domain, using an oscilloscope.



Time domain displays

The information displayed is amplitude (voltage) versus time, which is adequate for most low frequency audio and digital waveform measurements involving timing and phase.

However, time domain measurements are not usually adequate when studying RF devices such as amplifiers, oscillators, filters, mixers, modulators and antennas. The reasons for this are given below.

1. Oscilloscopes cannot normally view waveforms above several hundred MHz. Their internal amplifiers are not capable of amplifying many of the high frequency signals found in communications equipment. For example, AUSSAT signals return to earth at 12GHz (12000 MHz!!!). An oscilloscope can not be used to view these signals.

2. Oscilloscopes are often not sensitive enough to display the tiny signals found in communications equipment. For example, most oscilloscopes have 1mV/cm as the most sensitive range, and could not display signals with uV levels. It would be even more difficult to display a 1uV signal at the same time as a 10V signal.

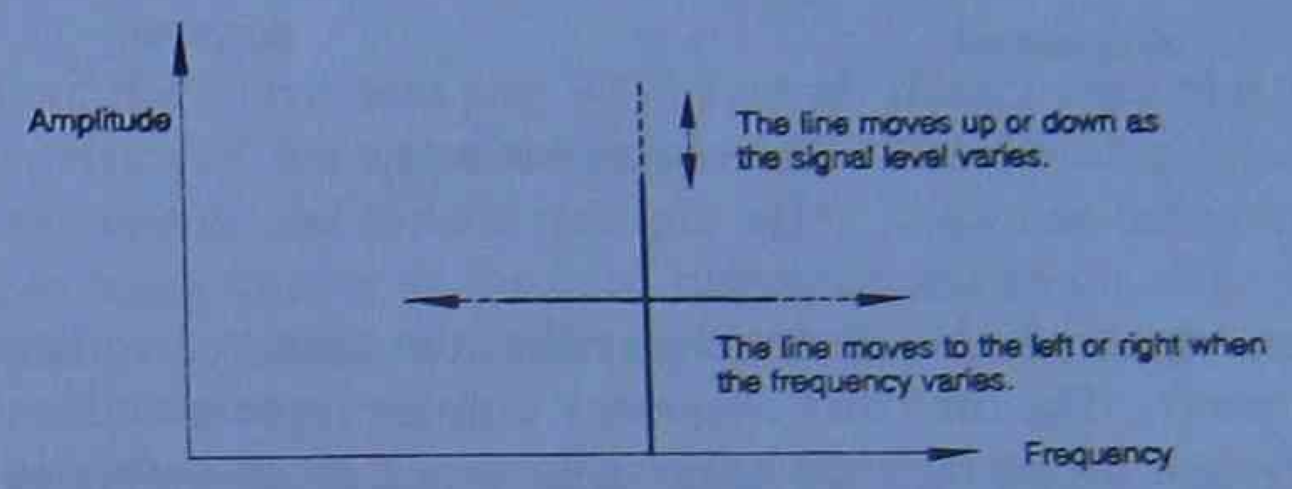
3. An oscilloscope cannot break a complex signal down into its constituent parts; it displays them all added together. Many signals are complex; that is, they are composed of more than one frequency component. It is impossible with an oscilloscope to examine individual components of a complex wave.

Are you ready to throw your oscilloscope away? Don't! Even though it can't do the things mentioned above, it is probably the most versatile general purpose laboratory instrument. Besides, the instrument that can do everything mentioned above may cost between $10,000 and $100,000. This expensive instrument is called a spectrum analyser.

Frequency domain

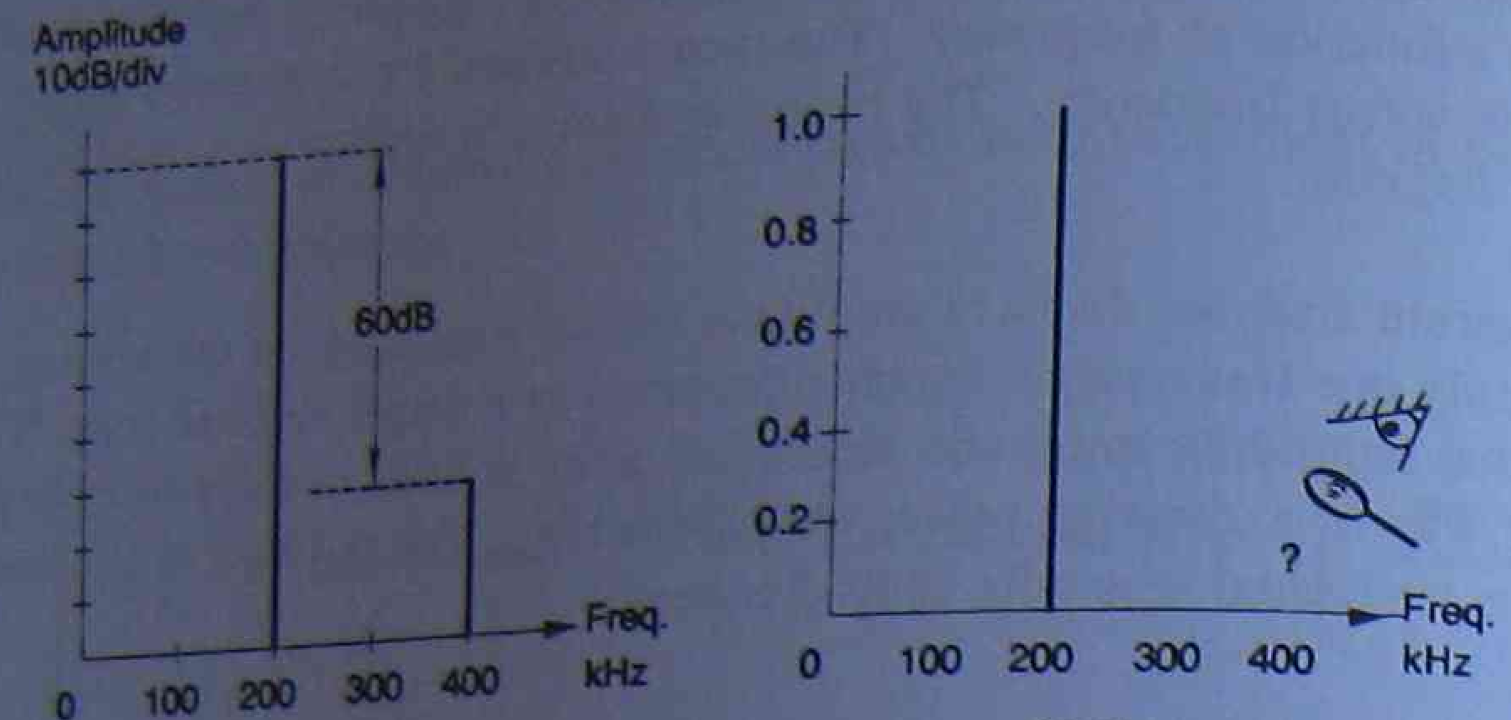
In general, the term frequency domain refers to any graph or measurement which is taken as a function of frequency. The most commonly encountered measurement is amplitude versus frequency. The resulting display is known as a spectrum or a spectral diagram.

The spectrum analyser displays amplitude versus frequency on the screen. A signal having only one frequency component appears as a single vertical line. The height of this line represents amplitude measured either in volts or milliwatts or dBm (another way of expressing power measurements). The position of this vertical line along the horizontal axis tells us its frequency.



Frequency domain display

One feature of the spectrum analyser is that it allows either linear or logarithmic (dB) scales. The logarithmic scale permits both large and small signals to be displayed simultaneously. For example a signal which is 60dB below another is $\frac{1}{1000,000}$ of that signal's power. On a linear scale only the larger signal would be seen. Viewed on an oscilloscope, the effect of the smaller signal would not be noticeable.



(a) Logarithmic

(b) Linear

Spectrum analyser display

In many cases a signal can be observed in either the time or frequency domains. The choice is yours. The time and frequency domain representations of a signal are complementary, and if one representation is known, then the other can be derived from it.

This field of mathematics is known as Fourier analysis, after Jean Baptiste Joseph, Baron de Fourier (1768-1830). He accompanied Napoleon on the Egyptian campaign in 1798, becoming Governor of Lower Egypt before returning to France where he produced his classic paper 'Theories Analytique de la Chaleur' (Analysis of the Flow of Heat). In it he evolved the mathematical series which bears his name today, and has found application in most branches of applied science.

Summary

- Time domain refers to signals and quantities viewed as a function of time.
- The oscilloscope displays signals in the time domain.
- Frequency domain refers to signals and quantities viewed as a function of frequency.
- The spectrum analyser displays signals in the frequency domain.

Fundamentals of Fourier Analysis

Stated in the simplest of terms, Fourier's theorem says:

> A complex periodic waveform may be analysed as a number of harmonically related sinusoidal waves.

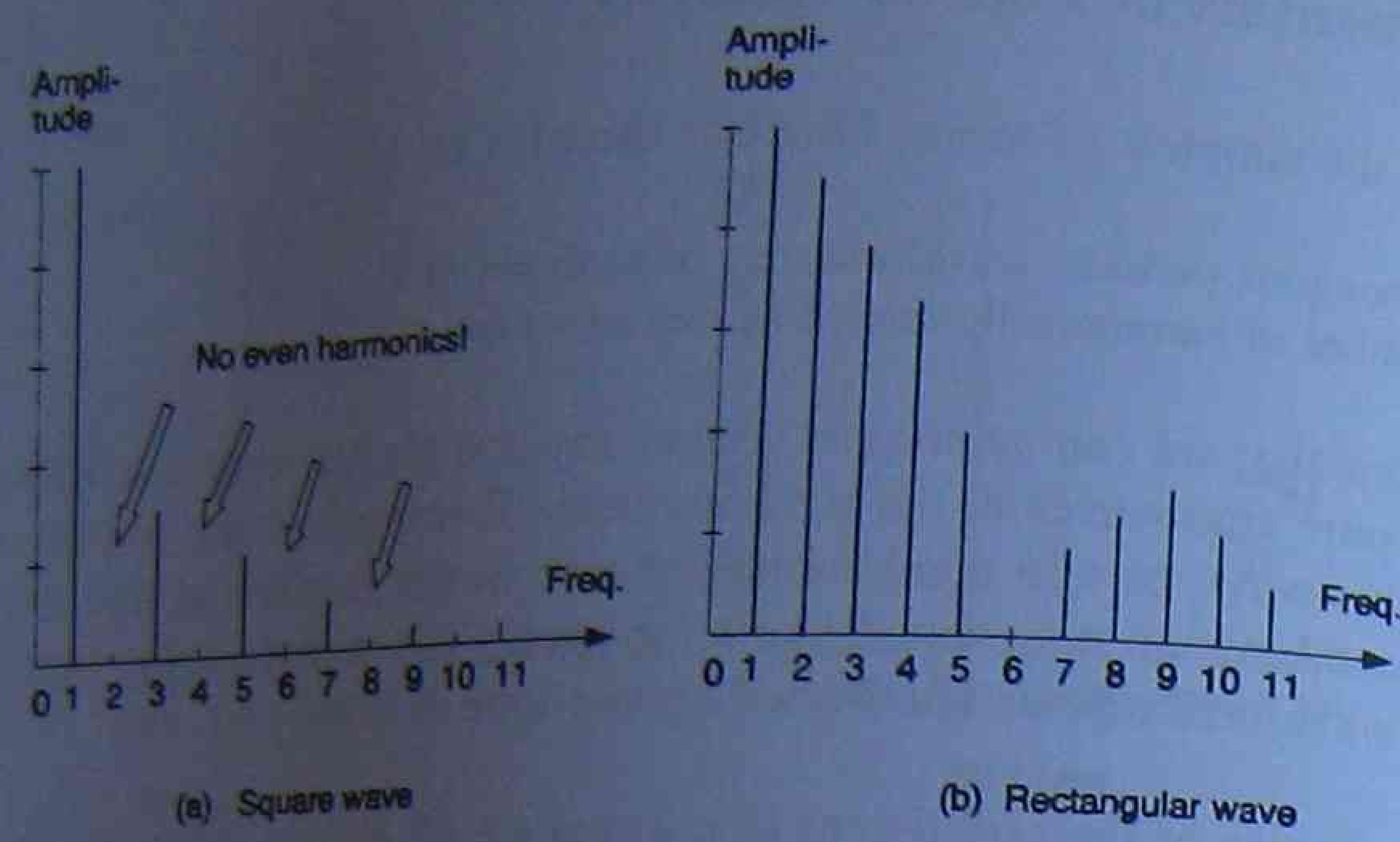
This means that we can synthesise (make) any complex periodic waveform by adding together pure sine waves in the right amounts. Electronic music can be created in exactly this way: certain combinations of sine waves may sound like a flute, while another combination may sound like a fog-horn. The term periodic simply means that the waveform repeats itself after a given time period T.

The frequencies of the constituent sine waves are all integer multiples of the fundamental frequency of the waveform concerned. These multiples are known as the harmonic frequencies, the second multiple being known as the second harmonic, the third multiple being known as the third harmonic and so on. The first harmonic is just the original frequency, and is referred to simply as the fundamental.

This applies to all complex periodic waveforms, such as square, triangle, pulsed and sawtooth signals. An ideal sine wave however, has only a fundamental component and no harmonics.

Generally speaking, the higher harmonics are weaker than the lower ones, although the individual amplitudes may vary in a complex manner. (Note that in the figure on the next page, the ninth harmonic is larger than the eighth harmonic). Also note that the fundamental or any harmonic(s) may have zero amplitude.

An example of this is the square wave which has only odd harmonics. Another property of the square wave is that the third harmonic has an amplitude $\frac{1}{3}$ that of the fundamental, the fifth harmonic has an amplitude $\frac{1}{5}$ that of the fundamental and so on.



Distribution of harmonics

Example

A waveform has a period T = 40mS. Calculate the frequency of the fundamental, and the second, third and fourth harmonics.

Fundamental frequenc= $\frac{1}{T}$

$= \frac{1}{40 \times 10^{-3}}$

$= 25$Hz

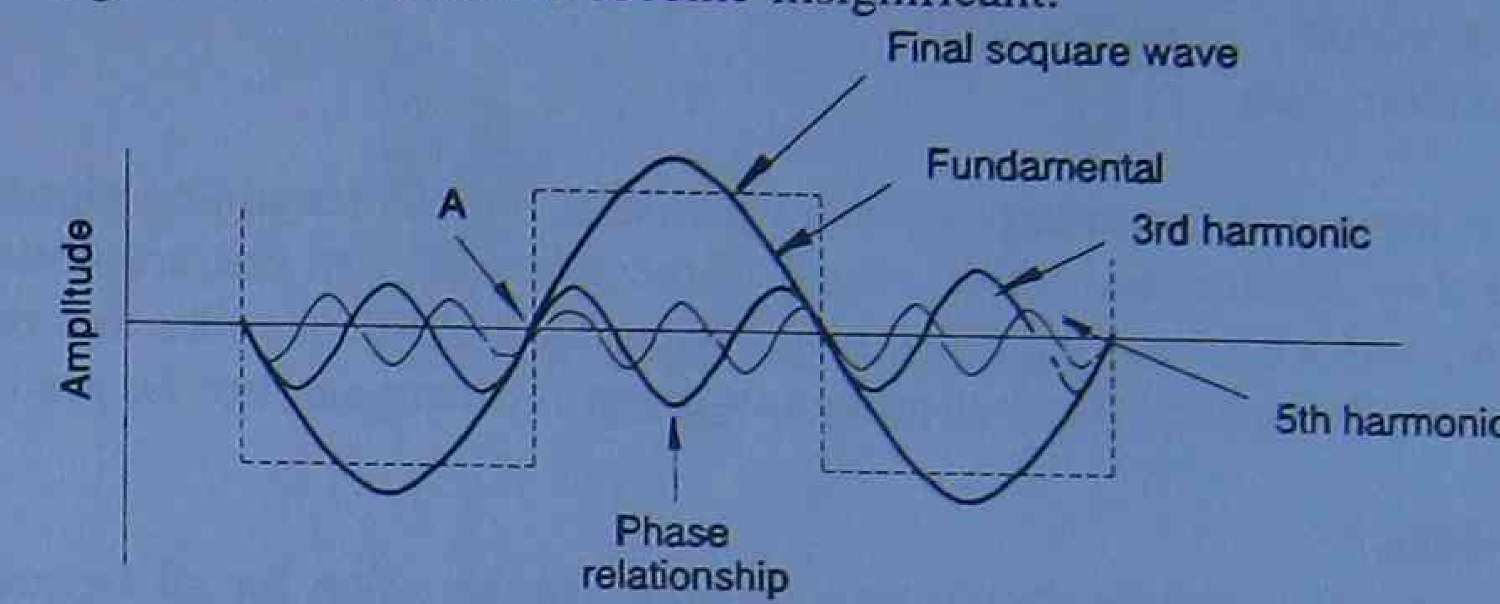
Second harmonic $= 2 \times 25$Hz
$= 50$Hz

Third harmonic $= 3 \times 25$Hz
$= 75$Hz

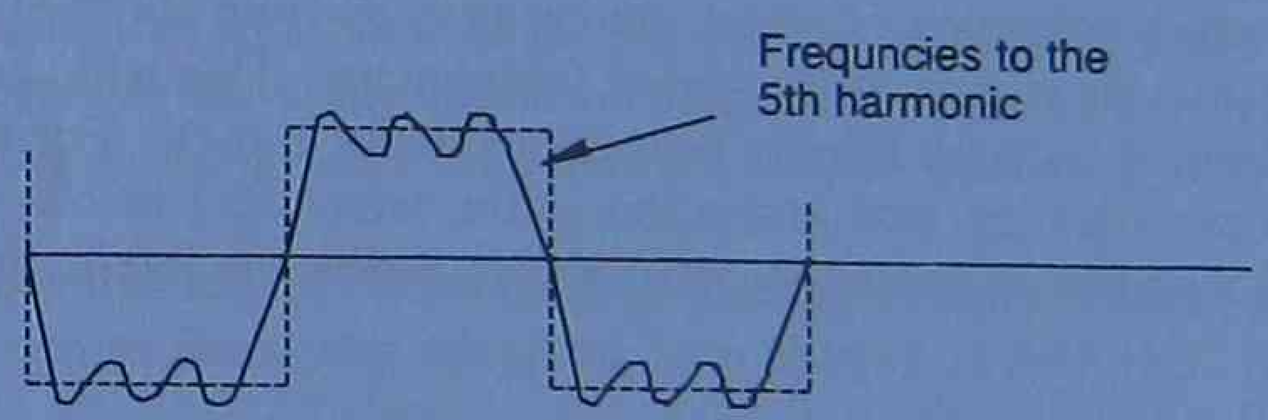
Fourth harmonic $= 4 \times 25$Hz
$= 100$Hz

Let us now see how a square wave can be made by adding a fundamental frequency and a number of harmonics. You will recall that only odd harmonics are required, and that the third harmonic has $\frac{1}{3}$ the amplitude of the fundamental and so on.

Figure (a) below shows the fundamental, the third and fifth harmonics and their phase relationships, and Figure (b) shows the resultant. Note that all odd harmonics to infinity must be considered to construct a perfect square wave, although in practice the higher order harmonics become insignificant.



(a) Components of a square wave



(b) Resultant of frequencies to fifth harmonic

Waveforms and spectral diagrams for common signals

You will now be introduced to some of the waveforms commonly found in communications equipment and systems. The waveforms are:

- the sine wave
- the square wave
- white noise
- speech
- music
- television
- random data.

The sine wave and the square wave are periodic, but the remaining signals are not, because they are not predictable. Each of these has its own characteristic frequency spectrum, and we know in general terms what each one looks like, but the precise detail of the frequency distribution at any given moment can not be predicted.

Bandwidth

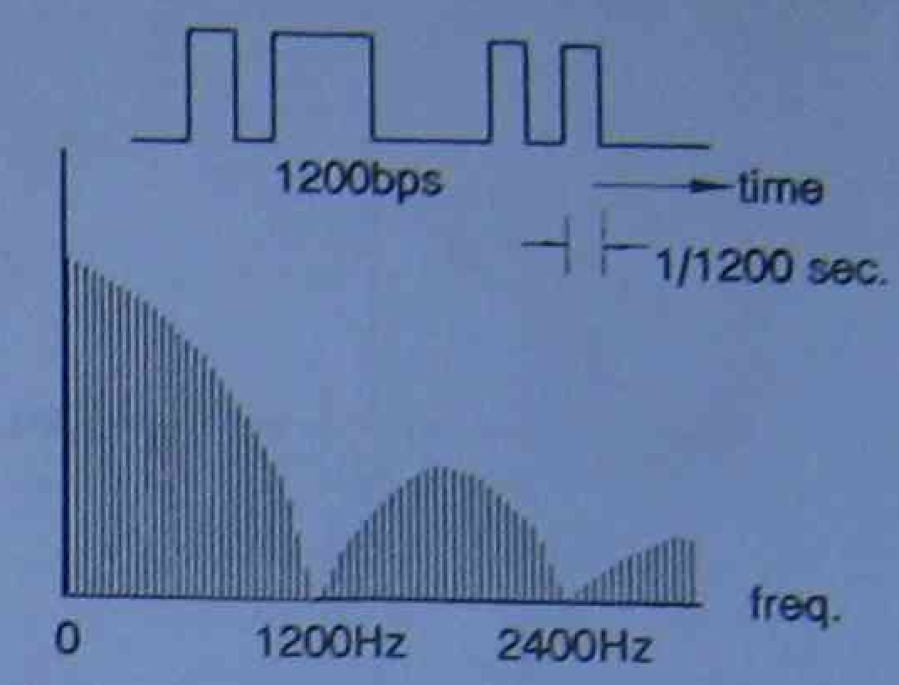
A communication system should provide good transmission for all frequencies where the signal power spectrum is significant.

Speech
For speech, the entire collection of vocal sounds extends from about 80Hz to 12kHz, with strongly decreasing energy at the higher frequencies. This wide range of frequencies is required for high fidelity broadcast quality speech. For most communications purposes (eg. taxi and police radio, telephone) such a wide range is unnecessary and it can be restricted to 300-3400Hz before the intelligibility suffers.

Music
For high fidelity music, the band from 30Hz to 15kHz is required.

Video
For video, frequencies from 0 to 5MHz are required.

Random data
For data, the required bandwidth is related to the bit rate. The higher the bit rate, the more bandwidth is required. For most applications we can say that the required bandwidth is equal to the reciprocal of duration of the narrowest pulse.



Spectrum of random data

The term bandwidth in relation to the various signals above, refers to the numerical difference between the upper and lower frequency limits of the signal. For example, the speech signal above has a minimum acceptable bandwidth of 3100Hz.

But what does this mean? Does it mean that the component frequencies in the speech that comes from your telephone stop sharply at 3400Hz?

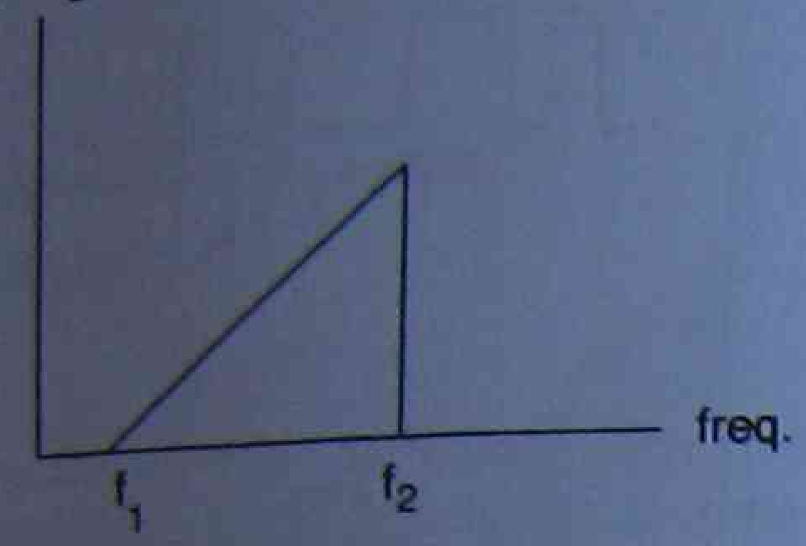
No! It means that outside the limits of 300Hz and 3400Hz, the spectral components are weaker by 3 decibels (dB) or more below the strongest spectral components within the 300 to 3400Hz band.

Note: The decibel is a logarithmic power ratio which is used throughout all fields of electronics. In particular, 3 dB means that the power has fallen by half.

In the case of speech which has been transmitted through a telephone network, the bandwidth will have been reduced so that the 3dB points are 300Hz and 3400Hz.

Symbolic representation of the baseband

Since baseband signals have varying spectral diagrams, it is necessary to adopt a standard representation for all of them, regardless of their actual spectra. The standard shape is show in the figure below, the hypotenuse of the triangle increasing in the direction of increasing frequency.



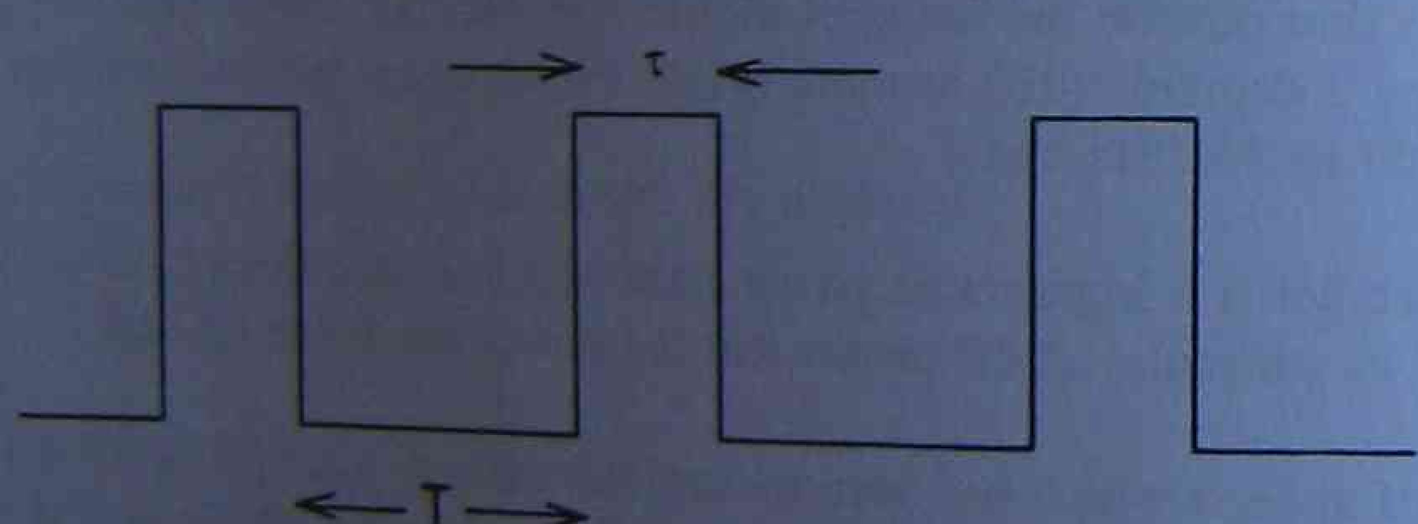
Symbolic representation of the baseband

Note that this symbolic shape usually bears no resemblance to the actual baseband signal in the system. Speech signals in fact have the opposite characteristic.

The significance of this representation will become apparent when we consider the topic Modulation in Section 4.

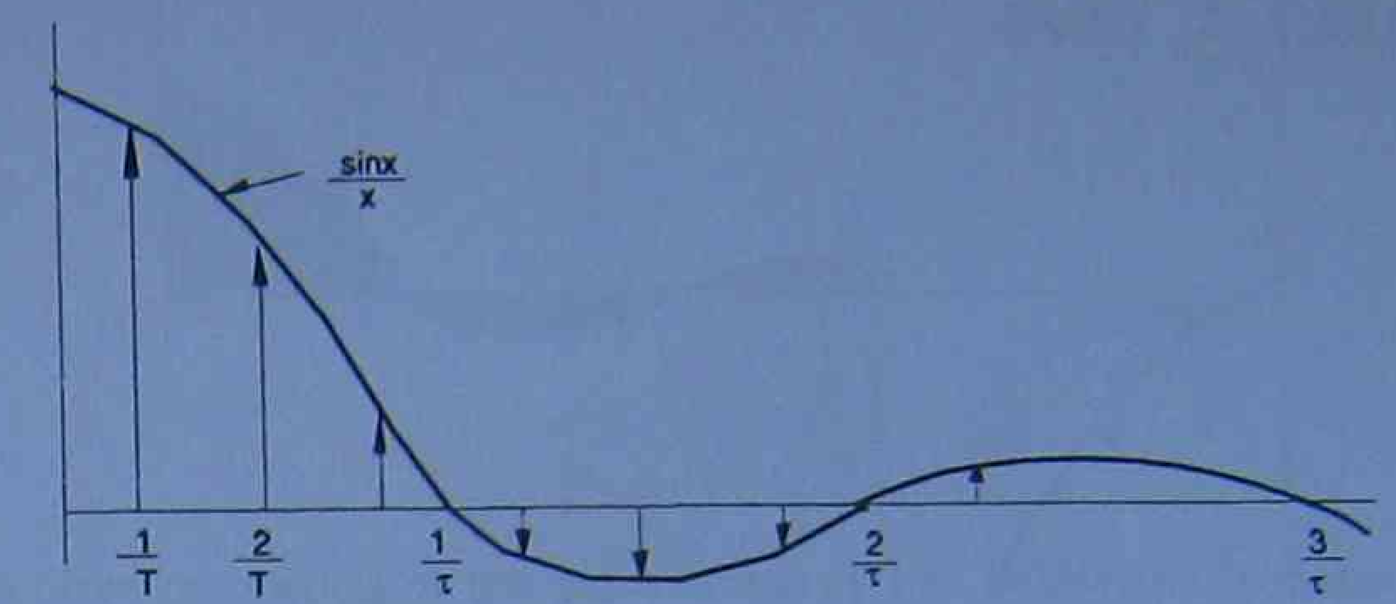
The spectrum of a pulse train

For a rectangular pulse train with period T and pulse width τ as shown below, the relative amplitudes of the spectral components can be found from the $\frac{sinx}{x}$ curve.



The first zero point on the $\frac{sinx}{x}$ curve corresponds to the frequency $\frac{1}{\tau}$.

The fundamental frequency is, of course, $\frac{1}{T}$.

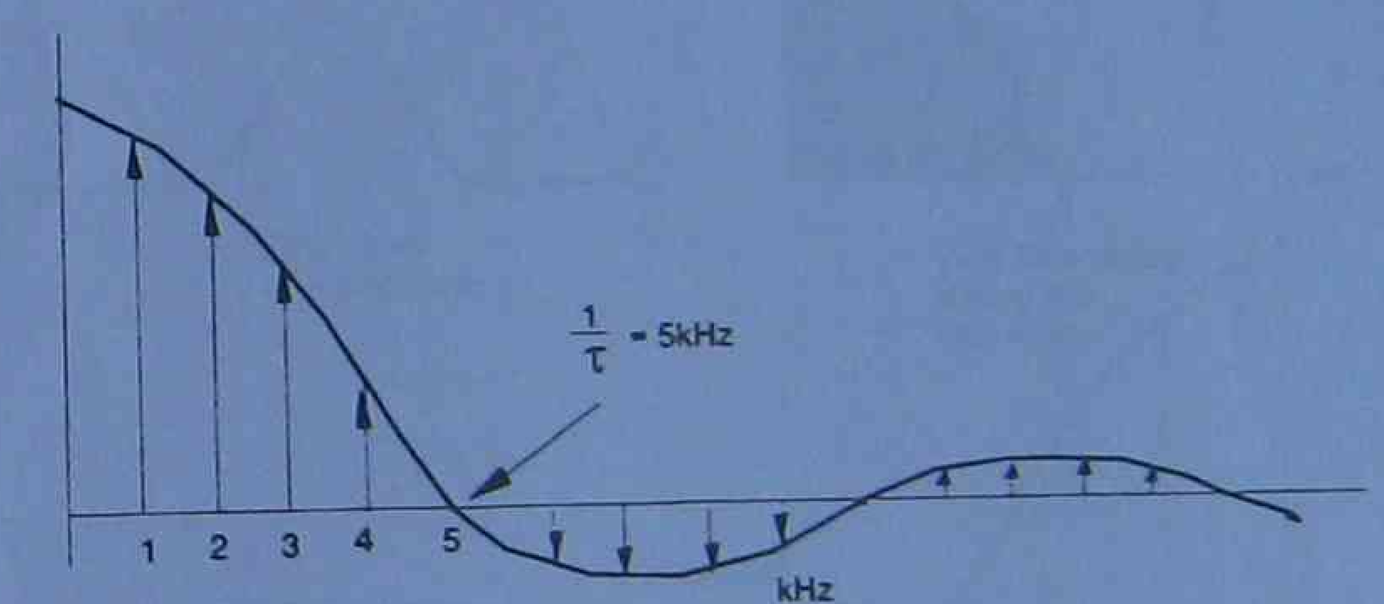


Example 1

T = 1ms, τ = 0.2ms

$\therefore \frac{1}{T} = 1kHz, \quad \frac{1}{\tau} = 5kHz$

Locate 5kHz at the first zero of the curve.



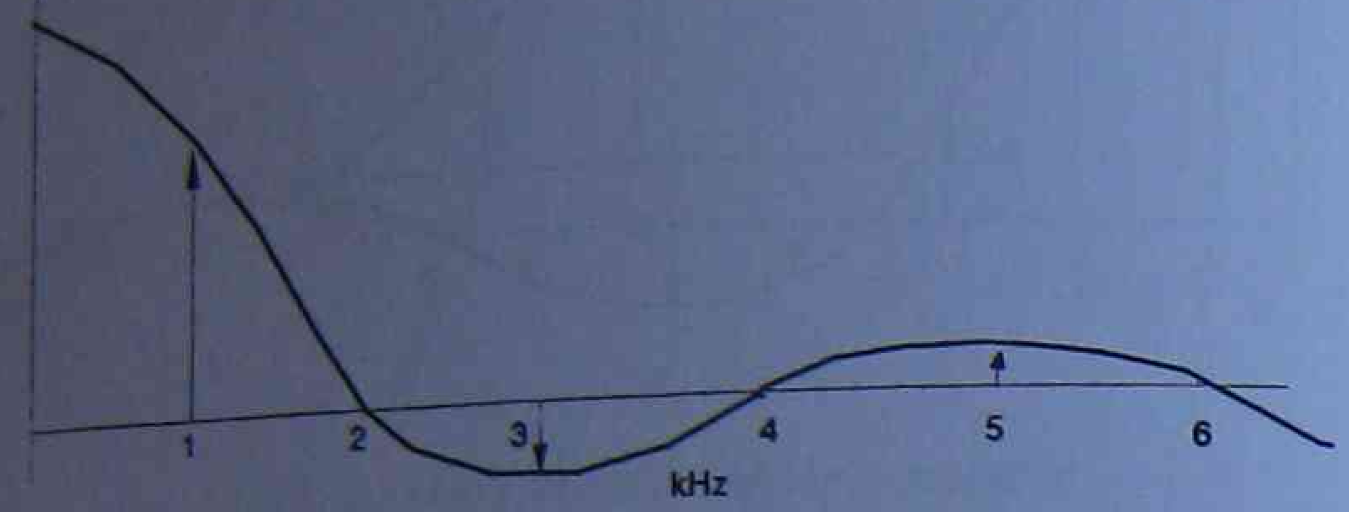
In this case the 5th harmonic (5kHz) has zero amplitude.

Note that when the above spectra are displayed on a spectrum analyzer, all the spectral lines are shown above the horizontal axis.

Example 2

T = 1ms, τ = 0.5ms (square wave)

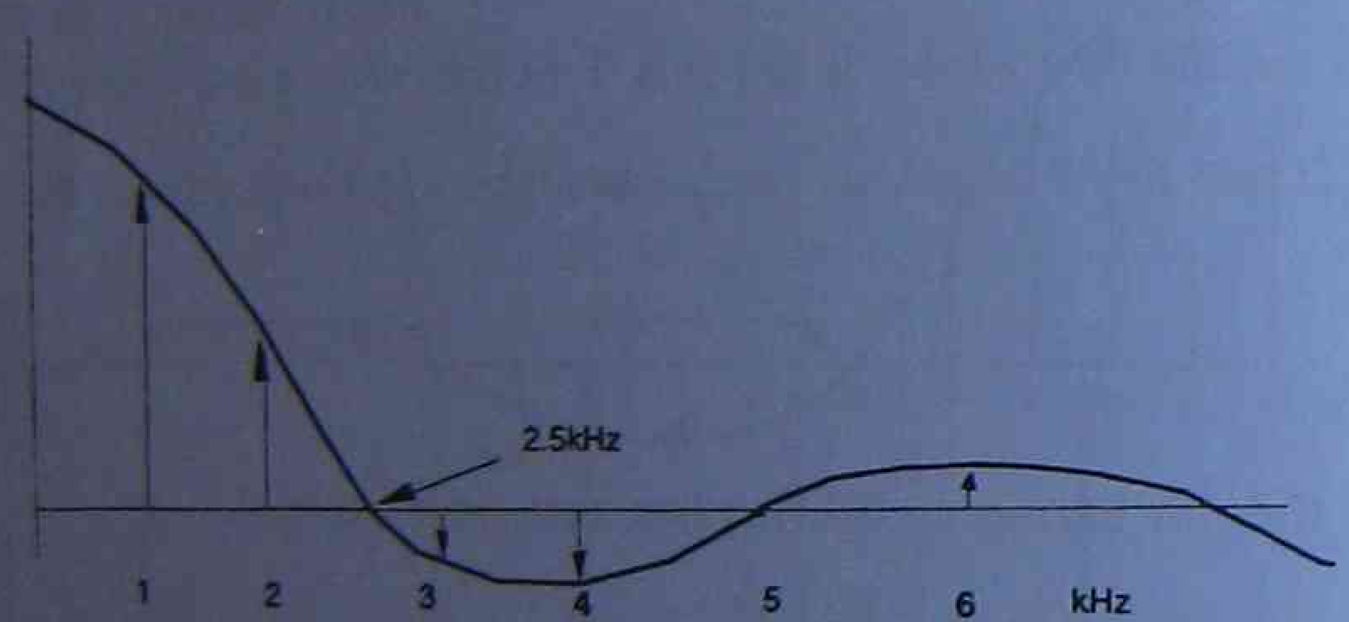
$\therefore \frac{1}{T} = 1kHz, \; \frac{1}{\tau} = 2kHz$



Example 3

T = 1ms, τ = 0.4ms

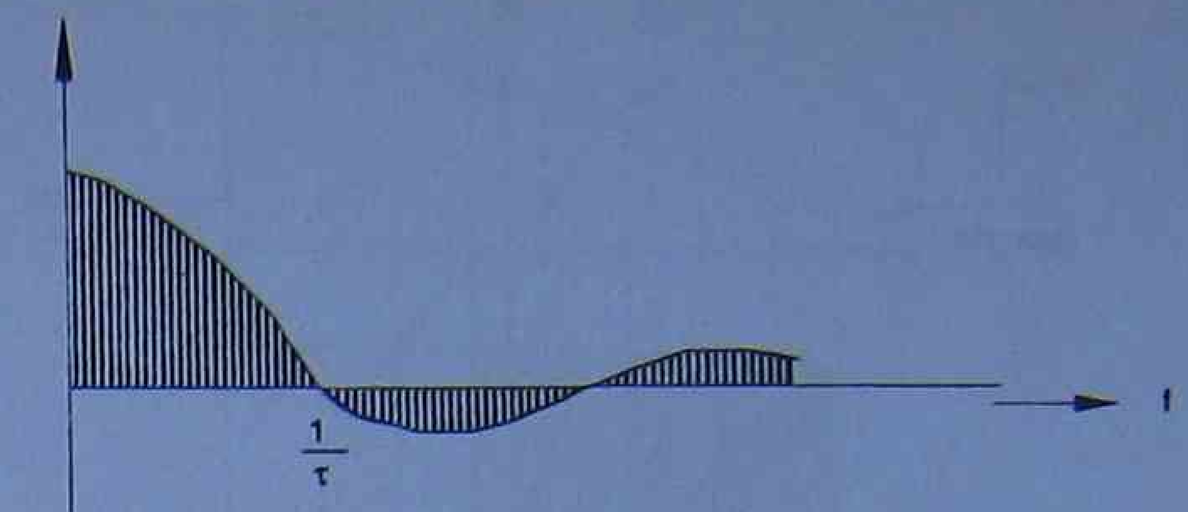
$\therefore \frac{1}{T} = 1kHz, \; \frac{1}{\tau} = 2.5kHz$



Note that in this case $\frac{1}{\tau}$ is not a harmonic, and that the first component to have zero amplitude is at 5kHz (the fifth harmonic).

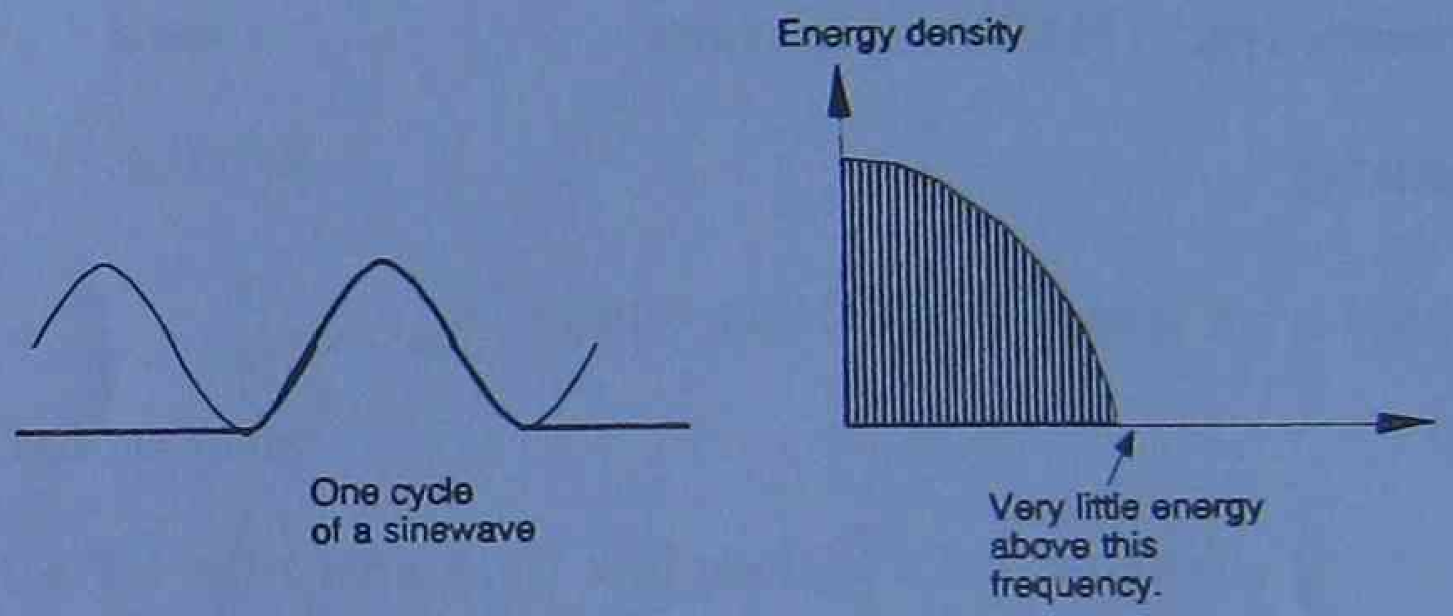
As T increases, the spectral lines get closer together.

If T $\rightarrow \infty$ (ie. we have only one pulse), the spectral lines form a continuum.

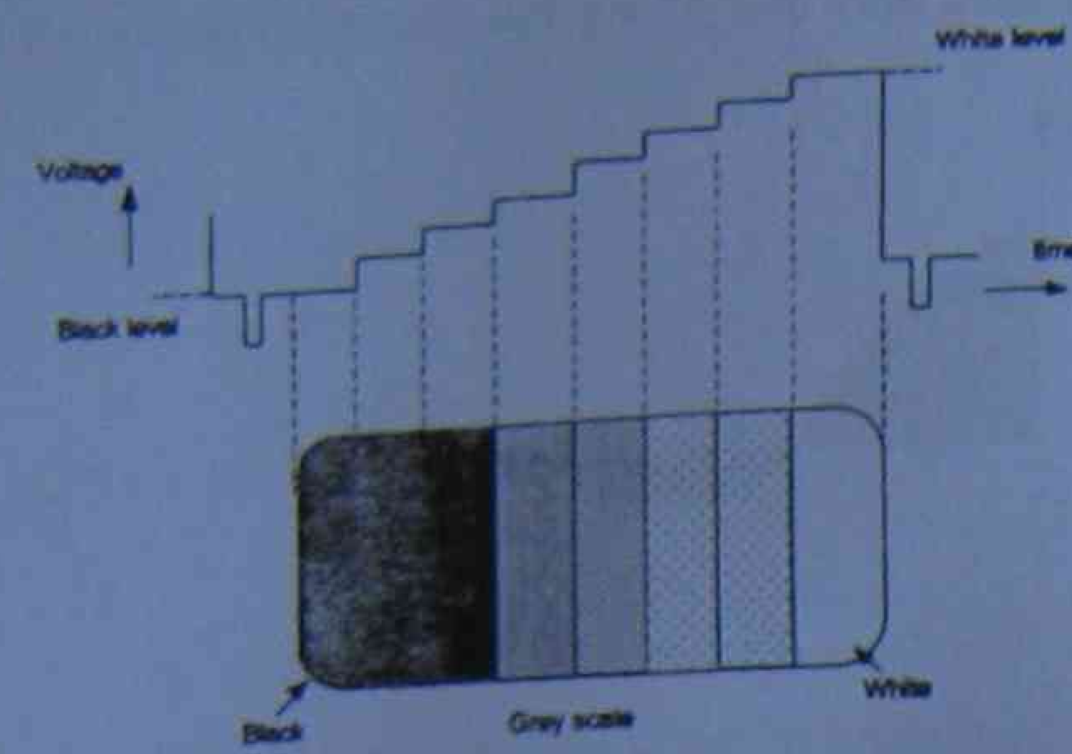


Note that a large proportion of the pulse's energy lies below the frequency $\frac{1}{\tau}$.

Even one cycle of a sinewave has a continuous spectrum. This is used as a test signal in Television.



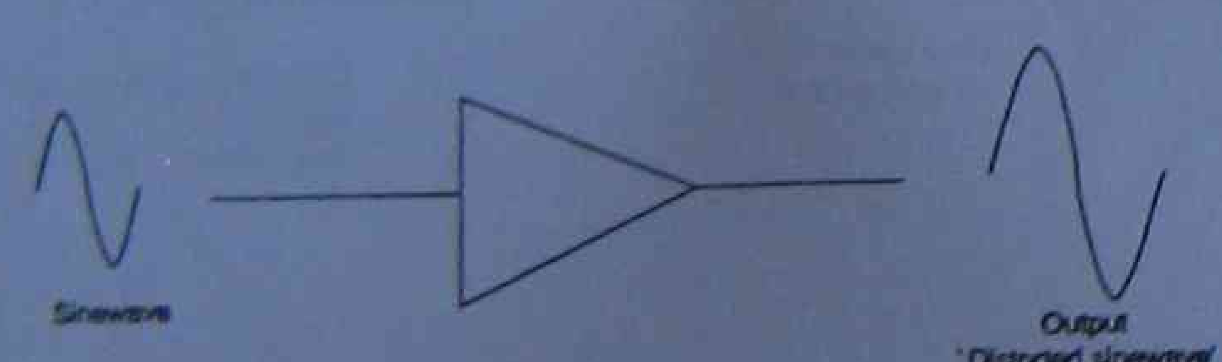
The video signal



Spectrum with varying signal:
- Luminance: 0 - 5MHz
- Chrominance: Centred on 4.43MHz
- Horizontal sync: 15625Hz + harmonics.

Non-linearity

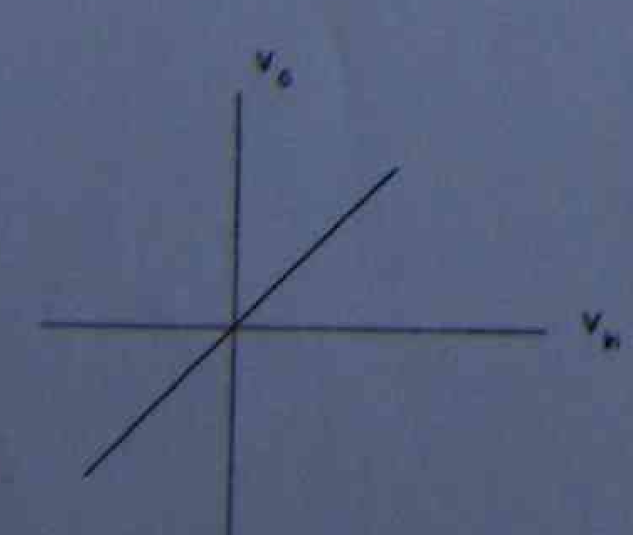
Output voltage is not proportional to input voltage.



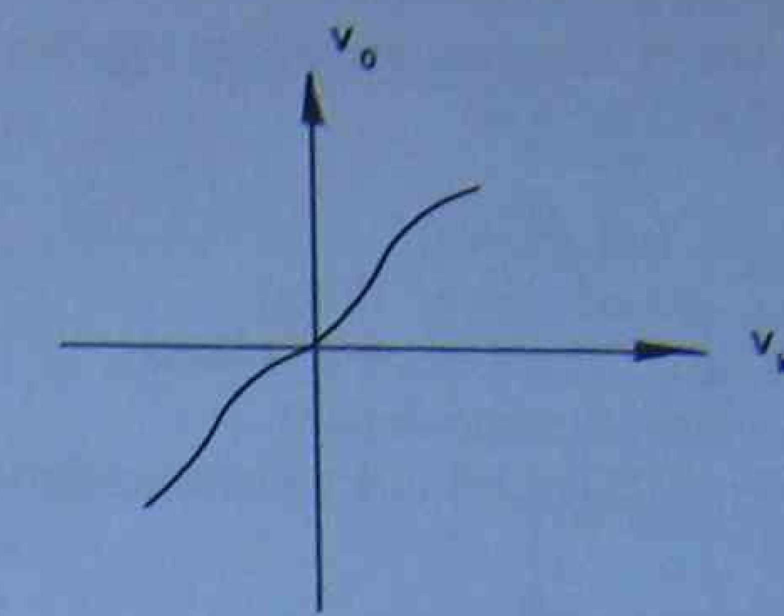
Output is not a sinewave but is still periodic. It therefore contains harmonics and has the same fundamental frequency as the input.

A linear device has $v_0 \,\alpha\, v_{in}$

$$v_0 = Av_{in}$$



Non-linear device



The curve can be represented by as power series:

$$v_0 = A_1 v_{in} + A_2 v_{in}^2 + A_3 v_{in}^3 + ...$$

Harmonic distortion

If we let $v_{in} = \sin\omega t$

then $A_2 v_{in}^2 = A_2 \sin^2\omega t$

$$= \frac{A_2}{2} - \frac{A_2}{2} cos2\omega t$$

$\cos 2\omega t$ = 2nd harmonic.

Likewise the 3rd order term $A_3 v_{in}^3$ will produce a 3rd harmonic.

Intermodualation distortion

Let the input consist of two sinewaves:

$$v_{in} = \sin\omega_1 t + \sin\omega_2 t$$

The second term in the power series above becomes
$A_2 V_{in}^2 = A_2 (\sin\omega_1 t + \sin\omega_2 t)^2$

A little trigonometry will show that this term produces not only the harmonics $2\omega_1$ and $2\omega_2$ but also sum and difference frequencies $\omega_1 \pm \omega_2$

The 3rd order term will produce $\omega_1 \pm 2\omega_2$ and $2\omega_1 \pm \omega_2$.

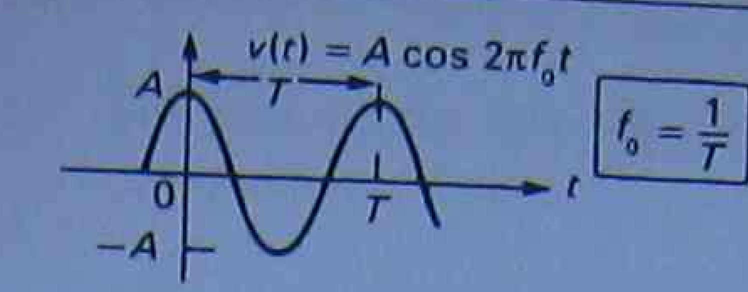
Example

A 1kHz sinewave and a 10kHz sinewave are added together and applied to a non-linear device. The output will include:

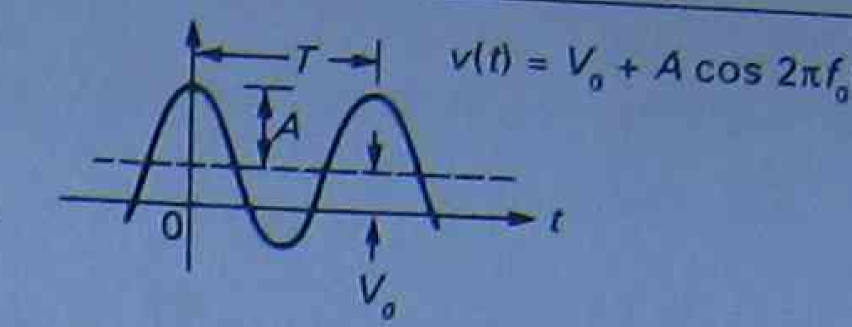
- 1kHz, 10kHz (original frequencies)
- 2kHz, 3kHz, 4kHz.....(harmonics of 1kHz)
- 20kHz, 30kHz, 40kHz ......(harmonics of 10kHz)
- 9kHz, 11kHz (2nd order intermodulation)
- 8kHz, 12kHz, 19kHz, 21kHz (3rd order intermodulation).

TABLE 3–1 Some Periodic Waveforms and Their Fourier Series Mathematical Expressions   wk 4+5

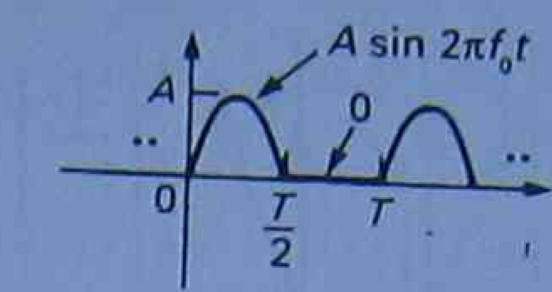
a.



b.

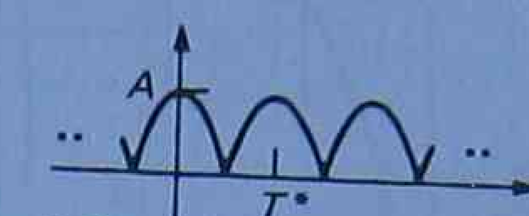


c.



$$v(t) = \frac{A}{\pi} + \frac{A}{2}\sin 2\pi f_0 t - \frac{2A}{3\pi}\cos 2\pi(2f_0)t - \frac{2A}{15\pi}\cos 2\pi(4f_0)t + \cdots$$
$$= \frac{A}{\pi} + \frac{A}{2}\sin 2\pi f_0 t + \sum_{n=2}^{\infty}\frac{A[1+(-1)^n]}{\pi(1-n^2)}\cos 2\pi(nf_0)t$$

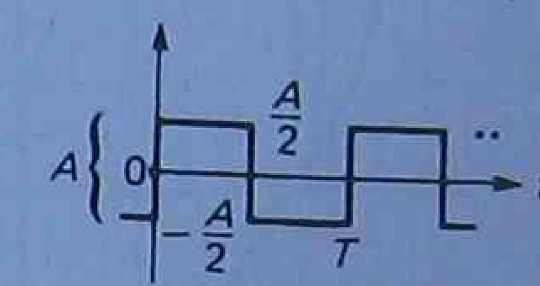
d.



* (the rectifier input signal will have a period of 2 T)

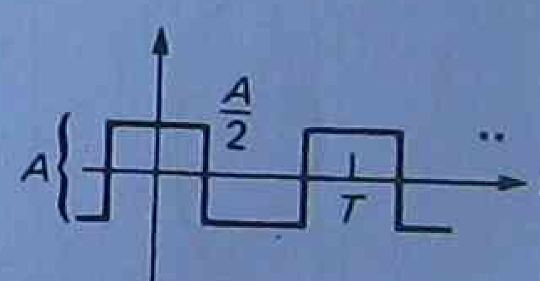
$$v(t) = \frac{2A}{\pi} + \frac{4A}{3\pi}\cos 2\pi f_0 t - \frac{4A}{15\pi}\cos 2\pi(2f_0)t + \cdots$$
$$= \frac{2A}{\pi} + \sum_{n=1}^{\infty}\frac{4A(-1)^n}{\pi[1-(2n)^2]}\cos 2\pi(nf_0)t$$

e.



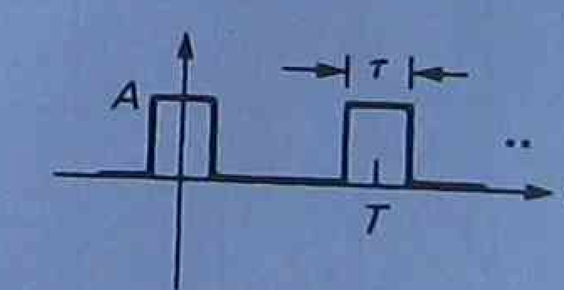
$$v(t) = \frac{2A}{\pi}\sin 2\pi f_0 t + \frac{2A}{3\pi}\sin 2\pi(3f_0)t + \cdots$$
$$= \sum_{n,\ \text{odd only}}^{\infty}\frac{2A}{n\pi}\sin 2\pi(nf_0)t$$

f.



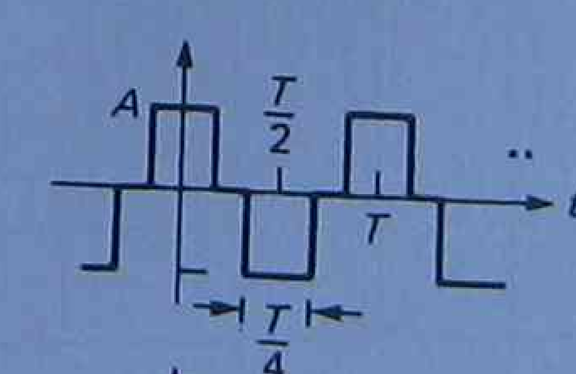
$$v(t) = \frac{2A}{\pi}\cos 2\pi f_0 t - \frac{2A}{3\pi}\cos 2\pi(3f_0)t + \frac{2A}{5\pi}\cos 2\pi(5f_0)t + \cdots$$
$$= \sum_{n=1}^{\infty}\left(A\frac{\sin n\pi/2}{n\pi/2}\right)\cos 2\pi(nf_0)t$$

g.



$$v(t) = \frac{A\tau}{T} + \sum_{n=1}^{\infty}\left(2A\frac{\tau}{T}\right)\left(\frac{\sin n\pi\tau/T}{n\pi\tau/T}\right)\cos 2\pi(nf_0)t$$

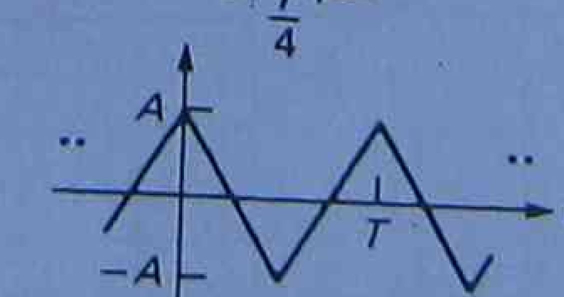
h.



$$v(t) = \sum_{n,\ \text{odd only}}^{\infty}\left(A\frac{\sin n\pi/4}{n\pi/4}\right)\cos 2\pi(nf_0)t$$

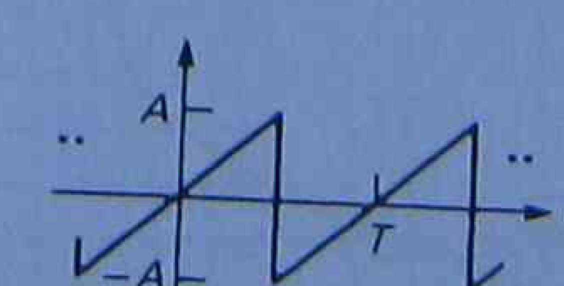
(special case of 50% "alternate inversion")

i.



$$v(t) = \frac{8A}{\pi^2}\cos 2\pi f_0 t + \frac{8A}{9\pi^2}\cos 2\pi(3f_0)t + \frac{8A}{25\pi^2}\cos 2\pi(5f_0)t + \cdots$$
$$= \sum_{n\ \text{odd}}^{\infty}\frac{8A}{(n\pi)^2}\cos 2\pi(nf_0)t$$

j.



$$v(t) = \frac{2A}{\pi}[\sin 2\pi f_0 t - \tfrac{1}{2}\sin 2\pi(2f_0)t + \tfrac{1}{3}\sin 2\pi(3f_0)t + \cdots]$$
$$= \sum_{n=1}^{\infty}[(-1)^{n+1}]\left(\frac{2A}{n\pi}\right)\sin 2\pi(nf_0)t$$

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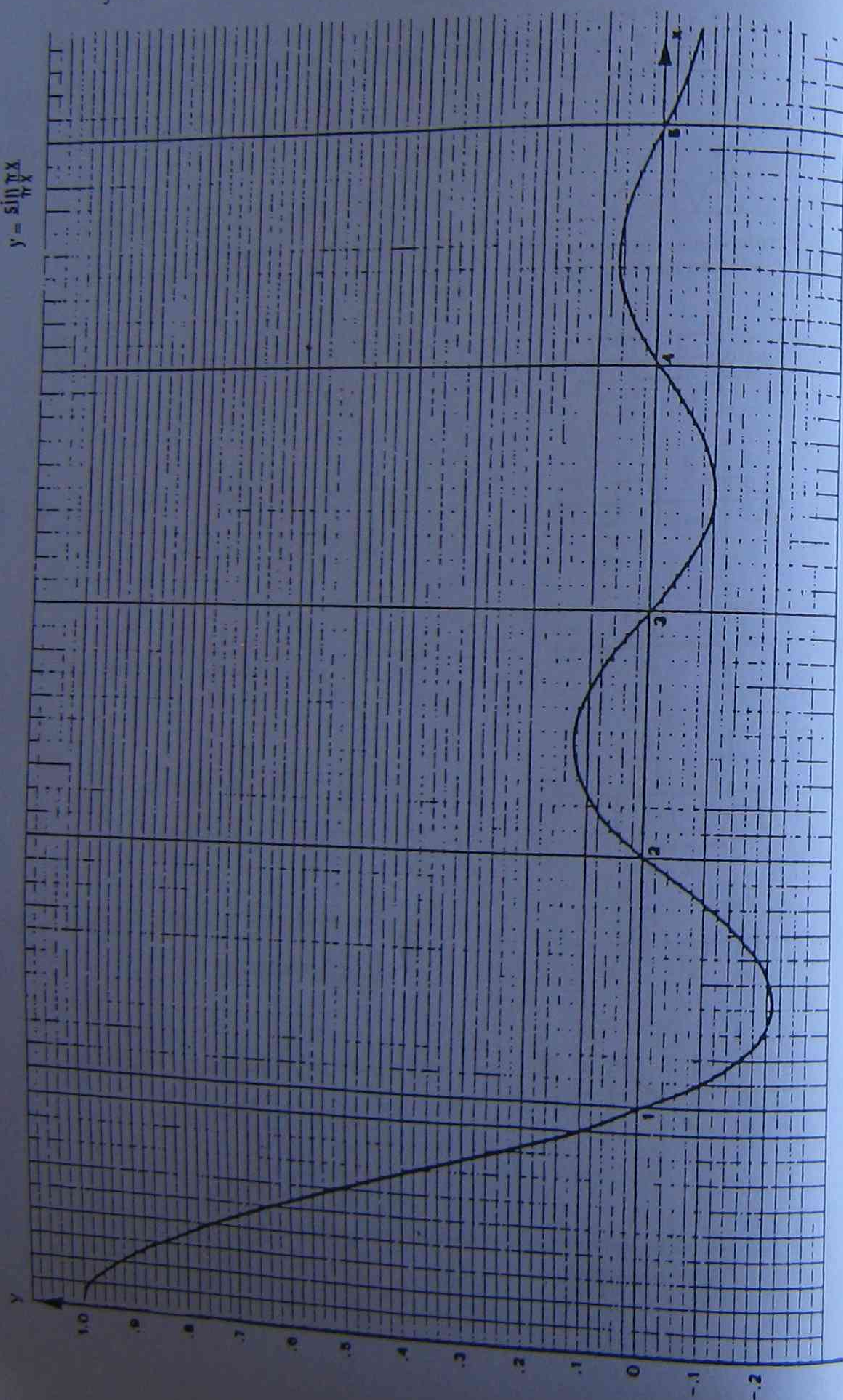
where $X = n\frac{\tau}{T}$

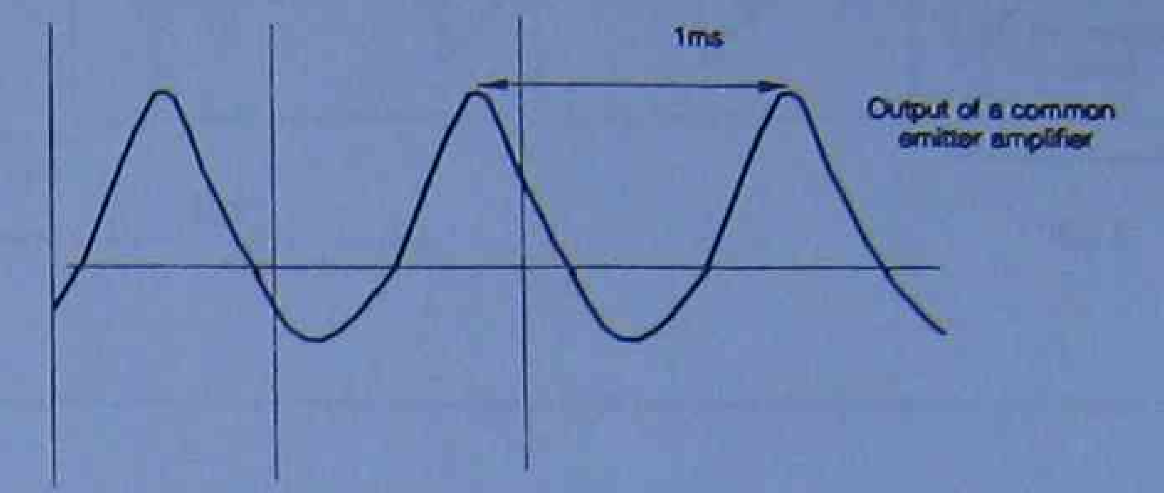


Review questions

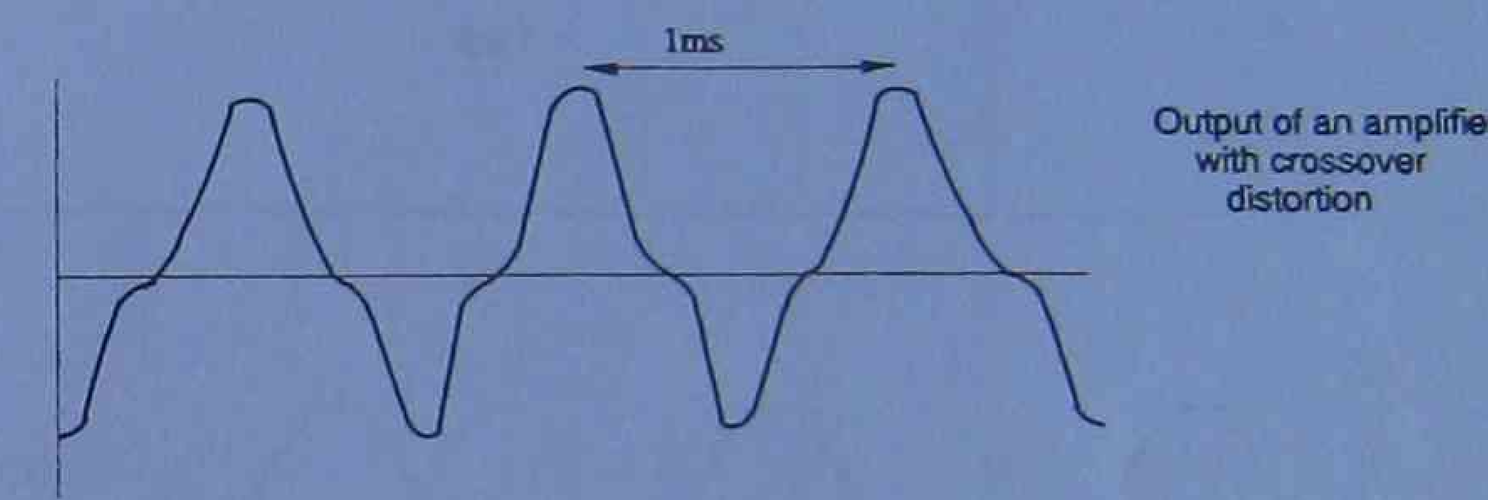
These questions will help you revise what you have learnt in Section 2.

1. For each of the following waveforms, state the frequencies of the first three components present.

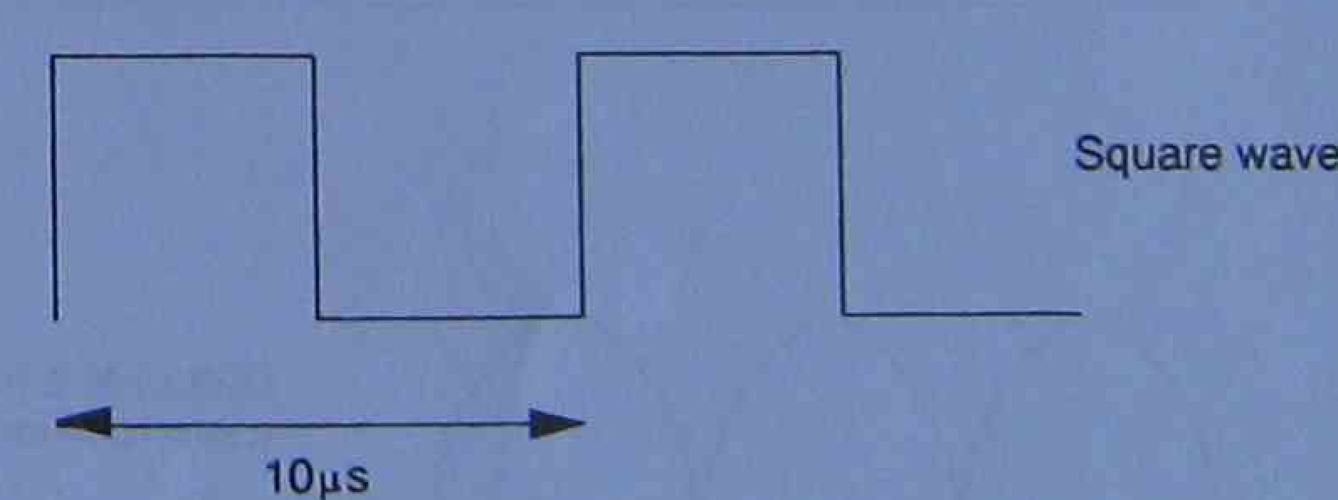
(a)



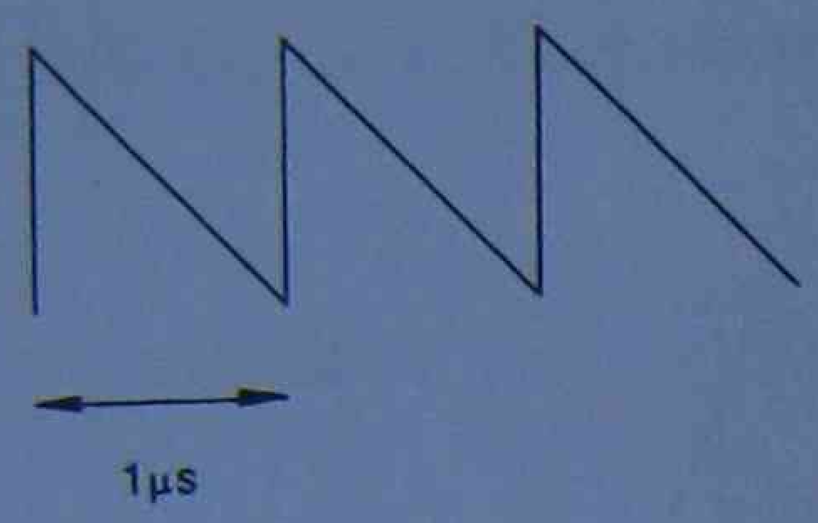
(b)



(c)

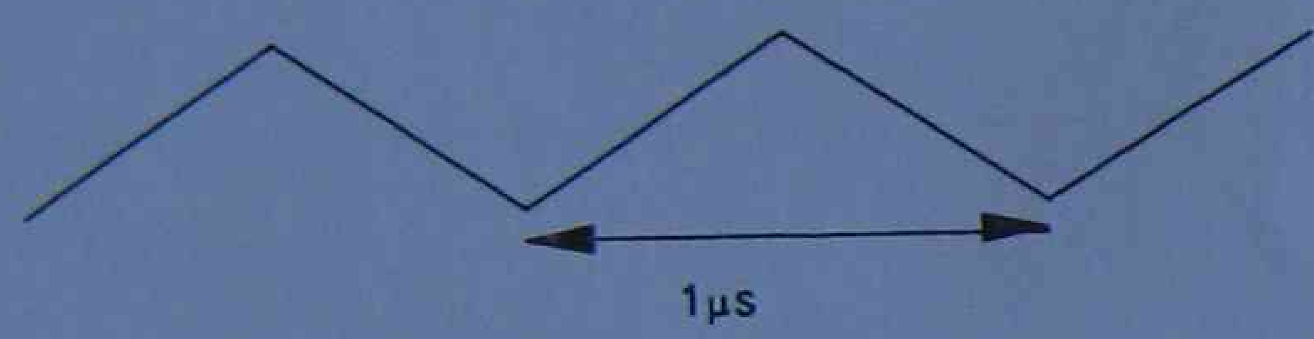


(d)



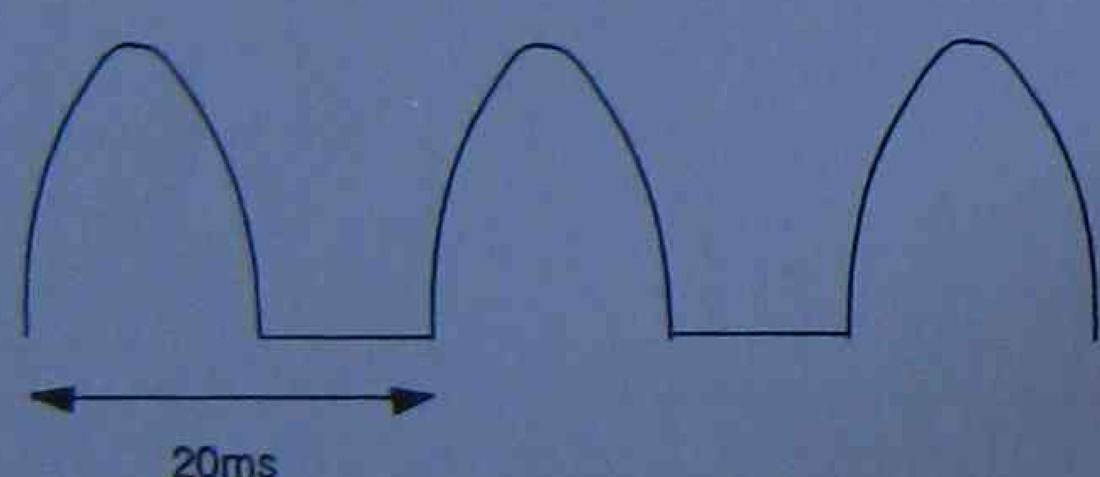
Sawtooth wave

(e)



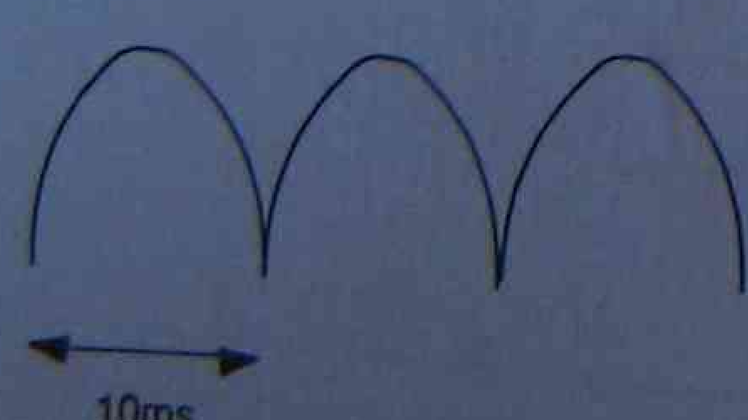
Triangular wave

(f)



Output of a half wave rectifier

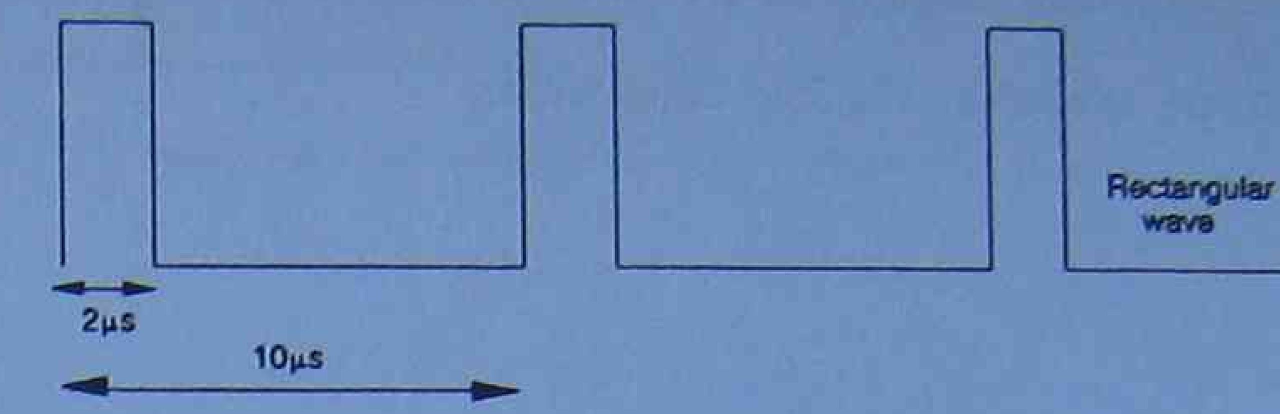
(g)



Output of a full wave rectifier

(Discuss this with your teacher!)

2. For the waveform shown below, will the 5th harmonic be present?



3. A rectangular pulse train has a peak amplitude of 5V, a pulse width of 2ms and a period of 5ms.

(a) Calculate the DC ('average') value of this waveform.

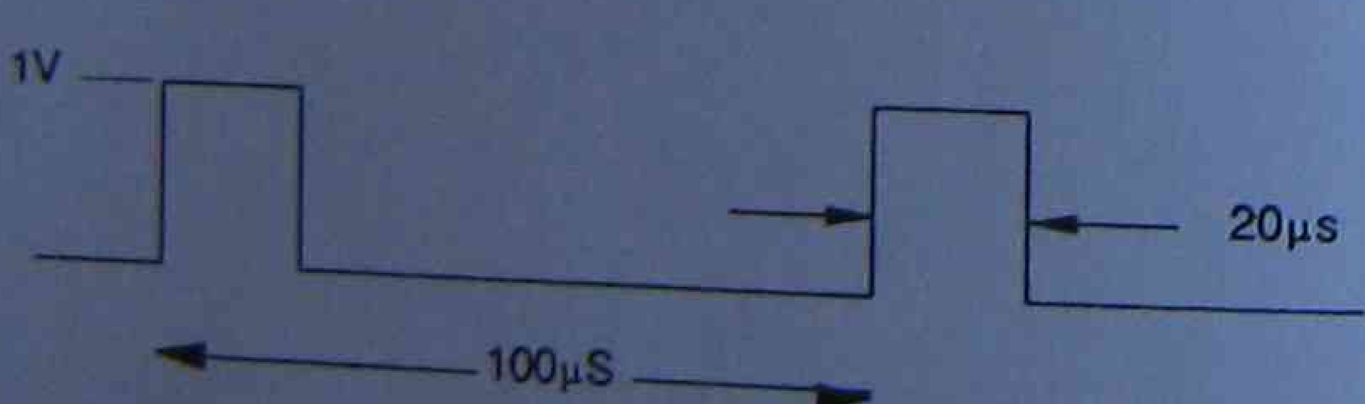
(b) Calculate the mark-space ratio.

(c) On a $\frac{1}{x}\sin x$ curve, sketch the five lowest frequency components in the waveform, excluding the DC component.

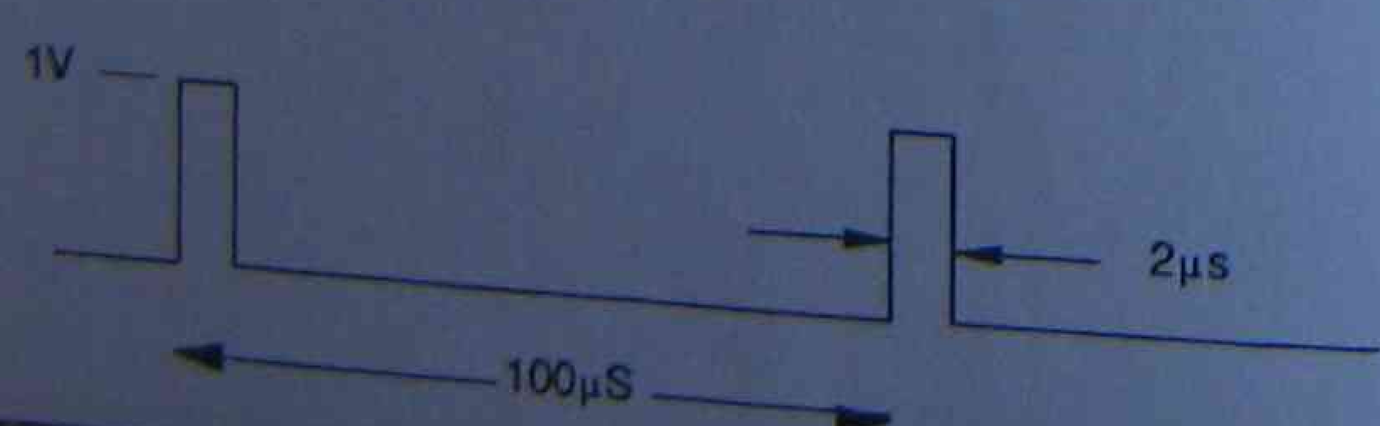
(d) Which harmonics have zero amplitude?

4. (a) Using a $\frac{1}{x}\sin x$ curve, sketch the spectra of each of the pulse trains below, marking the frequency order of the harmonic at the first zero of the curve.

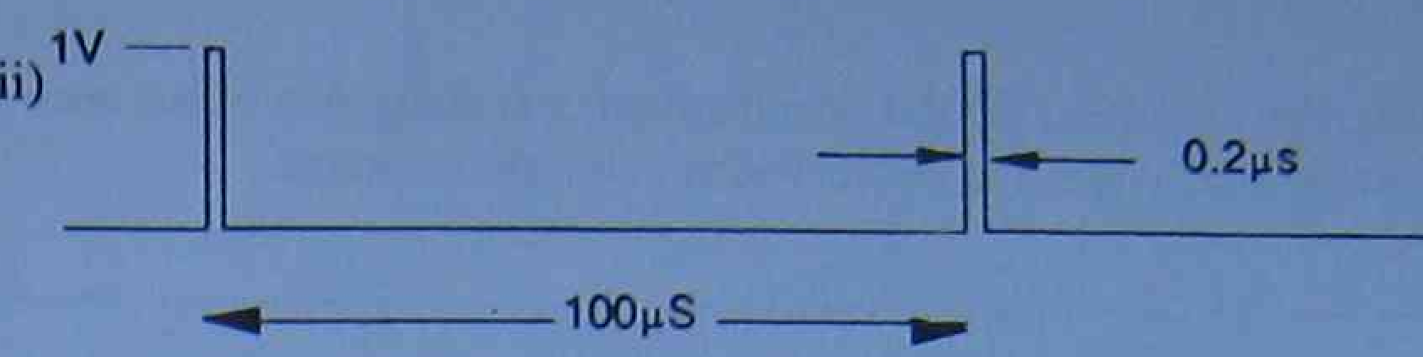
(i)



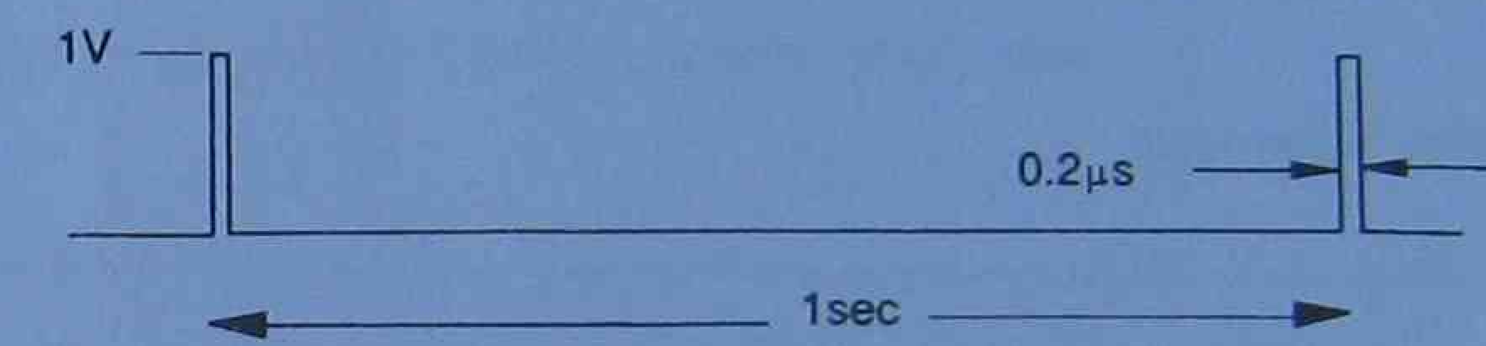
(ii)



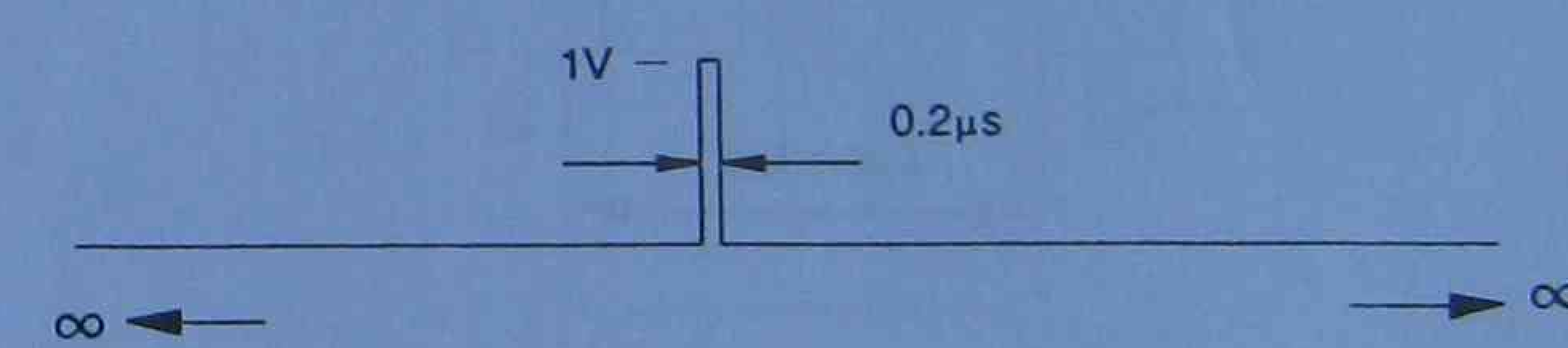
(iii)



(iv)



(v)



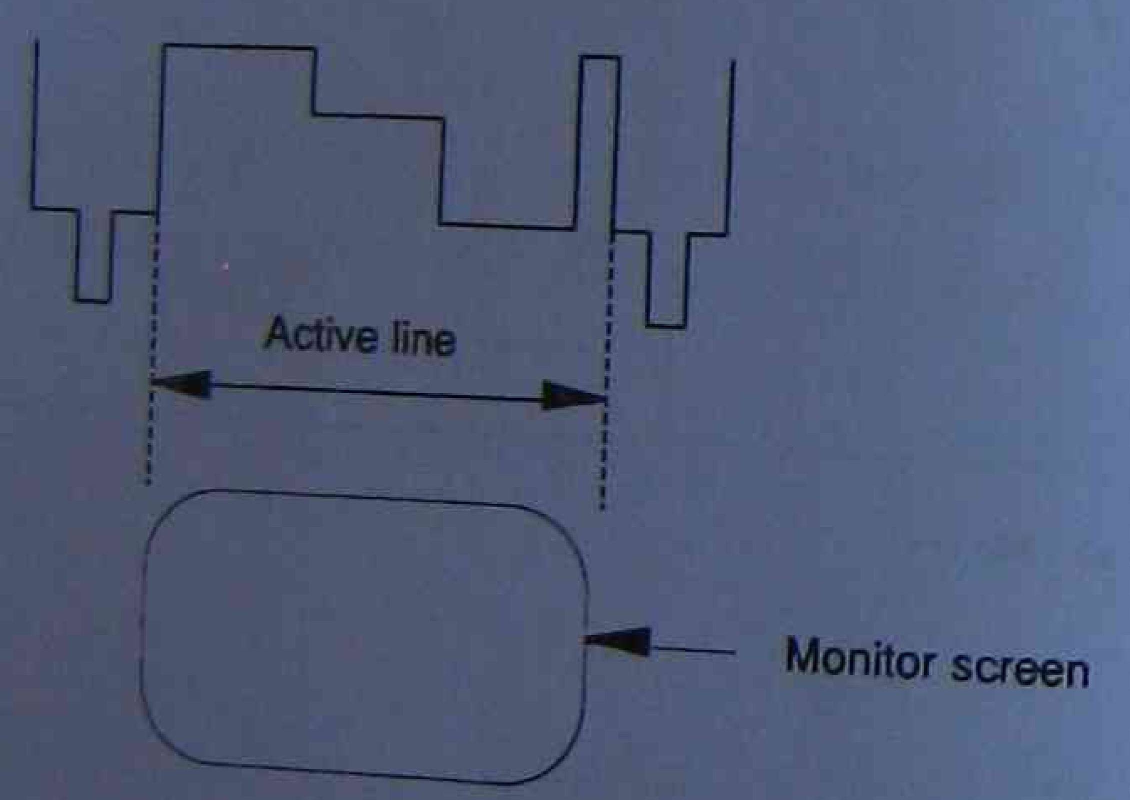
(b) Describe the effect on the spectrum of a rectangular pulse train if the pulse width is decreased, while the pulse period remains constant.

(c) Describe the effect on the spectrum of a rectangular pulse train if the period is increased, while the pulse width remains constant.

5. What range of frequencies is required for high quality transmission of music?

6. Define white noise.

7. Sketch the pattern that the following video signal will produce on a video monitor.



8. Sketch the spectrum of 2400bps random binary data.

9. A 50Hz sinewave and a 7kHz sinewave are applied to a non-linear amplifier. List the output frequencies produced by 2nd and 3rd order non-linearities.

10. Non-linearity in the 'front end' of radio receivers causes intermodulation products which may cause one station to be received at several points on the dial (and perhaps to block out your favourite station). Third order products can be particularly troublesome.

    For a FM receiver receiving stations on 100HMz, 101MHz and 102MHz, calculate all the third order products which fall within the 88 - 108MHz band.

Section 3: Filters

SUGGESTED DURATION	PREAMBLE
5 hrs 20 mins	To introduce you to simple filter types, filter parameters, and the effects of filters on various signals.
This section covers learning outcome 5 of the Module Descriptor.	

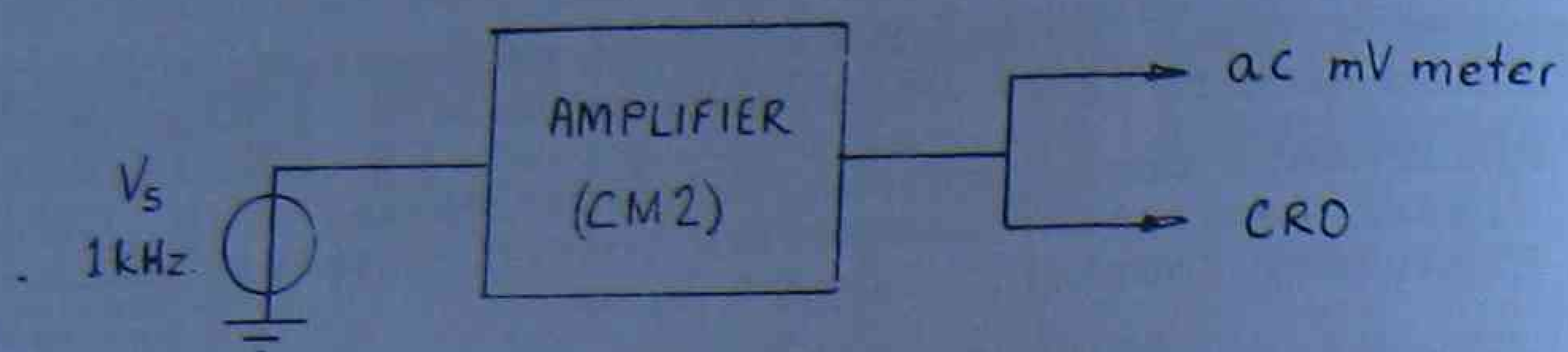
Objectives

At the end of this section you should be able to:

- ☐ use frequency domain diagrams to define both ideal and practical filters of the following types:
  - low-pass
  - high pass
  - band pass
  - band stop
- ☐ draw and identify block diagram symbols for ideal and practical filters
- ☐ for ideal and practical filters, use diagrams to explain what is meant by:
  - amplitude versus frequency response
  - phase versus frequency response
  - 3 dB bandwidth
  - insertion loss
  - roll-off slope
- ☐ describe the effects of both high and low-pass filters on audio signals
- ☐ describe the effect of low-pass filtering on video signals
- ☐ describe the effect of low-pass filtering on data signals
- ☐ give two reasons for using filters in communications equipment.

ESS Lab No 3

MEASURING AMPLIFIER SNR



AIM: To measure the Signal to Noise Ratio of an amplifier.

EQUIPMENT: Comms Trainer, CM2 Amplifier, mV meter, CRO, Decade Resistance Box, Signal Generator. 2×BNC/BNC, 2×BNC/4mm, BNC T piece (BNC/4mm adaptor)

PROCEDURE: 1) Set the amplifier volume control to maximum and leave it set in this position for the remainder of the lab. Switch the amplifier output to the internal load.

2) Set Vs to produce maximum undistorted output from the amplifier, using the CRO to monitor the output. Measure output level with the mV meter.

Max.O/P = ...V rms = ... dBV

) Place a decade box in series with the amplifier's input and djust it to halve the amplifier's output. The setting on the ecade box will equal the amplifier's input resistance, Ri.

Ri = ...

) Connect a resistance equal to Ri across the amplifier's input erminals and measure the amplifier's output with the mV meter. bserve the output on the CRO.

Noise O/P = ...V rms = ... dBV

) Calculate the amplifier's SNR.

SNR = ...dB.

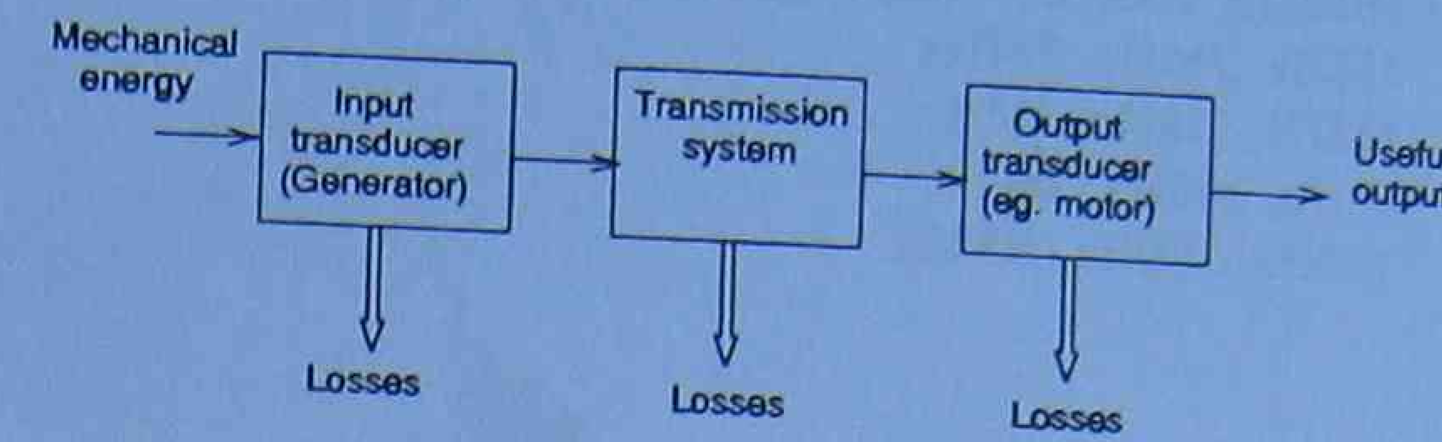
Repeat 2), 3) and 5) above, using the dB scale on the mV ter. Ask Teacher if not familar with this method.

SNR = ...dB.

NCLUSIONS: What errors are involved in the SNR measurement as rried out above? Are any of these significant? Explain.

Section 1

1.



2.
- In an electrical power transfer system, efficiency is the prime consideration. In an information transfer system, waveform integrity and signal-to-noise ratio are the prime considerations.
- In an electrical power transfer system, power is provided at the source. In an information transfer system, power is provide at various points.

3.
- The laser tracking system in a compact disk player
- an agc system in a radio receiver
- a crystal oven for a high stability oscillator
- The pressure controller in a steel rolling mill

Note that a remote controller for a model aeroplane transmits but does not receive. Therefore there can be no feedback to the controller itself, though there would be feedback in the plane.

4. The output voltage is sampled by a resistive divider. The sample is compared with the source and the difference (error signal) is amplified.

5.
- Open loop control: The motor speed will greatly depend on the motor load and supply voltage, as well as on the controller setting.
- Closed loop control: The effect of load and supply voltage will be greatly reduced by the feedback.

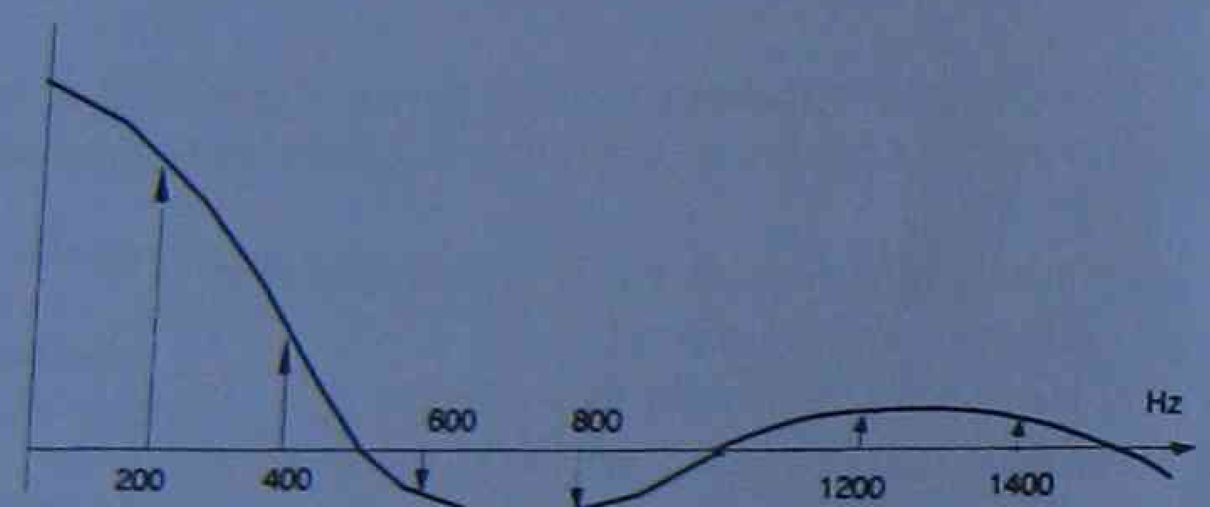
Section 2

1. (a) 1kHz, 2kHz, 3kHz
   (b) 1kHz, 3kHz, 5kHz
   (c) 100kHz, 300kHz, 500kHz
   (d) 1MHz, 2MHz, 3MHz
   (e) 1MHz, 3MHz, 5MHz
   (f) 50Hz, 100Hz, 200Hz
   (g) 100Hz, 200Hz, 300Hz.

2. No

3. (a) 2V
   (b) 2:3 or 0.667

   (c)



   (d) 5th, 10th 15th etc.

4. (a) (i) The 5th harmonic falls at the first zero of the curve.
   (ii) The 50th harmonic falls at the first zero of the curve.
   (iii) The 500th harmonic falls at the first zero of the curve.
   (iv) The 5 millionth harmonic falls at the first zero of the curve.
   (v) The spectrum is a continuum.

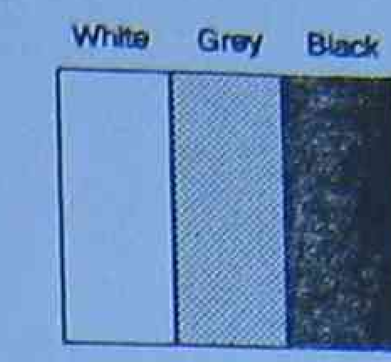
   (b) The spectral frequencies remain the same but more components fall in the first lobe of the sinx/x curve. That is, there are more harmonics with significant amplitude. Therefore, the signal requires more bandwidth.

   (c) The frequency corresponding to the first zero of the sinx/x curve remains unchanged. Therefore, the required bandwidth remains unchanged, even though there are more harmonics in the first lobe.

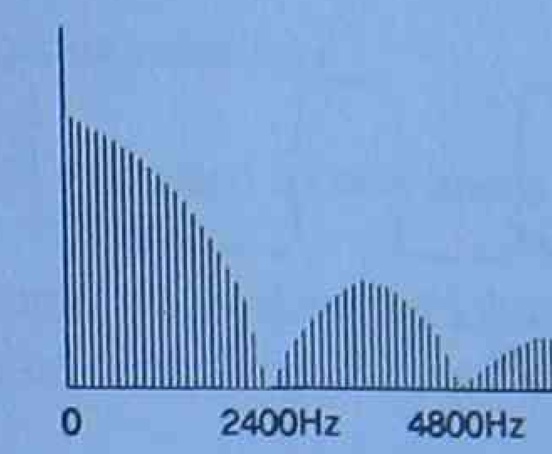
5. 30Hz - 15kHz (or perhaps 20kHz for young people with excellent hearing).

6. Noise with equal power per unit of bandwidth.

7.



8.



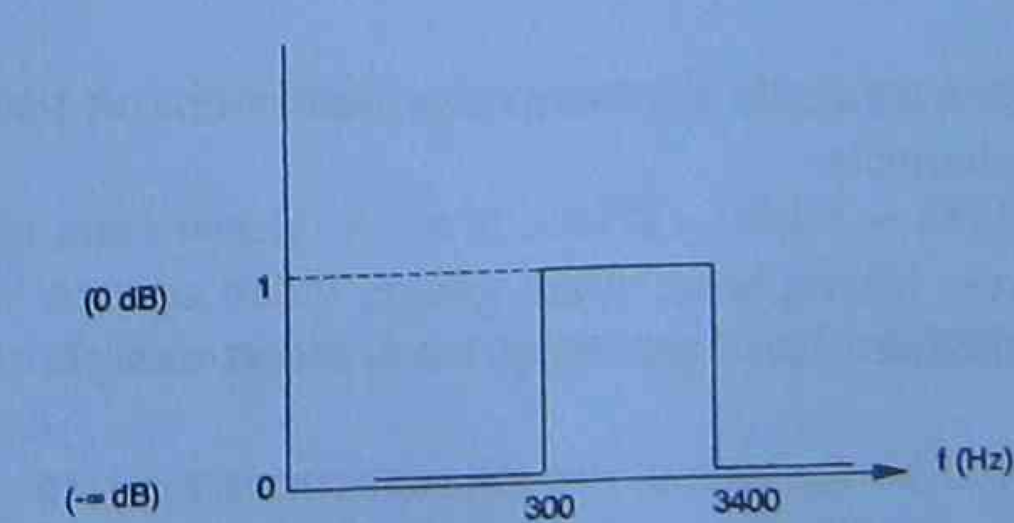
9. 2nd order:
   - 100Hz, 14kHz (Harmonics)
   - 6.95kHz, 7.05kHz (Intermod products)

   3rd order:
   - 150hz, 21kHz (Harmonics)
   - 6.9kHz, 7.1khz, 13.95kHz, 14.05kHz

10. 98, 99, 100, 102, 103 and 104MHz

Section 3

1.



2. 20°

3. Band stop filter. (Stop band = 100 - 200 kHz).

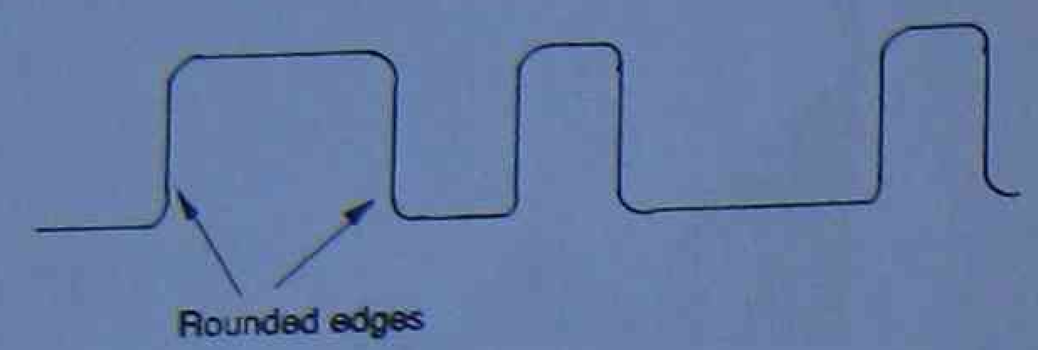
4. As the 3dB bandwidth.

5. (a) Band pass
   (b) 1dB
   (c) 4kHz (148 - 152kHz)

6. 6dB/octave.

7. ▪ loss of colour
   ▪ loss of definition

8.

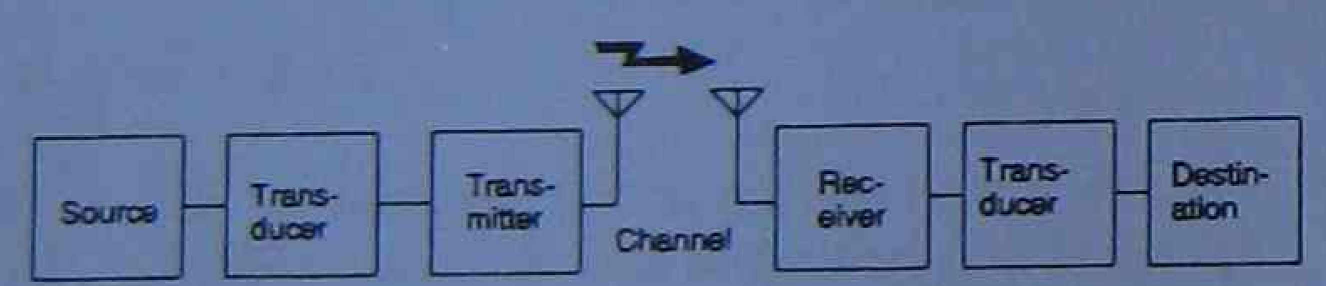


9. Both the phase/frequency response and the amplitude/frequency response affect the shape of the waveform. The shape of the waveform is very important for digital signals.

10. ▪ to select a particular signal
    ▪ to reject unwanted signals and noise

Section 4

1. (a)



   (b) A channel is a means of one-way communication. A bearer may carry a number of channels.

   (c) ▪ Source: Microphone, strain gauge, video camera
       ▪ Destination: Loudspeaker, printer, video monitor

2. No, the source would be the sound, the microphone is the transducer.

3. • To allow multiplexing
   • To allow use of a more suitable frequency range.

4. Any three of the following:
   ▪ noise
   ▪ crosstalk
   ▪ limited frequency response
   ▪ delay distortion
   ▪ non-linearities.

5. FDM and TDM

6. (a) VHF
   (b) UHF
   (c) UHF
   (d) HF

7. ▪ Super High Frequency
   ▪ 3GHz - 30GHz
   ▪ used for satellite communications

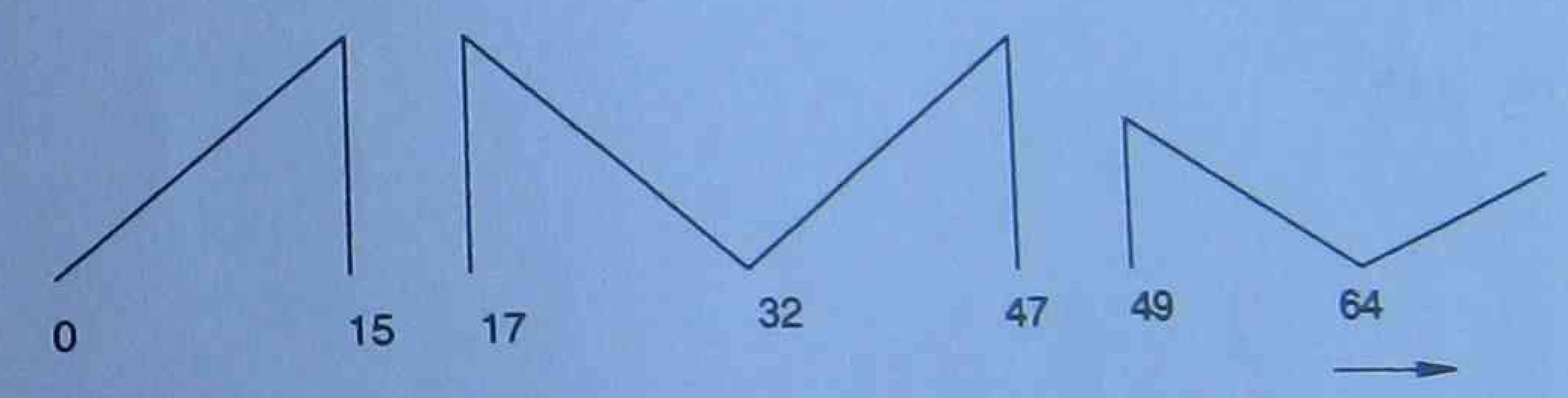
8. Its amplitude would be restricted to two levels.

9. A modem is a modulator/demodulator which converts a digital signal to a form suitable for transmission through an analog channel and reconverts to digital.

10. Codec.

Section 5

1. (a)



   (b) With a 15kHz LPF.

   (c) Yes, provided fairly sharp cut-off filters are used.

   (d) An alias would occur at 32kHz - 19kHz = 13kHz.

2. Analog to digital conversion (quantisation and encoding) and parallel to serial conversion.

3. (a) 256
   (b) 49.8dB

4. ▪ Serial to parallel conversion
   ▪ Digital to analog conversion
   ▪ Low-pass filtering

5. (a) 768 kbps
   (b) 384kHz

Section 6

1. Motor brushes, relays (virtually anything electrical!).

2. Lightning and cosmic radiation.

3. Thermal, shot and flicker.

4. To reduce shot noise. (Noise produced in the first stage of an amplifier is amplified by all the other stages.)

5. To minimise flicker noise.

6. No, it has more power at lower frequencies whereas white noise has uniform power over the spectrum.

7. White noise.

8. Impulse noise.

9. (a) 182nV.
   (b) 81dB.

10. 1uW.

wk 5+6

OPEN TRAINING & EDUCATION NETWORK TAFE

2832B Electrical Computations

Unit 4

FOURIER METHOD OF WAVEFORM ANALYSIS

CONTENTS

FOURIER METHOD OF WAVEFORM ANALYSIS

The response of linear networks to a.c. signals has been considered in other subjects within the Electrical Engineering Course. However, the analysis has been restricted to sinusoidal waveforms whereas non-sinusoidal waveforms do not have the same response characteristics.

In the consideration of the response of linear circuits to non-sinusoidal waveforms it is often convenient to express the waveform as a sum of sinusoidal functions. This method is referred to as the Fourier method of analysis.

4.1 TRIGONOMETRIC FOURIER SERIES

A periodic waveform can be expressed as a sum of pure sine waves of different frequencies and amplitudes. The component waveform with the same period as the waveform under analysis is referred to as the fundamental, the remaining components are referred to as harmonics and have frequencies which are integral multiples of the fundamental frequency.

A voltage waveform may be expressed as:

$$v(t) = V_{d.c.} + V_1 \sin(wt + \theta_1) + V_2 \sin(2wt + \theta_2) + V_3 \sin(3wt + \theta_3) + V_4 \sin(4wt + \theta_4) + \ldots \quad (1)$$

where:

$V_{d.c.}$ = d.c. component

$V_1 \sin(wt + \theta_1)$ = fundamental

$V_2 \sin(2wt + \theta_2)$ = second harmonic

$V_3 \sin(3wt + \theta_3)$ = third harmonic

$V_4 \sin(4wt + \theta_4)$ = fourth harmonic etc.

Alternatively the waveform may be expressed as:

$$v(t) = V_{d.c.} + A_1 \cos(wt) + A_2 \cos(2wt) + A_3 \cos(3wt) + \ldots + B_1 \sin(wt) + B_2 \sin(2wt) + B_3 \sin(3wt) + \ldots \quad (2)$$

where:

$A_1 = V_1 \sin\theta_1 \quad ; \quad B_1 = V_1 \cos\theta_1$

$A_2 = V_2 \sin\theta_2 \quad ; \quad B_2 = V_2 \cos\theta_2$

$A_3 = V_3 \sin\theta_3 \quad ; \quad B_3 = V_3 \cos\theta_3$

etc.

since

$$V_1 \sin(wt + \theta_1) = V_1 \sin(wt)\cos\theta_1 + V_1 \cos(wt)\sin\theta_1$$
$$= V_1 \sin\theta_1 \cos(wt) + V_1 \cos\theta_1 \sin(wt)$$
$$= A_1 \cos(wt) + B_1 \sin(wt)$$

To determine the component values of the harmonics we shall use the expression in equation (2) for ease of analysis.

The amplitudes of the component waveforms are given by:

$$A_n = \frac{1}{\pi}\int_0^{2\pi} v(t)\cos(n\theta)\,d\theta \quad \text{for } n = 1,2,3, \ldots$$

$$B_n = \frac{1}{\pi}\int_0^{2\pi} v(t)\sin(n\theta)\,d\theta \quad \text{for } n = 1,2,3, \ldots$$

Note that $\theta = wt$

4.2 SYMMETRICAL WAVEFORMS

Calculation of the amplitudes of component waveforms can be made easier if the waveform is symmetrical.

For an even function, all terms in the Fourier series are cosine terms with a possible constant value.

A voltage waveform $v(t)$ is even if $v(t) = v(-t)$.

Examples of even functions are given in figure 4.1.

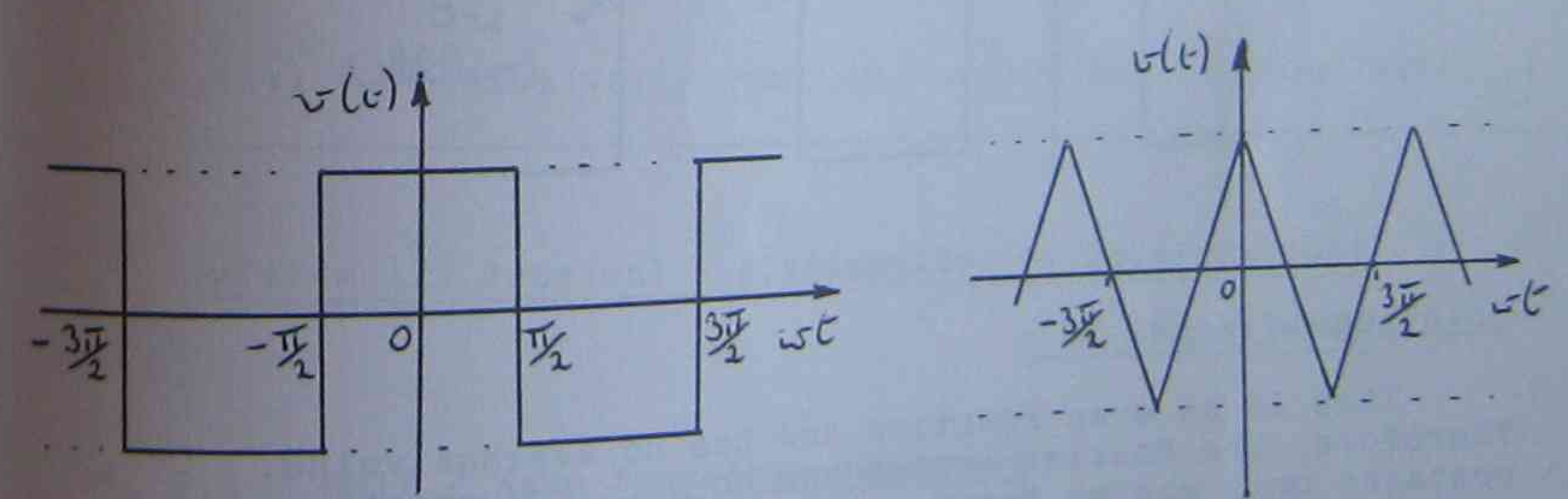


Figure 4.1

For an odd function, all terms in the Fourier series are sine terms.

A voltage waveform $v(t)$ is odd if $v(t) = -v(-t)$

Examples of odd functions are given in figure 4.2.

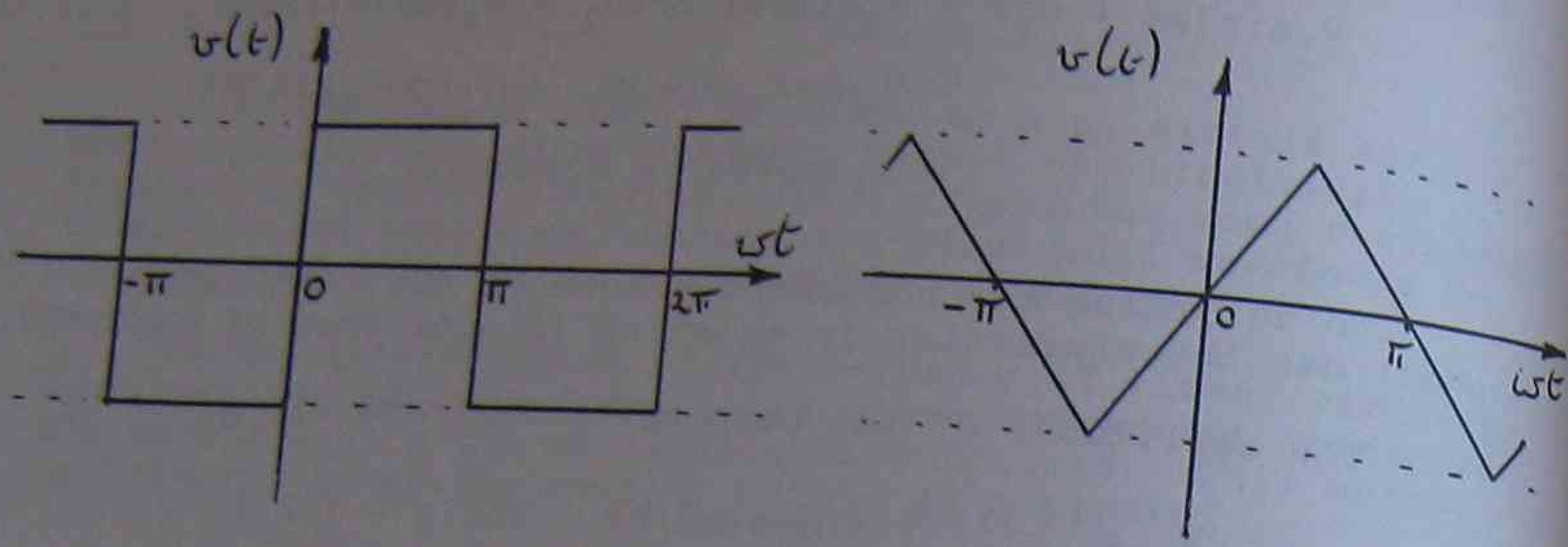


Figure 4.2

EXAMPLE 4.2.1

Find the first four terms in the trigonometric Fourier series for the square wave shown in figure 4.3.

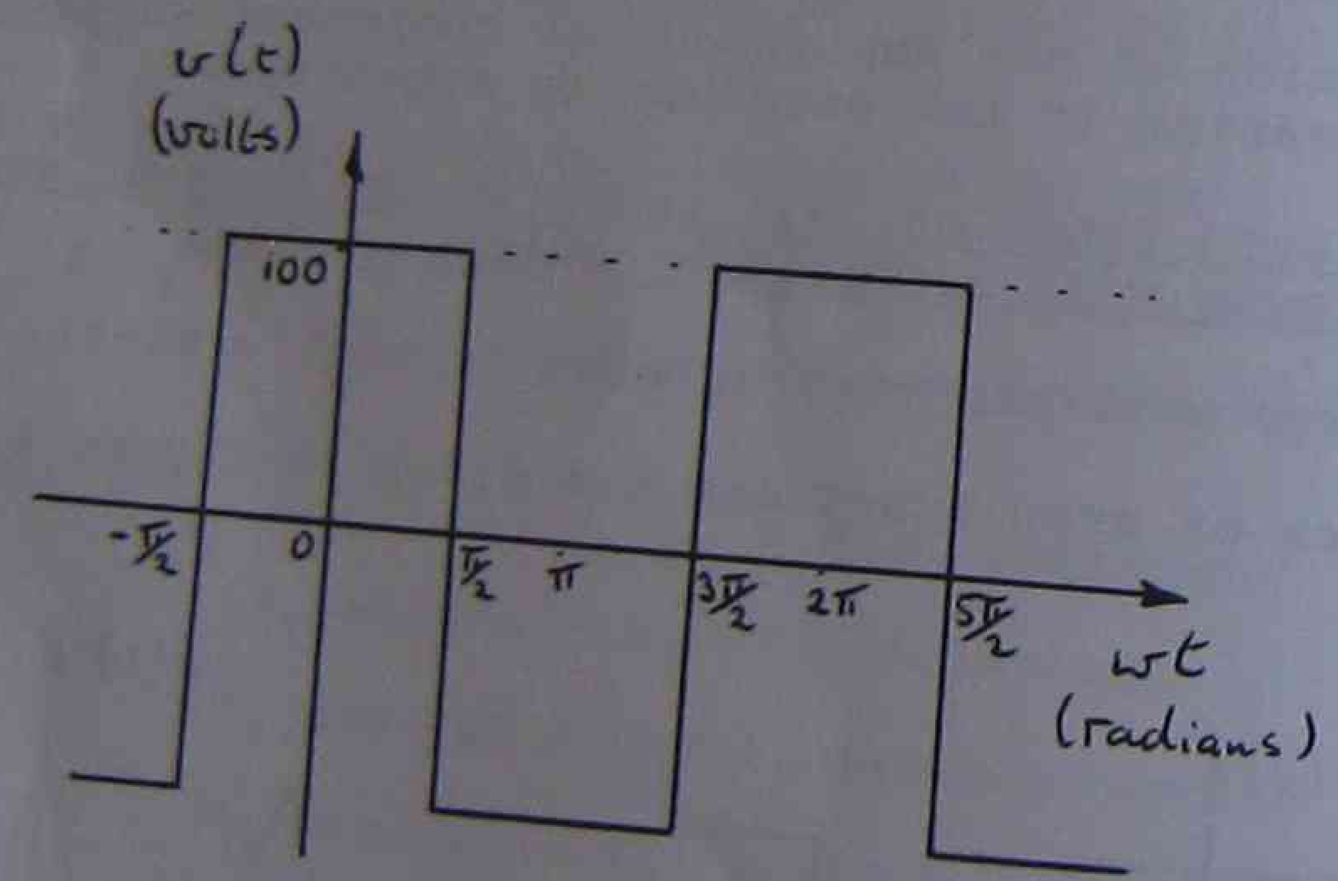


Figure 4.3

Solution:

This is an even function and has no average value. Therefore, the Fourier series has no d.c. component and contains only cosine terms.

$$v(t) = A_1\cos(wt) + A_2\cos(2wt) + A_3\cos(3wt) + \ldots$$

where $A_n = \frac{1}{\pi}\int_0^{2\pi} v(t)\cos(n\theta)\,d\theta$

$$= \frac{1}{\pi}\left\{\int_0^{\frac{\pi}{2}} 100\cos(n\theta)\,d\theta + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} -100\cos(n\theta)\,d\theta + \int_{\frac{3\pi}{2}}^{2\pi} 100\cos(n\theta)\,d\theta\right\}$$

$$= \frac{100}{\pi}\left\{\left[\frac{\sin(n\theta)}{n}\right]_0^{\frac{\pi}{2}} + \left[\frac{-\sin(n\theta)}{n}\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} + \left[\frac{\sin(n\theta)}{n}\right]_{\frac{3\pi}{2}}^{2\pi}\right\}$$

$$= \frac{100}{n\pi}\left\{\sin\left(\frac{n\pi}{2}\right) - \sin 0 - \sin\left(\frac{3n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right) + \sin(2n\pi) - \sin\left(\frac{3n\pi}{2}\right)\right\}$$

$$= \frac{200}{n\pi}\left\{\sin\left(\frac{n\pi}{2}\right) - \sin\left(\frac{3n\pi}{2}\right)\right\}$$

n = 1 ; $A_1 = \frac{200}{\pi} \times 2 = \frac{400}{\pi}$

n = 2 ; $A_2 = 0$

n = 3 ; $A_3 = \frac{200}{\pi} \times \left(\frac{-2}{3}\right) = \frac{-400}{\pi} \times \frac{1}{3}$

n = 4 ; $A_4 = 0$

n = 5 ; $A_5 = \frac{200}{\pi} \times \frac{2}{5} = \frac{400}{\pi} \times \frac{1}{5}$

n = 6 ; $A_6 = 0$

n = 7 ; $A_7 = \frac{200}{\pi} \times \left(\frac{-2}{7}\right) = \frac{-400}{\pi} \times \frac{1}{7}$

$$v(t) = \frac{400}{\pi}\left(\cos(wt) - \frac{1}{3}\cos(3wt) + \frac{1}{5}\cos(5wt) - \frac{1}{7}\cos(7wt) + \ldots\right)$$

$$v(t) = 127.3\cos(wt) - 42.44\cos(3wt) + 25.46\cos(5wt) - 18.19\cos(7wt) + \ldots$$

4. 2.1 Half - Wave Symmetry: Harmonics

The nth harmonic(n is an integer) is defined as An Cosnθ, BnSin nθ.
If $V(\theta + \pi) = +V(\theta)$ then the Fourier series for $V(\theta)$ will contain only even harmonics.
If $V(\theta + \pi) = -V(\theta)$ then the Fourier series for $V(\theta)$ will contain only odd harmonics.

$$B_n = \frac{50}{n\pi}\{2 - 2\cos(\frac{n\pi}{2})\}$$

$$= \frac{100}{n\pi}\{1 - \cos(\frac{n\pi}{2})\}$$

$n = 1$; $B_1 = \frac{100}{\pi}(1 - 0) = \frac{100}{\pi}$

$n = 2$; $B_2 = \frac{100}{2\pi}(1 + 1) = \frac{100}{\pi}$

$n = 3$; $B_3 = \frac{100}{3\pi}(1 - 0) = \frac{100}{\pi} \times \frac{1}{3}$

$n = 4$; $B_4 = \frac{100}{4\pi}(1 - 1) = 0$

$n = 5$; $B_5 = \frac{100}{5\pi}(1 - 0) = \frac{100}{\pi} \times \frac{1}{5}$

$n = 6$; $B_6 = \frac{100}{6\pi}(1 + 1) = \frac{100}{\pi} \times \frac{1}{3}$

$$\underline{v(t) = \frac{100}{\pi}\{\sin(wt) + \sin(2wt) + \frac{1}{3}\sin(3wt) + \frac{1}{5}\sin(5wt) + \frac{1}{3}\sin(6wt) + \ldots\}}$$

$$\underline{v(t) = 31.83\sin(wt) + 31.83\sin(2wt) + 10.61\sin(3wt) + 6.37\sin(5wt) + 10.61\sin(6wt) + \ldots.}$$

EXAMPLE 4.2.3

Determine the first four terms of the trigonometric Fourier series for the waveform shown in figure 4.5.

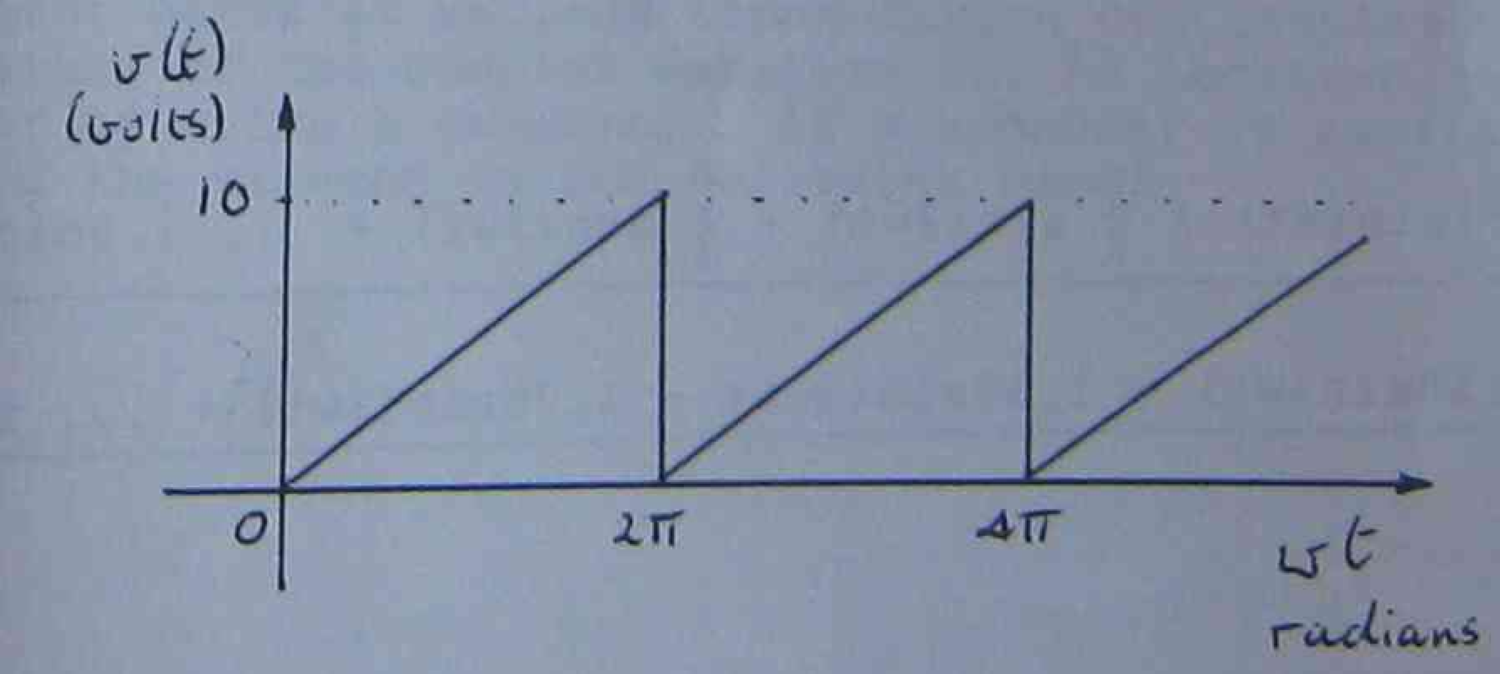


Figure 4.5

$$A_n = \frac{1}{\pi}\int_0^{2\pi} v(t)\cos(n\theta)\,d\theta$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{10}{2\pi}\, wt \cos(n\theta)\,d\theta$$

$$= \frac{5}{\pi^2}\int_0^{2\pi} \theta\cos(n\theta)\,d\theta \qquad \text{(Use integration by parts)} \quad \int u\,dv = uv - \int v\,du$$

$$= \frac{5}{\pi^2}\left\{\left[\frac{\sin(n\theta)}{n}\,\theta\right]_0^{2\pi} - \int_0^{2\pi} \frac{\sin(n\theta)}{n}\,d\theta\right\}$$

$$= \frac{5}{\pi^2}\left\{\frac{\sin(2n\pi)}{n} \times 2\pi - \frac{\sin 0}{n} \times 0 - \left[-\frac{\cos(n\theta)}{n^2}\right]_0^{2\pi}\right\}$$

$$= \frac{5}{\pi^2}\left\{\frac{0}{n} + \frac{\cos(2\pi n)}{n^2} - \frac{\cos 0}{n^2}\right\}$$

$$= 0 \text{ for all } n$$

There are no cosine terms in the series

$$V_{d.c.} = \frac{1}{2\pi}\int_0^{2\pi} v(t)\,d\theta = 5 \text{ volts} \quad [\text{i.e. average value}]$$

$$B_n = \frac{1}{\pi}\int_0^{2\pi} v(t)\sin(n\theta)\,d\theta$$

$$= \frac{5}{\pi^2}\int_0^{2\pi} \theta\sin(n\theta)\,d\theta \qquad \text{(Use integration by parts)}$$

$$= \frac{5}{\pi^2}\left\{\left[-\frac{\cos(n\theta)}{n}\,\theta\right]_0^{2\pi} - \int_0^{2\pi} -\frac{\cos(n\theta)}{n}\,d\theta\right\}$$

$$= \frac{5}{\pi^2}\left\{-\frac{\cos(2n\pi)}{n} \times 2\pi + 0 + \left[\frac{\sin(n\theta)}{n^2}\right]_0^{2\pi}\right\}$$

$$= \frac{5}{\pi^2}\left\{-\frac{2\pi}{n} + \frac{\sin(2n\pi)}{n^2} - 0\right\}$$

$$= -\frac{5}{\pi^2}\,\frac{2\pi}{n} + 0$$

$$B_n = -\frac{10}{\pi n}$$

$$\underline{v(t) = 5 - \frac{10}{\pi}\left\{\sin(wt) + \frac{1}{2}\sin(2wt) + \frac{1}{3}\sin(3wt) + \ldots\right\} \text{ volts}}$$

$$\underline{v(t) = 5 - 3.18\sin(wt) - 1.59\sin(2wt) - 1.06\sin(3wt) - \ldots \text{ volts}}$$

4.3 R.M.S. VALUE OF A COMPLEX WAVEFORM

The r.m.s. value of a complex waveform expressed as

$$v(t) = V_{d.c.} + V_1\sin(wt + \theta_1) + V_2\sin(2wt + \theta_2) + V_3\sin(3wt + \theta_3) + V_4\sin(4wt + \theta_4) + \ldots$$

is given by the following expression:-

$$V_{r.m.s.} = \sqrt{V_{d.c.}^2 + \frac{V_1^2}{2} + \frac{V_2^2}{2} + \frac{V_3^2}{2} + \frac{V_4^2}{2} + \ldots}$$

$$= \sqrt{\text{sum of the (r.m.s. values)}^2 \text{ of the component waveforms.}}$$

If the waveform is expressed in the form:

$$v(t) = V_{d.c.} + A_1\cos(wt) + A_2\cos(2wt) + A_3\cos(3wt) + \ldots + B_1\sin(wt) + B_2\sin(2wt) + B_3\sin(3wt) + \ldots$$

Then $V_{r.m.s.} = \sqrt{V_{d.c.}^2 + \frac{A_1^2}{2} + \frac{B_1^2}{2} + \frac{A_2^2}{2} + \frac{B_2^2}{2} + \ldots}$

Also $V_1^2 = A_1^2 + B_1^2$; $V_2^2 = A_2^2 + B_2^2$; etc.

thus again $V_{r.m.s.} = \sqrt{V_{d.c.}^2 + \frac{V_1^2}{2} + \frac{V_2^2}{2} + \frac{V_3^2}{2} + \ldots}$

4.4 WAVEFORM SYNTHESIS

A complex waveform may be synthesised by adding the component parts at various times during one complete cycle. The values of the complex waveform can be obtained in tabular form or by using a computer. If a computer is available try running the program on the following page:-

```
10 PRINT "THIS PROGRAMME PLOTS THE CURVE FOR A COMPLEX WAVEFORM"
20 PRINT "CONTAINING A SECOND AND THIRD HARMONIC."
30 PRINT "THE VOLTAGE WAVEFORM HAS THE EQUATION:"
40 PRINT "       E = E1*SIN(WT+A1) + E2*SIN(WT+A2) + E3*SIN(WT+A3)"
50 PRINT "ENTER THE MAGNITUDE AND PHASE ANGLE OF THE FUNDAMENTAL"
60 PRINT "E1 IN VOLTS, A1 IN RADIANS"
62 INPUT E1,A1
63 PRINT "ENTER THE MAGNITUDE AND PHASE ANGLE OF THE SECOND HARMONIC"
64 INPUT E2,A2
65 PRINT "ENTER THE MAGNITUDE AND PHASE ANGLE OF THE THIRD HARMONIC"
66 INPUT E3,A3
70 PRINT TAB(35);"O"
80 PRINT "--------------------------------------------------------------"
90 P1=ATN(1)*4
100 FOR T=0 TO 2*P1 STEP P1/10
110 E = E1*SIN(T+A1) + E2*SIN(2*T+A2) + E3*SIN(3*T+A3)
120 M = ABS(E1)+ABS(E2)+ABS(E3)
130 X = INT(35+E*35/M)
140 PRINT TAB(X);"*"
150 NEXT T
170 END
OK
```

```
THIS PROGRAMME PLOTS THE CURVE FOR A COMPLEX WAVEFORM
CONTAINING A SECOND AND THIRD HARMONIC.
THE VOLTAGE WAVEFORM HAS THE EQUATION:
       E = E1*SIN(WT+A1) + E2*SIN(WT+A2) + E3*SIN(WT+A3)
ENTER THE MAGNITUDE AND PHASE ANGLE OF THE FUNDAMENTAL
E1 IN VOLTS, A1 IN RADIANS
? 100,0
ENTER THE MAGNITUDE AND PHASE ANGLE OF THE SECOND HARMONIC
? 20,0.35
ENTER THE MAGNITUDE AND PHASE ANGLE OF THE THIRD HARMONIC
? -40,1.41
                                   O
--------------------------------------------------------------
                          *
                                      *
                                                     *
                                                              *
                                                               *
                                                        *
                                             *
                                     *
```

WORK TO BE FORWARDED FOR COMMENT

1. A waveform has a trigonometric Fourier series of

$$v(t) = 100(\sin(wt) - \tfrac{1}{9}\sin(3wt) + \tfrac{1}{25}\sin(5wt) \ \ldots\ ) \text{ volts.}$$

Write a BASIC programme that will give a sketch of the waveform between the intervals of 0 to 4π.

2. Find the trigonometric Fourier series for each waveform shown in figure 4.6.

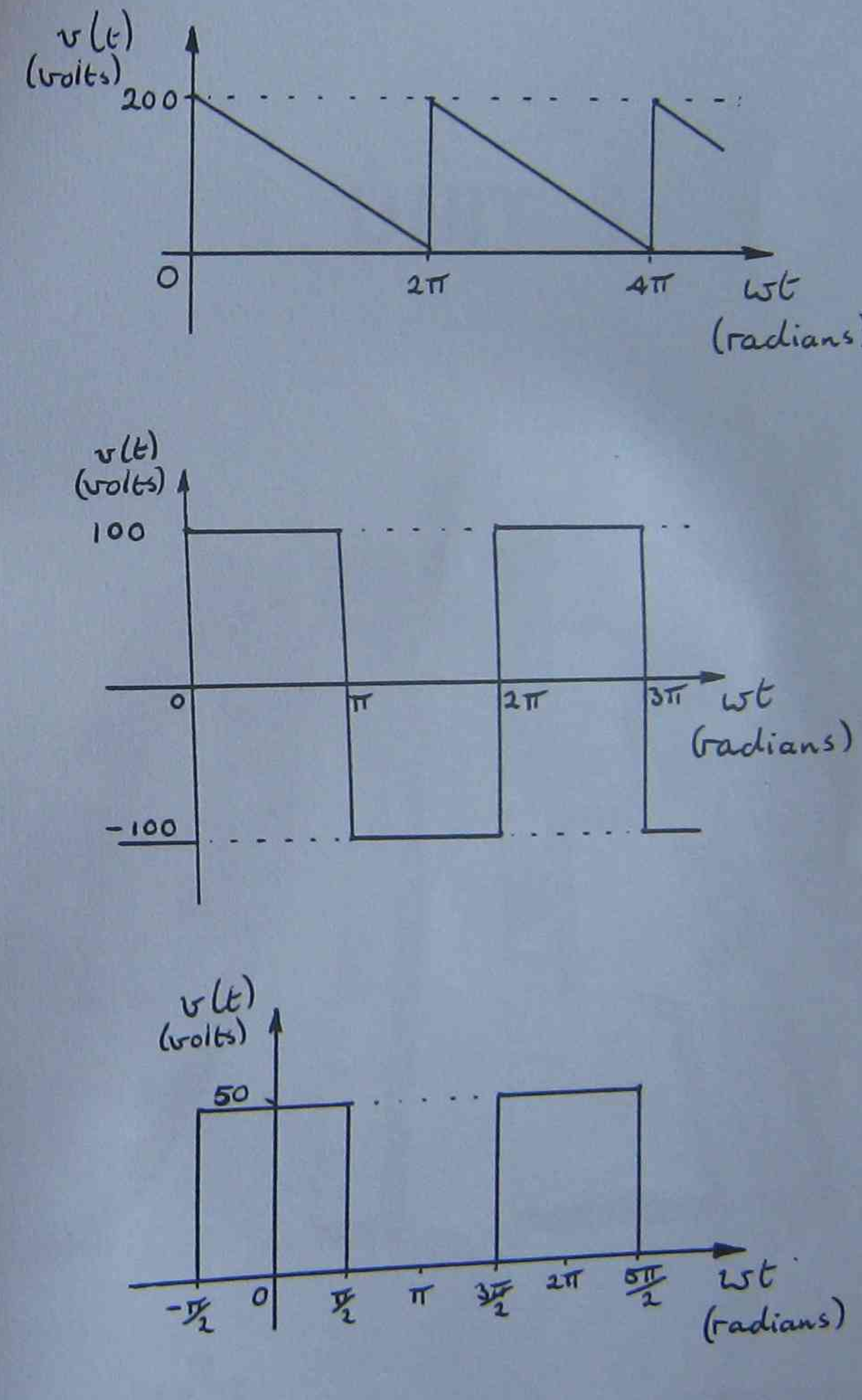


Figure 4.6

ANSWERS - UNIT 4

2 (i) $v(t) = 100 + \frac{200}{\pi}\left\{\sin(wt) + \frac{1}{2}\sin(2wt) + \frac{1}{3}\sin(3wt) + \ldots\right\}$

(ii) $v(t) = \frac{400}{\pi}\left\{\sin(wt) + \frac{1}{3}\sin(3wt) + \frac{1}{5}\sin(5wt) + \ldots\right\}$

(iii) $v(t) = 25 + \frac{100}{\pi}\left\{\cos(wt) - \frac{1}{3}\cos(3wt) + \frac{1}{5}\cos(5wt) - \frac{1}{7}\cos(7wt) + \ldots\right\}$

20

$$\mathcal{L}^{-1}\frac{5/2\, s}{s^2+25} = 5/2\,\mathcal{L}^{-1}\frac{s}{s^2+5^2} = \frac{5}{2}\sin 5t$$

$$\mathcal{L}^{-1}\frac{25/2}{s^2+25} = \frac{25}{2}\,\mathcal{L}^{-1}\frac{1}{s^2+5^2} = \frac{25}{2}\cos 5t$$

$$\mathcal{L}^{-1}\frac{5/2}{s+5} = 5/2\,\mathcal{L}^{-1}\frac{1}{s+5} = \frac{5}{2}e^{-5t}$$

$$I(t) = \frac{5}{2}e^{-5t} - \frac{5}{2}\cos 5t + \frac{5}{2}\sin 5t$$

\#

week 7 | 8 | 9

Ordinary Differential Equations

Integrations

Pb1 Find a general solution of the differential equation

$$\frac{dy}{dx} = 3x^2$$

$$\therefore \frac{dy}{dx} = 3x^2 \longrightarrow dy = 3x^2\,dx$$

$$\int dy = \int 3x^2\,dx$$

$$y = 3 \times \frac{x^{2+1}}{2+1} + C = 3\frac{y^3}{3} + C$$

$$y = y^3 + C$$

Pb2 Solve $y'' = 3x - 2$, $y(0) = 2$, $y'(1) = -3$

$$y'' = 3x - 2$$

$$\int y''\,dy = \int 3x\,dx - \int 2\,dx + c_1 = 3\frac{x^2}{2} - 2x + c_1 = y'$$

$$y'(1) = -3 \Longrightarrow -3 = 3\frac{1^2}{2} - 2\times 1 + c_1$$

$$\therefore c_1 = -3 - \frac{3}{2} + 2 = -1 - \frac{3}{2} = -\frac{5}{2}$$

$$\therefore y' = \frac{3x^2}{2} - 2x - \frac{5}{2}$$

$$\int y'\,dy = \int \frac{3x^2}{2}\,dx - \int 2x\,dx - \int \frac{5}{2}\,dx + C_2$$

$$= 3\frac{x^{2+1}}{2\times[2+1]} - 2\frac{x^{1+1}}{1+1} - \frac{5}{2}x + C_2$$

$$y = \frac{x^3}{2} - \frac{2x^2}{\cancel{2}} - \frac{5}{2}x + C_2$$

$$\text{But } y(0) = 2 \longrightarrow 2 = \frac{0^3}{2} - (0)^2 - \frac{5}{2}\times 0 + C_2$$

$$\therefore C_2 = 2$$

Separation of Variables

Pb3

(a) Find the general solution of $(4x + xy^2)dx + (y + x^2y)dy = 0$

(b) Find the particular solution of $y(1) = 2$

(a) $(4x + xy^2)dx + (y + x^2y)dy = 0$

$x(4 + y^2)dx + y(1 + x^2)dy = 0$

Divide by $(1+x^2) \times (4+y^2)$

$$\frac{x(4+y^2)\,dx}{(1+x^2)(4+y^2)} + \frac{y(1+x^2)\,dy}{(4+y^2)(1+x^2)} = 0$$

$$\frac{x\,dx}{1+x^2} + \frac{y\,dy}{4+y^2} = 0$$

Integrate $\displaystyle\int \frac{x\,dx}{1+x^2} + \int \frac{y\,dy}{4+y^2} = \int 0 = 0$

$d(1+x^2) = 2x\,dx \qquad d(4+y^2) = 2y\,dy$

$\therefore x\,dx = \dfrac{d(1+x^2)}{2}, \qquad y\,dy = \dfrac{d(4+y^2)}{2}$

Substitute $\displaystyle\int \frac{\frac{d(1+x^2)}{2}}{1+x^2} + \int \frac{\frac{d(4+y^2)}{2}}{4+y^2} = 0$

$$\frac{1}{2}\int \frac{d(1+x^2)}{1+x^2} + \frac{1}{2}\int \frac{d(4+y^2)}{4+y^2} = 0$$

$$\frac{1}{2}\ln(1+x^2) + \frac{1}{2}\ln(4+y^2) = C_1$$

$$\ln(1+x^2) + \ln(4+y^2) = 2C_1$$

$$\ln(1+x^2)\cdot(4+y^2) = 2C_1$$

$$\therefore (1+x^2)(4+y^2) = e^{2C_1} = C$$

For $y(1) = 2 \Rightarrow$ when $x = 1$, $y = 2$ $\quad (1+1)^2 + (4+2^2) = c$

$4 + 12 = c = 16$

$\boxed{(1+x^2)(4+y^2) = 16}$

3

4 Solve $\frac{dy}{dx} + 3y = 8, \quad y(0) = 2$

Find equation of dy & dx

$$\frac{dy}{dx} = 8 - 3y$$

$$dy = (8-3y)\,dx \longrightarrow dx = \frac{dy}{(8-3y)}$$

$$d(8-3y) = d8 - d3y = -3dy \longrightarrow dy = -\frac{d(8-3y)}{3}$$

$$\therefore dx = \frac{-\frac{d(8-3y)}{3}}{(8-3y)} = -\frac{1}{3}\,\frac{d(8-3y)}{8-3y}$$

$$\int dx = -\frac{1}{3}\int \frac{d(8-3y)}{(8-3y)} = -\frac{1}{3}\ln(8-3y) + c$$

$$x = -\frac{1}{3}\ln(8-3y) + c$$

$$y(0) = 2 \longrightarrow x = 0 \longrightarrow y = 2$$

$$(0) = -\frac{1}{3}\ln(8-3\times 2) + c$$

$$0 = -\frac{1}{3}\ln 2 + c$$

$$\therefore c = \frac{1}{3}\ln 2$$

$$\therefore x = -\frac{1}{3}\ln(8-3y) + \frac{1}{3}\ln 2$$

Pb 5 Solve $\frac{dy}{dx} = \sec y \tan x$

$$\frac{dy}{dx} = \frac{1}{\cos y}\,\frac{\sin x}{\cos x}$$

$$\cos y\,dy = \frac{\sin x}{\cos x}\,dx$$

$$d\cos x = -\sin x\,dx$$

$$\therefore \sin x\,dx = -d\cos x$$

$$\longrightarrow \cos y\,dy = -\frac{d\cos x}{\cos x}$$

$$\int \cos y\,dy = -\int \frac{d\cos x}{\cos x}$$

$$\sin y = -\ln\cos x + c$$

$$\therefore c = \sin y + \ln\cos x$$

$\therefore$ ~~Siny~~

#

Exact equation

$m(x,y)\,dx + n(x,y)\,dy = 0$

when $\dfrac{\partial m}{\partial y} = \dfrac{\partial N}{\partial x}$

Then $\displaystyle\int m\,\partial x + \int \left(N - \frac{\partial}{\partial y} \int m\,\partial x\right) dy = c$

pb 6 Show that $(3x^2 + y\cos x)\,dx + (\sin x - 4y^3)\,dy = 0$ is exact differential equation and find it's general solution

Solution

$(3x^2 + y\cos x)\,dx + (\sin x - 4y^3)\,dy = 0$

compare $m(x,y)\,dx + N(x,y)\,dy = 0$

$\therefore m = 3x^2 + y\cos x, \quad N = \sin x - 4y^3$

$\dfrac{\partial m}{\partial y} = \dfrac{\partial}{\partial y}(3x^2 + y\cos x) = \cancel{\dfrac{\partial 3x^2}{\partial y} + y\dfrac{\partial \cancel{\cos x}}{\partial y}}$

$= \cancel{3x^2}_0 + \cos x = \cancel{6x + y\sin x}$

$\dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial x}(\sin x - 4y^3) = \cos x$

$\therefore \dfrac{\partial m}{\partial y} = \dfrac{\partial N}{\partial x}$

Then $\displaystyle\int m\,\partial x + \int \left[N - \frac{\partial}{\partial y} \int m\,\partial x\right] \partial y = c$

$\displaystyle\int (3x^2 + y^2\cos x)\,\partial x + \int \left[(\sin x - 4y^3) - \frac{\partial}{\partial y}\left(\int (3x^2 + y\cos x)\,\partial x\right)\right] \partial y = c$

$\displaystyle 3\frac{x^{2+1}}{2+1} + y^2\sin x + \int \left[(\sin x - 4y^3) - \cancel{3x\,x^2 + \frac{\partial}{\partial y}[y\sin x]}\right.$

$\displaystyle \left. - \frac{\partial}{\partial y}\left(3\frac{x^{2+1}}{2+1} + y\sin x\right)\right] \partial y = c$

$$x^3 + y\sin x + \int \left[(\sin x - 4y^3) - \frac{\partial}{\partial y}(x^3 + y\sin x) \right] \partial y$$

$$x^3 + y\sin x + \int (\sin x - 4y^3) - 0 - \sin x \,] \partial y$$

$$x^3 + y\sin x + \cancel{\int \sin x \, \partial y}_{\,0} - \int 4y^3 \partial y - \cancel{\int \sin x \, \partial y}_{\,0}$$

$$x^3 + y\sin x - 4\frac{y^{3+1}}{3+1} = c$$

$$x^3 + y\sin x - 4\frac{y^4}{4} = x^3 + y\sin x - y^4 = c$$

Integrating factor

$$m(x,y)\,dx + N(x,y)\,dy = 0$$

when $\frac{\partial m}{\partial y} \neq \frac{\partial N}{\partial x}$

Apply Integrating factor.

$$\mu m\,dx + \mu N(x,y)\,dy = 0$$

$$\frac{\partial \mu m}{\partial y} = \frac{\partial \mu N}{\partial x}$$

Integrating factor $\mu = e^{\int f(x)\,dx}$

Integrating factor

$m(x,y)\,dx + N(x,y)\,dy = 0$

where $\frac{\partial m}{\partial y} \neq \frac{\partial N}{\partial x}$

Integrating factor

(a) $\frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right)$

(b) $\frac{x\,dy - y\,dx}{y^2} = -d\left(\frac{x}{y}\right)$

(c) $\frac{x\,dy - y\,dx}{x^2+y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$

(d) $\frac{x\,dy - y\,dx}{x^2-y^2} = \frac{1}{2}\,d\left(\ln\frac{x-y}{x+y}\right)$

(e) $\frac{x\,dx + y\,dy}{x^2+y^2} = \frac{1}{2}\,d\left(\ln(x^2+y^2)\right)$

Pb 7 Solve (a) $(y + x^4)\,dx - x\,dy = 0$

$y\,dx + x^4\,dx - x\,dy = 0$

$y\,dx - x\,dy + x^4\,dx = 0$

Integrating factor $= \frac{1}{x^2}$

$\therefore$ divide by $x^2 \Rightarrow \frac{y\,dx - x\,dy}{x^2} + \frac{x^4\,dx}{x^2} = 0$

$\frac{y\,dx}{\cancel{x^2}} - \frac{x\,dy}{x^2} + x^2\,dx = 0$

$\frac{d}{dx}\left(\frac{y}{x}\right) - \cancel{d\left(\frac{y}{x}\right)} + d\left(\frac{x^3}{3}\right)$

$$-\frac{y}{x} + \frac{x^3}{3} = 0$$

Pb 8 $(x^3 + xy^2 - y)\,dx + x\,dy = 0$

$\times \frac{1}{x^2+y^2} \Rightarrow \cancel{\frac{(x^3+xy^2-y)\,dx}{x^2+y^2}} +$

$$x^3\,dx + xy^2\,dx - y\,dx + x\,dy = 0$$

$$x^3\,dx + xy^2\,dx - (y\,dx - x\,dy) = 0$$

Divide by $\frac{1}{x^2+y^2}$ $\quad \cancel{x\,(x^2\,dx + y}^2 d$

$$x\,dx\,(x^2+y^2) + x\,dy - y\,dx = 0$$

divide by $x^2+y^2 \Rightarrow \frac{x\,dx\,(x^2+y^2)}{(x^2+y^2)} + \frac{x\,dy - y\,dx}{x^2+y^2} = 0$

$$x\,dx + \frac{x\,dy - y\,dx}{x^2+y^2} = 0$$

$$x\,dx + d\left(\tan^{-1}\frac{y}{x}\right) = 0$$

$$\cancel{\int x\,dx + \int \frac{d}{dx}\left(\frac{y}{x}\right) = 0}$$

$$\cancel{\frac{x^2}{2} + y \int \frac{d}{dx} \frac{1}{x}}$$

$$x\,dx + \frac{d}{dx}\tan^{-1}\frac{y}{x} = 0$$

$$\frac{d}{dx}\left(\frac{x^2}{2}\right) + \frac{d}{dx}\left(\tan^{-1}\frac{y}{x}\right) = 0$$

$$\frac{x^2}{2} + \tan^{-1}\frac{y}{x} = 0$$

Linear Equation

solve $\frac{dy}{dx} + P(x)y = Q(x)$

$$y = e^{-\int P dx} \int Q\, e^{\int P dx}\, dx + c\, e^{-\int P dx}$$

Pb 4 solve $x\frac{dy}{dx} - 2y = x^3 \cos 4x$

Compare with $\frac{dy}{dx} + P(x)y = Q(x)$

Change $\frac{dy}{dx} - \frac{1}{x} \times 2y = x^2 \cos 4x$

Then $P = -\frac{2}{x}$, $Q = x^2 \cos 4x$

$$\therefore y = e^{-\int P dx} \int Q\, e^{\int P dx}\, dx + c\, e^{-\int P dx}$$

$$= e^{-\int \frac{2}{x} dx} \int x^2 \cos 4x\, e^{\int -\frac{2}{x} dx}\, dx + c\, e^{-\int -\frac{2}{x} dx}$$

$$= e^{-2 \ln x} \int x^2 \cos 4x \cdot e^{-\ln x}\, dx + c\, e^{2 \ln x}$$

$$= e^{\ln x^{-2}} \int x^2 \cos 4x\, e^{\ln x^{-2}}\, dx + c\, e^{\ln x^2}$$

$$= \frac{1}{x^2} \int x^2 \cos 4x \times \frac{1}{x^2}\, dx + c \times x^2$$

$$= \frac{1}{x^2} \int \cos 4x\, dx + c\, x^2$$

$$= \frac{1}{x^2} \times \frac{1}{4} \sin 4x + c\, x^2$$

$\boxed{e^{\ln x} = x}$

$e^{\ln x^{-2}} = \cancel{x}^2$

$= \cancel{x}$

$= 1/x^2$

Homogeneous Equation

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

Then $\frac{y}{x} = v$ or $y = vx$

Then $v + x\frac{dv}{dx} = F(v)$ (or) $x\,dv + (F(v) - v)\,dx = 0$

$$\ln x = \int \frac{dv}{F(v) - v} + c$$

where $v = \frac{y}{x}$ $\longrightarrow$ Solution $y = cx$

Pb (10) solve $(2x^3 + y^3)\,dx - 3xy^2\,dy = 0$

Rewrite $\frac{dy}{dx} = \frac{2x^3 + y^3}{3xy^2}$

$$\frac{2x^3 + y^3}{3xy^2} = \frac{2x^3}{3xy^2} + \frac{y^3}{3xy^2} = \frac{2x^2}{3y^2} + \frac{y}{3x}$$

$$\frac{dy}{dx} = \frac{2}{3\,(y/x)^2} + \frac{1}{3}\left(\frac{y}{x}\right)$$

$$\frac{dy}{dx} = \frac{2}{3\,(y/x)^2} + \frac{1}{3}\left(\frac{y}{x}\right) = F(y/x)$$

Homogeneous equation,

Then $\frac{y}{x} = v$ (or) $y = vx$

Then $\ln x = \int \frac{dv}{F(v) - v} + c$

$$\cancel{\ln x = \int \frac{dv}{\frac{2}{3v^2} +}}$$

$$\ln x = \int \frac{dv}{\frac{2}{3v^2} + \frac{1}{3}v} + c$$

$$= \int \frac{dv}{\frac{1}{3}\left[\frac{2}{v^2} - v\right]} + c$$

$$= \int \frac{1}{\frac{1}{3}\left(\frac{2 - v^3}{v^2}\right)} + c$$

$$= \int \frac{3v^2\,dv}{2 - v^3} + c$$

$$d(2 - v^3) = -3v^2\,dv$$

$$\therefore \ln x = \int \frac{-d(2 - v^3)}{2 - v^3} + c$$

$$\ln x = -\ln(2 - v^3) + c$$

$$\ln x = -\ln\left(2 - \left(\frac{y}{x}\right)^3\right) + c$$

$$\ln x + \ln\left(2 - \frac{y^3}{x^3}\right) = c$$

$$\cancel{\ln \frac{x}{2 - \frac{y^3}{x^3}}} \neq \ln x\left(2 - \frac{y^3}{x^3}\right) = c$$

$$\cancel{\ln \frac{}{x^4}}$$

$$\ln 2x - \frac{y^3}{x^2} = c$$

$$\frac{2x^3 - y^3}{x^2} = e^c = c_1$$

11

$2x^3 - y^3 = c_1 x^2$

Pb (11) Solve $\dfrac{dy}{dx} = \dfrac{5x+y}{2x-y}$

$$\frac{dy}{dx} = \frac{5 + y/x}{2 - y/x}$$

$\therefore$ RHS is function of LHS

Homogeneous equation

Then $\dfrac{y}{x} = v \qquad y = vx$

$$\ln x = \int \frac{dv}{F(v) - v} + c$$

$$\ln x = \int \frac{dv}{\frac{5+v}{2-v} - v} + c$$

$$\ln x = \int \frac{dv}{\frac{5+v-2+v^2}{2-v}} + c$$

$$= \int \frac{(2-v)(dv)}{3+v+v^2} + c$$

$d(3+v+v^2) = (\cancel{1+2v)dv}\ \ 1 + 2v\,dv \Rightarrow v\,dv = \dfrac{d(3+v+v^2)}{2}$

$\therefore\ v = \dfrac{d(3+v+v^2) - 1}{2}$

$2 - v = 2 - \dfrac{d(3+v+v^2) - 1}{2}$

$y = vx$ (2

$\cancel{x \frac{dv}{dx} + v}$

$$\frac{d}{dx}(vx) = \frac{5+4v}{2-v}$$

$$x\frac{dv}{dx} + v = \frac{5+4v}{2-v}$$

$$\frac{x\,dv}{dx} = \frac{5+4v}{2-v} - v$$

$$\frac{x\,dv}{dx} = \frac{5+4v-2v+v^2}{2-v}$$

$$\frac{x\,dv}{dx} = \frac{v^2+2v+5}{2-v}$$

$$\frac{dx}{x\,dv} = \cancel{\frac{v^2+2v+5}{2-v}} \quad \frac{2-v}{v^2+2v+5}$$

$$\frac{dx}{x} = \frac{2-v}{v^2+2v+5}\,dv$$

$$\frac{dx}{x} - \frac{2-v}{v^2+2v+5}\,dv = 0$$

$$\frac{dx}{x} + \frac{(v-2)}{v^2+2v+5}\,dv = 0$$

$$\cancel{\int \frac{dx}{x} + \frac{v\,dv}{v^2+2v+5} - \frac{2}{v^2+2v+5}\,dv = 0}$$

$$\cancel{\int \frac{dx}{x} + \frac{\frac{1}{2}d(v^2+2v+5)-2}{v^2+2v+5}}$$

$$\frac{dx}{x} + \int \frac{(v+1)\,dv}{v^2+2v+5} + \int \frac{-3\,dv}{v^2+2v+5}$$

$$\frac{dx}{x} + \frac{1}{2}\ln(v^2+2v+5) + \int \frac{-3\,dv}{(v+1)^2+4}$$

$$\frac{dx}{x} + \frac{1}{2}\ln(v^2+2v+5) - \frac{3}{2}\tan^{-1}\left(\frac{v+1}{2}\right)$$

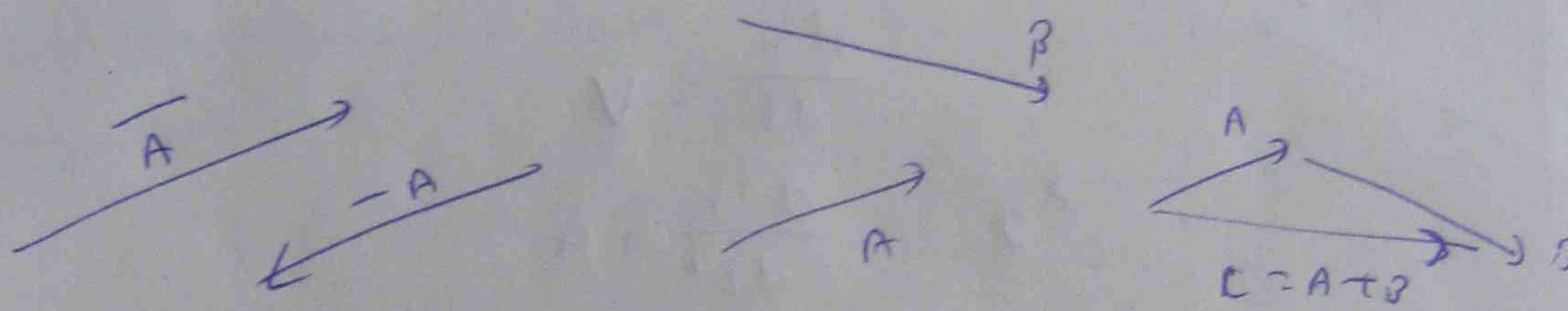
$$d(v^2+2v+5)$$

$$= 2v\,dv + 2$$

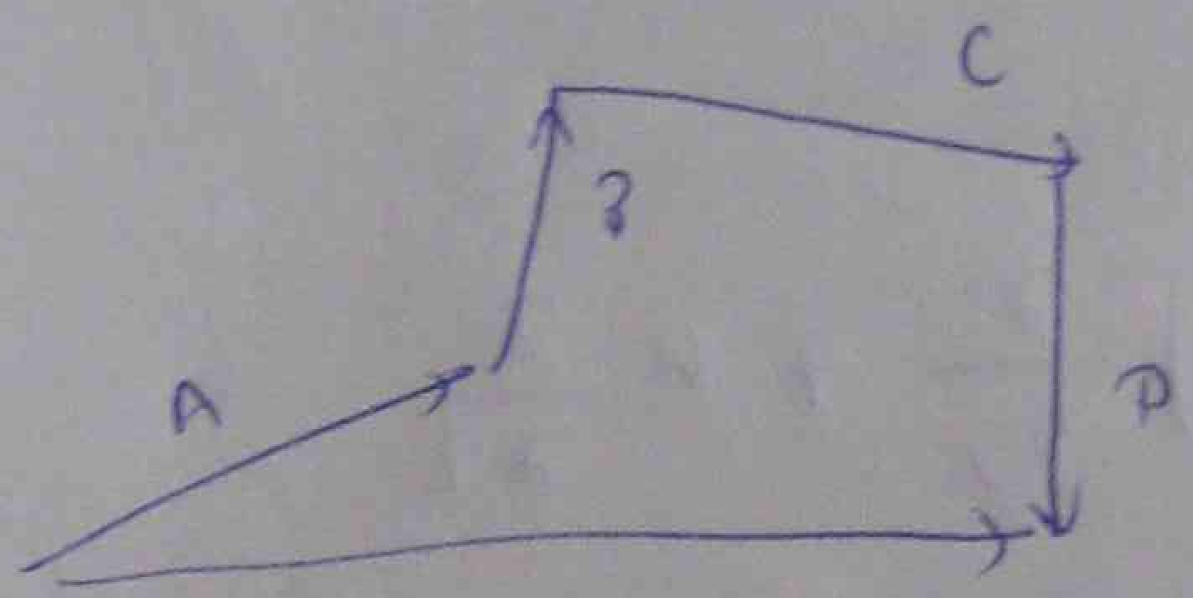
$$\therefore d(v^2+2v+5) - 2 = 2v\,dv$$

$$v\,dv = \frac{1}{2}\left(d(v^2+2v+5) - 2\right.$$

Vector Analysis



$C = A + B$



$E = A + B + C + D$

Law of vector algebra

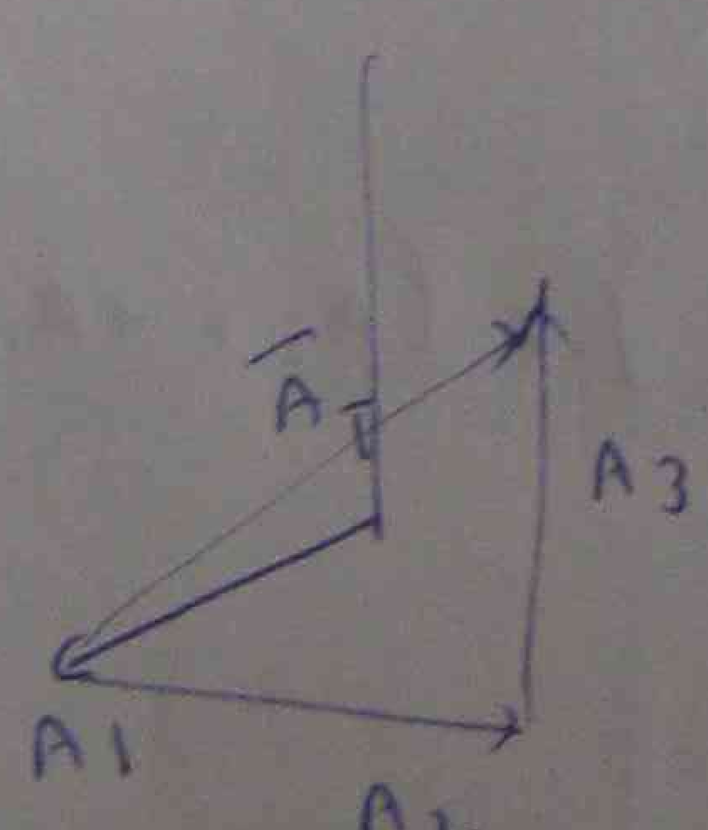
$A + B = B + A$

$A + (B + C) = (A + B) + C$

$m(nA) = (mn)A$

$(m+n)A = mA + nA$

$m(A+B) = mA + mB$



$A_T = A_1\hat{i} + A_2\hat{j} + A_3 k$

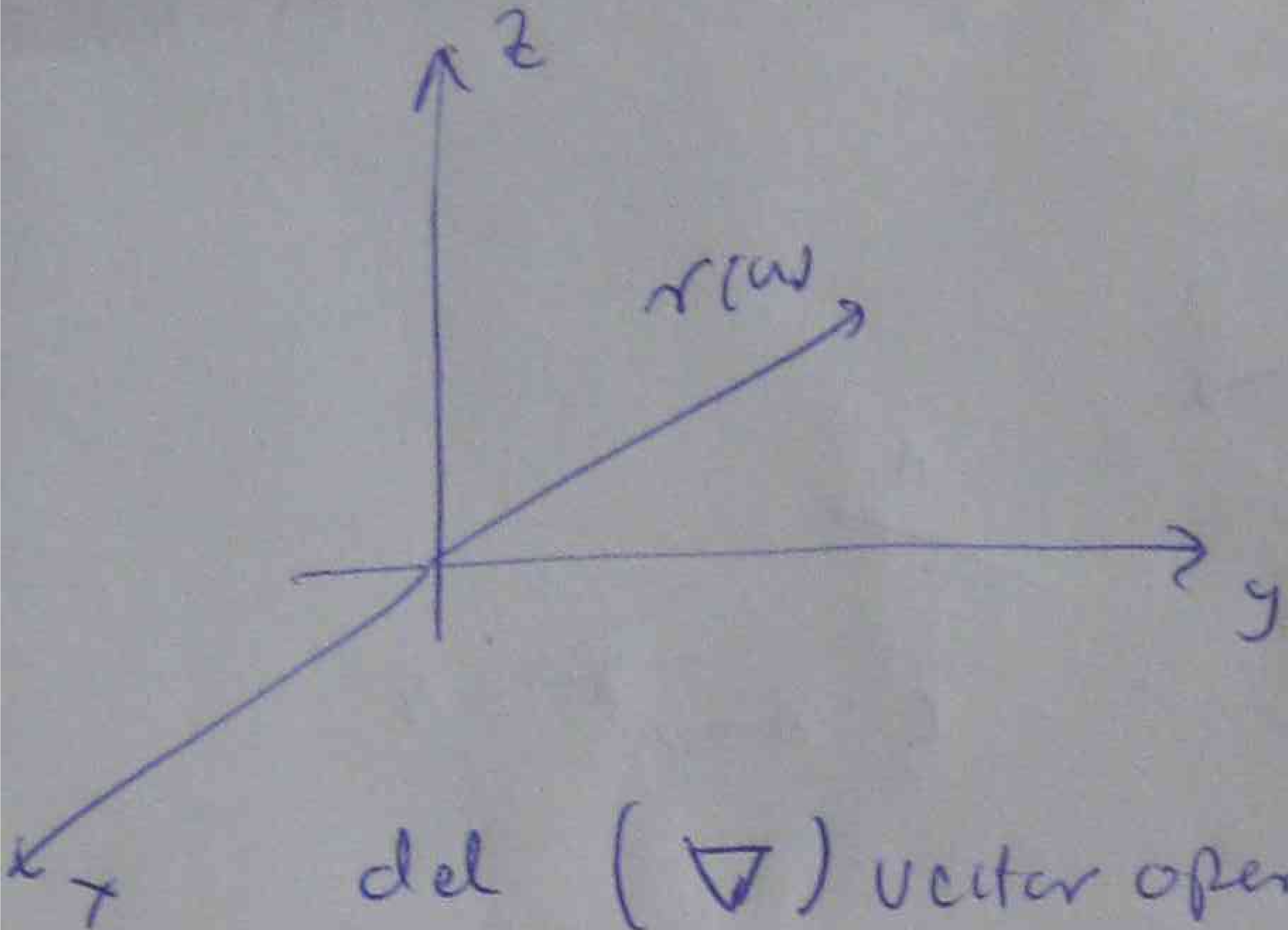
$A_T = |A| = \sqrt{A_1^2 + A_2^2 + A_3^2}$

Cross vector product

$$A \times B = \begin{vmatrix} \hat{i} & \hat{j} & k \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$A \cdot (B \times C) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

2



$$\frac{dr}{dt} = V$$

$$\frac{d^2r}{dt^2} = a$$

del (∇) vector operator

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

Gradient ϕ $\qquad \phi(x, y, z)$

$$\text{Grad}\,\phi = \nabla\phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\phi$$

Divergence

$$\text{div}\,A = \nabla \cdot A = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\left(A_1 i + A_2 j + A_3 k\right)$$

$$= \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

Curl

$$\text{curl}\,A = \nabla \times A = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\times\left(A_1 i + A_2 j + A_3 k\right)$$

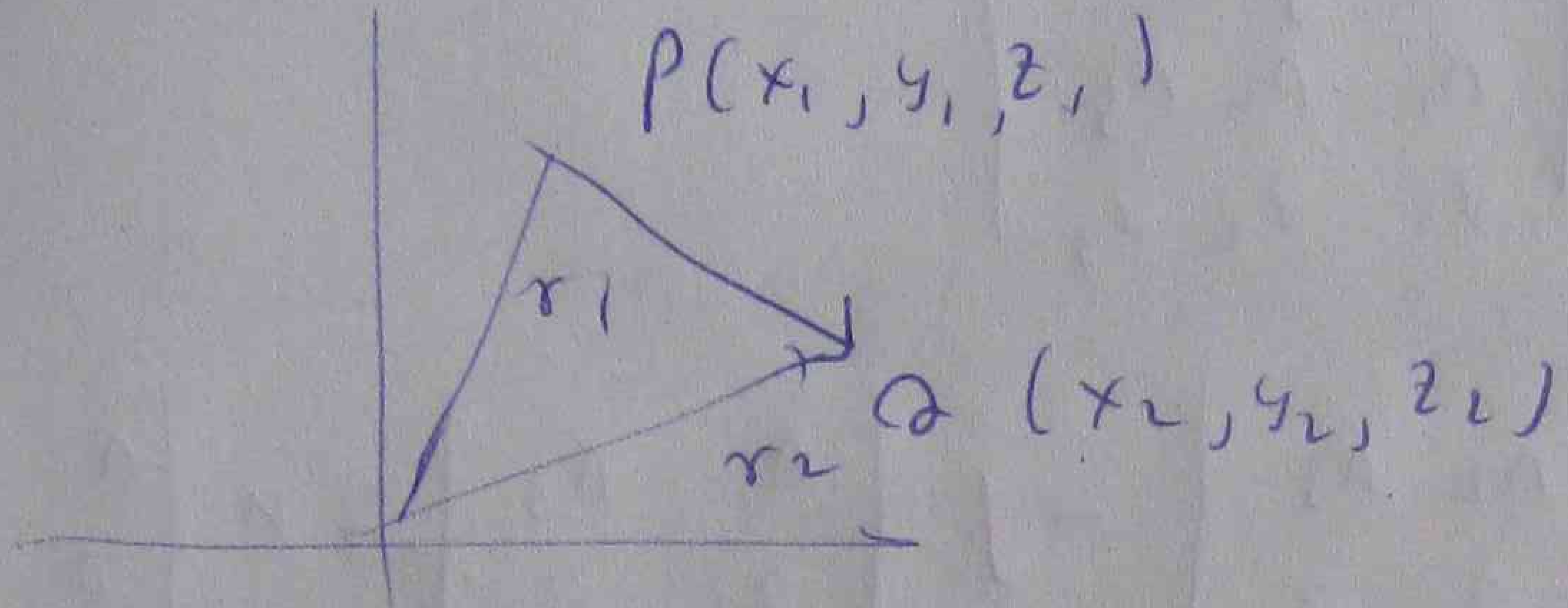
$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)i + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)j$$

$$+ \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)k$$

lu

pb1 determine the vector having initial point $P(x_1, y_1, z_1)$ and the terminal point $Q(x_2, y_2, z_2)$ and find its magnitude.



$$PQ = r_2 - r_1 = (x_2 i + y_2 j + z_2 k) - (x_1 i + y_1 j + z_1 k)$$
$$= (x_2 - x_1) i + (y_1 - y_1) j + (z_2 - z_1) k$$

magnitude of PQ

$$\overline{PQ} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$\boxed{\text{dot (or) scalar product}}$

pb2 If $A = A_1 i + A_2 j + A_3 k$ & $B = B_1 i + B_2 j + B_3 k$

Prove $A \cdot B = A_1 B_1 + A_2 B_2 + A_3 B_3$

$$A \cdot B = (A_1 i + A_2 j + A_3 k) \cdot (B_1 i + B_2 j + B_3 k)$$
$$= A_1 i \cdot (B_1 i + B_2 j + B_3 k) + A_2 j (B_1 i + B_2 j + B_3 k)$$
$$+ A_3 k (B_1 i + B_2 j + B_3 k)$$
$$= A_1 B_1 i \cdot i + A_1 B_2 i j + A_1 B_3 i \cdot k + B_2 B_1 j \cdot i + A_2 B_2 j \cdot j$$
$$+ A_3 B_3 j \cdot k + A_3 B_1 k \cdot j + A_3 B_3 k \cdot k$$

$i \cdot i = 1$, $j \cdot j = 1$, $k \cdot k = 1$ | $i \cdot j = j \cdot k = k \cdot i = 0$ | $= A_1 B_1 + A_2 B_2 + A_3 B_3$

Cross Product

~~$i \times i \neq 1$~~ $\quad$ $i \times i = j \times j = k \times k = 0$

~~$i \times j \neq 0$~~ $\quad$ $i \times j = k, \quad j \times k = i, \quad k \times i = j$

Pb 3

If $A = A_1 i + A_2 j + A_3 k$ and $B = B_1 i + B_2 j + B_3 k$

Prove that $A \times B = \begin{vmatrix} i & j & k \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$

$$\begin{vmatrix} i & j & k \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = i \begin{vmatrix} A_2 & A_3 \\ B_2 & B_3 \end{vmatrix} - j \begin{vmatrix} A_1 & A_3 \\ B_1 & B_3 \end{vmatrix} + k \begin{vmatrix} A_1 & A_2 \\ B_1 & B_2 \end{vmatrix}$$

$$= i\,(A_2 B_3 - A_3 B_2) - j\,(A_1 B_3 - B_1 A_3) + k\,(A_1 A_2 - B_1$$

$$A \times B = (A_1 i + A_2 j + A_3 k) \times (B_1 i + B_2 j + B_3 k)$$

$$= A_1 i \times (B_1 i + B_2 j + B_3 k) + A_2 j \times (B_1 i + B_2 j + B_3 k)$$
$$+ A_3 k \times (B_1 i + B_2 j + B_3 k)$$

$$= A_1 B_1 \underbrace{i \times i}_{0} + A_1 B_2 i \times j + A_1 B_3 i \times k + A_2 B_1 j \times i$$
$$+ A_2 B_2 \underbrace{j \times j}_{0} + A_2 B_3 j \times k + A_3 B_1 k \times i + A_3 B_2 k \times j$$
$$+ A_3 B_3 k \times k$$

$= \cancel{A_1 B_2 k +}$

$$= A_1 B_2 i \times j + A_1 B_3 i \times k + A_2 B_1 j \times i + A_2 B_3 j \times k$$
$$+ A_3 B_1 k \times i + A_3 B_2 k \times j$$

$$= \cancel{A_1 B_2 k} + A_1 B_3 j + A_2 B_1 k + A_2 B_3 i + A_3 B_1 j$$
$$+ A_3 B_2 i$$

$i \times \hat{j} = k$, $\hat{j} \times i = -k$

$\hat{j} \times k = i$ $\quad k\hat{j} = -i$

$k \times i = \hat{j}$ $\quad \hat{i}k = -\hat{j}$

$= A_1 B_2 k + A_1 B_2 (-\hat{j}) + A_2 B_1 (-k) + A_2 B_3 i +$

$A_3 B_1 (\hat{j}) + A_3 B_2 (-i)$

$= A_1 B_2 k - A_1 B_2 \hat{j} - A_2 B_1 k + A_2 B_3 i + A_3 B_1 \hat{j}$

$- A_3 B_2 i$

$= A_2 B_3 i - A_3 B_2 i + A_3 B_1 \hat{j} - A_1 B_2 \hat{j} + A_1 B_2 k$

$- A_2 B_1 k$

$= A_2 B_3 i - A_3 B_2 i - A_1 B_2 \hat{j} + A_3 B_1 \hat{j} + A_1 B_2 k - A_2 B_1 k$

$= (A_2 B_3 - A_3 B_2) i - (A_1 B_2 - A_3 B_1) \hat{j} + (A_1 B_2 - A_2 B_1) k$

pb 4

If $A = 3i - \hat{j} + 2k$, $B = 2i + 3\hat{j} - k$

Find $A \times B$

$$A \times B = \begin{vmatrix} i & \hat{j} & k \\ 3 & -1 & 2 \\ 2 & 3 & -1 \end{vmatrix} = i \begin{vmatrix} -1 & 2 \\ 3 & -1 \end{vmatrix} - \hat{j} \begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} + k \begin{vmatrix} 3 & -1 \\ 2 & 3 \end{vmatrix}$$

$= i \,| -1 \times -1 - 2 \times 3 | - \hat{j} \,| 3 \times (-1) - 2 \times 2 | + k \,| 3 \times 3 - 2(-1)]$

$= i \,[1 - 6] - \hat{j} \,[-3 - 4] + k \,[9 + 2]$

$= -5i + 7\hat{j} + 11k$

Ans

If $A = i + \hat{j}$, $B = 2i - 3\hat{j} + k$, $C = 4\hat{j} - 3k$

Find (a) $(A \times B) \times C$ (b) $A \times (B \times C)$

(a) $A \times B = \begin{vmatrix} i & \hat{j} & k \\ 1 & 1 & 0 \\ 2 & -3 & 1 \end{vmatrix} = i \begin{vmatrix} 1 & 0 \\ -3 & 1 \end{vmatrix} - \hat{j} \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} + k \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix}$

$= i\,[1 \times 1 - (-3) \times 0] - \hat{j}\,[1 \times 1 - 2 \times 0] + k\,[1(-3) - 2 \times 1]$

$= \hat{i}\,[1] - \hat{j}\,[1] + k\,(-3-2)$

$A \times B = \hat{i} - \hat{j} - 5k$

$(A \times B) \times C = (\hat{i} - \hat{j} - 5k) \times (4\hat{j} - 3k)$

$= \begin{vmatrix} \hat{i} & \hat{j} & k \\ 1 & -1 & -5 \\ 0 & 4 & -3 \end{vmatrix}$

$= \hat{i}\,[-1 \times (-3) - (-5) \times 4)] - \hat{j}\,[1(-3) - 0 \times (-5)$

$+ k\,[1 \times 4 - 0(-1)]$

$= \hat{i}\,[3 + 20] - \hat{j}\,[-3] + k \times 4$

$= 23\hat{i} + 3\hat{j} + 4k$

7

(i) $B \times C = \begin{array}{c} \\ B \\ C \end{array} \begin{vmatrix} i & j & k \\ 2 & -3 & 1 \\ 0 & 4 & -3 \end{vmatrix} = i \begin{vmatrix} -3 & 1 \\ 4 & -3 \end{vmatrix} - j \begin{vmatrix} 2 & 1 \\ 0 & -3 \end{vmatrix} + k \begin{vmatrix} 2 & -3 \\ 0 & 4 \end{vmatrix}$

$= i(-3 \times (-3) - 4 \times 1) - j(2 \times (-3) - 0 \times 1) + k(2 \times 4 - 0 \times -3)$

$= i(9 - 4) - j(-6) + k \times 8$

$= 5i + 6j + 8k$

$A \times (B \times C) = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 5 & 6 & 8 \end{vmatrix} \begin{array}{c} \\ A \\ B \times C \end{array}$

$= i \begin{vmatrix} 1 & 0 \\ 6 & 8 \end{vmatrix} - j \begin{vmatrix} 1 & 0 \\ 5 & 8 \end{vmatrix} + k \begin{vmatrix} 1 & 1 \\ 5 & 6 \end{vmatrix}$

$= i(1 \times 8 - 6 \times 0) - j(1 \times 8 - 5 \times 0) + k(1 \times 6 - 5 \times 1)$

$= 8i - j(8) + k(1)$

$= 8i - 8j + k$

Gamma, Beta & other Special Functions

$$\Gamma(m) = \int_0^\infty x^{m-1} e^{-x}\,dx$$

$$\Gamma(m+1) = m\,\Gamma(m)$$

$$\Gamma(m+1) = m!,$$

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

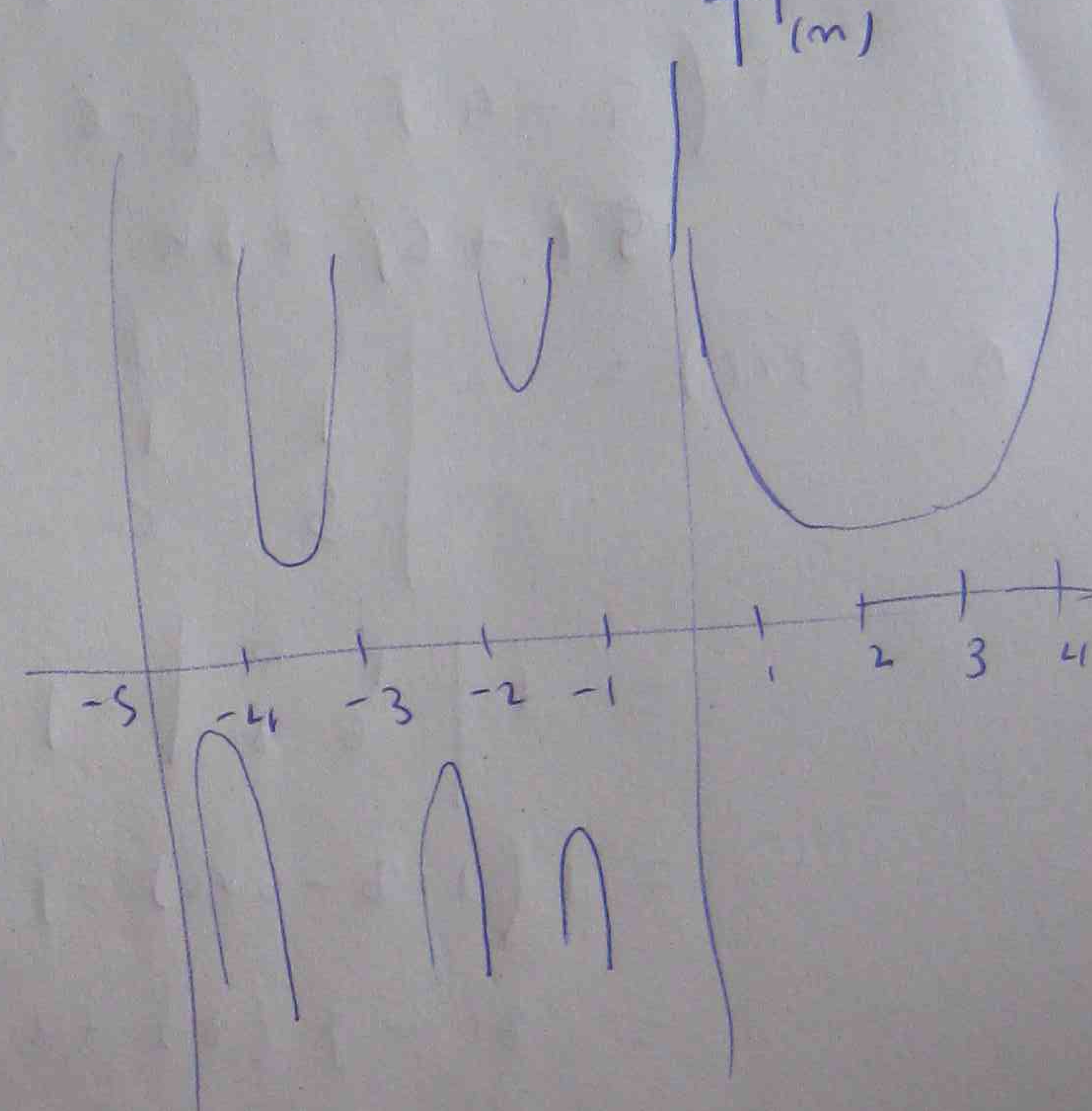


Table of Gamma Function

m	Γ(m)
1.00	1.00
1.10	0.9514
1.20	0.9182
1.30	0.8975
1.40	0.8873
1.50	0.8862
1.60	0.8935
1.70	0.9086
1.80	0.9314
1.90	0.9618
2.00	1.00

Pb (1) Evaluate each of the followings

(a) $\dfrac{\Gamma(6)}{2\Gamma(3)}$ (b) $\dfrac{\Gamma(\frac{5}{2})}{\Gamma(\frac{1}{2})}$ (c) $\dfrac{\Gamma(3)\,\Gamma(2.5)}{\Gamma(5.5)}$

(d) $\dfrac{6\,\Gamma(\frac{8}{3})}{5\,\Gamma(\frac{2}{3})}$

(a) $\dfrac{\Gamma(6)}{2\Gamma(3)}$ $\Gamma(m+1) = m!$ $\dfrac{\Gamma(5+1)}{2\Gamma(2+1)} = \dfrac{5!}{2\times 2!} = \dfrac{5\times4\times3\times2\times1}{2\times2\times1}$

4

b) $\dfrac{\Gamma(\frac{5}{2})}{\Gamma(\frac{1}{2})}$ $\qquad \Gamma(m+1) = m\,\Gamma(m)$

$$\frac{\Gamma(\cancel{1}+\frac{3}{2})}{\Gamma(\frac{1}{2})} = \frac{\frac{3}{2}\,\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2})} = \frac{\frac{3}{2}\,\Gamma(1+\frac{1}{2})}{\Gamma(\frac{1}{2})}$$

$$= \frac{\frac{3}{2} \times \frac{1}{2}\,\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2})} = \frac{3}{2} \times \frac{1}{2} = \frac{3}{4}$$

(c) $$\frac{\Gamma(3) \times \Gamma(2.5)}{\Gamma(5.5)} = \frac{\Gamma(3) \times \Gamma(\frac{5}{2})}{\Gamma(\frac{11}{2})}$$

$$\Gamma(3) = \Gamma(2+1) = 2!$$

$$= \Gamma(5/2) = \Gamma(1+\tfrac{3}{2}) = \tfrac{3}{2}\,\Gamma(\tfrac{3}{2}) = \tfrac{3}{2}\,\Gamma(1+\tfrac{1}{2})$$

$$= \tfrac{3}{2} \times \tfrac{1}{2}\,\Gamma(\tfrac{1}{2})$$

$$\Gamma(5.5) = 1.5 \times 0.5\,\Gamma(0.5)$$

$$\Gamma(\tfrac{11}{2}) = \Gamma(1+\tfrac{9}{2}) = \tfrac{9}{2}\,\Gamma(\tfrac{9}{2}) = 4.5 \times \Gamma(1+\tfrac{7}{2})$$

$$= 4.5 \times \tfrac{7}{2}\,\Gamma(\tfrac{7}{2}) = 4.5 \times 3.5 \times \Gamma(1+5/2)$$

$$= 4.5 \times 3.5 \times \tfrac{5}{2}\,\Gamma(5/2)$$

$$= 4.5 \times 3.5 \times 2.5 \times \Gamma(1+3/2)$$

$$= 4.5 \times 3.5 \times 2.5 \times \tfrac{3}{2}\,\Gamma(3/2)$$

$$= 4.5 \times 3.5 \times 2.5 \times 1.5 \times \Gamma(1+1/2)$$

$$= 4.5 \times 3.5 \times 2.5 \times 1.5 \times \tfrac{1}{2}\,\Gamma(1/2)$$

$$= 4.5 \times 3.5 \times 2.5 \times 1.5 \times 0.5\,\Gamma(0.5)$$

$$\frac{2! \times 1.5 \times 0.5 \times \cancel{\Gamma(0.5)}}{4.5 \times 3.5 \times 2.5 \times 1.5 \times 0.5 \times \cancel{\Gamma(0.5)}} = \frac{2 \times 1 \times 1.5 \times 0.5}{4.5 \times 3.5 \times 2.5 \times 1.5 \times 0.5}$$

$$= \frac{16}{315}$$

(d) $$\frac{6\,\Gamma(8/3)}{5\,\Gamma(2/3)} = \frac{6\,\Gamma(1+5/3)}{5\,\Gamma(2/3)} = \frac{6 \times \frac{5}{3}\,\Gamma(5/3)}{5\,\Gamma(2/3)}$$

$$= \frac{6 \times \frac{5}{3} \times \Gamma(1+2/3)}{5 \times \Gamma(2/3)} = \frac{6 \times \frac{5}{3} \times \frac{2}{3}\,\Gamma(2/3)}{5 \times \Gamma(2/3)}$$

$$= \frac{6 \times 5 \times 2}{3 \times 3 \times 5} = \frac{4}{3}$$

Pb(2) find $\int_0^\infty x^3 e^{-x} dx = \int_0^\infty x^{4-1} e^{-x} dx = \Gamma(m) = \Gamma(4)$

$\Gamma(m) = (m-1)! = \cancel{\Gamma}(4-1)! = 3! = 3 \times 2 = 6$

Beta Function

$$B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

$$B(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

Pb 1 $$\int_0^1 x^4 (1-x)^3 dx = ?$$

$$\int_0^1 x^{5-1} (1-x)^{4-1} dx = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

$$= \frac{\Gamma(5)\,\Gamma(4)}{\Gamma(5+4)} = \frac{\Gamma(5)\,\Gamma(4)}{\Gamma(9)}$$

$$= \frac{(5-1)!\,(4-1)!}{(9-1)!} = \frac{4! \times 3!}{8!} = \frac{4 \times 3 \times 2 \times 1 \times 3 \times 2 \times 1}{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{1}{280}$$

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$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{1}{2} B(m,n) = \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)}$$

Pb ② Solve $\int_0^2 \frac{x^2 dx}{\sqrt{2-x}}$

when $x = 2v$ Limit is divide by 2

$0 \longrightarrow {}_0\int^2 = {}^0/_2\int^{2/2} = \int_0^1$

let $x = 2v$

$$\int_0^1 \frac{(2v)^2 \, d(2v)}{\sqrt{2-2v}} = \int_0^1 \frac{4v^2 2dv}{\sqrt{2}(\sqrt{1-v})} = 4\sqrt{2}\int_0^1 \frac{v^2}{\sqrt{1-v}} dv$$

$$= 4\sqrt{2}\int_0^1 v^2 (1-v)^{-\frac{1}{2}} dv$$

$$\int_0^1 x^{m-1}(1-x)^{n-1} dv = \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)}$$

$m-1 = 2 \Rightarrow m = 1+2 = 3$

$n-1 = -\frac{1}{2} \quad n = 1 - \frac{1}{2} = \frac{1}{2}$

$$= \frac{\Gamma(3)\,\Gamma(\frac{1}{2})}{2\,\Gamma(3+\frac{1}{2})} = \frac{(3-1)!\quad \Gamma(\frac{1}{2})}{2\,\Gamma(7/2}$$

$$= \frac{2!\;\Gamma(\frac{1}{2})}{2\Gamma(1+5/2)} = \frac{2!\;\Gamma(\frac{1}{2})}{2 \times \frac{5}{2}\Gamma(5/2)} = \frac{2!\;\Gamma(\frac{1}{2})}{2 \times \frac{5}{2} \times \Gamma(1+3/2)}$$

$$= \frac{2!\;\Gamma(\frac{1}{2})}{5 \times \frac{3}{2}\Gamma(3/2)} = \frac{2!\;\Gamma(\frac{1}{2})}{7.5 \times \Gamma(1+\frac{1}{2})} = \frac{2!\;\cancel{\Gamma(\frac{1}{2})}}{7.5 \times \frac{1}{2}\cancel{\Gamma(\frac{1}{2})}}$$

$$= \frac{2 \times 1}{7.5 \times 0.5} =$$

Pb (3) Evaluate $\int_0^{\pi/2} \sin^6 \theta \, d\theta$

compare $\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

$2m-1 = 6$, $2n-1 = 0$ because $\cos^{2n-1}\theta = 0$ $\therefore 2n-1 = 0$

$\therefore m = 7/2 \quad n = 1/2$

$$\therefore \int_0^{\pi/2} \sin^6\theta \, d\theta = \frac{\Gamma(m)\,\Gamma(n)}{2\times\Gamma(m+n)} = \frac{\Gamma(7/2)\,\Gamma(\frac{1}{2})}{2\times\Gamma(\frac{7}{2}+\frac{1}{2})}$$

$$= \frac{\Gamma(7/2)\,\Gamma\frac{1}{2}}{2\times\Gamma(4)} = \frac{\Gamma(1+5/2)\,\Gamma\frac{1}{2}}{2\times(4-1)!}$$

$$= \frac{\frac{5}{2}\,\Gamma(5/2)\,\Gamma(\frac{1}{2})}{2\times 3!} = \frac{2.5\times\Gamma(1+3/2)\times\Gamma(\frac{1}{2})}{2\times 3\times 2}$$

$$= \frac{2.5\times\frac{3}{2}\,\Gamma(3/2)\times\Gamma(\frac{1}{2})}{2\times 6} = \frac{2.5\times 1.5\times\Gamma(1+\frac{1}{2})\Gamma(\frac{1}{2})}{2\times 6}$$

$$= \frac{2.5\times 1.5\times\frac{1}{2}\,\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})}{2\times 6}$$

$$= \frac{2.5\times 1.5\times 0.5}{2\times 6}\times\sqrt{\pi}\times\sqrt{\pi} = \frac{2.5\times 1.5\times 0.5\times\pi}{2\times 6} = \frac{3\pi}{6}$$

$$= \frac{5/2\times 3/2\times 1/2\,\pi}{12}$$

$$= \frac{5\times\cancel{3}\times\pi}{2\times 2\times 2\times\cancel{12}_4} = \frac{5\pi}{32}$$

(1) $\displaystyle\int_0^{\pi/2} \sin^4\theta \cos^5\theta \, d\theta = ?$

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

$2m-1 = 4 \Rightarrow m = 5/2$

$2n-1 = 5 \Rightarrow n = 6/2 = 3$

$\therefore \frac{1}{2}\left(B(m,n)\right) = \frac{1}{2} B(5/2, 3)$

$$= \frac{\Gamma(\frac{5}{2}) \times \Gamma(3)}{2 \times \Gamma(5/2+3)}$$

$$= \frac{\Gamma(1+3/2) \times (3-1)!}{2 \times \Gamma(\frac{11}{2})}$$

$$= \frac{\frac{3}{2}\Gamma(\frac{3}{2}) \times 2!}{\Gamma(1+\frac{9}{2})} = \frac{\frac{3}{2} \times \Gamma(1+\frac{1}{2}) \times 2}{\frac{9}{2}\Gamma(9/2)}$$

$$= \frac{\frac{3}{2} \times \frac{1}{2}\Gamma(\frac{1}{2}) \times 2}{\frac{9}{2} \times \Gamma(1+7/2)} = \frac{1.5 \times 0.5 \times 2 \quad \Gamma(\frac{1}{2})}{4.5 \times \frac{7}{2}\Gamma(\frac{7}{2})}$$

$$= \frac{1.5 \times 0.5 \times 2 \quad \Gamma(\frac{1}{2})}{4.5 \times 3.5 \quad \Gamma(1+5/2)} = \frac{1.5 \times \Gamma(\frac{1}{2})}{4.5 \times 3.5 \times 5/2\,\Gamma(5/2)}$$

$$= \frac{1.5 \times \Gamma(\frac{1}{2})}{4.5 \times 3.5 \times 2.5 \quad \Gamma(1+3/2)} = \frac{1.5 \times \sqrt{\pi}}{4.5 \times 3.5 \times 2.5 \times \frac{3}{2}\Gamma(3/2)}$$

$$\frac{1.5 \times \sqrt{\pi}}{4.5 \times 3.5 \times 2.5 \times 1.5 \times \Gamma(1+1/2)} = \frac{1.5 \times \sqrt{\pi}}{4.5 \times 3.5 \times 2.5 \times 1.5 \times \frac{1}{2}\Gamma\frac{1}{2}}$$

$$= \frac{1.5 \times \cancel{\sqrt{\pi}}}{4.5 \times 3.5 \times 2.5 \times 1.5 \times 0.5 \times \cancel{\sqrt{\pi}}} = \frac{2}{31.5}$$

Other special functions

Error function $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^3} du$

$$= 1 - \frac{2}{\pi} \int_x^\infty e^{-u^2} du$$

Exponential integral $Ei(x) = \int_x^\infty \frac{e^{-u}}{u} du$

Sine Integral $= Si(x) = \int_0^x \frac{\sin u}{u} du = \frac{\pi}{2} - \int_x^\infty \frac{\sin u}{u} du$

cosine integral $Ci(x) = \int_x^\infty \frac{\cos u}{u} du$

~~Fresns~~ fresnel sine integral $S(x) = \sqrt{\frac{2}{\pi}} \int_0^x \sin u^3 du$

$$= 1 - \sqrt{\frac{2}{\pi}} \int_0^x \sin u^3 du$$

~~Fresgnel~~

fresnel cosine integral $=$

$$C(x) = \sqrt{\frac{2}{\pi}} \int_0^x \cos^2 u\, du = 1 - \sqrt{\frac{2}{\pi}} \int_x^\infty \cos u^2 du$$

Laplace Transform

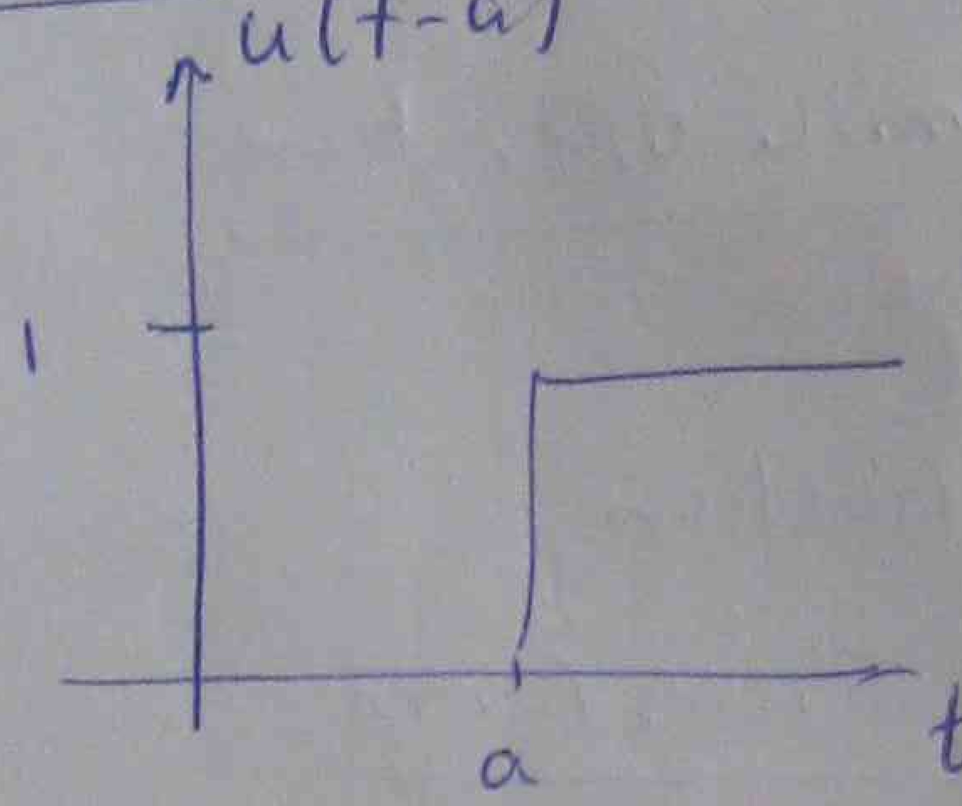
Numerical operation	Logarithmic operation
multiply	Addition
Divide	Substraction

Calculus	Laplace Transform
Integration	multiplication
differentiation	Division

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\alpha} e^{-st} f(t)\,dt$$

$f(t)$	$\mathcal{L}\, f(t) = F(s)$		$\mathcal{L}^{-1} F(s)$	$f(t)$
1	$\frac{1}{s}$	$s > 0$	$\mathcal{L}^{-1} \frac{1}{s}$	1
t^n $n = 1, 2, 3, 4 \ldots$	$\frac{n!}{s^{n+1}}$	$s > 0$	$\mathcal{L}^{-1} \frac{n!}{s^{n+1}}$	t^n
t^p $p > -1$	$\frac{p(p+1)}{s^{p+1}}$	$s > 0$	$\mathcal{L}^{-1} \frac{p(p+1)}{s^{p+1}}$	t^p
e^{at}	$\frac{1}{s-a}$	$s > a$	$\mathcal{L}^{-1} \frac{1}{s-a}$	e^{at}
$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$	$s > 0$	$\mathcal{L}^{-1} \frac{s}{s^2 + \omega^2}$	$\cos \omega t$
$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$	$s > 0$	$\mathcal{L}^{-1} \frac{\omega}{s^2 + \omega^2}$	$\sin \omega t$
$\cosh at$	$\frac{a}{s^2 - a^2}$		$\mathcal{L}^{-1} \frac{a}{s^2 - a^2}$	$\cosh at$

The unit step function



$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$$

$$\mathcal{L}^{-1}\, \frac{e^{-as}}{s} = u(t-a)$$

Pb (1) Prove that $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$ if $s > a$

$$\mathcal{L} f(t) = F(s) = \int_0^{\infty} e^{-st} f(t)\, dt$$

$$\mathcal{L}\, e^{at} = \int_0^{\infty} e^{-st} \cdot e^{at}\, dt = \int_0^{\infty} e^{-(s-a)t}\, dt$$

$$\boxed{\int e^u\, du = e^u \qquad \int e^u\, du = e^u}$$

$$d(-(s-a)t) = -(s-a)\, dt$$

$$\therefore dt = \frac{d(-(s-a)t)}{-(s-a)}$$

$$\int_0^{\infty} \frac{e^{-(s-a)t}\, d(-(s-a)t)}{-(s-a)}$$

$$\frac{e^{-(s-a)t}}{-(s-a)} \Bigg|_0^{\infty} = \cancel{\frac{e^{-(s-a)\infty}}{-(s-a)}}_{\to 0} - \left[\frac{e^{-(s-a)\times 0}}{-(s-a)}\right]$$

$$= +\frac{1}{s-a} \qquad s > a$$

No 2

Find the Laplace transforms of each of the followings.

(a) $3e^{\cancel{-4t}}$ (b) $2t^2$ (c) $4\cos 5t$ (d) $\sin \pi t$ (e) $-3/\sqrt{t}$

$\mathcal{L}\, e^{at} = \dfrac{1}{s-a}$

(a) $\mathcal{L}\, 3e^{-4t} = 3\,\mathcal{L}\, e^{-4t} = 3 \times \dfrac{1}{s-(-4)} = \dfrac{3}{s+4}$

$s > -4$

(b) $\mathcal{L}\, t^n = \dfrac{n!}{s^{n+1}}$

$\mathcal{L}\, 2t^2 = 2\,\mathcal{L}\, t^2 = \dfrac{2!}{s^{2+1}} = \dfrac{2\times 2\times 1}{s^3} = \dfrac{4}{s^3}$

(c) $\mathcal{L} \cos wt = \dfrac{s}{s^2+w^2}$ $\quad s > 0$

$\mathcal{L}\, 4\cos 5t = 4\,\mathcal{L} \cos 5t = 4 \times \dfrac{s}{s^2+5^2} = \dfrac{4s}{s^2+25}$

$s > 0$

(d) $\mathcal{L} \sin wt = \dfrac{w}{s^2+w^2}$

$\mathcal{L} \sin \pi t = \dfrac{\pi}{s^2+\pi^2}$ $\quad s > 0$

(e) $\cancel{\mathcal{L}\, e^{at} = \dfrac{1}{s-a}}$ $\quad \mathcal{L}\, t^n = \dfrac{n!}{s^{n+1}}$

$\mathcal{L}\, \dfrac{-3}{\sqrt{t}} = \mathcal{L}\, -3\, t^{-1/2} = \cancel{-3\,\mathcal{L}\, t^{-1/2}\ 3} \times$

$= \dfrac{-3 \times (-1/2)\,(-1/2+1)}{s^{-\frac{1}{2}+1}}$

$= \dfrac{3/2 \times 1/2}{s^{1/2}} = \dfrac{3}{4\sqrt{s}}$

Pb3 Find Laplace transform of $5\sin 2t - 3\cos 2t$

$$\mathcal{L}\sin\omega t = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}\sin 2t = \frac{2}{s^2+2^2} = \frac{2}{s^2+4}$$

$$\mathcal{L}\cos 2t = \frac{s}{s^2+2^2} = \frac{s}{s^2+4}$$

$$\mathcal{L}(5\sin 2t - 3\cos 2t) = 5\times\frac{2}{s^2+4} - 3\times\frac{s}{s^2+4} = \frac{10-3s}{s^2+4}$$

Pb4 Find $\mathcal{L}(\sin t\cos t)$

$$\mathcal{L}\sin\omega t = \frac{\omega}{s^2+\omega^2}$$

$\sin 2t = 2\sin t\cos t$

$\therefore \sin t\cos t = \frac{\sin 2t}{2}$

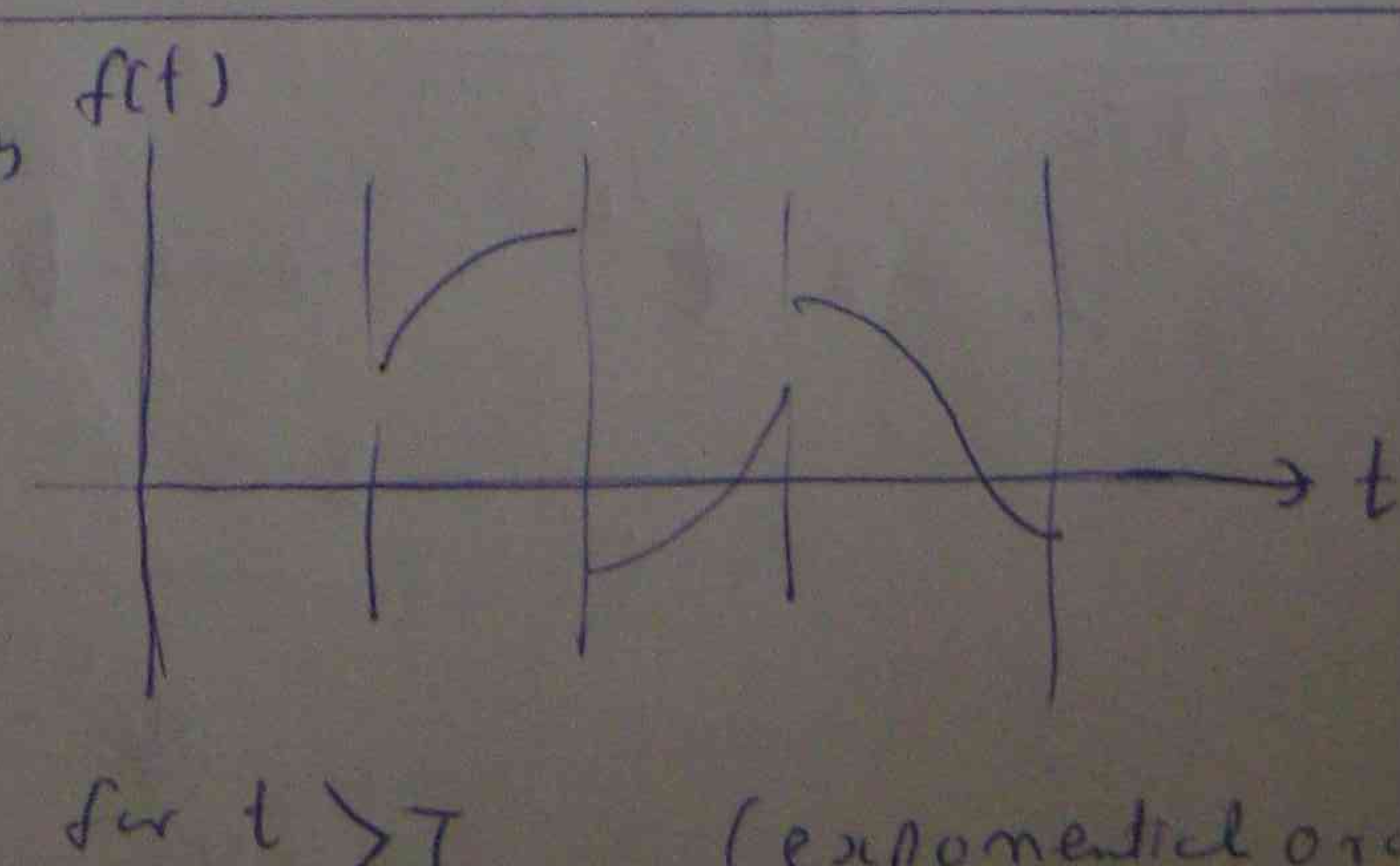
$$\mathcal{L}\sin t\cos t = \mathcal{L}\frac{\sin 2t}{2} = \frac{1}{2}\mathcal{L}\sin 2t = \frac{1}{2}\times\frac{2}{s^2+2^2}$$

$$= \frac{1}{2}\times\frac{2}{s^2+4}$$

$$= \frac{1}{s^2+4}$$

Sufficient conditions for existance of Laplace Transform

(1) Piecewise continuity



(2) $f(t) \leq m e^{\alpha t}$ for $t > T$ (exponential order)

Some special Theorems on Laplace Transform

$$\mathcal{L}\, F(t) = F(s)$$

$$\mathcal{L}\left\{ e^{at} f(t) \right\} = F(s-a)$$

$$\mathcal{L}\left\{ u(t-a)\, f(t-a) \right\} = e^{-as} F(s)$$

$$\mathcal{L}\, f(t) = F(s)$$

$$\mathcal{L}\, f(at) = \frac{1}{a} F\left(\frac{s}{a}\right)$$

$$\mathcal{L}\, t^n f(t) = (-1)^n \frac{d^n F}{ds^n} = (-1)^n F^n(s)$$

periodic function $\quad p > 0$

$$\mathcal{L}\, f(t) = \frac{\int_0^p e^{-st} f(t)\,dt}{1 - e^{-sp}}$$

Integration $\quad \mathcal{L} \int_0^t f(u)\,du = \frac{F(s)}{s}$

$$\mathcal{L}^{-1} F(s) = f(t) \quad \text{then} \quad \mathcal{L}^{-1} \frac{F(s)}{s} = \int_0^t f(u)\,du$$

$$\mathcal{L}\, \frac{f(t)}{t} = \int_0^{\infty} F(u)\,du$$

If $\mathcal{L}\{f(t)\} = F(s)$, $\mathcal{L}\{g(t)\} = G(s)$

Then $\mathcal{L}\left\{\int_0^t f(u)\, g(t-u)\, du\right\} = F(s)\, G(s)$

If $\mathcal{L}^{-1}\{F(s)\} = f(t)$, $\mathcal{L}^{-1}\{G(s)\} = g(t)$

Then $\mathcal{L}^{-1} F(s)\, G(s) = \int_0^t f(u)\, g(t-u)\, du$

Pb 5 Find (a) $\mathcal{L}^{-1} \frac{5}{s+2}$ (b) $\mathcal{L}^{-1} \frac{4s-3}{s^2+4}$

(c) $\mathcal{L}^{-1} \frac{2s-5}{s^2}$ ~~(d)~~ $\cancel{\mathcal{L}^{-1} \frac{1}{s^u}}$ $u>0$

(a) $\mathcal{L}^{-1} \frac{5}{s+2}$ To apply $\mathcal{L}^{-1} \frac{1}{s-a} = e^{at}$

modify $\mathcal{L}^{-1} \frac{1}{s+a} = e^{-at}$

$\therefore \mathcal{L}^{-1} \frac{5}{s+2} = 5\, \mathcal{L}^{-1} \frac{1}{s+2} = 5 \times e^{-2t}$

(b) $\mathcal{L}^{-1} \frac{4s-3}{s^2+4} = \mathcal{L}^{-1} \frac{4s}{s^2+4} - \mathcal{L}^{-1} \frac{3}{s^2+4}$

use $\mathcal{L}^{-1} \frac{s}{s^2+\omega^2} = \cos \omega t$, $\mathcal{L}^{-1} \frac{\omega}{s^2+\omega^2} = \sin \omega t$

$\mathcal{L}^{-1} \frac{4s}{s^2+4} = 4\, \mathcal{L}^{-1} \frac{s}{s^2+2^2} = 4 \cos 2t$

$\mathcal{L}^{-1} \frac{3}{s^2+4} = 3\, \mathcal{L}^{-1} \frac{1}{s^2+2^2} = \frac{3}{2}\, \mathcal{L}^{-1} \frac{2}{s^2+2^2} = \frac{3}{2} \sin 2t$

$\therefore \mathcal{L}^{-1} \frac{4s}{s^2+4} - \mathcal{L}^{-1} \frac{3}{s^2+4} = 4\cos 2t - \frac{3}{2} \sin 2t$

$$\boxed{\mathcal{L}^{-1}\frac{1}{s^{p}} = \frac{t^{p-1}}{\Gamma(p)}}$$

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma(n+1) = n\Gamma(n)$$

$$\Gamma\left(1+\tfrac{1}{2}\right) = 1\ \Gamma(1/2) = 1 \times \sqrt{\pi} = \sqrt{\pi}$$

Thus $\Gamma(3/2) = \sqrt{\pi}$

Ph 6 Find $\mathcal{L}^{-1}\dfrac{4-5s}{s^{3/2}}$

$$\mathcal{L}^{-1}\frac{4}{s^{3/2}} - \mathcal{L}^{-1}\frac{5s}{s^{3/2}}$$

$$4\mathcal{L}^{-1}\frac{1}{s^{3/2}} - 5\mathcal{L}^{-1}\frac{1}{s^{1/2}}$$

$$4 \times \frac{t^{3/2-1}}{\Gamma(3/2)} - 5\frac{t^{1/2-1}}{\Gamma(1/2)}$$

$$4\frac{t^{1/2}}{\sqrt{\pi}} - 5\frac{t^{-1/2}}{\sqrt{\pi}} = \frac{4t^{1/2} - 5t^{-1/2}}{\sqrt{\pi}}$$

Ph 7 Find $\mathcal{L}^{-1}\dfrac{1}{s^{k}}$, $k > 0$

To cover fraction $\mathcal{L}^{-1}\dfrac{1}{s^{k}} = \dfrac{t^{k-1}}{\Gamma(k)}$ #

t

(c) $\mathcal{L}^{-1} \frac{2s-5}{s^2} = \mathcal{L}^{-1} \frac{2s}{s^2} - \mathcal{L}^{-1} \frac{5}{s^2}$

$\boxed{\mathcal{L}^{-1} \frac{n!}{s^{n+1}} = t^n}$ $= \mathcal{L}^{-1} \frac{2}{s} - \mathcal{L}^{-1} \frac{5}{s^2}$

$0!$ $\quad s^{1+1}$

$s^1 = s^{0+1}$

$\mathcal{L} \frac{0!}{s^{0+1}} = t^1$ $\quad = 2 \mathcal{L}^{-1} \frac{0!}{s^{0+1}} - 5 \mathcal{L} \frac{1!}{s^{1+1}}$

$0! = 1$ $\quad = 2\, t^0 - 5\, t^1$

$= 2 - 5t$

Pb 6

(d) $\mathcal{L}^{-1} \frac{4-5s}{s^{3/2}}$ $\quad$ (h) $\mathcal{L}^{-1} \frac{1}{s^2+2s}$

Gamma Function

$$\Gamma(p+1) = \int_0^\infty x^p e^{-x} dx \qquad p > -1$$

$$\Gamma(p+1) = p\,\Gamma(p) \cdot \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi} \quad \Gamma(1) = 1$$

used for ~~integer power~~ / non integer power / fractional power

p

$\boxed{\mathcal{L}\, t^p = \frac{\Gamma(p+1)}{s^{p+1}}} \iff \boxed{\mathcal{L}^{-1} \frac{\Gamma(p+1)}{s^{p+1}} = t^p}$

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Pb 10 Find $\mathcal{L}^{-1} \dfrac{1}{s^2+2s}$

$$\mathcal{L}^{-1} \frac{1}{s^2+2s} = \mathcal{L}^{-1} \frac{1}{s(s+2)}$$

$$\frac{1}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2}$$

$$\frac{1}{s(s+2)} = \frac{A(s+2) + Bs}{(s)(s+2)}$$

$\therefore A(s+2) + Bs = 1$

If $s=0 \Rightarrow A(2) + B(0) = 1$

$\therefore 2A + 0 = 1 \Rightarrow 2A = 1 \rightarrow A = 1/2$

If $s=1 \Rightarrow A(1+2) + B(1) = 1$

$3A + B = 1$

$3 \times 1/2 + B = 1 \Rightarrow B = 1 - 3/2 = -1/2$

$$\therefore \frac{1}{s(s+2)} = \frac{1/2}{s} + \frac{(-1/2)}{s+2}$$

$$= \frac{1}{2} \times \frac{1}{s} - \frac{1}{2} \times \frac{1}{(s+2)}$$

$$\mathcal{L}^{-1} \frac{1}{s(s+2)} = \mathcal{L}^{-1} \frac{1}{2} \times \frac{1}{s} - \mathcal{L}^{-1} \frac{1}{2} \times \frac{1}{s+2}$$

$$= \frac{1}{2} \mathcal{L}^{-1} \frac{1}{s} - \frac{1}{2} \mathcal{L}^{-1} \frac{1}{s+2}$$

$$= \frac{1}{2} \times 1 - \frac{1}{2} e^{-2t} = \frac{1}{2}(1 - e^{-2t})$$

#

$\mathcal{L}^{-1} \dfrac{1}{s+a} = e^{-at}$

Laplace Tranform Derivatives

$$\mathcal{L}\, f'(t) = \int_0^{\infty} e^{-st} f'(t)\, dt$$

If $u(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$

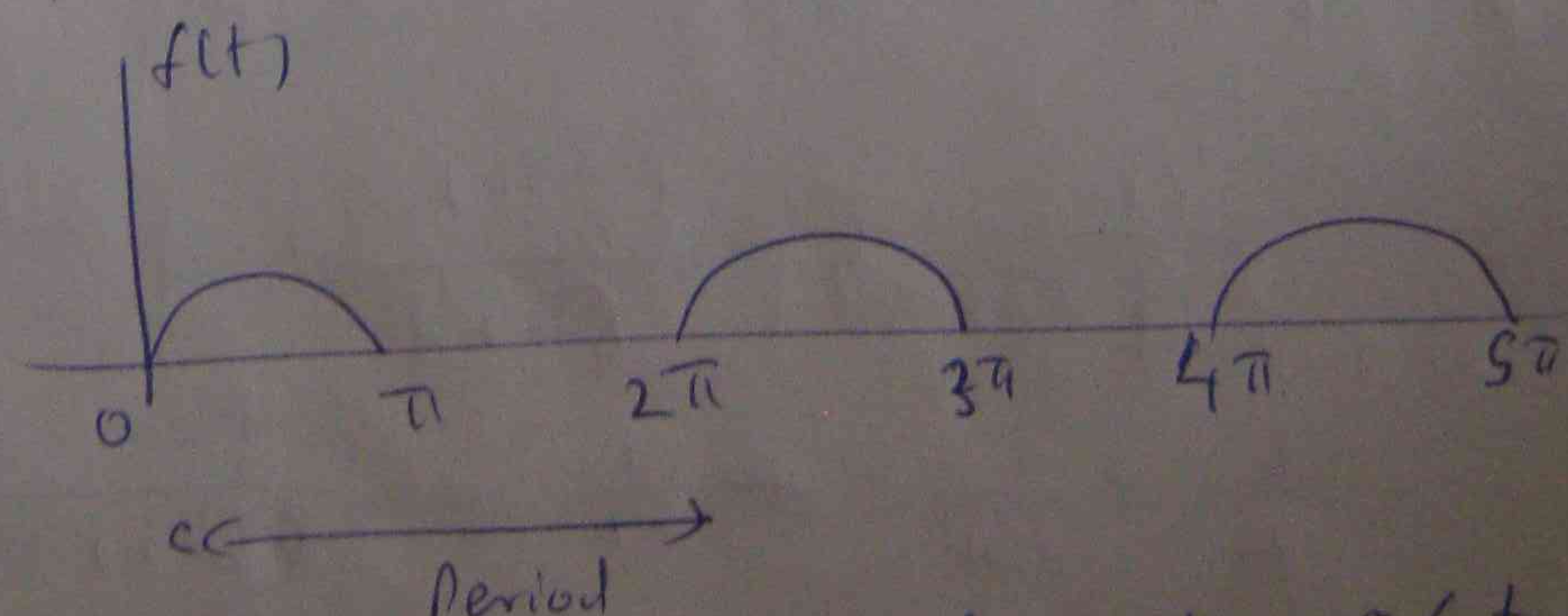
Then $\mathcal{L}\, u(t-a) = \dfrac{e^{-as}}{s}$

If ~~u(t-a)~~ $f(t) = \begin{cases} \sin t & t < \pi \\ t & t > \pi \end{cases}$

Then $\mathcal{L}\, f(t) = \mathcal{L} \sin t + e^{-\pi s}\left[\dfrac{\pi}{s} + \dfrac{1}{s^2} + \dfrac{1}{s^2+1}\right]$

$$= \frac{1}{s^2+1} + e^{-\pi s}\left[\frac{\pi}{s} + \frac{1}{s^2} + \frac{1}{s^2+1}\right]$$

Pb 4 Find the function of following graph



$0 \leq t \leq 2\pi \longrightarrow f(t) = \begin{cases} \sin t & 0 \leq t \leq \pi \\ 0 & \pi \leq t \leq 2\pi \end{cases}$

$f(t)$ has Period $p > 0$ $\quad f(t+p) = f(t)$

Then $\mathcal{L}\, f(t) = \int^{p} e^{-st} f(t)\, dt$

$$\mathcal{L}(f(t)) = \frac{\int_0^{2\pi} e^{-st} \sin t \, dt}{1 - e^{-s \times p}} \qquad \text{where } p = 2\pi$$

$$= \frac{\int_0^{\pi} e^{-st} \sin t \, dt}{e^{-s \times 2\pi}} + \cancel{\frac{\int_{\pi}^{2\pi} e^{-st} \times 0 \, dt}{e^{-s \times 2\pi}}} \to 0$$

$$= \frac{\int_0^{\pi} e^{-st} \sin t \, dt}{1 - e^{-2\pi s}}$$

$$\int_0^{\pi} e^{-st} \sin t \, dt = \int_0^{\pi} u$$

$$\mathcal{L} \sin t \, \cancel{dt} = \frac{1}{s^2 + 1^2}$$

$$\mathcal{L} \sin 1t \, \cancel{dt} = \frac{1}{s^2 + 1}$$

$$\mathcal{L} \, e^{+st} \sin \omega t = \frac{(1 + e^{+\pi \times s})\,\omega}{\cancel{s^2 + \omega}\; s^2 + \omega^2}$$

$$\therefore \cancel{\mathcal{L} \, e^{-st} \sin t \, dt} = \int_0^{\pi} e^{-st} \sin t \, dt = \mathcal{L} \, e^{-st} \sin t = \frac{1 + e^{\pi(-s)}}{} \times \frac{1}{s^2 + 1^2}$$

$$= \frac{1 + e^{-\pi s}}{s^2 + 1}$$

$$\therefore = \frac{\frac{1 + e^{-\pi s}}{s^2 + 1}}{1 - e^{-2\pi s}} = \frac{1 + e^{-\pi s}}{(1 - e^{-2\pi s})(s^2 + 1)}$$

$$= \frac{1 + e^{-\pi s}}{(1^2 - (e^{-\pi s})^2)(s^2 + 1)} = \frac{\cancel{1 + e^{-\pi s}}}{\cancel{(1 + e^{-\pi s})}(1 - e^{-\pi s})(s^2 + 1)}$$

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$$= \frac{1}{(s^2+1)(1-e^{-\pi s})}$$

Solution to differential equations

Pb 12

Solve $y''(t) + y(t) = 1$ given $y(0) = 1$, $y'(0) = 0$

$y''(t) + y(t) = 1$

$\mathcal{L}\, y''(t) + \mathcal{L}\, y(t) = \mathcal{L}\, 1$

$\mathcal{L}\, 1 = \frac{1}{s}$

s^2

$\mathcal{L}\, f^{(n)}(t) = s^n \mathcal{L}\, f(t) - s^{n-1} \mathcal{L}\, f(0)$

$= - s^{n-2} \mathcal{L}\, f'(0)$

$n=2$

$\mathcal{L}\, y''(t) = s^2 \underline{\mathcal{L}\, y(t)} - s^{2-1} \mathcal{L}\, y(0) - s^{2-2} \mathcal{L}\, f'(0)$

$= s^2 Y - s \times y(0) - s^0 \times y'(0)$

$= s^2 Y - s \times 1 - 1 \times 0$

$= s^2 Y - s$

$\mathcal{L}\, y(t) = Y$

$\therefore \mathcal{L}(y''(t) + \mathcal{L}\, y(t) = \mathcal{L}\, 1$

$s^2 Y - s + Y = \frac{1}{s}$

$s^2 Y + Y = s + \frac{1}{s}$

$(s^2+1) Y = \frac{s^2+1}{s}$

$Y = \frac{1}{s}$

$y = \mathcal{L}^{-1} Y = \mathcal{L}^{-1} \frac{1}{s}$

$= 1$

#

$$Y = \frac{2s^2 - 5s - 5}{(s^2 - 3s + 2)(s+1)}$$

(14) $s^2 - 3s + 2 = \begin{matrix} s & -2 \\ s & -1 \end{matrix}$

$= (s-2)(s-1)$

$$= \frac{2s^2 - 5s - 5}{(s-2)(s-1)(s+1)}$$

$$\frac{2s^2 - 5s - 5}{(s-2)(s-1)(s+1)} = \frac{A}{(s-2)} + \frac{B}{(s-1)} + \frac{C}{(s+1)}$$

$$\frac{2s^2 - 5s - 5}{(s-2)(s-1)(s+1)} = \frac{A(s-1)(s+1) + B(s-2)(s+1) + C(s-2)(s-1)}{(s-2)(s-1)(s+1)}$$

$$2s^2 - 5s - 5 = A(s-1)(s+1) + B(s-2)(s+1) + C(s-2)(s-1)$$

$s = 1$

$$2(1)^2 - 5\times 1 - 5 = \cancel{A(1-1)(1+1)}_{0} + B(1-2)(1+1) + \cancel{C(1-2)(1-1)}_{0}$$

$$2 - 5 - 5 = B(-1)(2)$$

$$-8 = -2B \qquad \therefore B = 4$$

$s = -1$

$$2(-1)^2 - 5(-1) - 5 = \cancel{A(-1-1)(-1+1)}_{0} + \cancel{B(-1-2)(-1+1)}_{0}$$

$$+ C(-1-2)(-1-1)$$

$$2 + 5 - 5 = C(-3)(-2)$$

$$2 = C(6) \qquad \therefore C = 1/3$$

Pb 13 Solve $y'' - 3y' + 2y = 2e^{-t}$

where $y(0) = 2$, $y'(0) = -1$

$$\mathcal{L}(y'' - 3y' + 2y) = \mathcal{L}\,2e^{-t}$$

$$\boxed{\begin{aligned} \mathcal{L}y'' &= s^2Y - sy(0) - y'(0) \\ \mathcal{L}y' &= sY - y(0) \\ \mathcal{L}y &= Y \end{aligned}}$$

$\mathcal{L}\,2e^{-t} = 2\mathcal{L}e^{-t} = \frac{2}{s+1}$

$\mathcal{L}e^{at} = \frac{1}{s-a}$ | $\mathcal{L}e^{-t} = \frac{1}{s+1}$

$\mathcal{L}e^{-at} = \frac{1}{s+a}$

$$\left[s^2Y - sy(0) - y'(0)\right] - 3\left[sY - y(0)\right] + 2Y = \frac{2}{s+1}$$

$$\left[s^2Y - s \times 2 - (-1)\right] - 3\left[sY - 2\right] + 2Y = \frac{2}{s+1}$$

$$s^2Y - 2s + 1 - 3sY + 6 + 2Y = \frac{2}{s+1}$$

$$s^2Y - 2s - 3sY + 2Y + 7 = \frac{2}{s+1}$$

$$s^2Y - 3sY + 2Y - 2s + 7 = \frac{2}{s+1}$$

$$Y(s^2 - 3s + 2) = \frac{2}{s+1} + 2s - 7$$

$$Y(s^2 - 3s + 2) = \frac{2 + (2s-7)(s+1)}{(s+1)}$$

$$Y(s^2 - 3s + 2) = \frac{2 + 2s^2 - 7s + 2s - 7}{(s+1)}$$

$s = 2$

$$2(2)^2 - 5(2) - 5 = A(2-1)(2+1) + B\cancel{(2-2)}(2+1) + C\cancel{(2-2)}(2-1)$$

$$2 \times 4 - 10 - 5 = A \times 3$$

$$8 - 15 = 3A$$

$$3A = -7 \longrightarrow A = -7/3$$

$$\therefore \mathcal{L}^{-1} \frac{2s^2 - 5s - 5}{(s-2)(s-1)(s+1)} = \mathcal{L}^{-1} \frac{-7/3}{s-2} + \mathcal{L}^{-1} \frac{4}{s-1} + \mathcal{L}^{-1} \frac{1/3}{s+1}$$

$$= -7/3\, e^{2t} + 4e^{t} + \frac{1}{3} e^{-t}$$

$$= \frac{1}{3} e^{-t} + 4e^{t} - 7/3\, e^{2t}$$

Solving Electrical problems

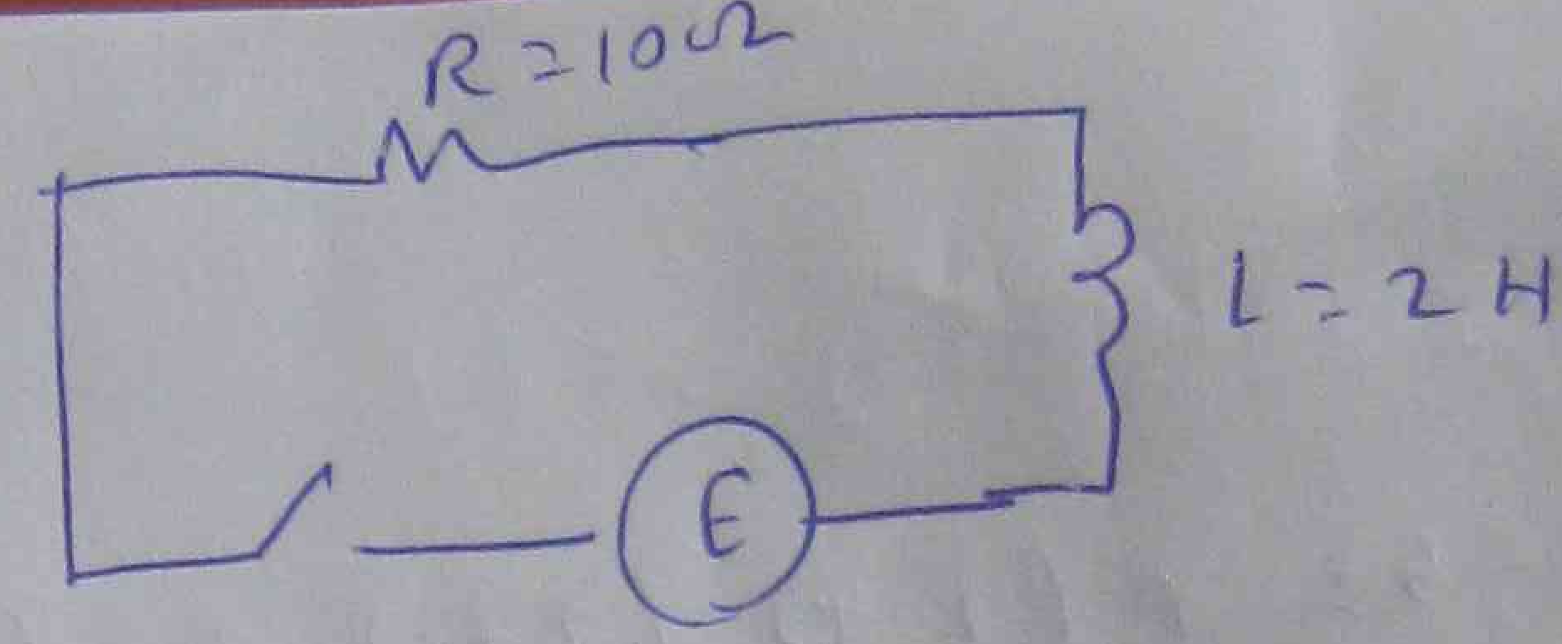
Pb 14 A resistor of $R = 10\Omega$, an inductor of $L = 2\,H$ and a battery of E volts are connected in series with a switch "S". At $t = 0$ the switch is closed and the current $I = 0$

Find I for $t > 0$ if (a) $E = 40V$

(b) $E = 20e^{-3t}$

(c) $E = 50 \sin 5t$

$I(0) = 0$



$I(0)=0$

$E=40\,V$

$$\frac{dI}{dt}+5I=\frac{E}{2}$$

$$I'+5I=\frac{E}{2}$$

$$\mathcal{L}I'+\mathcal{L}5I=\mathcal{L}\frac{E}{2}$$

$$\mathcal{L}I'+\mathcal{L}5I=\mathcal{L}\frac{40}{2}=\mathcal{L}20$$

$$sI(s)-I(0)+5I(s)=\frac{20}{s}$$

$$I(0)=0$$

$$s\,I(s)+5I(s)=\frac{20}{s}$$

$$I(s)\,(s+5)=\frac{20}{s}$$

$$I(s)=\frac{20}{(s+5)\,s}$$

$$\frac{20}{(s+5)\,s}=\frac{A}{s+5}+\frac{B}{s}$$

$$\frac{20}{(s+5)\,s}=\frac{As+B(s+5)}{(s+5)\,s}$$

$$\therefore As+B(s+5)=20$$

$s=0 \qquad B\times 5=20 \longrightarrow B=4$

$s=-5 \longrightarrow -5A+B(-5+5)=20 \quad \therefore A=-4$

$$\frac{20}{(s+5)(s)} = \frac{-4}{s+5} + \frac{4}{s}$$

$$\mathcal{L}^{-1}\frac{20}{(s+5)(s)} = \mathcal{L}^{-1}\frac{-4}{s+5} + \mathcal{L}^{-1}\frac{4}{s}$$

$$= -4e^{-5t} + 4$$

$$I = 4(1 - e^{-5t})$$

(b) If $E = 20e^{-3t}$

$$I(s)\,(s+5) = \mathcal{L}\,\frac{20e^{-3t}}{2} = 10 \times \frac{1}{s+3}$$

$$I(s) = \frac{10}{(s+5)\,(s+3)}$$

$$\frac{20}{(s+5)\,(s+3)} = \frac{A}{s+5} + \frac{B}{s+3}$$

$$20 = A(s+3) + B(s+5)$$

$s = -3$ $\quad 20 = A(-3+3) + B(-3+5)$

$$\therefore 2B = 10 \longrightarrow B = \cancel{10}\,5$$

$s = -5$ $\quad 20 = A(-5+3) + B(-5+5)$

$$-2A = 10 \longrightarrow \cancel{A = 10}$$

$$A = -5$$

$$= \frac{-\cancel{10}\,5}{s+5} + \frac{5}{s+3} \Rightarrow \mathcal{L}^{-1} \Rightarrow -10(e^{-5t} + e^{-3t})$$

Ans

$(s)\,I(s+5) = \mathcal{L}\,\frac{50}{2} \ldots$

$= \frac{50}{2}\,\mathcal{L}\,\sin 5t$

$= \frac{50}{2} \times \frac{5}{s^2+5^2}$

$I(s)\,(s+5) = \frac{125}{s^2+25}$

$I(s) = \frac{\cancel{250}\;125}{(s^2+25)(s+5)}$

$\frac{125}{(s^2+25)(s+5)} = \frac{As+B}{s^2+25} + \frac{C}{s+5}$

$125 = (As+B)(s+5) + C(s^2+25)$

$s = 0$

$125 = (A \times 0 + B)(0+5) + C(0+25)$

$125 = 5B + 25C \longrightarrow (1)$

$s = -5$

$125 = (A(-5)+B)\cancel{(-5+5)} + C((-5)^2+25)$

$125 = 50C \longrightarrow C = \frac{125}{50} = \frac{5}{2}$

$\therefore 125 = 5B + 25 \times \frac{5}{2}$

$125 = 5B + \frac{125}{2}$

$5B = \cancel{+55}\ 125 - \frac{125}{2} = \frac{125}{2}$

$B = \frac{125}{10} = \frac{25}{2}$

$\underline{\underline{s = 1}}$

$125 = (A + B)(1+5) + C(1+25)$

$125 = (A + \frac{25}{2})\cancel{(1+1}\times 6 + \frac{5}{2} \times 26$

$125 = (A + \frac{25}{2}) \times 6 + 65$

$(A + \frac{25}{2}) \times 6 = 60$

$A + \frac{25}{2} = 10$

$A = 10 - \frac{25}{2} = -5/2$

$\cancel{\frac{20}{(s+5)(s)}} = \cancel{-5/2\, s} +$

$$\frac{125}{(s^2+25)(s+5)} = \frac{-5/2\, s + \frac{25}{2}}{s^2+25} + \frac{5/2}{s+5}$$

$$= \frac{-5/2\, s}{s^2+25} + \frac{25/2}{s^2+25} + \frac{5/2}{s+5}$$

Legendre functions and other orthogonal functions (1)

$(1-x^2)y'' - 2xy' + n(n+1)y = 0$ ← Legendre's differential equation

General solution

$$y = c_1 P_n(x) + c_2 Q_n(x)$$

$$y = c_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 - \cdots\right] + c_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 \cdots\right]$$

Recurrence formula

(1) $P_{n+1}(x) = \dfrac{2n+1}{n+1}\, x\, P_n(x) - \dfrac{n}{n+1} P_{n-1}(x)$

(2) $P'_{n+1}(x) - P'_{n-1}(x) = (2n+1) P_n(x)$

Pb Given that $P_0(x) = 1$, $P_1(x) = x$ find (a) $P_2(x)$ and (b) $P_3(x)$

$$P_{(n+1)}(x) = \frac{2n+1}{n+1}\, x\, P_n(x) - \frac{n}{n+1} P_{n-1}(x)$$

For $P_2(x)$, substitute $n = 1$

$$\therefore P_{(1+1)}(x) = \frac{2\times 1+1}{1+1}\, x\, P_1(x) - \frac{1}{1+1} P_{1-1}(x)$$

$$P_2(x) = \frac{3}{2} x \times x - \frac{1}{2} P_0(x)$$

$$= \frac{3}{2}x^2 - \frac{1}{2} = \frac{1}{2}(3x^2 - 1)$$

For $P_3(x)$ substitute $n=2$

$$P_{(2+1)}(x) = \frac{2\times 2+1}{2+1} x P_2(x) - \frac{2}{2+1} P_{2-1}(x)$$

$$P_3(x) = \frac{5}{3} \times x \times \left\{\frac{1}{2}(3x^2-1)\right\} - \frac{2}{3} P_1(x)$$

$$= \frac{5}{3} x \times \frac{(3x^2-1)}{2} - \frac{2}{3} \times x$$

$$= \frac{5}{6}(3x^2-1)x - \frac{2}{3}x$$

$$= \frac{2x}{3}\left[\frac{5}{2}(3x^2-1) - 1\right]$$

$$= \frac{2x}{3}\left[\frac{5(3x^2-1) - 2}{2}\right]$$

$$= \frac{2x}{3}\left[\frac{15x^2-5-2}{2}\right]$$

$$\frac{2x}{3}\left[\frac{15x^2-7}{2}\right]$$

\#

Biomomial theorem

$$(P+v)^n = 1 + Pv + \frac{P(P-1)}{2!}v^2 + \frac{P(P-1)(P-2)}{3!}v^3$$

Generative function

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n$$

$$\left(\frac{1}{\sqrt{1-2xt+t^2}}\right)^2 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} P_m(x) P_n(x) t^{m+n}$$

Pb Prove that $\int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1}$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left\{\int_{-1}^{1} P_m(x) P_n(x)\,dx\right\} t^{m}$$

$$= \sum_{n=0}^{\infty}\left\{\int_{-1}^{1} P_n^2(x)\,dx\right\} t^{2}$$

$$\frac{1}{\sqrt{1-2tx+t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n$$

Squaring both sides

$$\frac{1}{1-2tx+t^2} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} P_m(x)\, P_n(x)\, t^{m+n}$$

By integrating -1 to 1

$$\int_{-1}^{1} \frac{1}{1-2tx+t^2}\,dx = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \left\{\int_{-1}^{1} P_m(x)\, P_n(x)\,dx\right\} t^{m+n}$$

$$d(1-2tx+t^2) = \underset{0}{\cancel{d1}} - d\,2tx + \underset{0}{\cancel{dt^2}} =$$

$$= -2t\,dx$$

$$\therefore dx = -\frac{1}{2t}\,d(1-2tx+t^2)$$

$$\int_{-1}^{1} -\frac{1}{2t}\,\frac{d(1-2tx+t^2)}{1-2tx+t^2} = \sum_{n=0}^{\infty}\left\{\int_{-1}^{1} P_n^2(x)\,dx\right\} t^{2n}$$

$$-\frac{1}{2t}\ln(1-2tx+t^2)\Big|_{-1}^{1} = \sum_{n=0}^{\infty}\left\{\int_{-1}^{1} P_n^2(x)\,dx\right\} t^{2n}$$

LHS

$$-\left[\frac{1}{2\times t}\ln(1-2\times t+t^2)\right] + \left[\frac{1}{2\times(t)}\ln(1-2\times t(-1)+(t)^2)\right]$$

$$-\frac{1}{2t}\ln(1-t)^2 + \frac{1}{2t}\ln(1+t)^2$$

$$-\frac{1}{2t}\left\{\cancel{\ln 2(1-t)}\right\} + \frac{1}{2t}\left\{\cancel{\ln 2(1+t)}\right\} \quad -\frac{1}{2t}\times 2\ln(1-t) + \frac{1}{2t}\times 2\ln(1+t)$$

$$-\frac{1}{2t}\ln 2\left\{\cancel{\ln(1-t)} + \ln(1+t)\right\} \quad \frac{1}{t}\ln\left(\frac{1+t}{1-t}\right)$$

(4)

$$+\frac{1}{2t}\cancel{\ln 2\left\{\ln(1+t)-\ln(1-t)\right\}}$$

$$\cancel{\frac{1}{}\ \ln\left(\frac{1+t}{1-t}\right)}$$

$$\frac{1}{t}\ln\left(\frac{1+t}{1-t}\right)=\sum_{n=0}^{\infty}\left\{\int_{-1}^{1}P_n^2(x)\,dx\right\}t^{2n}$$

$$\boxed{\frac{1}{t}\ln\left(\frac{1+t}{1-t}\right)=\sum_{n=0}^{\infty}\frac{2t^{2n}}{2n+1}}$$

$$\therefore \sum_{n=0}^{\infty}\frac{2t^{2n}}{2n+1}=\sum_{n=0}^{\infty}\left\{\int_{-1}^{1}P_n^2(x)\,dx\right\}t^{2n}$$

$$\therefore \int_{-1}^{1}P_n^2(x)\,dx=\frac{2}{2n+1}$$

#

Series of Legendre polynomials

If $f(x) = \sum_{k=0}^{\infty} A_k P_k(x) , \quad -1 < x < 1$

Then $A_k = \frac{2k+1}{2} \int_{-1}^{1} P_k(x) \, f(x) \, dx$

$\boxed{\text{Rodrigue's formula}}$ $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$

Pb Expand the function

$$f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & -1 < x < 0 \end{cases}$$

in a series of the form $\sum_{k=0}^{\infty} A_k P_k(x)$

$$A_k = \frac{2k+1}{2} \int_{-1}^{1} P_k(x) \, f(x) \, dx$$

$$= \frac{2k+1}{2} \int_{-1}^{0} P_k(x) \, 0 \, dx + \frac{2k+1}{2} \int_{0}^{1} P_k(x) \, (1) \, dx$$

$$A_k = \frac{2k+1}{2} \int_{0}^{1} P_k(x) \, dx$$

$k=0 \rightarrow \quad A_0 = \frac{2\times 0+1}{2} \int_0^1 P_0(x) \, dx \quad \frac{1}{2} \int_0^1 1 \, dx = \frac{1}{2} [x]_0^1$

$$= \frac{1}{2}(1-0) = \frac{1}{2}$$

$k=1 \quad A_1 = \frac{2\times 1+1}{2} \int_0^1 P_1(x) \, dx$

Rodrigue's formula

$$P_m(x) = \frac{1}{2^m m!} \frac{d^n}{dx^n} (x^2-1)^n$$

$$P_0(x) = \frac{1}{2^0 \times 0!} \frac{d^0}{dx^0} (x^2-1)^0$$

$= 1$

(6)

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$

$$P_1(x) = \frac{1}{2^1 \times 1!} \frac{d}{dx} (x^2-1) = \frac{1}{2} \times 2x$$

$$= x$$

$$\therefore A_1 = \frac{3}{2} \int_0^1 x\,dx = \frac{3}{2} \left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{3}{2} \left[\frac{1-0}{2}\right] = \frac{3}{4}$$

$$P_2(x) = \frac{1}{2^2\, 2!} \frac{d^2}{dx^2} (x^2-1)^2$$

$$= \frac{1}{4 \times 2} \frac{d^2}{dx^2} (x^2-1)^2$$

$$\frac{d}{dx} (x^2-1)^2$$

$$= 2(x^2-1) \frac{d}{dx} x^2$$

$$\cancel{\frac{d^2}{dx^2} = 2 \times 2}$$

$$= 2(x^2-1) \times 2x$$

$$= 4(x^2-1)x$$

$$\frac{d^2}{dx^2}(\ ) = \frac{d}{dx}\, 4x^3 - 4x$$

$$= 4 \times 3x^2 - 4 = 4(3x^2-1)$$

$$= \frac{1}{4 \times 2}\, 4(3x^2-1) = \frac{3x^2-1}{2}$$

$$\therefore A_2(x) = \frac{2 \times 2 + 1}{2} \int_0^1 P_2(x)\,dx$$

$$= \frac{5}{2} \times \int_0^1 \frac{3x^2-1}{2}\,dx$$

$$= \frac{5}{2} \left[\frac{3}{2} \times \frac{x^3}{3} \Big|_0^1 - \left[\frac{x}{2}\right]_0^1 \right.$$

$$\frac{5}{2} \left[\frac{3}{2} \times \frac{1}{3} - \frac{1}{2}\right]$$

$$= \frac{5}{2} \times \left[\frac{1}{2} - \frac{1}{2}\right]$$

$$\frac{5}{2} \times 0 = 0$$

$$A_3 = \frac{2\times 3+1}{2} \int_0^1 P_3(x)\,dx$$

$$= \frac{7}{2} \int_0^1 P_3(x)\,dx$$

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$

$$P_3(x) = \frac{1}{2^3 \times 3!} \frac{d^3}{dx^3} (x^2-1)^3$$

$$= \frac{1}{8\times 3\times 2} \frac{d^3}{dx^3} (x^2-1)^3$$

$$= \frac{1}{48} \frac{d^3}{dx^3} (x^2-1)^3$$

$$\frac{d^3}{dx^3}(x^2-1)^3 = ? \qquad \frac{d}{dx}(x^2-1)^3 = 3(x^2-1)^{3-1} \frac{d}{dx}(x^2-1)$$

$$= 3(x^2-1)^2 \times 2x$$

$$= 6x(x^2-1)^2$$

$$= 6x(x^4 - 2x^2 + 1) = 6x^5 - 12x^3 + 6x$$

$$\frac{d^2}{dx^2}(x^2-1)^3 = \frac{d}{dx}(6x^5 - 12x^3 + 6x)$$

$$= 6\times 5\,x^4 - 12\times 3\,x^2 + 6$$

$$= 30x^4 - 36x^2 + 6$$

$$\frac{d^3}{dx^3}(x^2-1)^3 = 30\times 4\,x^3 - 36\times 2\,x + 0$$

$$= 120x^3 - 72x$$

(8)

$$A_3 = \frac{7}{2}\int_0^1 P_3(x)\,dx = \cancel{\frac{7}{2}}$$

$$P_3(x)\,dx = \frac{1}{48}\frac{d^3}{dx^3}(x^2-1)^3 = \frac{1}{48}\left[120x^3 - 72x\right]$$

$$= \frac{5x^3 - 3x}{2}$$

$$A_3 = \frac{7}{2} \times \int_0^1 \frac{5x^3 - 3x}{2}\,dx = \frac{7}{2}\left[\frac{5}{2}\left(\frac{x^{3+1}}{3+1}\right)_0^1 - \frac{3}{2}\left(\frac{x^2}{2}\right)_0^1\right]$$

$$= \frac{7}{2}\left[\frac{5}{2}\left[\frac{x^3}{\cancel{3}4}\right]_0^1 - \frac{3}{2}\left[\frac{x^2}{2}\right]_0^1\right]$$

$$= \frac{7}{2}\left[\frac{5}{2} \times \frac{1}{4} - \frac{3}{2} \times \frac{1}{2}\right]$$

$$= \frac{7}{2}\left[\frac{5}{8} - \frac{3}{4}\right] = \frac{7}{2}\left[\frac{\cancel{10}5 - 6}{8}\right]$$

$$= \frac{7}{2} \times \frac{-1}{8} = -\frac{7}{16}$$

$$f(x) = A_0 P_0(x) + A_1 P_1(x) + A_2 P_2(x) + A_3 (P_3(x)$$

$$= \frac{1}{2} P_0(x) + \frac{3}{4} P_1(x) + 0\,P_2(x) + -\frac{7}{16} P_3(x)$$

$$= \frac{1}{2} P_0(x) + \frac{3}{4} P_1(x) - \frac{7}{16} P_3(x)$$

Partial differential equation

pb If $v = F(y-3x)$

Prove $\frac{\partial v}{\partial x} + 3\frac{\partial v}{\partial y} = 0$

$v = F(y-3x)$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x} F(y-3x) = f'(y-3x)\frac{\partial}{\partial x}(y-3x)$$

$$= f'(y-3x)\left[\cancel{\frac{\partial}{\partial x} y}_{\to 0} - \frac{\partial}{\partial x}3x\right]$$

$$= -f'(y-3x) \times 3$$

$$= -3f'(y-3x) \quad \text{--- (i)}$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y} f(y-3x) = F'(y-3x)\frac{\partial}{\partial y} \times (y-3x)$$

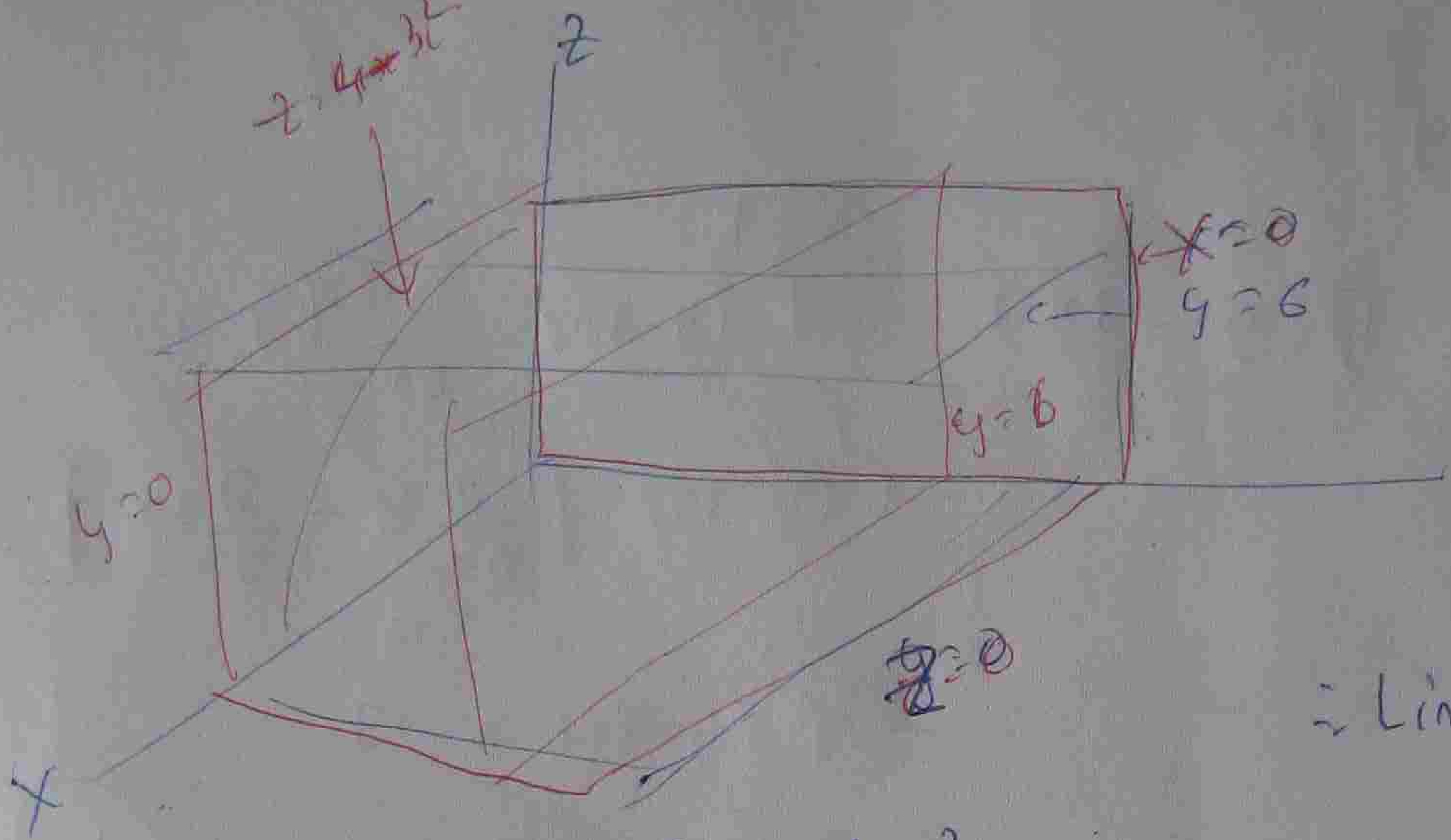
$$= f'(y-3x)\left[\frac{\partial y}{\partial y} - \cancel{\frac{\partial 3x}{\partial y}}_{\to 0}\right]$$

$$= f'(y-3x) \quad \text{--- (ii)}$$

$$\frac{\partial v}{\partial x} + 3\frac{\partial v}{\partial y} = -3f'(y-3x) + 3f'(y-3x)$$

$$= 0$$

Pb Find the volume of the region R bounded by the parabolic cylinder $z = 4 - x^2$ and the planes $x = 0$ $y = 0$ $y = 6$ $z = 0$



when

$z = 0$

$z = 4 - x^2$

$0 = 4 - x^2$

$\therefore x^2 = 4$

$x = \pm 2$

$y = 0 \to 6$

$\therefore$ Limit $z = 0 \longrightarrow 4 - x^2$

$y = 0 \longrightarrow 6$

$x = 0 \longrightarrow 2$

$$\int_{x=0}^{2} \int_{y=0}^{6} \int_{z=0}^{4-x^2} f(z)\, dz\, dy\, dx$$

$$\int_{x=0}^{2} \int_{y=0}^{6} \int_{z=0}^{4-x^2} \underline{\underline{f(z)\, dz}}\, dy\, dx$$

$z = 4 - x^2$

$$\int_{x=0}^{2} \int_{y=0}^{6} (4 - x^2)\, dy\, dx$$

$$\int_{x=0}^{2} \Big[(4 - x^2) y \Big]_{y=0}^{6} dx$$

$$\int_{x=0}^{2} (4 - x^2) 6\, dx$$

$$6 \int_0^2 (4 - x^2)\, dx$$

$$6 \times \left\{ 4[x]_0^2 - \left[\frac{x^3}{3}\right]_0^2 \right\}$$

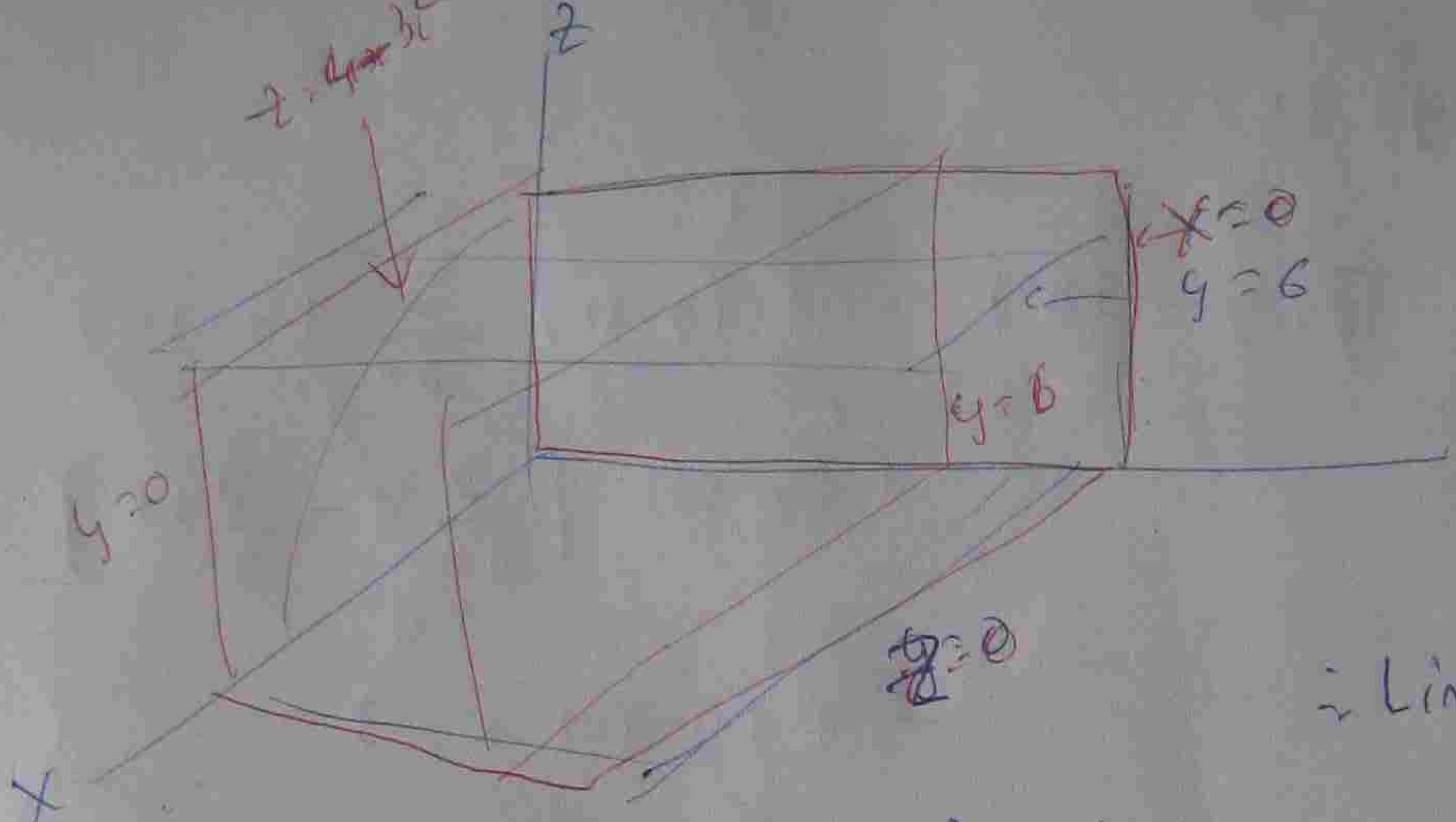
$$6 \left\{ 4 \times 2 - \frac{2^3}{3} \right\}$$

$$6 \left\{ 8 - \frac{8}{3} \right\} = 6 \times \frac{16}{3}$$

$$= 32$$

Multiple Integration

Pb Find the volume of the region R bounded by the parabolic cylinder $z = 4 - x^2$ and the planes $x = 0$, $y = 0$, $y = 6$, $z = 0$



when $z = 0$

$z = 4 - x^2$

$0 = 4 - x^2$

$\therefore x^2 = 4$

$x = \pm 2$

$y = 0 \to 6$

$\therefore$ Limit $z = 0 \longrightarrow 4 - x^2$

$y = 0 \longrightarrow 6$

$x = 0 \longrightarrow 2$

$$\int_{x=0}^{2} \int_{y=0}^{6} \int_{z=0}^{4-x^2} f(z)\, dz\, dy\, dx$$

$$\int_{x=0}^{2} \int_{y=0}^{6} \int_{z=0}^{4-x^2} \underline{f(z)\, dz}\, dy\, dx$$

$z = 4 - x^2$

$$\int_{x=0}^{2} \int_{y=0}^{6} (4 - x^2)\, dy\, dx$$

$$\int_{x=0}^{2} \Big[(4 - x^2) y \Big]_{y=0}^{6} dx$$

$$\int_{x=0}^{2} (4 - x^2) 6\, dx$$

$$6 \int_0^2 (4 - x^2)\, dx$$

$$6 \times \left\{ 4[x]_0^2 - \left[\frac{x^3}{3}\right]_0^2 \right\}$$

$$6 \left\{ 4 \times 2 - \frac{2^3}{3} \right\}$$

$$6 \left\{ 8 - \frac{8}{3} \right\} = 6 \times \frac{16}{3} = 32$$

Matrices

ph①

$$A = \begin{bmatrix} 2 & 1 & 4 \\ -3 & 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & -5 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 2+3 & 1+(-5) & 4+1 \\ -3+2 & 0+1 & 2+3 \end{bmatrix} = \begin{bmatrix} 5 & -4 & 5 \\ -1 & 1 & 5 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 2-3 & 1-(-5) & 4-1 \\ -3-2 & 0-1 & 2-3 \end{bmatrix} = \begin{bmatrix} -1 & 6 & 3 \\ -5 & -1 & -1 \end{bmatrix}$$

$$4A = 4\begin{bmatrix} 2 & 1 & 4 \\ -3 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 4\times 2 & 4\times 1 & 4\times 4 \\ -3\times 4 & 0\times 4 & 2\times 4 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 4 & 16 \\ -12 & 0 & 8 \end{bmatrix}$$

$$A^T = \begin{pmatrix} 2 & -3 \\ 1 & 0 \\ 4 & 2 \end{pmatrix}$$

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = r_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = r_2$$

$$\text{---------------------------------}$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = r_m$$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}$$

Pb2

$$A = \begin{pmatrix} 2 & 1 & 4 \\ -3 & 0 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 5 \\ 2 & -1 \\ 4 & 2 \end{pmatrix}$$

1st matrix

2nd matrix x

$$A \times B = \begin{bmatrix} (\text{Row 1} \times \text{colum 1} & \text{Row}_1 \text{ col 2} \\ \text{Row2} \times \text{colum 2}) & \text{Row2 col}_2 \end{bmatrix}$$

$$= \begin{bmatrix} (2\times 3 + 1\times 2 + 4\times 4) & (2\times 5 + 1\times(-1) + 4\times 2 \\ (-3\times 5 + 0\times -1 + 2\times 2) & (-3\times 5 + 0\times -1 + 2\times 2) \end{bmatrix}$$

$$= \begin{bmatrix} 24 & 17 \\ -1 & -11 \end{bmatrix}$$

Pb3 Determinant

$$\begin{vmatrix} 3 & -2 & 2 \\ 1 & 2 & -3 \\ 4 & 1 & 2 \end{vmatrix} = ?$$

$$3\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} - (-2)\begin{vmatrix} 1 & -3 \\ 4 & 2 \end{vmatrix} + 2\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix}$$

$$= 3\,[2\times 2 - (-3)\times 1] + 2\,[1\times 2 - (4\times -3)] + 2\,[1\times 1 - 4\times 2]$$

$$= 3\,[4 + 3] + 2\,[2 + 12] + 2\,[1 - 8]$$

$$= 3\times 7 + 2\times 14 + 2\times(-7) = 35$$

Trace of matrix

$$\begin{pmatrix} 5 & 2 & 0 \\ 3 & 1 & -2 \\ -1 & 4 & 2 \end{pmatrix} \quad \text{Trace of matrix} = 5+1+2=8$$

Cofactor

$$\begin{bmatrix} 2 & -1 & 1 & 3 \\ -3 & 2 & 5 & 0 \\ 1 & 0 & -2 & 2 \\ 4 & -2 & 3 & 1 \end{bmatrix}$$

cofactor of 5

$$\begin{bmatrix} 2 & -1 & 1 & 3 \\ -3 & 2 & 5 & 0 \\ 1 & 0 & -2 & 2 \\ 4 & -2 & 3 & 1 \end{bmatrix}$$

Remove -

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 2 \\ 4 & -2 & 1 \end{bmatrix}$$

pb 4

$$A = \begin{bmatrix} 3 & -2 & 2 \\ 1 & 2 & -3 \\ 4 & 1 & 2 \end{bmatrix}$$

matrix of cofactor A $(A_{jk}) =$? $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

$$A_{11} = (-1)^{1+1} \begin{bmatrix} 3 & -2 & 2 \\ 1 & 2 & -3 \\ 4 & 1 & 2 \end{bmatrix} = (-1)^{1+1} \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 2\times2-(1)(-3) = 7$$

$$A_{12} = (-1)^{1+2}\begin{bmatrix} 3 & -2 & 2 \\ 1 & 2 & -3 \\ 4 & 1 & 2 \end{bmatrix} = (-1)^{1+2}\begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix} = -[1\times2-(4)(-3)]$$
$$= -[2+12]$$
$$= -14$$

$$A_{13} = (-1)^{1+3}\begin{bmatrix} 3 & -2 & 2 \\ 1 & 2 & -3 \\ 4 & 1 & 2 \end{bmatrix} = (-1)^{1+3}\begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix} = [1\times1-4\times2]$$
$$= -7$$

$$A_{21} = (-1)^{2+1}\begin{bmatrix} 3 & -2 & 2 \\ 1 & 2 & -3 \\ 4 & 1 & 2 \end{bmatrix} = (-1)^{2+1}\begin{bmatrix} -2 & 2 \\ 1 & 2 \end{bmatrix} = -[-2\times2-2\times1]$$
$$= -[-4-2]$$
$$= -(-6) = 6$$

$$A_{22} = \begin{bmatrix} 3 & -2 & 2 \\ 1 & 2 & -3 \\ 4 & 1 & 2 \end{bmatrix} = (-1)^{2+2}\begin{bmatrix} 3 & 2 \\ 4 & 2 \end{bmatrix} = +[3\times2-4\times2]$$
$$= +[6-8] = -2$$

$$A_{23} = (-1)^{2+3}\begin{bmatrix} 3 & -2 & 2 \\ 1 & 2 & -3 \\ 4 & 1 & 2 \end{bmatrix} = -1\begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix} = -1[3-(4)(-2)]$$

row col

$$= -1[3+8] = -11$$

$$A_{31} = (-1)^{3+1}\begin{bmatrix} 3 & -2 & 2 \\ 1 & 2 & -3 \\ 4 & 1 & 2 \end{bmatrix} = 1\begin{bmatrix} -2 & 2 \\ 2 & -3 \end{bmatrix} = 1[-2\times-3-2\times2]$$
$$= [+6-4]$$
$$= 2$$

$${}_{32} = (-1)^{3+2} \begin{vmatrix} 3 & -2 & 2 \\ 1 & 2 & -3 \\ 4 & 1 & 2 \end{vmatrix} = -1 \begin{bmatrix} 3 & 2 \\ 1 & -3 \end{bmatrix} = -1 \left[3(-3) - 2\times 1 \right]$$

$$= -1[-9-2]$$

$$= 11$$

$$A_{33} = (-1)^{3+3} \begin{bmatrix} 3 & -2 & 2 \\ 1 & 2 & -3 \\ 4 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix} = 3\times 2 - (1)(-2)$$

$$= 6 + 2 = 8$$

cofactor

$$(A_{jk}) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{array}{l} \begin{bmatrix} 7 & -14 & -7 \\ 6 & -2 & -11 \\ 2 & 11 & 8 \end{bmatrix} \end{array} \begin{array}{l} \to \text{Row 1} \\ \to \text{Row 2} \\ \to \text{Row 3} \end{array}$$

Row 1 → column 1
Row 2 — column 2
Row 3 → column 3

$$(A_{jk})^T = \begin{bmatrix} 7 & 6 & 2 \\ -14 & -2 & 11 \\ -7 & -11 & 8 \end{bmatrix}$$

Transpose of matrix

$$A^{-1} = \frac{(A_{jk})^T}{\det A}$$

$\det A = ?$

$$\det \begin{vmatrix} 7 & 6 & 2 \\ -14 & -2 & 11 \\ -7 & -11 & 8 \end{vmatrix}$$

$$= +7 \begin{bmatrix} -2 & 11 \\ -11 & 8 \end{bmatrix} - 6 \begin{bmatrix} -14 & 11 \\ -7 & 8 \end{bmatrix} + 2 \begin{bmatrix} -14 & -2 \\ -7 & -11 \end{bmatrix}$$

$$\det A = 7 \left[-2\times 8 - (-11)11 \right] - 6 \left[-14\times 8 - (-7)\times 11 \right] + 2 \left[-14\times -11 - (-2)(-7) \right]$$

$$= 7 \left[-16 + 121 \right] - 6 \left[-112 + 77 \right] + 2 \left\{ 154 - 14 \right\}$$

$$\cancel{= 7 \left[10\,5\right] - 6 \times (-35) + 2 \times 140}$$

$$= \cancel{28}$$

$$= \cancel{735} -$$

$$A^{-1} = \frac{(Aju)^T}{\det A}$$

$$\det A = \det \begin{vmatrix} \overset{+}{3} & \overset{-}{-2} & \overset{+}{2} \\ 1 & 2 & -3 \\ 4 & 1 & 2 \end{vmatrix}$$

$$3\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} - (-2)\begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix} + 2\begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}$$

$$= 3\left[4 - (-3) \times 1\right] + 2\left[1 \times 2 - 4(-3)\right] + 2\left[1 \times 1 - 4 \times 2\right]$$

$$= 3\left[4 + 3\right] + 2\left[2 + 12\right] + 2\left[1 - 8\right]$$

$$= 3 \times 7 + 2 \times 14 + 2(-7) =$$

$$= 21 + 28 - 14 = 35$$

$$\therefore A^{-1} = \frac{(Aju)^T}{\det A} = \frac{\begin{bmatrix} 7 & 6 & 2 \\ -14 & -2 & 11 \\ -7 & -11 & 8 \end{bmatrix}}{35} = \frac{1}{35}\begin{bmatrix} 7 & 6 & 2 \\ -14 & -2 & 11 \\ -7 & -11 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{5} & \frac{6}{35} & \frac{2}{35} \\ \frac{-2}{5} & \frac{-2}{35} & \frac{11}{35} \\ -\frac{1}{5} & \frac{-11}{35} & \frac{8}{35} \end{bmatrix}$$

Pb 5

Refer Pb 4 and solve the following equations

$$3x_1 - 2x_2 + 2x_3 = 10$$
$$x_1 + 2x_2 - 2x_3 = -1$$
$$4x_1 + x_2 + 2x_3 = 3$$

$$\begin{pmatrix} 3 & -2 & 2 \\ x_1 & 2 & -2 \\ 4 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 10 \\ -1 \\ 3 \end{pmatrix}$$

$$A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 10 \\ -1 \\ 3 \end{pmatrix}$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{pmatrix} 10 \\ -1 \\ 3 \end{pmatrix} \times A^{-1} = A^{-1} \begin{pmatrix} 16 \\ -1 \\ 3 \end{pmatrix}$$

According to Pb 4 $\quad A^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{6}{35} & \frac{2}{35} \\ -\frac{2}{5} & -\frac{2}{35} & \frac{11}{35} \\ -\frac{1}{5} & \frac{-11}{35} & \frac{8}{35} \end{bmatrix}$

$$= \begin{pmatrix} \frac{1}{5} & \frac{6}{35} & \frac{2}{35} \\ -\frac{2}{5} & -\frac{2}{35} & \frac{11}{35} \\ -\frac{1}{5} & \frac{-11}{35} & \frac{8}{35} \end{pmatrix} \begin{pmatrix} 10 \\ -1 \\ 3 \end{pmatrix} = \begin{bmatrix} \frac{1}{5} \times 10 + \frac{6}{35}(-1) + \frac{2}{35} \times 3 \\ -\frac{2}{5} \times 10 + \left(-\frac{2}{35}\right)(-1) + \frac{11}{35}(+3) \\ -\frac{1}{5} \times 10 + \left(\frac{-11}{35}\right)(-1) + \frac{8}{35}(3) \end{bmatrix}$$

$$\begin{bmatrix} \frac{7}{35}\times 10 - \frac{6}{35} + \frac{6}{35} \\ -\frac{14}{35}\times 10 + \frac{2}{35} - \frac{11}{35} \\ -\frac{7}{35}\times 10 + \frac{11}{35} + \frac{24}{35} \end{bmatrix} = \begin{bmatrix} \frac{70}{35} \cancel{- \frac{6}{35}} \cancel{+ \frac{6}{35}} \\ -\frac{140}{35} + \frac{2}{35} + \frac{38}{35} \\ -\frac{70}{35} + \frac{11}{35} + \frac{24}{35} \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{-140+2+38}{35} \\ \frac{-70+11+24}{35} \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$x_1 = 2,\ x_2 = 3,\ x_3 = 1$

---

ph 6

Solve
$$2x_1 + 5x_2 - 3x_3 = 3$$
$$x_1 - 2x_2 + x_3 = 2$$
$$7x_1 + 4x_2 - 3x_3 = -4$$

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = m_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = m_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = m_3$$

$$x_1 = \frac{\begin{bmatrix} m_1 & a_{12} & a_{13} \\ m_2 & a_{22} & a_{23} \\ m_3 & a_{32} & a_{33} \end{bmatrix}}{\Delta},\quad x_2 = \frac{\begin{bmatrix} a_{11} & m_1 & a_{13} \\ a_{21} & m_2 & a_{23} \\ a_{31} & m_3 & a_{33} \end{bmatrix}}{\Delta},\quad x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & m_1 \\ a_{21} & a_{22} & m_2 \\ a_{31} & a_{32} & m_3 \end{vmatrix}}{\Delta}$$

$$\Delta = \begin{bmatrix} 2 & 5 & -3 \\ 1 & -2 & 1 \\ 7 & 4 & -3 \end{bmatrix} = 2\begin{bmatrix} -2 & 1 \\ 4 & -3 \end{bmatrix} - 5\begin{bmatrix} 1 & 1 \\ 7 & -3 \end{bmatrix} + (-3)\begin{bmatrix} 1 & -2 \\ 7 & 4 \end{bmatrix}$$

$$2(2) - 5(-10) - 3(18)$$

$$4 + 50 - 54 = 0$$

$\therefore$ No solution

Pn 7 solve

$$3x_1 - 2x_2 + 2x_3 = 10$$
$$x_1 + 2x_2 - 3x_3 = -1$$
$$4x_1 + x_2 + 2x_3 = 3$$

$$x_1 = \frac{\begin{bmatrix} m_1 & a_{12} & a_{13} \\ m_2 & a_{22} & a_{23} \\ m_3 & a_{32} & a_{33} \end{bmatrix}}{\Delta} \qquad x_2 = \frac{\begin{bmatrix} a_{11} & m_1 & a_{13} \\ a_{21} & m_2 & a_{23} \\ a_{31} & m_3 & a_{33} \end{bmatrix}}{\Delta}$$

$$x_3 = \frac{\begin{bmatrix} a_{11} & a_{12} & m_1 \\ a_{21} & a_{22} & m_2 \\ a_{31} & a_{32} & m_3 \end{bmatrix}}{\Delta}$$

$$\Delta = \begin{bmatrix} 3 & -2 & 2 \\ 1 & 2 & -3 \\ 4 & 1 & 2 \end{bmatrix} = 3\begin{pmatrix} 2 & -3 \\ 1 & 2 \end{pmatrix} - (-2)\begin{pmatrix} 1 & -3 \\ 4 & 2 \end{pmatrix} + 2\begin{pmatrix} 1 & 2 \\ 4 & 1 \end{pmatrix}$$

$$= 3(4+3) + 2(2+12) + 2(1-8)$$

$$= 3 \times 7 + 2 \times 14 - 2 \times 7 = 21 + 28 - 14 = 35$$

$$x_1 = \frac{\begin{bmatrix} 10 & -2 & 2 \\ -1 & 2 & -3 \\ 3 & 1 & 2 \end{bmatrix}}{35} = \frac{10\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} - (-2)\begin{bmatrix} -1 & -3 \\ 3 & 2 \end{bmatrix} + 2\begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}}{35}$$

$$= \frac{10[4+3] + 2[-2+9] + 2[-1-6]}{35}$$

$$= \frac{10 \times 7 + 14 - 14}{35} = 2$$

$$x_2 = \frac{\begin{vmatrix} 3 & 10 & 2 \\ 1 & -1 & -3 \\ 4 & 3 & 2 \end{vmatrix}}{35}$$

$$= \frac{3\begin{vmatrix} -1 & -3 \\ 3 & 2 \end{vmatrix} - 10\begin{vmatrix} 1 & -3 \\ 4 & 2 \end{vmatrix} + 2\begin{vmatrix} 1 & -1 \\ 4 & 3 \end{vmatrix}}{35}$$

$$= \frac{3[-2+9] - 10[2+12] + 2[3+4]}{35}$$

$$= \frac{3\times 7 - 10\times 14 + 2\times 7}{35} = \frac{21-140+14}{35} = \frac{-105}{35}$$
$$= -3$$

$$x_3 = \frac{\begin{bmatrix} 3 & -2 & 10 \\ 1 & 2 & -1 \\ 4 & 1 & 3 \end{bmatrix}}{35} = \frac{3\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} - (-2)\begin{vmatrix} 1 & -1 \\ 4 & 3 \end{vmatrix} + 10\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix}}{35}$$

$$= \frac{3[6+1] + 2[3+4] + 10[1-8]}{35}$$

$$= \frac{3\times 7 + 2\times 7 - 70}{35} = \frac{35-70}{35} = \frac{-35}{35} = -1$$

$x_1 = 2, \quad x_2 = -3, \quad x_3 = -1$

(1)(a) $y = \underline{x^2 + 2x + 1}$

2.1. DIFFERENTIATION

2.1.1 Differentiation of Powers

If $y = x^n$,

then $$\boxed{\frac{dy}{dx} = nx^{n-1}}$$

for all values of n. Th index n may be positive, negative, integral or fractional. For example, +2, -2, $-\frac{1}{2}$ and $+\frac{1}{2}$.

Also if $y = ax^n$

then $$\boxed{\frac{dy}{dx} = nax^{n-1}}$$

2.1.1.1 Examples

Differentiate the following functions with respect to x

(a) x^5

(b) $0.6x^7$

(c) $2x^{1.5}$

(c)

Solutions

Let "y" equal each function in turn.

(a) In this case n = 5 and a = 1 so that

$$\frac{dy}{dx} = 5\,(x^{5-1}) = 5x^4$$

(b) Here n = 7 and a = 0.6

$$\frac{dy}{dx} = 0.6\,(7x^{7-1}) = 4.2x^6$$

(c) For $y = 2x^{1.5}$, n = 1.5 and a = 2 so that

$$\frac{dy}{dx} = 2\,(1.5x^{1.5-1}) = 3x^{0.5}$$

2.1.2 Differentiation of a Sum of Functions

The differentiation of a sum of functions is equal to the sum of the individual differentiations of the functions.

In symbols, if $y = f_1(x) + f_2(x) + f_3(x)$,

then $$\frac{dy}{dx} = \frac{d}{dx}[f_1(x)] + \frac{d}{dx}[f_2(x)] + \frac{d}{dx}[f_3(x)]$$

2.1.2.1 Examples

Differentiate the following sums of functions with respect to x

(i) $y = 5x^3 + 6x^2 + 7$

(ii) $y = \sin x + \cos x$

Solutions

(i) $$\frac{dy}{dx} = \frac{d}{dx}(5x^3) + \frac{d}{dx}(6x^2) + \frac{d}{dx}(7)$$
$$= 15x^2 + 12x$$

(ii) $$\frac{dy}{dx} = \frac{d}{dx}(\sin x) + \frac{d}{dx}(\cos x)$$
$$= \cos x - \sin x$$

2.1.3 Differentiation of a Product of Functions

Consider two functions of x, namely u(x) and v(x). Let $y = u(x).v(x)$, that is, the product of the two functions.

Then $\frac{dy}{dx} = u(x) \frac{d}{dx} [v(x)] + v(x) \frac{d}{dx} [u(x)]$ or more simp

$$\boxed{\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}}$$

where $u = u(x)$ and $v = v(x)$.

2.1.3.1 Examples

Differentiate the following product of functions with respect to x

(i) $y = (x + 1)(x + 3)$

(ii) $y = (x + 1)^2 (x + 3)^3$

Solutions

(i) Let $u = x + 1$ and $v = x + 3$

$\frac{du}{dx} = 1$ and $\frac{dv}{dx} = 1$

From above $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

$= (x + 1)(1) + (x + 3)(1)$

$= x + 1 + x + 3$

$= 2x + 4$

(ii) Let $u = (x + 1)^2$ and $v = (x + 3)^3$

$\frac{du}{dx} = 2(x + 1)$ and $\frac{dv}{dx} = 3(x + 3)^2$

Then $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

$= (x + 1)^2 [3(x + 3)^2] + (x + 3)^3 [2(x + 1)]$

$= 3(x + 1)^2 (x + 3)^2 + 2(x + 3)^2 (x + 1)$

2.1.4 Differentiation of a Quotient of Functions

Consider two functions of x, namely u(x) and v(x). Let $y = \frac{u(x)}{v(x)}$, that is the quotient of functions.

Then it can be shown that:

$$\frac{dy}{dx} = \frac{v(x) \frac{d}{dx} [u(x)] - u(x) \frac{d}{dx} [v(x)]}{[v(x)]^2}$$

or more simply

$$\boxed{\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}}$$

An easy way to recall this relation is to note tha v occurs first in the numeration and is squared in the denominator. This is emphasised below.

$$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

2.1.4.1 Examples

Differentiate the following quotient of functions with respect to x

(i) $\frac{x}{x + 1}$ (ii) $\frac{x^2 + 2x + 1}{x^2 - 2x + 1}$

Solutions

Let y equal each function.

(i) $y = \frac{x}{x + 1}$

Let $u = x$; $v = x + 1$. Note that u and v are fixed in this case and cannot be interchanged as in the product rule.

$$\frac{du}{dx} = 1 \quad ; \quad \frac{dv}{dx} = 1$$

$$\frac{dy}{dx} = \frac{(x + 1)(1) - (x)(1)}{(x + 1)^2}$$

$$= \frac{x + 1 - x}{(x + 1)^2}$$

$$\frac{dy}{dx} = \frac{1}{(x + 1)^2}$$

(ii) $y = \frac{x^2 + 2x + 1}{x^2 - 2x + 1}$

$u = x^2 + 2x + 1$ and $v = x^2 - 2x + 1$

$\frac{du}{dx} = 2x + 2$ and $\frac{dv}{dx} = 2x - 2$

$$\frac{dy}{dx} = \frac{(x^2 - 2x + 1)(2x + 2) - (x^2 + 2x + 1)(2x - 2)}{(x^2 - 2x + 1)^2}$$

$$= \frac{2x^3 - 4x^2 + 2x + 2x^2 - 4x + 2 - 2x^3 - 4x^2 - 2x + 2x^2 + 4x + 2}{(x^2 - 2x + 1)^2}$$

$$= \frac{-4x^2 + 4}{(x^2 - 2x + 1)^2}$$

Note that usually a fair amount of simplification can be achieved in the numerator of the differential coefficient.

2.1.5 Differentiation of Trigonometric Functions

It is necessary to know the standard forms of the differential coefficients of trigonometric functions and apply the rules of differentiation in order to differentiate trigonometric functions.

Standard Forms

1. $\frac{d}{dx}(\sin x) = \cos x$
2. $\frac{d}{dx}(\cos x) = -\sin x$
3. $\frac{d}{dx}(\tan x) = \sec^2 x$
4. $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$
5. $\frac{d}{dx}(\sec x) = \sec x \tan x$
6. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

2.1.6 Differentiation of Exponential Functions

The differential coefficient of the exponential function can be found by the basic method of differential calculus, namely

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$$

where $y = e^x$

By this method it can be shown that

$$\boxed{\frac{dy}{dx} = e^x}$$

This function, e^x is the only mathematical function which when differentiated does not change.

The differential coefficient of

$$y = e^{ax}$$

is using rule 2.1.9 $\quad \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$

$$\boxed{\frac{dy}{dx} = ae^{ax}}$$

In general, if

$$y = e^{f(x)} \quad \text{then}$$

$$\boxed{\frac{dy}{dx} = f'(x)e^{f(x)}}$$

where $\quad f'(x) = \frac{d}{dx}\,[f(x)]$

For example, if $\quad y = e^{\frac{1}{2}bx^2 + x}$

then

$$\frac{dy}{dx} = \frac{d}{dx}\,[\tfrac{1}{2}\,bx^2 + x]\; e^{\frac{1}{2}bx^2 + x}$$

$$= (bx + 1)e^{\frac{1}{2}bx^2 + x}$$

2.1.7 Differentiation of $\log_e x$

Let $\quad y = \log_e x$

Then by definition

$$x = e^y$$

Differentiate both sides with respect to y.

Then $\quad \frac{dx}{dy} = e^y$

By inversion $\quad \frac{dy}{dx} = \frac{1}{e^y}$

But $\quad x = e^y$

$$\therefore \quad \boxed{\frac{dy}{dx} = \frac{1}{x}}$$

where $\quad y = \log_e x$

In general it can be show that if

$$y = \log_e f(x)$$

$$\boxed{\frac{dy}{dx} = \frac{f'(x)}{f(x)}}$$

where $\quad f'(x) = \frac{d}{dx}\,[f(x)]$

2.1.7.1 Example

From the differential coefficient of the following functions:

(a) $\log_e x^2$

(b) $\log_e (x^2 - 1)$

(c) $\log_e \sin x$

Solution

(a) Let $y = \log_e x^2$

where $f(x) = x^2$

then $\frac{dy}{dx} = (\frac{2x}{x^2}) = \frac{2}{x}$

Alternatively:

$$y = \log_e x^2 = 2 \log_e x$$

Then $\frac{dy}{dx} = 2\ (\frac{1}{x}) = \frac{2}{x}$

(b) Let $y = \log_e (x^2 - 1)$

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(x^2 - 1)}{x^2 - 1} = \frac{2x}{x^2 - 1}$$

(c) Let $y = \log_e \sin x$

Then $\frac{dy}{dx} = \frac{\frac{d}{dx}(\sin x)}{\sin x}$

$$= \frac{\cos x}{\sin x}$$

$$= \cot x$$

2.1.8 Successive Differentiation

Consider the expression $x^3 + 3x^2 + 4$

Let $y = x^3 + 3x^2 + 4$

Then $\frac{dy}{dx} = 3x^2 + 6x$

Obviously $\frac{dy}{dx}$ is a function of x and can itself be differentiated with respect to x.

Then $\frac{d}{dx}\ (\frac{dy}{dx}) = \frac{d}{dx}\ (3x^2 + 6x)$

$$= 6x + 6$$

$\frac{d}{dx}\ (\frac{dy}{dx})$ is written as $\frac{d^2y}{dx^2}$ (read "d squared y d x squared")

Then $\frac{d^2y}{dx^2} = 6x + 6$

Likewise $\frac{d^2y}{dx^2}$ is a function of x and can be differentiated with respect to x.

Then $\frac{d}{dx}\ (\frac{d^2y}{dx^2}) = \frac{d^3y}{dx^3} = 6$

Some functions can be differentiated successively indefinitely without the differential coefficient becomeing zero.

An example is: $y = \frac{1}{x}$

Then $\frac{dy}{dx} = \frac{-1}{x^2}$, $\frac{d^2y}{dx^2} = \frac{2}{x^3}$, $\frac{d^3y}{dx^3} = \frac{-6}{x^4}$ and so on.

2.1.8.1 Example

Find the first three differential coefficients of the function

$$(\frac{1}{2x + 1})$$

Let $y = \frac{1}{2x + 1} = (2x + 1)^{-1}$

Then $\frac{dy}{dx} = (-1)(2)(2x + 1)^{-2} = \frac{-2}{(2x + 1)^2}$

$$\frac{d^2y}{dx^2} = (2)(-2)(-2)(2x+1)^{-3} = \frac{8}{(2x+1)^3}$$

$$\frac{d^3y}{dx^3} = (8)(+2)(-3)(2x+1)^{-4} = \frac{-48}{(2x+1)^4}$$

2.1.9 Differentiation of a Function of a Function

Mathemtically sin x, e^x, log x and $x^2 + 1$ are all functions of x. However, consider such functions as:

$$\sin^2(x^2+1), \quad e^{x^2}, \quad e^{\sin x} \quad \text{and} \quad \log_e \sin x$$

These functions are certainly functions of x but they contain two functions or are "function of a function" expressions.

Obviously a function $\log_e \sin x$ contains a logarithmic and and trigonometric function.

The differentiation of a "function of a function" expression can be difficult and a two-step differentiating process has been developed.

2.1.9.1 Consider the function of a function expression

$$y = \sin^3(2x^2 - 1)$$

Let $u = 2x^2 - 1$ $\qquad \frac{du}{dx} = 4x$

then $y = \sin^3 u$ $\qquad \frac{dy}{du} = 3\sin^2 u \cos u$

To combine these two differential coefficients the following relation is used:

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

Then $\frac{dy}{dx} = 4x \cdot 3\sin^2 u \cos u$

But $u = 2x^2 - 1$

Hence $\frac{dy}{dx} = 12x \sin^2(2x^2 - 1)\cos(2x^2 - 1)$

2.1.10 Differentiation of Implicit Functions

By definition an IMPLICIT function is one in which y is not expressed in terms of x EXPLICITLY.

The function $y = x^2 + x + 1$ is an EXPLICIT function, that is, y is expressed explicity as a function of x.

However, the equation

$$y^2 + 3xy + x^2 = 0$$

is an IMPLICIT function since y is not expressed in terms of x only. The equation implies that y is a function of x.

Some implicit functions can be made explicit by solving for y. For example,

$x^2 + y^2 = 9$ is an implicit function

and $y = \sqrt{9 - x^2}$ is an explicit function.

However, many implicit functions cannot be changed to explicit functions. For example,

$$y^2 + xy + x^2 = 0$$

To differentiate an implicit function use is made of the following relation:

If f(y) is a function of y and implicitly a function of x

$$\boxed{\frac{d}{dx}[f(y)] = \frac{d}{dy}[f(y)]\frac{dy}{dx}}$$

For example, if $f(y) = y^2$

$$\frac{d}{dx}[y^2] = \frac{d}{dy}(y^2)\frac{dy}{dx}$$

$$= 2y\frac{dy}{dx}$$

2.1.10.1 Example

Differentiate the following implicit functions with respect to x.

(a) $x^2 + y^2 = 4$ (b) $y \log_e x = 2$

Solution

(a)

$$2x + \frac{d}{dx}(y^2) = 0$$

$$2x + \frac{d}{dx}(y^2)\frac{dy}{dx} = 0$$

$$2x + 2y\frac{dy}{dx} = 0$$

Solve for $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$$

(b)

$$\frac{d}{dx}(y \log_e x) = \frac{d}{dx}(2)$$

Now $y \log_e x$ is a product of functions and must be differentiated as such.

Then $$\frac{d}{dx}(y) . \log_e x + y\frac{d}{dx}(\log_e x) = 0$$

$$\frac{dy}{dx}\log_e x + y . \frac{1}{x} = 0$$

$$\frac{dy}{dx} = -\frac{y}{x} . \frac{1}{\log_e x}$$

2.2 INTEGRATION

2.2.1 Introduction

Integration is the other half of the story of the differential and integral calculus. The integral calculus has many fundamental and important applications in science and engineering, particularly electrical engineering. It would be no understatement to say that electrical engineering could not have developed without the integral calculus.

Integration may be approached in three different ways. Firstly, the integral of a function f(x) can be considered to be THE AREA UNDER THE CURVE of f(x). This approach allows a basic insight into the underlying principles of integration.

It is also the GEOMETRICAL INTERPRETATION of the integral of a function f(x) taken between two limits.

Secondly, the process of integration may be considered as a MATHEMATICAL PROCESS in its own right. The fundamental idea of integration is the synthesis of a large number of small quantitites to make a whole.

Thirdly, integration may be considered to be the reverse process of differentiation. Then integration is said to be ANTIDIFFERENTIATION. This is the extremely important connection between integration and differentiation. It means that given the differential coefficient of an unknown function, the function itself may be found by integrating the differential coefficient. This idea of integration as antidifferentiation leads up to the subject of differential equations which is of central importance in electrical engineering.

2.2.2 Notation

In words the $\int_a^b f(x)\,dx$ is THE DEFINITE INTEGRAL OF f(x) WITH RESPECT TO x FROM LIMIT a TO LIMIT b.

Evaluating the integral is the process of INTEGRATION. The function f(x) is called the INTEGRAND. The x = a to x = b are the BOUNDARIES or LIMITS of integration.

2.2.2.1 Example

Find the area under the curve of the function $f(x) = 3 \sin x + 10$ between the limits of integration $x = \pi$ and $x = 2\pi$.

Solution

Sketch this function as shown in Figure 2.1 and shade the required area.

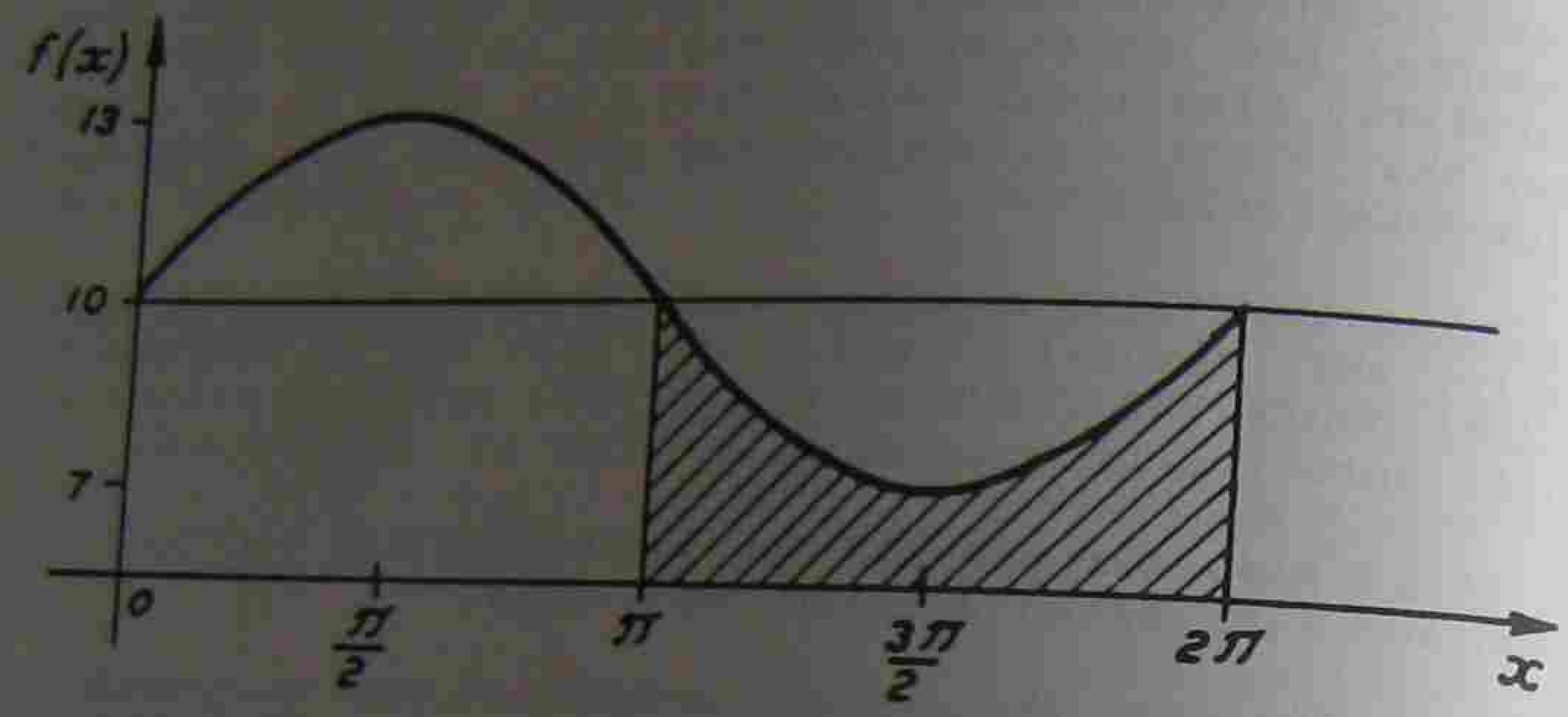


Figure 2.1 The required area under $y = 3 \sin x + 10$ **between limits** π **and** 2π

The exact area under the curve is

$$A = \int_{\pi}^{2\pi} (3 \sin x + 10)\, dx$$

The evaluation of integrals will be discussed in more detail later in this unit.

However, this integral becomes

$$\int_{\pi}^{2\pi} (3 \sin x + 10)\, dx = [-3 \cos x + 10x]_{\pi}^{2\pi}$$

$$= -3 \cos 2\pi + 10(2\pi) - \{-3 \cos \pi + 10\pi\}$$

$$= -3 \times 1 + 20\pi + 3 \times (-1) - 10\pi$$

$$= -6 + 10\pi$$

$$= 10\pi - 6$$

This area may be evaluated as accurately as desired. Take

$$\pi = 3.1416$$

Then $A = 10 \times 3.1416 - 6$

$$= 31.416 - 6$$

$$\underline{A = 25.416 \text{ sq. units}}$$

2.2.3 Integration of a Power of x

$$\boxed{\int x^n\, dx = \left(\frac{1}{n+1}\right) x^{n+1} + C}$$

where $n \neq -1$

The index n may be positive, negative or fractional. Check the integration by differentiation.

$$\frac{d}{dx}\left[\left(\frac{1}{n+1}\right) x^{n+1} + C\right] = \frac{n+1}{n+1} x^{n+1-1}$$

$$= x^n$$

Note that if $n = -1$, $\frac{1}{n+1}$ becomes $\frac{1}{0}$ which is indeterminate. Hence, the special case $x = -1$ is not permissible in this standard integral.

2.2.3.1 Examples would be:

$$\int x^7\, dx = \frac{1}{7+1} x^{7+1} + C$$

$$= \frac{1}{8} x^8 + C$$

$$\int x^{\frac{1}{5}}\, dx = \frac{1}{\frac{1}{5}+1} x^{\frac{1}{5}+1} + C$$

$$= \frac{5}{6} x^{\frac{6}{5}} + C$$

$$\int x^{-3}\,dx = \frac{1}{-3+1}\,x^{-3+1} + C$$

$$= -\frac{1}{2}\,x^{-2} + C$$

$$\int x^{-\frac{1}{2}}\,dx = \frac{1}{-\frac{1}{2}+1}\,x^{-\frac{1}{2}+1} + C$$

$$= 2x^{\frac{1}{2}} + C$$

Other general examples would be:

$$\int ax^{n}\,dx = a\int x^{n}\,dx = \frac{ax^{n+1}}{n+1} + C$$

Note that a constant inside the integral sign may always be taken outside the integral sign.

$$\int (x^{n} + x^{m})\,dx = \frac{1}{n+1}\,x^{n+1} + \frac{1}{m+1}\,x^{m+1} + C$$

This means that when integrating a sum of functions, the funcitons may be integrated individually. For example:

$$\int x^{4} + 2x^{3}\,dx = \frac{1}{4+1}\,x^{4+1} + 2\cdot\frac{1}{3+1}\,x^{3+1} + C$$

$$= \frac{1}{5}\,x^{5} + \frac{1}{2}\,x^{4} + C$$

2.2.4 The Standard Integral of $(ax + b)^{n}$ is

$$\boxed{\int (ax+b)^{n}\,dx = \frac{1}{a(n+1)}\,(ax+b)^{n+1} + C}$$

where $n \neq -1$

Again n may be positive, negative or fractional.

2.2.4.1 For example

$$\int (2x+3)^{3}\,dx = \frac{1}{2\,(3+1)}\,(2x+4)^{3+1} + C$$

$$= \frac{1}{8}\,(2x+4)^{4} + C$$

$$\int (-3x+2)^{-\frac{1}{3}}\,dx = \frac{1}{(-3)\,(-\frac{1}{3}+1)}\,(-3x+2)^{-\frac{1}{3}+1} + C$$

$$= \frac{1}{(-3)\,(+\frac{2}{3})}\,(-3x+2)^{+\frac{1}{3}} + C$$

$$= -\frac{1}{2}\,(-3x+2)^{\frac{2}{3}} + C$$

$$\int (5x+8)^{-2}\,dx = \frac{1}{5\,(-2+1)}\,(5x+8)^{-2+1} + C$$

$$= -\frac{1}{5}\,(5x+8)^{-1} + C$$

$$= \frac{1}{5\,(5x+8)} + C$$

2.2.5 Integration of Trigonometric Functions

The standard integrals are:

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \tan x \, dx = \ln \sec x + C$$

$$\int \cot x \, dx = \ln \sin x + C$$

$$\int \cos nx \, dx = \frac{1}{n} \sin nx + C$$

$$\int \sin nx \, dx = -\frac{1}{n} \cos nx + C$$

From the differentiation of trigonometric functions, the following integrals are obtained:

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + C$$

$$\int \sec x \tan x \, dx = \sec x + C$$

$$\int \operatorname{cosec}^2 x \, dx = -\cot x + C$$

2.2.6 Integration of Trigonometric Functions by Trigonometric Identities

By using trigonometric identities many trigonometric functions can be integrated.

Then since

$$\boxed{\cos 2x = 2\cos^2 x - 1}$$

$$\cos^2 x \, dx = \frac{\cos 2x + 1}{2}$$

Then

$$\int \cos^2 x \, dx = \int \frac{\cos 2x + 1}{2} dx$$

$$= \frac{1}{2} \int (\cos 2x + 1) \, dx$$

$$= \frac{1}{2} \left(\frac{1}{2} \sin 2x + x\right) + C$$

$$= \frac{1}{4} \sin 2x + \frac{1}{2} x + C$$

Also since

$$\boxed{\cos 2x = 1 - 2\sin^2 x}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

Then

$$\int \sin^2 x \, dx = \int \frac{1 - \cos 2x}{2} dx$$

$$= \frac{1}{2} \int (1 - \cos 2x) \, dx$$

$$= \frac{1}{2} \left(x - \frac{1}{2} \sin 2x\right) + C$$

$$= \frac{1}{2} x - \frac{1}{4} \sin 2x + C$$

In a similar way

$$\int \sin^2 2x \, dx = \int \frac{1 - \cos 4x}{2} \, dx$$

$$= \frac{1}{2} \int (1 - \cos 4x) \, dx$$

$$= \frac{1}{2} \{x - \frac{1}{4} \sin 4x\} + C$$

$$= \frac{1}{2} x - \frac{1}{8} \sin 4x + C$$

and $$\int \cos^2 2x = \int \frac{1 + \cos 4x}{2} \, dx$$

$$= \frac{1}{2} \int (1 + \cos 4x) \, dx$$

$$= \frac{1}{2} (x + \frac{1}{4} \sin 4x) + C$$

$$= \frac{1}{2} x + \frac{1}{8} \sin 4x + C$$

Using the identity

$$\boxed{\sec^2 x = \tan^2 x + 1}$$

$$\int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx$$

$$= \tan x - x + C$$

Other identities are

$$\boxed{\sin 2x = 2 \cos x \sin x}$$

$$\boxed{\operatorname{cosec}^2 x = \cot^2 x + 1}$$

Then

$$\int \cos x \sin x \, dx = \int \frac{1}{2} \sin 2x \, dx$$

$$= -\frac{1}{4} \cos 2x + C$$

$$\int \cot^2 x \, dx = \int (\operatorname{cosec}^2 x - 1) \, dx$$

$$= - \cot x - x + C$$

By use of the multiple angle formula, products of trigonometric functions can be integrated.

2.2.6.1 These are

$$\boxed{\begin{aligned} \sin A \cos B &= \frac{1}{2} \{\sin (A + B) + \sin (A - B)\} \\ \cos A \sin B &= \frac{1}{2} \{\sin (A + B) - \sin (A - B)\} \\ \cos A \cos B &= \frac{1}{2} \{\cos (A + B) + \cos (A - B)\} \\ \sin A \sin B &= \frac{1}{2} \{\cos (A - B) - \cos (A + B)\} \end{aligned}}$$

Then

$$\sin 3x \cos 4x = \frac{1}{2} \{\sin (3x + 4x) + \sin (3x - 4x)\}$$

$$= \frac{1}{2} \{\sin 7x - \sin x\}$$

Then

$$\int \sin 3x \cos 4x \, dx = \int \frac{1}{2} \{\sin 7x - \sin x\} \, dx$$

$$= -\frac{1}{14} \cos 7x + \frac{1}{2} \cos x + C$$

2.2.7 Integration of Exponential Functions

From the derivative

$$\frac{d}{dx} e^{ax} = ae^{ax}$$

$$\boxed{\int e^{ax}\, dx = \frac{1}{a} e^{ax} + c}$$

Likewise

$$\int be^{ax}\, dx = \frac{b}{a} e^{ax} + c$$

The following general relation is important:

$$\boxed{\int f^1(x)\, e^{f(x)}\, dx = e^{f(x)} + c}$$

where

$$f^1(x) = \frac{d}{dx} [\ f(x)\ ]$$

For example,

$$\int xe^{x^2}\, dx = \int \frac{d}{dx} (\tfrac{1}{2} x^2)\, e^{x^2}\, dx$$

$$= \frac{1}{2} e^{x^2} + c$$

2.2.8 Integration of Functions which Result in a Logarithmic Function

Since $\qquad \frac{d}{dx} [\ \ell n\ f(x)\ ] = \frac{f'(x)}{f(x)}$

then

$$\boxed{\int \frac{f'(x)}{f(x)}\, dx = \ell n\ f(x) + C}$$

2.2.8.1 For example,

$$\int \frac{4x\, dx}{2x^2 + 3} = \ell n\ (2x^2 + 3) + C$$

$$\int \frac{x\, dx}{2x^2 + 3} = \frac{1}{4} \int \frac{4x}{2x^2 + 3}\, dx$$

$$= \frac{1}{4}\ \ell n\ (2x^2 + 3) + C$$

$$\int \frac{\sin x}{\cos x}\, dx = -\ \ell n\ \cos x + C$$

$$= \ell n\ (\cos x)^{-1} + C$$

$$= \ell n\ \frac{1}{\cos x} + C$$

$$= \ell n\ \sec x + C$$

2.2.9 Integration by Change of Variable

Let $\qquad I = \int x \sqrt{2x + 1}\ dx$

This expression $x \sqrt{2x + 1}$ cannot be integrated directly by any standard form.

let $\qquad u = \sqrt{2x + 1}$

then $\qquad u^2 = 2x + 1$

Also $\qquad 2x = u^2 - 1$

$$x = \frac{1}{2} (u^2 - 1)$$

Then $\qquad \frac{dx}{du} = u$

$$dx = u\ du$$

Substitute in the above integral

then $$I = \int \frac{1}{2} (u^2 - 1)\ u\ .\ u\ du$$

$$= \frac{1}{2} \int (u^4 - u^2)\ du$$

$$= \frac{1}{2} \{\frac{1}{5} u^5 - \frac{1}{3} u^3\} + C$$

Now $$u = (2x + 1)^{\frac{1}{2}}$$

$$u^5 = (2x + 1)^{\frac{5}{2}}$$

$$u^3 = (2x + 1)^{\frac{3}{2}}$$

$$\therefore \quad I = \frac{1}{2} [\ \frac{1}{5} (2x + 1)^{\frac{5}{2}} - \frac{1}{3} (2x + 1)^{\frac{3}{2}}\ ] + C$$

Another example would be:

$$I = \int \frac{x}{\sqrt{5 - x}}\ dx$$

$$= \int x\ (5 - x)^{-\frac{1}{2}}\ dx$$

Let $$u = 5 - x$$

$$\therefore \quad x = 5 - u$$

$$\frac{dx}{du} = -1$$

$$dx = (-du)$$

Then $$I = \int (5 - u)(u^{-\frac{1}{2}})(-du)$$

$$= \int (-5u^{-\frac{1}{2}} + u^{\frac{1}{2}})\ du$$

$$= (\frac{-5}{\frac{1}{2}}) u^{\frac{1}{2}} + \frac{2}{3} u^{\frac{3}{2}} + C$$

$$= -10\ u^{\frac{1}{2}} + \frac{2}{3} u^{\frac{3}{2}} + C$$

$$= -10\ (5 - x)^{\frac{1}{2}} + \frac{2}{3} (5 - x)^{\frac{3}{2}} + C$$

2.2.10 Evaluation of the Definite Integral

It can be shown that in general

$$\boxed{\int_a^b f(x)\ dx = F(b) - F(a)}$$

where F(b) and F(a) are substituted instances of the integrated function F(x), that is

$$\int f(x)\ dx = F(x)$$

2.2.10.1 For example,

$$\int_2^4 x^2\ dx = \frac{1}{3} (4)^3 - \frac{1}{3} (2)^3$$

since $$\int x^2\ dx = \frac{1}{3} x^3$$

The usual notation is shown below.

$$\int_2^4 x^2\,dx = \left[\frac{1}{3} x^3 \right]_2^4$$

$$= \frac{1}{3}(4)^3 - \frac{1}{3}(2)^3$$

$$= \frac{1}{3}(64 - 8)$$

$$= \frac{56}{3}$$

WORK TO BE FORWARDED FOR COMMENT

PART 1 - DIFFERENTIATION

1. Find the gradient to the curve $y = x^2 + \sin x$ where $x = \frac{\pi}{4}$ radians.

2. Find the gradient to the curve $y = x^3 - x^2 + 2$ when $x = 2$ and $x = -2$.

3. Find the differential coefficient of the following functions with respect to x:

(i) $y = x^{10}$

(ii) $y = x^3$

(iii) $y = mx^k$

(iv) $y = nx^{(s + 1)}$

(v) $y = (x + 4)^3$

(vi) $y = (x - 5)^2$

(vii) $y = 15(x + 4)^7$

(viii) $y = 2(x - 3)^4$

(ix) $y = 4x^{-3.2}$

(x) $y = 2.1x^{-\frac{1}{2}}$

(xi) $y = 3(x - 2)^{-\frac{1}{2}}$

(xii) $y = 3.2(2x + 3)^{\frac{1}{5}}$

4. (i) The eddy current loss of an electrical machine is given by:

$$P = k_1 f^2 t^2 B^2 V$$

where P = power

and f = frequency

k_1, t, B and V are all constants. Differentiate P with respect to f.

(ii) The hysteresis loss of an electrical machine is given by:

$$P = k_2 f B^{1.6} W$$

where P = power

and B = maximum flux density

k_2, f and W may be considered as constants. Determine $\frac{dP}{dB}$

5. Differentiate the following functions with respect to x.

(i) $y = 2x^{\frac{3}{2}} + x^{-\frac{3}{2}}$

(ii) $y = 7x^6 + 6x^5 + 4x^4 + 3x^2 + 2x + 1$

(iii) $y = \frac{1}{\sqrt{x}} + \frac{1}{\sqrt[3]{x^2}}$

(iv) $y = -x^{-2} + \frac{1}{x} + x^2$

(v) $y - ax^n + bx^m + c$

6. (i) The loss in an electrical machine is given by:

$$P = af + bf^2$$

where P = power

and f = frequency and "a" and "b" are constants

Differentiate P with respect to f.

(ii) The relation of induced e.m.f. E of a direct current machine to field current I_f in the windings was found by digital computer to be:

$$E = 0.58 + 681.5\, I_f - 461.8\, I_f^{\,2} + 46.3\, I_f^{\,3}$$

This is the magnetisation curve of the d.c. machine.

Find $\frac{dE}{dI_f}$

7. Differentiate the following product of functions.

(i) $x^2(x^2 - 1)$

(ii) $(x + 2)(x - 7)$

(iii) $(x + 2)(x^2 + 4)$

(iv) $(x + 1)^{-\frac{1}{2}}(x - 5)$

(v) $(2x + 7)(4x^2 - 5)$

(vi) $(2x + 7)^3(4x^2 - 5)^2$

(vii) $(5x^2 + 6x + 3)(5x - 1)$

8. Differentiate the following quotient of functions.

(i) $\frac{x + 1}{x + 2}$

(ii) $\frac{x}{x + 1}$

(iii) $\frac{4 - x}{x - x^2}$

(iv) $\frac{x^{\frac{1}{4}}}{x^{\frac{1}{2}} - 1}$

9. Differentiate the following trigonometric functions:

(i) $\sin(10x + 4) + \cos(7x + 1)$

(ii) $\tan^2 3\theta$

(iii) $\sec x \tan x$

(iv) $\operatorname{cosec}^4(x^2 + 1)$

(v) $\cot 5x \sin 6x$

(vi) $i = 10 \sin 10t + 5 \sin 20t + 2.5 \sin 30t$

10. (a) The potential difference across an inductor of self-inductance L is:

$$v_L = L\frac{di}{dt}$$

If $i = 10 \sin(314t + 60^o)$ find the potential difference v_L.

(b) The self inductance of a rotor winding of a salient pole synchronous machine is:

$$L = L_0 + L_2 \cos 2\theta$$

where L_0 and L_2 are constant and θ is the angular position of the rotor. Find the rate of change of inductance with angular position.

11. Differentiate the following exponential, logarithmic and power functions:

(i) e^{-ax}

(ii) $e^{(x^2 + 2x)}$

(iii) $\log_e(x^2 + 2x + 3)$

(iv) $a^{x^2 + 1}$

(v) $x^{\sin x}$

12. The current growth in a resistive-inductive circuit from a suddenly applied battery e.m.f. is:

$$i = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

where E, R and L are constants, t is time. Find the rate of change of current with respect to time.

13. Find the first two differential coefficients of the following functions:

(i) $\log_e (x^2 - 1)$

(ii) $\sin^2 x$

(iii) $e^{\frac{1}{2} x^2}$

(iv) a^x

14. Differentiate the following functions using the function of a function rule:

(a) $e^{-\sin x}$

(b) $\log \cot x$

(c) $\tan (\sin^2 \theta)$

WORK TO BE FORWARDED FOR COMMENT

PART 2 - INTEGRATION

1. Find the area under the curve:

$$y = 4 \sin x + 3$$

between the limits π and 2π

2. Integrate the following functions:

(i) x^5

(ii) $\frac{1}{3} x^{-\frac{1}{2}}$

(iii) $6x^{-2}$

(iv) $3x^{\frac{1}{5}}$

(v) $\frac{1}{2} x^{-\frac{1}{3}} + x^2$

(vi) $(2x + 3)^3$

(vii) $(1 + x)^{-4}$

(viii) $(3 - x)^{\frac{1}{2}}$

(ix) $(5x + 6)^{-\frac{1}{3}}$

(x) $(3x - 2)^{-\frac{3}{2}}$

3. Integrate the following trigonometric functions:

(i) $\sec^2 x$

(ii) $\tan x$

(iii) $\cos 3x$

(iv) $\sin 6x$

(v) $\frac{1}{5}\sin^2 2x$

(vi) $\cot x$

(vii) $\operatorname{cosec} x \cot x$

(viii) $\sec x \tan x$

(ix) $\frac{1}{5}\cos^2 5x$

(x) $\operatorname{cosec}^2 x$

(xi) $\tan^2 3x$

(xii) $\sin x \cos x$

(xiii) $\cot^2 x$

(xiv) $\sin 3x \cos 4x$

(xv) $\sin 6x \cos x$

(xvi) $\cos 3x \cos 5x$

(xvii) $\sin x \sin 3x$

4. Integrate the following exponential functions:

(i) e^{-3x}

(ii) $2e^{4x}$

(iii) $5e^{-\frac{1}{3}x}$

(iv) $3xe^{x^2}$

(v) $-3x^2e^{-x^3}$

(vi) $\cos x\, e^{\sin x}$

5. Integrate the following logarithmic functions:

(i) $\dfrac{x}{3x^2 + 2}$

(ii) $\dfrac{e^{ax}}{e^{ax} + 4}$

(iii) $\dfrac{\cos x}{\sin x + 4}$

(iv) $\dfrac{\sec^2 x + 1}{\tan x + x}$

6. Integrate the following by change of variable:

(i) $x\sqrt{5x + 4}$

(ii) $\dfrac{x}{\sqrt{3x + 4}}$

7. Evaluate the following integrals:

(i) $\int_{-3}^{2} \frac{1}{3} x^2 \, dx$

(ii) $\int_{1}^{4} \frac{1}{x} \, dx$

(iii) $\int_{0}^{\frac{\pi}{2}} \sin x \, dx$

(iv) $\int_{1}^{1.4} e^{-2x} \, dx$

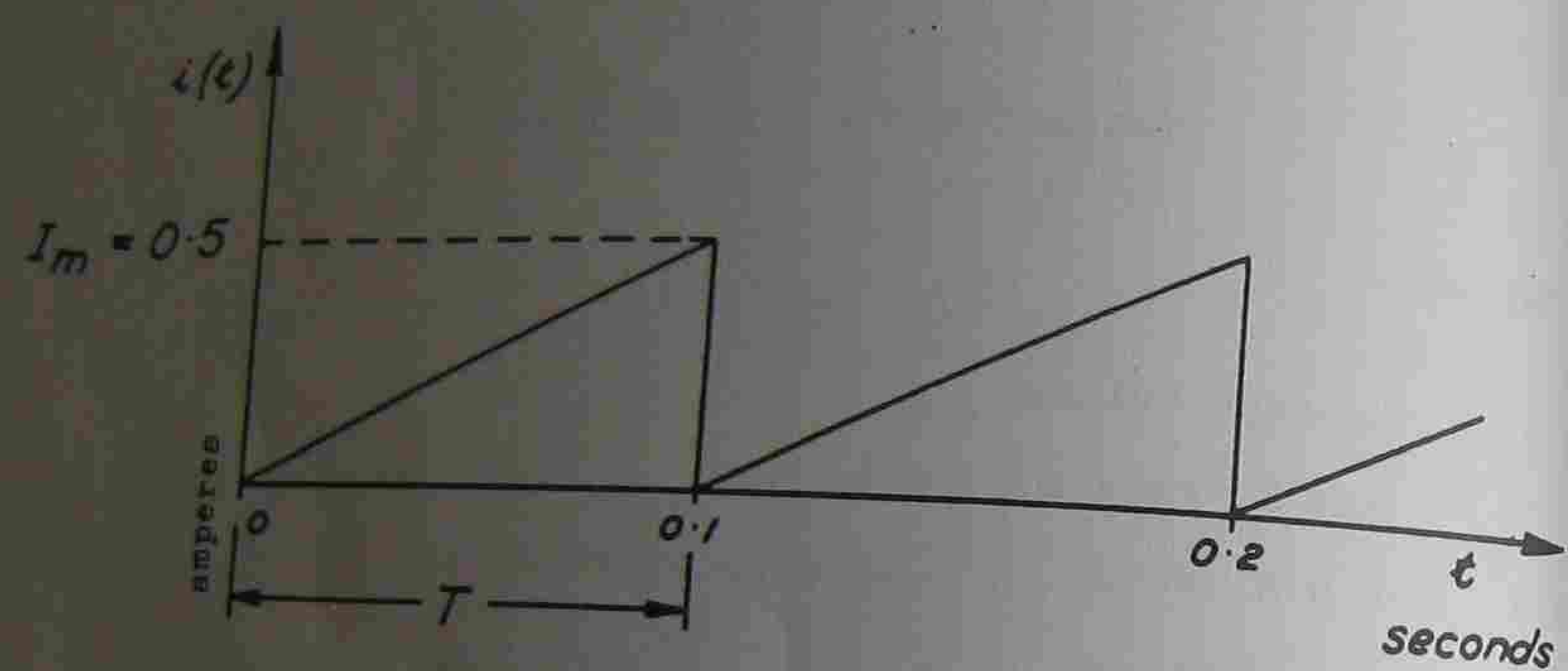
(v) $\int_{-4}^{2} (3x^2 + 4) \, dx$

AVERAGE AND R.M.S. VALUES

There are many physical applications of the definite integral in electrical engineering, but one of the most basic is the Root-Mean-Square (R.M.S.) value and the Direct Current for average value, usually referred to as the D.C. value. This may apply to a current or voltage wave.

3.1. AVERAGE VALUES

Consider a current wave as shown in Figure 3.1.



Example 3.1.1 Figure 3.1 A triangular wave of current

This current wave i(t) is periodic after every 0.1 seconds, that is, it repeats itself after each 0.1 seconds. In general, the time period is shown as T seconds. Hence, the average value needs only to be determined over one time period, as it will be the same for all time periods.

Then by definition:

$$\boxed{I_{AVE} = \frac{1}{T}\int_0^T i(t)\,dt}$$

where I_{AVE} = average or D.C. value of current.

In this particular case

$$\boxed{i(t) = \left(\frac{I_m}{T}\right) t \quad \text{in the range} \quad 0 \le t \le T}$$

Then

$$I_{AVE} = \frac{1}{T}\int_0^T \left(\frac{I_m}{T}\right) t\,dt$$

$$= \frac{I_m}{T^2}\left[\frac{1}{2}\,t^2\right]_0^T$$

$$= \frac{I_m}{2}$$

Note that the integral

$$\int_0^T i(t)\,dt$$

is, in fact, the area under the graph so that by dividing by the period T the height of a rectangle is determined with base T. The area of the rectangle is the same as the area as given by the integral. In the above example then, the area of the triangle is

$$\frac{1}{2}\,T\,I_m$$

Then dividing by the base length T gives

$$I_{AVE} = \frac{\frac{1}{2}\,T\,I_m}{T} = \frac{1}{2}\,I_m$$

which is the same result given by the integral formula. This should be kept in mind as a check or short cut method. The above basic principle is illustrated in Figure 3.2.

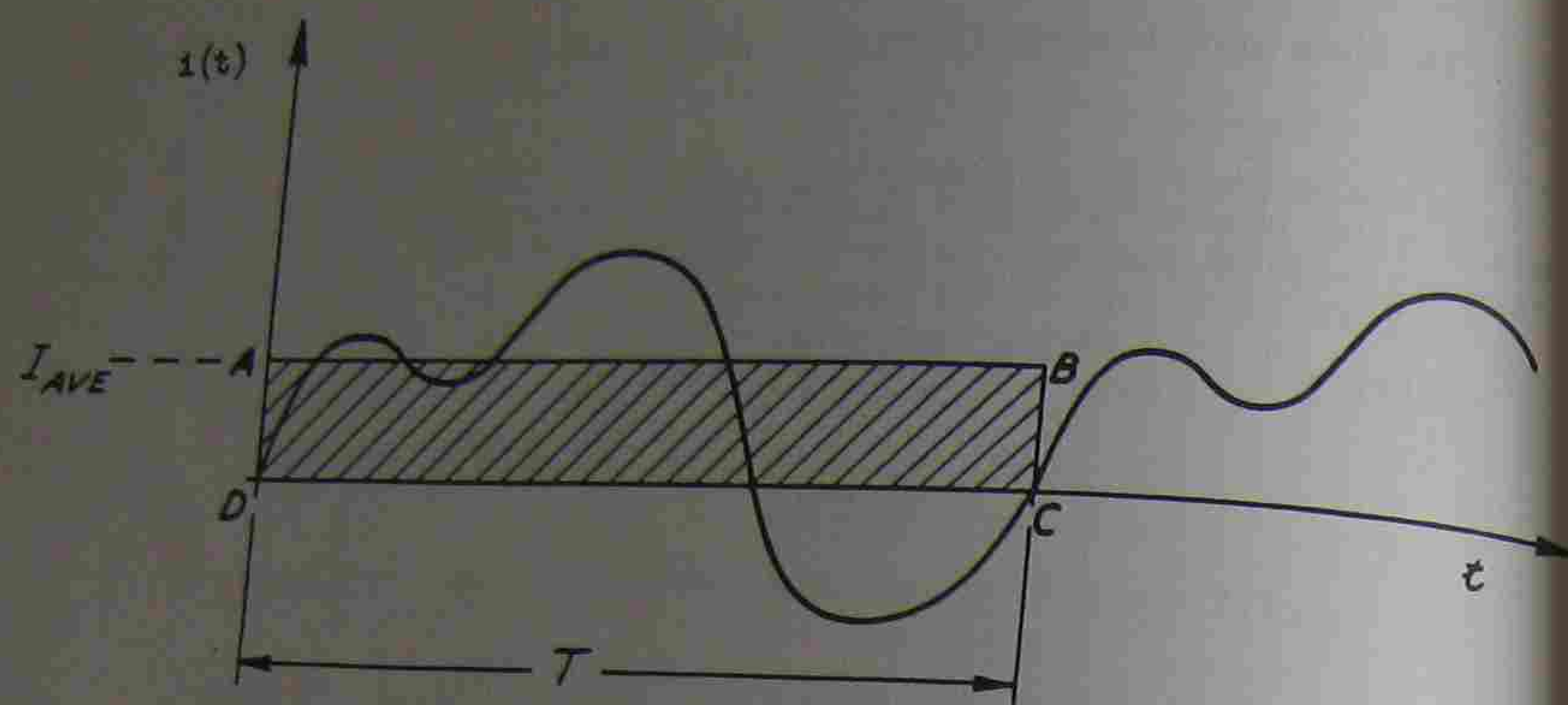


Figure 3.2 General case of the average value

For Figure 3.2

$$I_{AVE} = \frac{1}{T}\int_0^T i(t)\,dt = \frac{\text{AREA under } i(t) \text{ over period } T}{\text{BASE } T}$$

$$= \text{average height of } i(t)$$

and the integral

$$\int_0^T i(t)\,dt = \text{the area ABCD}$$

Note that the area below the X-axis would be subtracted from the area above the X-axis.

Example 3.1.2

Another case of particular interest is the rectified sine wave,

$$\boxed{i(\omega t) = I_m \sin \omega t} \qquad 0 \le \omega t \le \pi$$

and this is shown in Figure 3.3. Note for simplicity the variable of integration in this case has been changed from t to ωt.

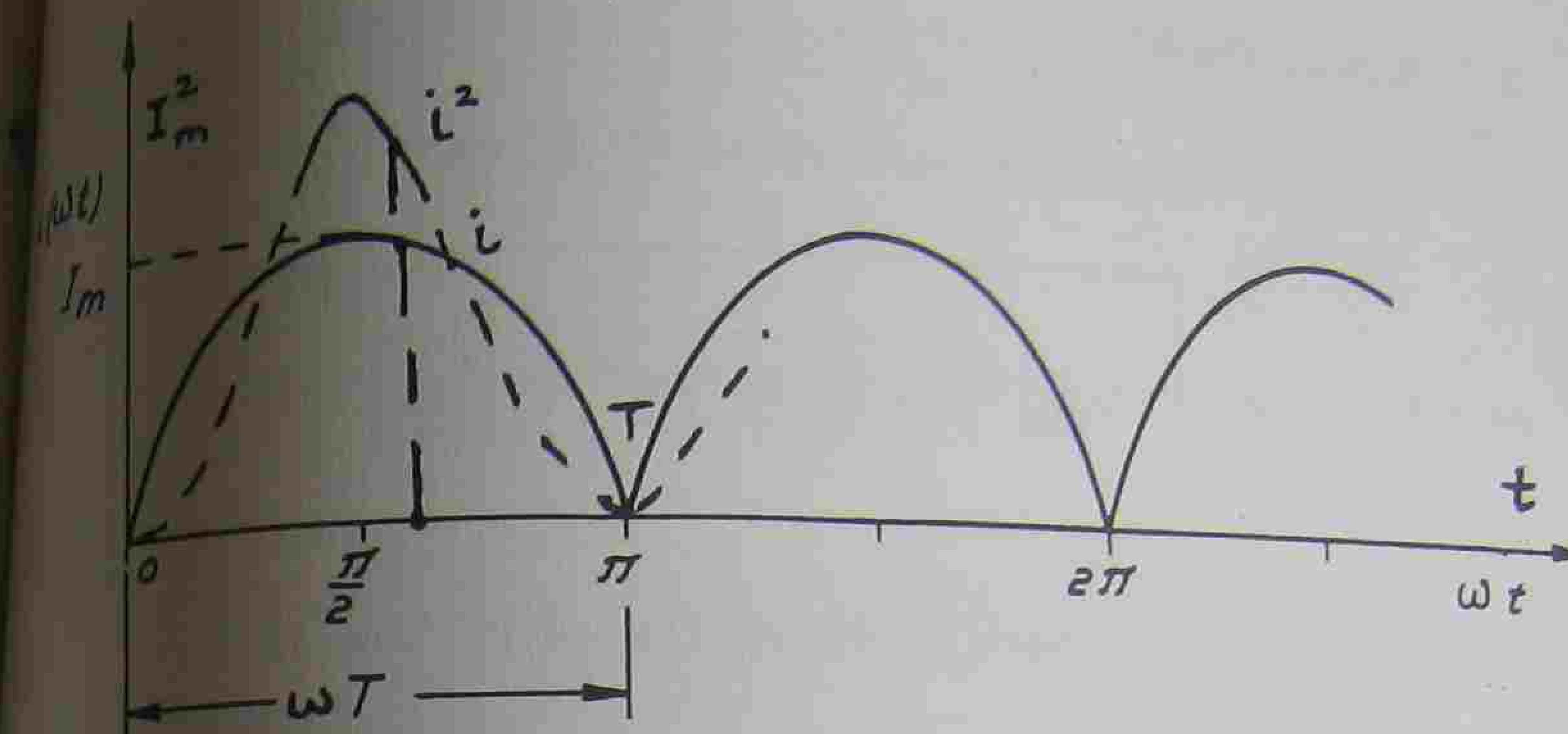


Figure 3.3. Rectified sine wave of current

In this case

$$I_{AVE} = \frac{1}{T}\int_0^T i(\omega t)\,dt$$

$$= \frac{1}{\pi}\int_0^\pi I_m \sin \omega t\, d(\omega t)$$

$$= \frac{I_m}{\pi}\left[-\cos \omega t\right]_0^\pi$$

$t = \frac{\pi}{\omega}$

Then

$$I_{AVE} = \frac{I_m}{\pi}\left[-\cos \pi + \cos 0°\right]$$

$$= \frac{I_m}{\pi}\left[1 + 1\right]$$

$$I_{AVE} = \frac{2\, I_m}{\pi}$$

$$= 0.636\, I_m$$

3.2 R.M.S. VALUES

For the Root-Mean-Square value of current, by definition

$$\boxed{I_{RMS} = \sqrt{\frac{1}{T}\int_0^T [i(t)]^2\, dt}}$$

This is usually more conveniently expressed as:

$$I_{RMS}^2 = \frac{1}{T}\int_0^T [i(t)]^2\, dt = \frac{\text{AREA under } i^2 \text{ over period } T}{\text{BASE } T}$$

$$= \text{mean height of } i^2$$

After the integration is performed, the square root can be taken.

Example 3.2.1

Then for the <u>triangular wave form</u>

$$I_{RMS}^2 = \frac{1}{T}\int_0^T \left(\frac{I_m}{T}\right)^2 t^2\, dt$$

$$\frac{I_m^2}{T^3}\left[\frac{1}{3}\, t^3\right]_0^T$$

$$I_{RMS}^2 = \frac{I_m^2}{3}$$

$$I_{RMS} = \sqrt{\frac{I_m^2}{3}}$$

$$I_{RMS} = \frac{I_m}{\sqrt{3}}$$

The result in itself is not important; but the method of arriving at a solution is. The R.M.S. current is the equivalent direct current which will give the same heating effect as the complex wave form. Since heating effect is $I_{DC}^2 R$,

that is, proportional to D.C. equivalent current squared, then the average is found of the SQUARE of the current wave form.

Example 3.2.2

For the <u>rectified sine wave</u> (or for an unrectified sine wave, as the result is the same)

$$I_{RMS}^2 = \frac{1}{\pi}\int_0^\pi I_m^2 \sin^2 \omega t\; d(\omega t)$$

Now $\qquad \sin^2 \omega t = \dfrac{1 - \cos 2\,\omega t}{2}$

Then $\qquad I_{RMS}^2 = \dfrac{I_m^2}{\pi}\displaystyle\int_0^\pi \left(\frac{1 - \cos 2\,\omega t}{2}\right) d(\omega t)$

$$= \frac{I_m^2}{2\,\pi}\left[\,\omega t - \frac{1}{2}\sin 2\,\omega t\,\right]_0^\pi$$

$$= \frac{I_m^2}{2\,\pi}\left[\,(\pi - 0) - (0 - 0)\,\right]$$

$$I_{RMS}^2 = \frac{I_m^2}{2}$$

$$I_{RMS} = \frac{I_m}{\sqrt{2}}$$

or the well-known result

$$I_{RMS} = 0.707\, I_m$$

3.3 R.M.S. VALUE OF A COMPLEX WAVEFORM

It can be show that a periodic waveform of complex shape may be expressed as a series of sine waves and cosine waves plus a possible d.c. component as indicated below:

Complex waveform Period $T = \frac{2\pi}{\omega}$ which is the period of the fundamental (first) harmonic.

$$i(t) = I_o + A_1 \cos \omega t + A_2\ 2\ \omega t + A_3 \cos 3\ \omega t$$
$$+ A_4 \cos 4\ \omega t + \ldots..$$
$$+ B_1 \sin \omega t + B_2 \sin 2\ \omega t + B_3 \sin 3\ \omega t$$
$$+ B_4 \sin 4\ \omega t + \ldots..$$

The R.M.S. value of this series is given by:

$$I_{RMS} = \sqrt{I_o^2 + \frac{1}{2}(A_1^2 + A_2^2 + A_3^2 + A_4^2 + \ldots.) + \frac{1}{2}(B_1^2 + B_2^2 + B_3^2 + B_4^2 + \ldots.)}$$

$$= \sqrt{(I^2_{DC} + A_1{}^2_{RMS} + A_2{}^2_{RMS} + \ldots + B_1{}^2_{RMS} + B_2{}^2_{RMS} + \ldots)}$$

$$= \sqrt{\text{Sum of (RMS)}^2 \text{ of Component Waveforms}}$$

Example 3.3.1

Determine the R.M.S. value for the periodic waveform given by the following equation:

$$i(t) = 10 + 15 \sin 100t + 5 \sin 200t \quad \text{amperes}$$

Solution:

$$I_{RMS} = \sqrt{10^2 + \frac{1}{2}(15^2 + 5^2)}$$

$$= \boxed{15 \text{ amperes}}$$

3.4 FORM FACTOR

The form factor of a waveform is the ratio of the effective (R.M.S.) value to the average value.

$$\text{FORM FACTOR} = \frac{\text{R.M.S. value}}{\text{Average value}}$$

$$= \frac{I_{RMS}}{I_{AVE}}$$

This factor gives some indication of the shape of a waveform and is of some use in industrial processes.

Example of form factors are given below:

Waveshape	Form Factor
Square wave (rectified)	1
Sine wave (rectified)	1.11
Triangular wave	1.15

Example 3.4.1

A voltage waveform is represented by the equation:

$$v(t) = 10 - 10 \sin \omega t \text{ volts}$$
$$= \text{D.C. component + a.c. component}$$

Determine the following:

(a) the average value;

(b) the R.M.S. vlaue

(i) by use of integration;
(ii) by use of formula.

(c) the form factor

Solution:

(a) $V_{AVE} = \frac{1}{T}\int_0^T v(t)\ dt$ (where T is the period of the wave $\omega T = 2\pi$)

$= \frac{1}{T}\int_0^T (10 - 10 \sin \omega t)\ dt$

$= \frac{1}{T}\left[10t + \frac{10}{\omega} \cos \omega t \right]_0^T$

$= \frac{1}{T}\left[(10T + \frac{10}{\omega} \cos \omega T) - (0 + \frac{10}{\omega} \cos 0) \right]$

$= \frac{1}{T}\left[10T + \frac{10}{\omega} \cos 2\pi - \frac{10}{\omega} \cos 0 \right]$

$= \underline{10 \text{ volts}}$ = D.C. component of $v(t)$

Alternatively V_{AVE} = SUM OF AVE VALUES OF COMPONENT WAVEFORMS

Alternatively we have

$V_{AVE} = \frac{1}{2\pi}\int_0^{2\pi} v(\theta)\ d\theta$ (where $\theta = \omega t$)

$= \frac{1}{2\pi}\int_0^{2\pi} (10 - 10 \sin \theta)\ d\theta$

$= \frac{1}{2\pi}\left[10\ \theta + 10 \cos \theta \right]_0^{2\pi}$

$= \frac{1}{2\pi}\left[(20\pi + 10 \cos 2\pi) - (0 + 10 \cos 0) \right]$

$\underline{V_{AVE} = 10 \text{ volts}}$

(b) (i) $V_{RMS} = \sqrt{\frac{1}{T}\int_0^T [\ v(t)\ ]^2\ dt}$

Alternatively

$V^2_{RMS} = \frac{1}{2\pi}\int_0^{2\pi} [\ v(\theta)\ ]^2\ d\theta$

$= \frac{1}{2\pi}\int_0^{2\pi} [\ 10(1 - \sin \theta)\ ]^2\ d\theta$

$= \frac{100}{2\pi}\int_0^{2\pi} (1 - 2 \sin \theta + \sin^2 \theta)\ d\theta$

$= \frac{50}{\pi}\int_0^{2\pi} [\ 1 - 2 \sin \theta + \frac{1}{2}\ (1 - \cos 2\theta)\ ]\ d\theta$

$= \frac{50}{\pi}\left[\theta + 2 \cos \theta + \frac{\theta}{2} - \frac{\sin 2\theta}{4} \right]_0^{2\pi}$

$= \frac{50}{\pi}\left[(\frac{3}{2} \times 2\pi + 2 - 0) - (0 + 2 - 0) \right]$

$= 150$

$\underline{V_{RMS} = 12.25 \text{ volts}}$

(ii) $V_{RMS} = \sqrt{\text{SUM OF (RMS)}^2 \text{ OF COMPONENT WAVEFORMS}}$

$= \sqrt{10^2 + (\frac{10}{\sqrt{2}})^2}$

$= \sqrt{150}$

$\underline{V_{RMS} = 12.25 \text{ volts}}$

(c) FORM FACTOR $= \dfrac{V_{RMS}}{V_{AVE}}$

$$= \frac{12.25}{10}$$

$$= \underline{1.225}$$

CIRCUIT TRANSIENTS

When a circuit is switched from one condition to another, the period when the currents and voltages are changing from one steady state condition to another steady state condition is referred to as the transient.

In this section we shall consider the circuit transients associated with series circuits containing resistance, inductance and capacitance. The linear differential equation with constant coefficients that describes the reaction to a circuit change has a two part general solution, the sum of the complimentary function and the particular function.

$$i = i_c + i_p$$

where $\quad = \text{C.F.} + \text{P.I.}$

i_c = complimentary function (i.e. the transient) contains the arbitrary constants

i_p = particular integral function (i.e. the steady state component) or 'final' current

5.1 RESPONSE OF RL AND RC CIRCUITS TO DC VOLTAGES

5.1.1 RL CIRCUIT

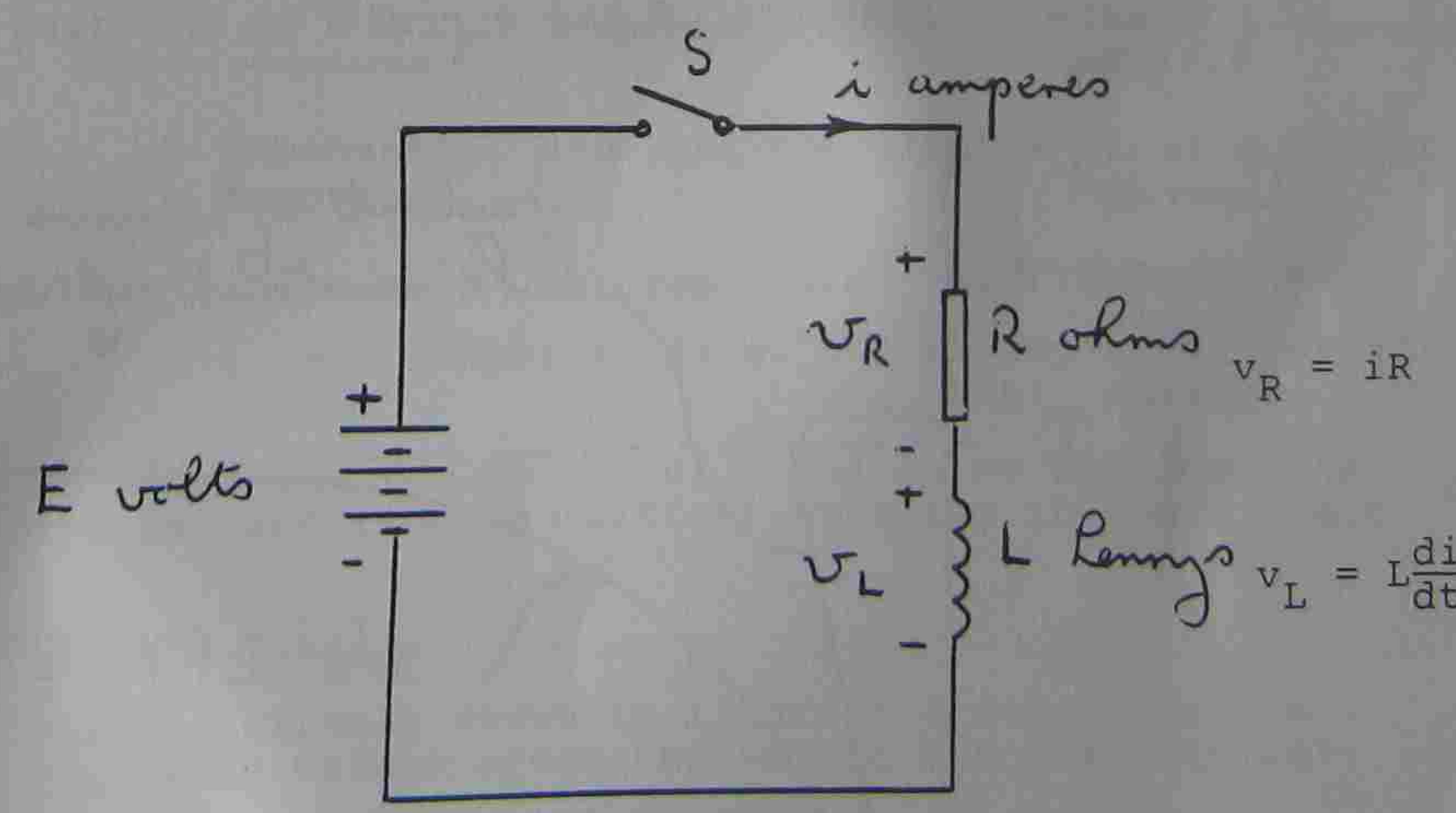


Figure 5.1

After closing switch S in figure 5.1 we have the following:

$$v_R + v_L = E$$

or $$\boxed{iR + L\frac{di}{dt} = E}$$

Initial condition
$v_L = E \quad v_R = 0$
$i = 0$ when $t = 0$
defines the arbitrary constant.

The solution to this first order differential equation is

$$\boxed{i = I(1 - e^{-\frac{R}{L}t})} \quad \text{amperes}$$

$iR + L\frac{di}{di} = E$

$i = I(1 - e^{-R/L\,t})$

where i = instantaneous value of current
I = final value of current = i_p
t = time after closing switch S.

$$i_c = -Ie^{-\frac{R}{L}t}$$

The solution of the differential equation is detailed in example 5.1.3.2

Time Constant

The time constant of a circuit gives an indication of the rate of response of the circuit to changing conditions.

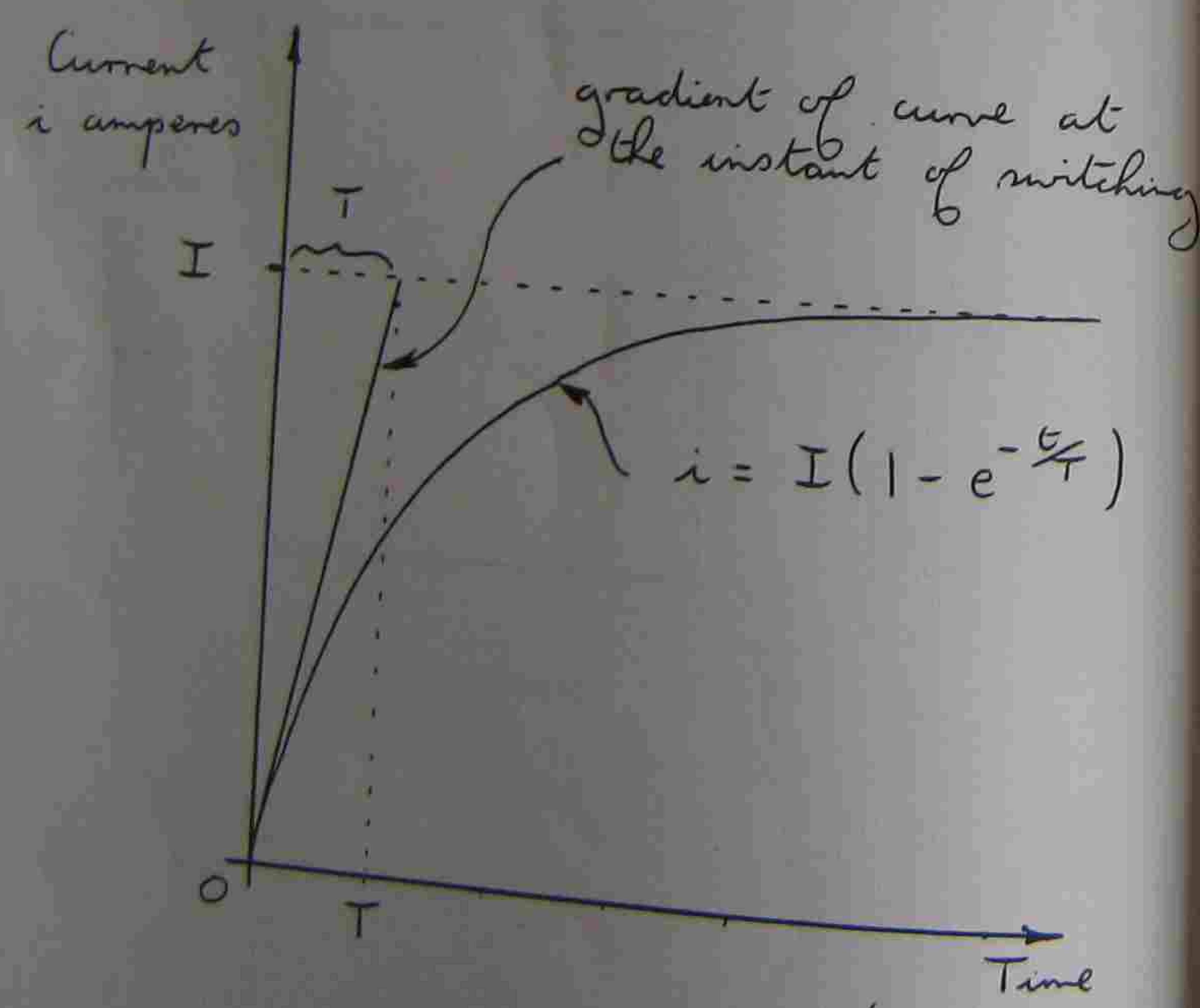


Figure 5.2

Time constant = T = measure of the initial rate of change of current.

For a circuit containing resistance and inductance the time constant is given by:-

$$\boxed{T = \frac{L}{R}} \quad \text{seconds}$$

Since $\frac{I}{T} = \frac{di}{dt}$ initially

$$= \frac{v_L}{L} = \frac{E}{L} = \frac{IR}{L}$$

thus $T = \frac{L}{R}$

The General Solution

The equation of the current in a circuit will depend upon the switching arrangements and will be of the following form:-

$$\boxed{i = Ae^{-\frac{t}{T}} + I_p}$$ amperes. A, is the arbitrary constant

= Exponential decaying component + final constant component

where $i_c = Ae^{-\frac{t}{T}}$ amperes (i.e. the transient)
$i_p = I_p$ amperes (i.e. the steady state current or final current)

To determine the values of A and I_p we consider the initial and final conditions existing in the circuit.

Example 5.1.1.1

For the circuit shown in figure 5.3 determine the following values after the switch has been closed:-

(a) the final value of current;
(b) the initial value of current;
(c) the time constant of the circuit;
(d) the equation of the current;
(e) the initial rate of change of current.

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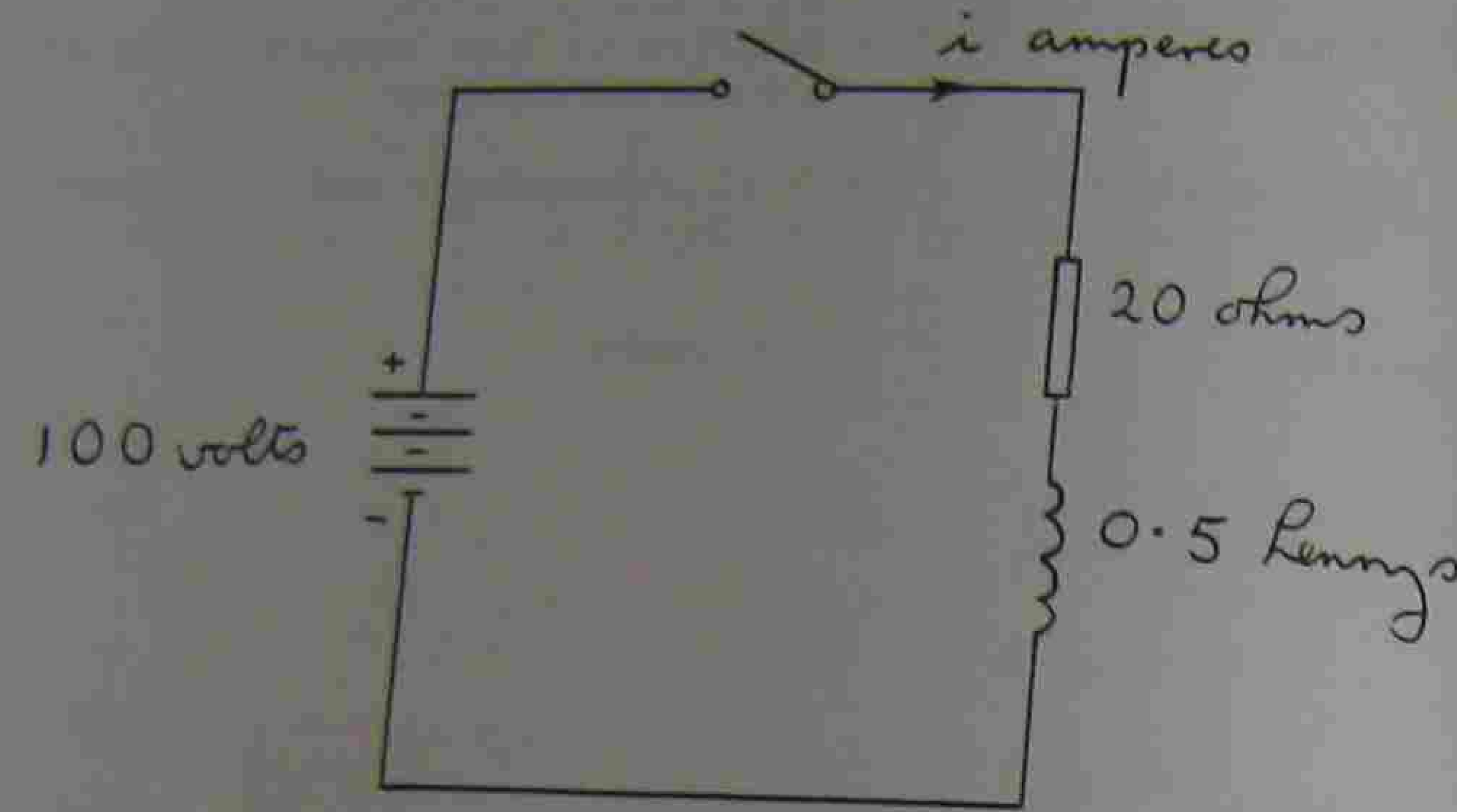

Figure 5.3

Solution:

(a) Final value of current $I = \frac{E}{R} = \frac{100V}{20\Omega} = \underline{5A}$

(b) Initial value of current = $\underline{0A}$

(c) Time constant = $\frac{L}{R} = \frac{0.5H}{20\Omega} = \underline{25ms}$

(d) $i = I(1 - e^{-\frac{t}{T}})$ amperes

$\therefore$ $\underline{i = 5(1 - e^{-40t})}$ amperes.

(e) Rate of change of current = $\frac{di}{dt}$

$\frac{di}{dt} = -5 \times (-40)\ e^{-40t}$

$= 200e^{-40t}$ A/s

Initial rate of change of current

$= \underline{200\text{ A/s}} = \frac{100V}{0.5H}$

$iR + L\frac{di}{dt} = E$

$\underline{t=0}$ $i=0$ $\therefore L\frac{di}{dt} = E$

$\frac{di}{dt} = \frac{E}{L} = \frac{100}{0.5H}$

Example 5.1.1.2

(a) Determine the equation of the current in figure 5.4 after switching to position 2. Assume that the steady state current had been attained in position 1.

(b) Sketch the current on suitable axes.

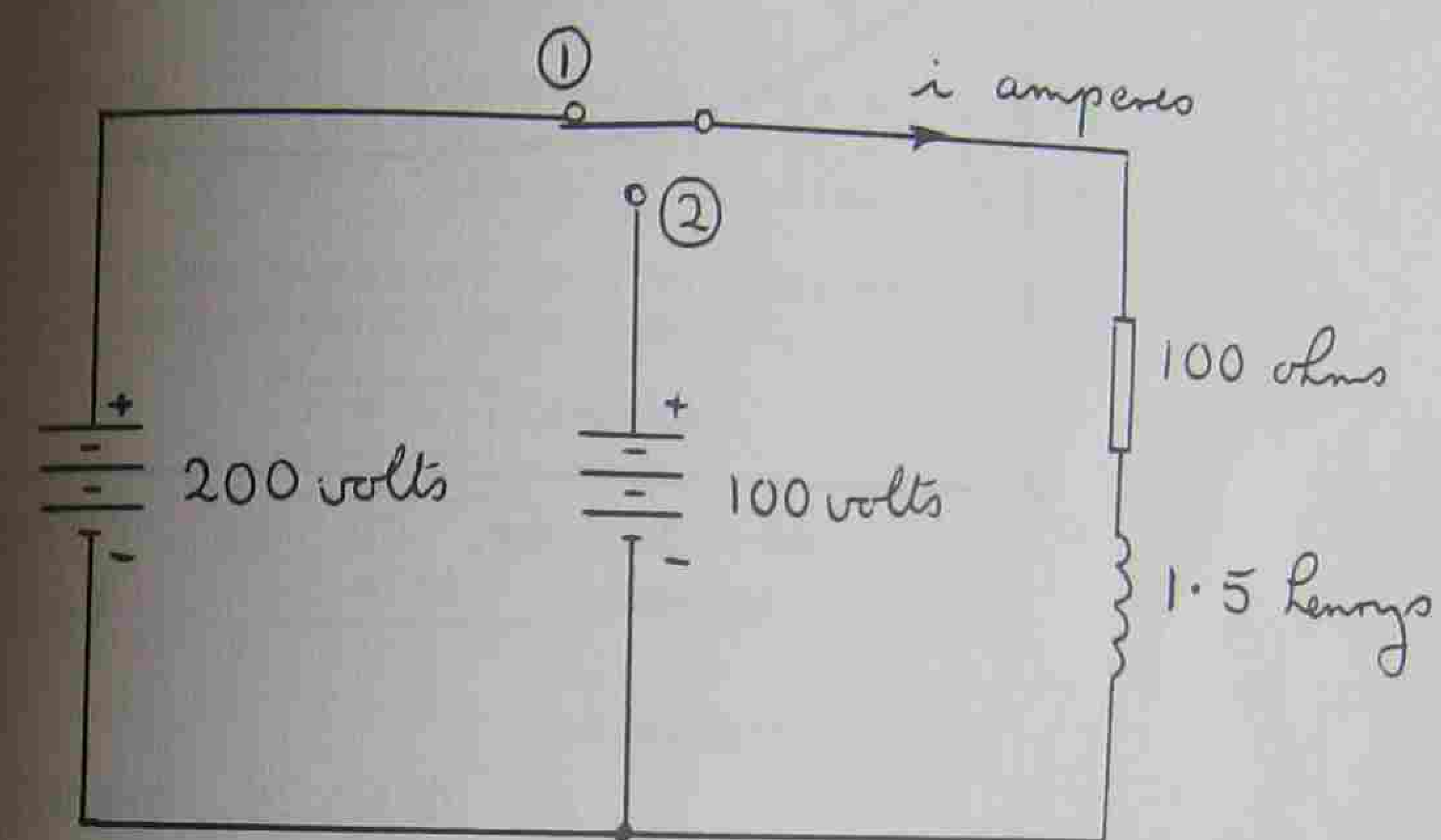

Figure 5.4

Solution

(a) In general:

$$\boxed{i = Ae^{-\frac{t}{T}} + I_p} \text{ amperes.}$$

$T = \frac{L}{R} = \frac{1.5H}{100\Omega} = 15ms$

$i = Ae^{-66.67t} + I_p$ amperes

Initial current $= \frac{200V}{100\Omega} = 2$ amperes

Final current $= \frac{100V}{100\Omega} = 1$ ampere $= I_p$

i.e. $2 = A + I_p$

$\therefore$ $A = I_p = 1$ ampere

$\underline{i = 1 + e^{-66.67t}}$ amperes

(a) $Y = \underline{x^2 + 2x + 1}$

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(b)

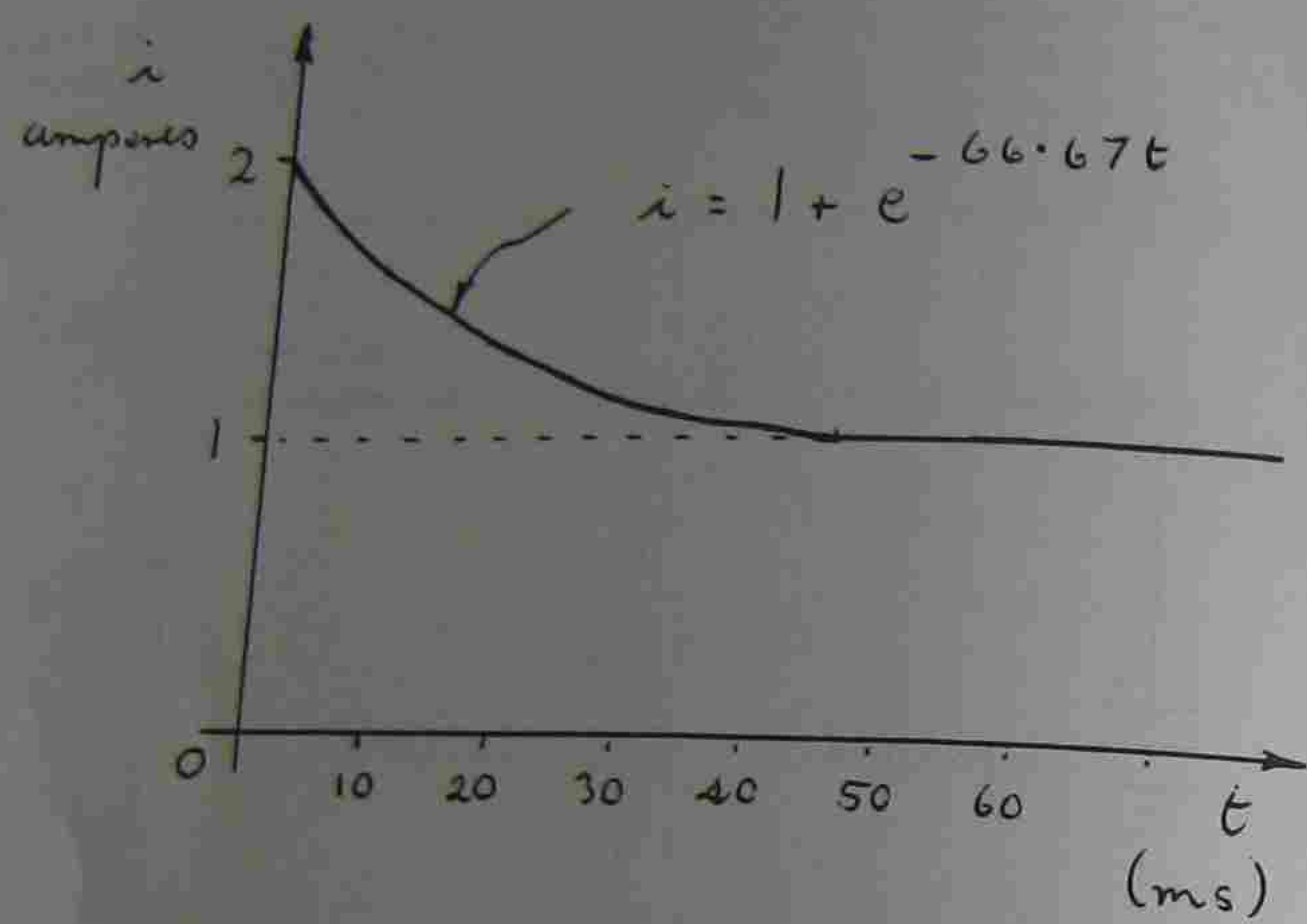
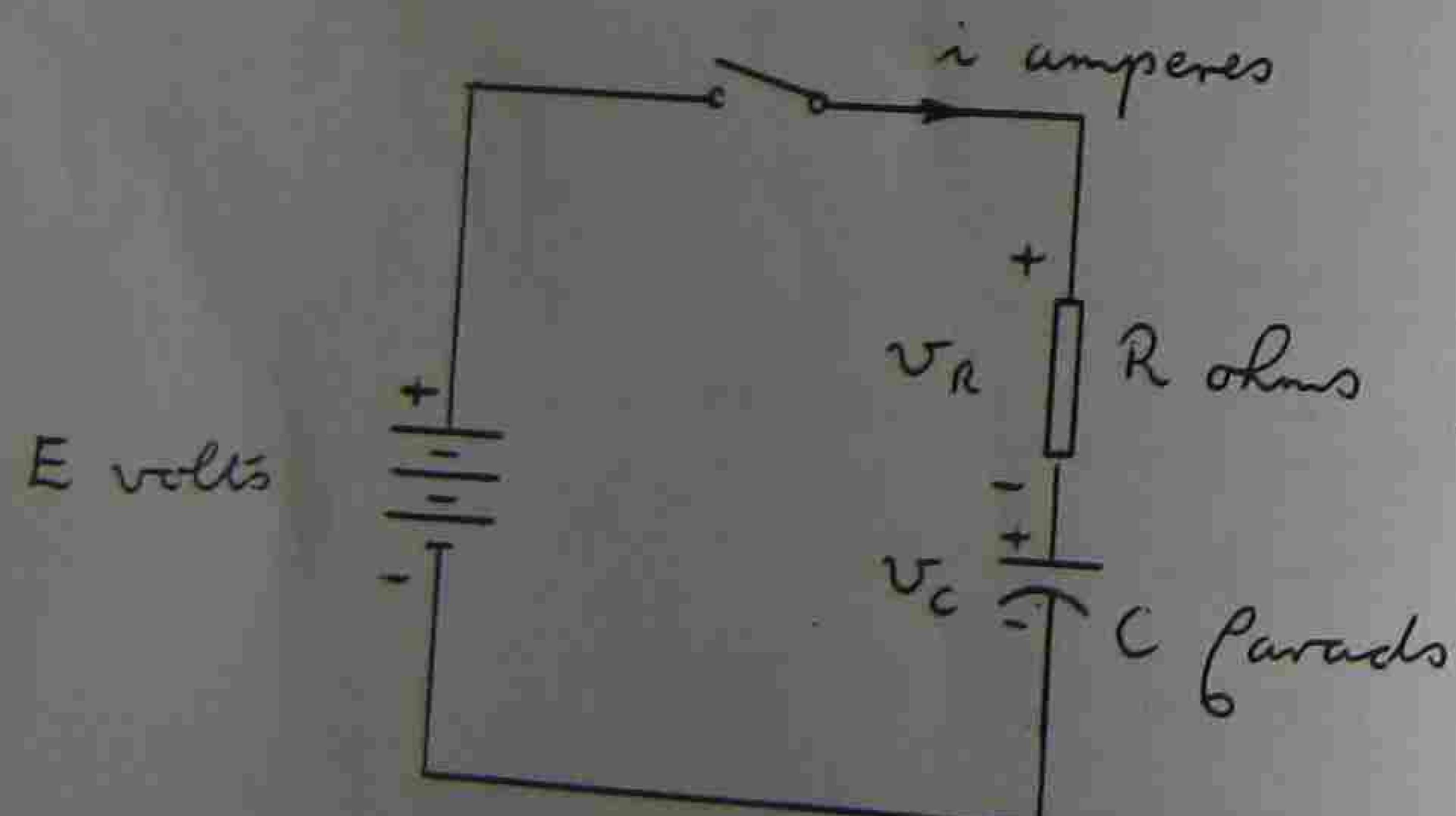


Figure 5.5

5.1.2 RC CIRCUIT



$v_R = iR$

$v_c = \frac{q}{C}$

$v_c = \frac{1}{C}\int i\,dt$

Figure 5.6

After closing switch S in figure 5.6 we have the following:

$$v_R + v_c = E$$

or $$iR + \frac{1}{C}\int i\,dt = E$$

Initial condition

$v_c = 0 \quad v_R = E$

$i = \frac{E}{R} = I$ when $t = 0$

Differentiate with respect to time.

$$R\frac{di}{dt} + \frac{1}{C}i = 0$$

The solution to this first order differential equation is

$$\boxed{i = Ie^{-\frac{t}{RC}}} \quad \text{amperes}$$

where
- i = instantaneous value of current
- I = initial value of current
- t = time after closing switch S.

The time constant in a circuit containing resistance and capacitance is given by:

$$\boxed{T = RC} \quad \text{amperes.}$$

$$\boxed{i = Ie^{-\frac{t}{T}}} \quad \text{amperes}$$

General Solution

In the RC circuit there is no steady state current and therefore the particular function is zero.

i.e. $$\boxed{i_p = 0}$$

and $$\boxed{i = i_c = A\ e^{-\frac{t}{T}}} \quad \text{amperes.}$$

The value of A_1 is determined by the initial conditions.

Example 5.1.2.1

For the circuit shown in fig. 5.6 E is 200 volts, R is 1000 ohms and C is 100 microfarads. If the initial charge on the capacitor is 2 millicoulombs, determine the following after the switch S is closed:-

(a) the initial current;
(b) the time constant of the circuit;
(c) the equation of the current;
(d) the equation of the voltage across the capacitor;
(e) the time required after closing the switch for the capacitor voltage to be 100 volts;
(f) sketch the voltage across the capacitor.

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Solution:

(a) Initial current $= \dfrac{E - v_c}{R}$

$$= \frac{200 - \dfrac{2 \times 10^{-3}}{100 \times 10^{-6}}}{1000} = \frac{(200 - 20)V}{1000\Omega}$$

$A = \underline{0.18 \text{ amperes}}$

(b) Time constant $= T = RC$
$= \underline{0.1 \text{ seconds}}$

(c) $i = Ae^{-\frac{t}{T}}$

$\boxed{i = 0.18e^{-10t}}$ amperes.

(d) Voltage across capacitor $= v_c = E - v_R$
$= E - iR$

$v_c = 200 - 0.18e^{-10t} \times 1000$

$\boxed{v_c = 200 - 180e^{-10t}}$ volts

(e) When $v_c = 100$

$100 = 200 - 180e^{-10t}$

$180e^{-10t} = 200 - 100$

$e^{-10t} = \dfrac{100}{180}$

$-10t = \ln\left(\dfrac{100}{180}\right)$ seconds

$\underline{t = 58.8 \text{ ms}}$

(f)

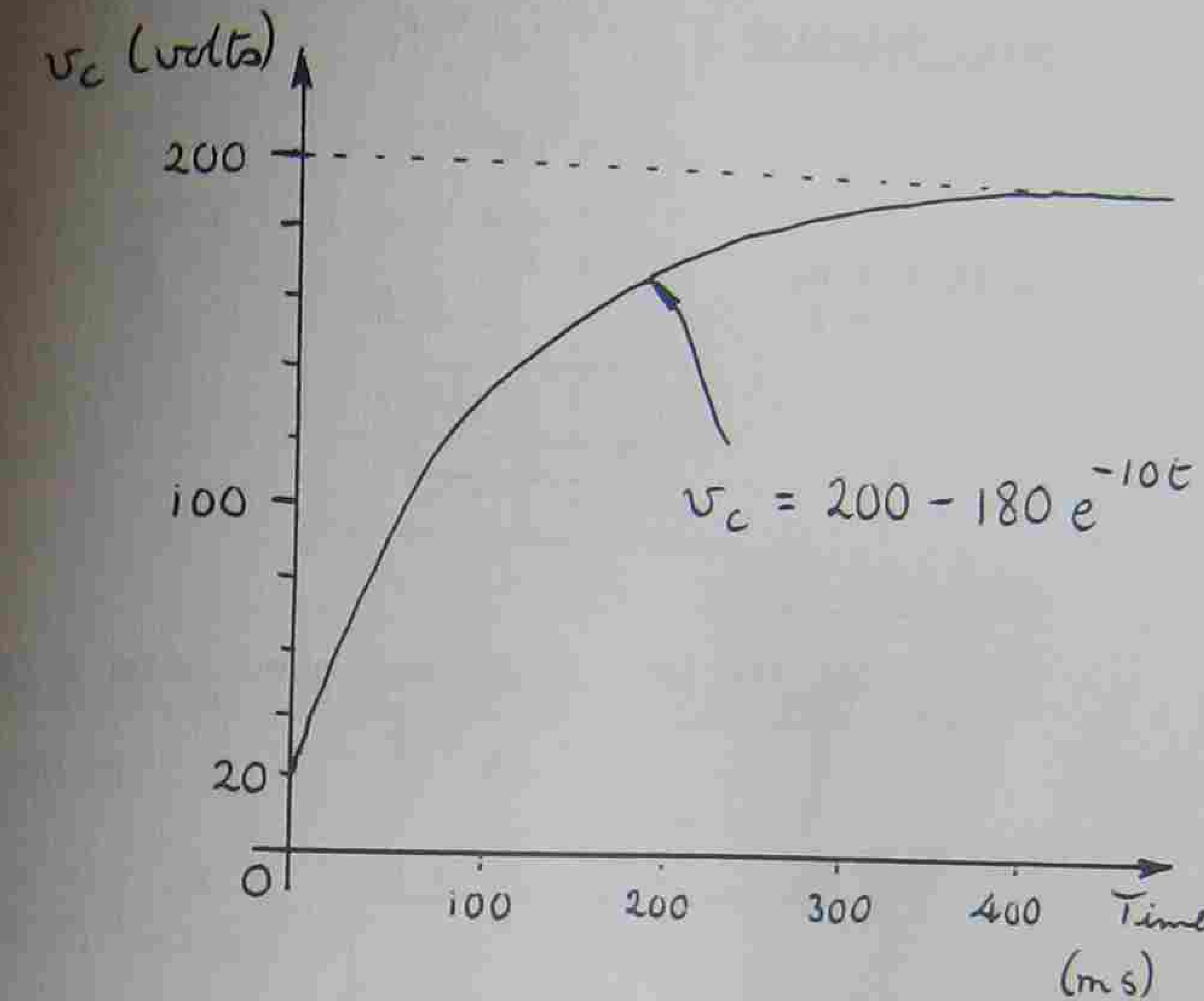


Figure 5.7

5.1.3 SOLVING SIMPLE FIRST ORDER DIFFERENTIAL EQUATIONS BY THE CF + PI METHOD

Example 5.1.3.1

Reconsider the CR CIRCUIT switched onto a DC SUPPLY

$v_R + v_c = E$

$iR + \frac{1}{C}\int i\,dt = E$

$iR + \frac{i}{C} t = E$

Solving DE $\boxed{R\frac{di}{dt} + \frac{1}{C}i = 0}$ At $t = 0$
$i = \frac{E}{R} = I$

$t = 0$

$iR = E$

$\therefore i = \frac{E}{R} = I$

PI i_p = final current $I_p = 0$

CF Subst. $i = Ae^{mt}$ A is the arbitrary constant.

$0 = R\frac{di}{dt} + \frac{1}{C}i$

$= (Rm + \frac{1}{C})\ Ae^{mt}$

$0 = R\frac{d}{dt}Ae^{mt} + \frac{1}{C}Ae^{mt}$

$0 = Rm + \frac{1}{C}$

$0 = (Rm + \frac{1}{C})\ Ae^{mt}$

$m = -\frac{1}{CR}$

$\therefore m = -\frac{1}{RC}$

$i_c = Ae^{-\frac{1}{CR}t} = Ae^{-\frac{t}{T}}$

$\therefore i_c = Ae^{mt} = Ae^{-\frac{t}{RC}}$

$t = 0,\ i = I \Rightarrow I = A$

Total Solution $\quad i = i_c + i_p$

$$= Ae^{-\frac{t}{T}}$$

Subst $t = o$, $i = I$ $\qquad A = I$

$$\boxed{i = Ie^{-\frac{t}{T}}}$$

Example 5.1.3.2

Reconsider the LR CIRCUIT switched onto a DC SUPPLY.

$$v_L + v_R = E$$

Solving DE $\quad L\frac{di}{dt} + Ri = E \qquad$ At $t = o$, $i = o$

CF

Subst. $i = Ae^{mt}$ into DE with RHS = o
A is the arbitrary constant.

$$o = L\frac{di}{dt} + Ri$$

$$= (Lm + R)\,Ae^{mt}$$

$$o = Lm + R$$

$$m = -\frac{R}{L}$$

$$i_c = Ae^{-\frac{R}{L}t} = Ae^{-\frac{t}{T}}$$

PI

Since the RHS of the DE is a constant voltage E, the forcing function, then assume a solution of the form.

$$i_p = \text{final current } I_p$$

then $\quad \frac{di}{dt} = o$

Subst. into DE $\quad L(o) + R(I_p) = E$

$$RI_p = E$$

$$i_p = I_p = \frac{E}{R} = I$$

Total Solution

$$i = i_c + i_p$$

$$= Ae^{-\frac{t}{T}} + I$$

Subst. $t = o$, $i = o$. $\qquad o = A + I$

$$\underline{A = -I}$$

$$i = -Ie^{-\frac{t}{T}} + I$$

$$i = I(1 - e^{-\frac{t}{T}})$$

5.2 RESPONSE OF RL AND RC CIRCUITS TO AC VOLTAGES

5.2.1 RL CIRCUIT

$e = E_m \sin(wt + \Theta)$ volts

$v_R = iR$

$v_L = L\frac{di}{dt}$

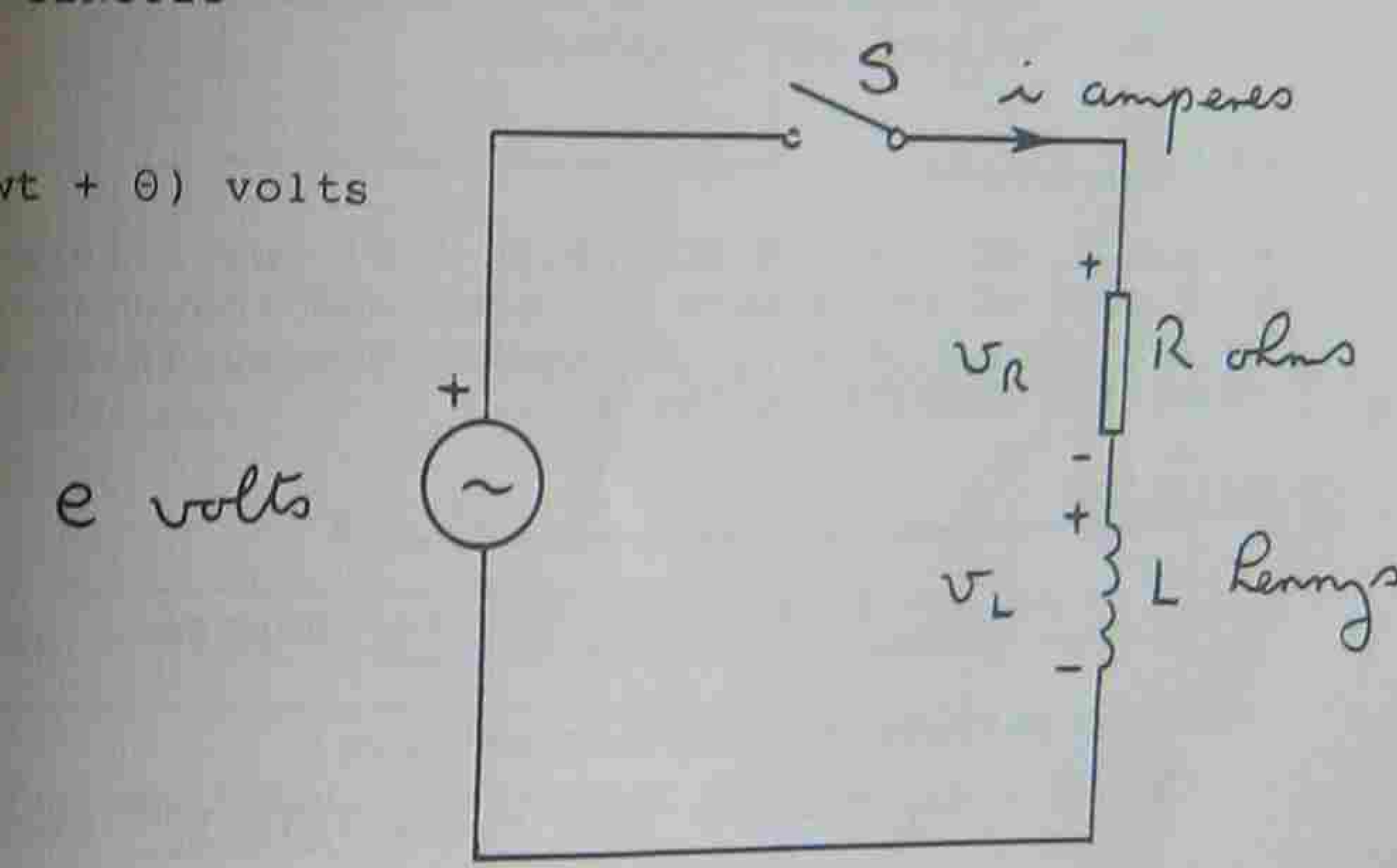


Figure 5.8

After closing the switch S in figure 5.8, we have the following:

$$v_R + v_L = e$$

$$\boxed{iR + L\frac{di}{dt} = E_m \sin(wt + \Theta)}$$

The solution to this first order differential equation has a complimentary function and a particular function.

$$i = i_c + i_p$$

where $i_c = Ae^{-\frac{t}{T}}$ amperes

$i_p = I_p \sin(wt + \theta - \varphi)$ amperes.

The constant A is determined by the conditions that apply at the time of switching and the particular function (i_p) is the steady state current in the circuit.

A General Solution

$$\boxed{i = Ae^{-\frac{t}{T}} + I_p \sin(wt + \theta - \varphi)} \text{ amperes}$$

= exponential decaying component + final sinusoidal component

Example 5.2.1.1

An e.m.f. of e = 100 sin (314t + θ) volts is applied to a coil of resistance 200 ohms and inductance 0.5 henrys when θ is 30°. Determine the equation of the resulting current in the coil.

Solution

$$\boxed{i = Ae^{-\frac{t}{T}} + I_p \sin(wt + \theta - \varphi)} \text{ amperes}$$

For the particular function we have:

$$\bar{Z} = R + jX_L = 200 + j157 = 254.3\underline{/38.13^\circ} \text{ ohms}$$

$$\bar{E}_m = 100\underline{/30^\circ} \text{ volts} \qquad \text{note } \tan\varphi = \frac{X}{R}$$

$$\bar{I}_p = \frac{\bar{E}_m}{\bar{Z}} = \frac{100\underline{/30^\circ}}{254.3\underline{/38.13^\circ}} = 0.3932\underline{/-8.13^\circ} \text{ amperes.}$$

$\therefore i_p = 0.3932 \sin(314t - 8.13^\circ)$ amperes.

Also, $T = \frac{L}{R} = \frac{0.5}{200}$ s = 2.5 ms

$\therefore i_c = Ae^{-400t}$

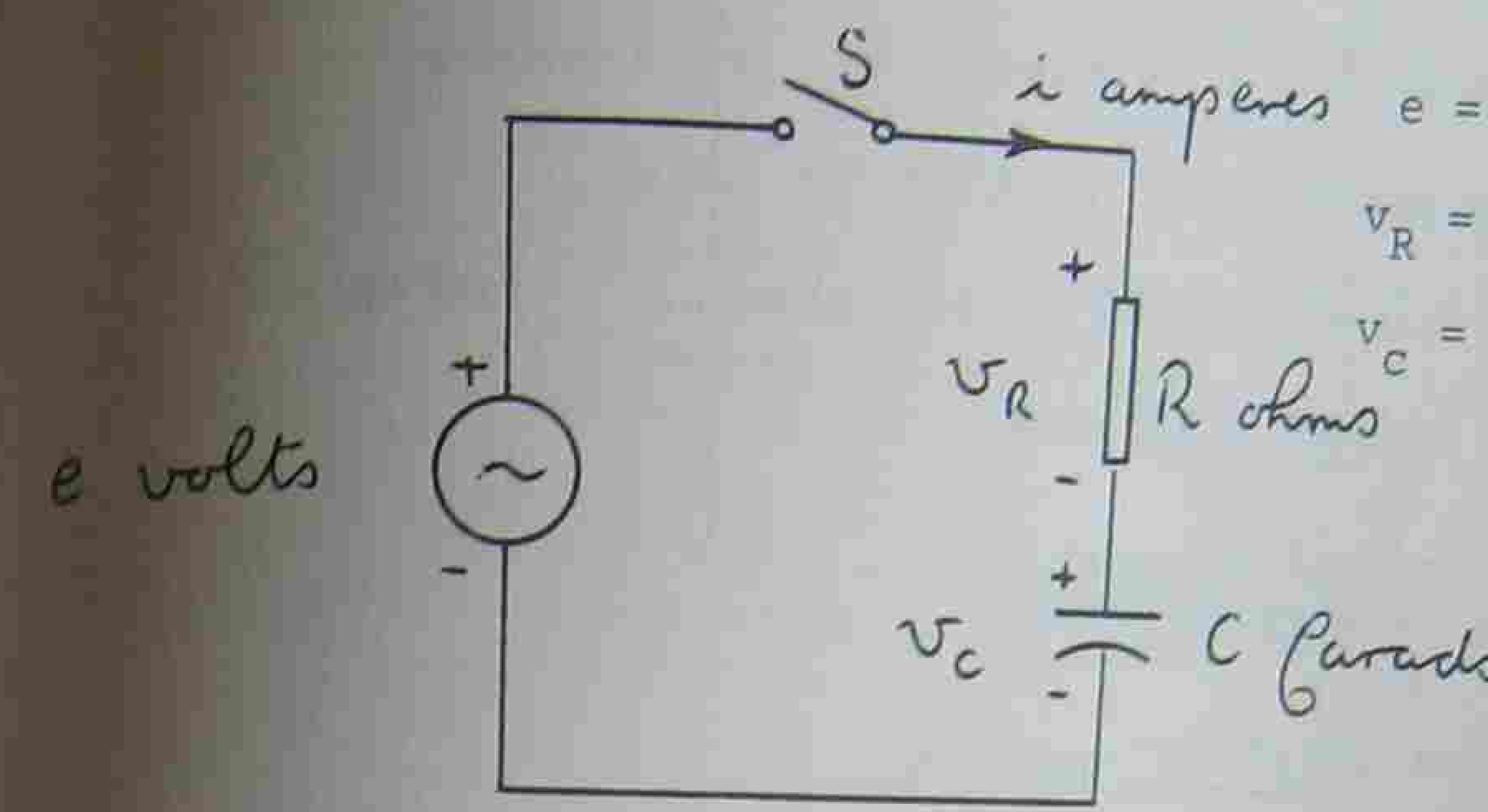
At t = 0; i = 0

$\therefore 0 = A + 0.3932 \sin(-8.13^\circ)$

$A = 0.0556$

$$i = 0.0556e^{-400t} + 0.3932 \sin(314t - 8.13^\circ) \text{ amperes.}$$

5.2.2 RC CIRCUIT



$e = E_m \sin(wt + \theta)$ volts

$v_R = iR$

$v_C = \frac{1}{C}\int idt$

Figure 5.9

After closing the switch S in figure 5.9 we have the following:

$$v_R + v_C = e$$

$$\boxed{iR + \frac{1}{C}\int idt = E_m \sin(wt + \theta)}$$

Differentiate with respect to time.

$$R\frac{di}{dt} + \frac{1}{C}i = E_m w \cos(wt + \theta)$$

The solution to this first order differential equation is the same as for the RL circuit.

$$i = i_c + i_p$$

where $i_c = Ae^{-\frac{t}{T}}$ amperes

$i_p = I_p \sin(wt + \theta - \varphi)$ amperes.

The General Solution

$$\boxed{i = Ae^{-\frac{t}{T}} + I_p \sin(wt + \theta - \varphi)} \text{ amperes.}$$

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Example 5.2.2.1

An e.m.f. of $e = 200 \sin(500t + \theta)$ volts is applied to an RC circuit where R = 300 ohms and C = 5 microfarads when θ is 60°. Determine the equation of the current in the circuit if the initial charge on the capacitor is 250 microcoulombs.

Solution:

$$i = Ae^{-\frac{t}{T}} + I_p \sin(wt + \theta - \varphi) \text{ amperes.}$$

For the particular function we have:

$$\bar{Z} = R - jX_C = 300 - j400 \text{ ohms} \qquad \tan\varphi = \frac{X}{R}$$

$$= 500\underline{/-53.13°} \text{ ohms.}$$

$$\bar{I}_p = \frac{\bar{E}_m}{\bar{Z}} = \frac{200\underline{/60°}}{500\underline{/-53.13°}}$$

$$= 0.4\underline{/113.13°} \text{ amperes.}$$

Also $T = RC = 1.5 \times 10^{-3}$ s = 1.5 ms

$$\therefore \quad i = Ae^{-666.7t} + 0.4 \sin(500t + 113.13°)$$

Initial voltage across capacitor $= \frac{q}{C} = 50$ volts.

$$\text{Initial current} = \frac{E_m \sin 60° - 50}{R}$$

$$= 0.4107 \text{ amperes.}$$

$\therefore$ At t = 0; $0.4107 = A + 0.4 \sin(113.13°)$

$$\therefore \quad A = 0.0433$$

$$\boxed{i = 0.0433e^{-666.7t} + 0.4 \sin(500t + 113.13°)} \text{ amperes.}$$

TRANSIENTS IN SERIES RLC CIRCUITS - DC VOLTAGE

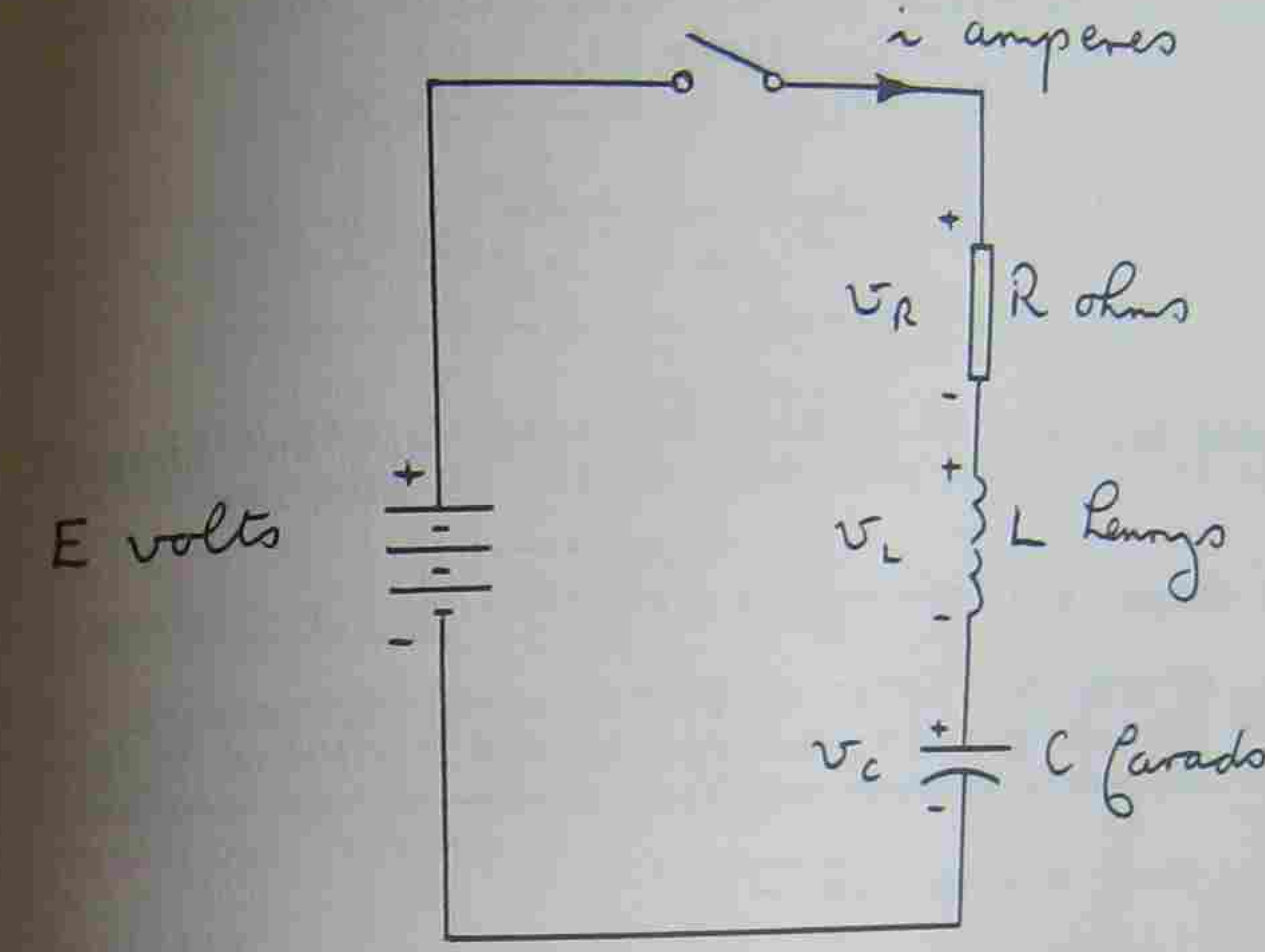


Figure 5.10

In figure 5.10 by applying Kirchhoff's Voltage Law after the switch is closed, we have:

$$v_R + v_L + v_C = E$$

$$v_R = Ri \;; \quad v_L = L\frac{di}{dt} \;; \quad v_C = \frac{q}{C} = \frac{1}{C}\int i\,dt$$

$$\therefore \quad L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = E$$

Differentiate with respect to time.

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}\,i = 0$$

Divide throughout by L.

$$\boxed{\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}\,i = 0}$$

Compare form

$$\boxed{a\frac{d^2i}{dt^2} + b\frac{di}{dt} + ci = 0}$$

This is a second order differential equation and there are three possible solutions:

$$i = i_c + i_p \quad \text{since} \quad \boxed{i_p = 0}$$

$$i = i_c \quad \text{subst.} \quad i_c = Ae^{mt}$$

$$am^2 + bm + c = 0$$

Roots for $m = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ are real and unequal, equal, or complex.

By comparison $m = -\dfrac{R}{2L} \pm \sqrt{\left(\dfrac{R}{2L}\right)^2 - \dfrac{1}{LC}}$

Let $\boxed{\alpha = \dfrac{R}{2L}}$; $\boxed{w_o = \dfrac{1}{\sqrt{LC}}}$; $\boxed{\beta = \sqrt{|\alpha^2 - w_o^2|}}$

then $m = -\alpha \pm \sqrt{\alpha^2 - w_o^2}$

$m = (-\alpha \pm \beta)$ or $(-\alpha \pm j\beta)$

<u>Condition</u> <u>General Solutions</u>

1) Overdamped $\alpha > w_o$ $i = e^{-\alpha t}(A_1e^{\beta t} + A_2e^{-\beta t})$
contains 2 arbitrary constants A_1 and A_2.

2) Critically damped $\alpha = w_o$ $i = e^{-\alpha t}(A_1 + A_2t)$

3) Underdamped $\alpha < w_o$ $i = e^{-\alpha t}(A_1\sin\beta t + A_2\cos\beta t)$
(w_o is the natural frequency)

The current in the circuit may take the shape of the curves shown in figure 5.11.

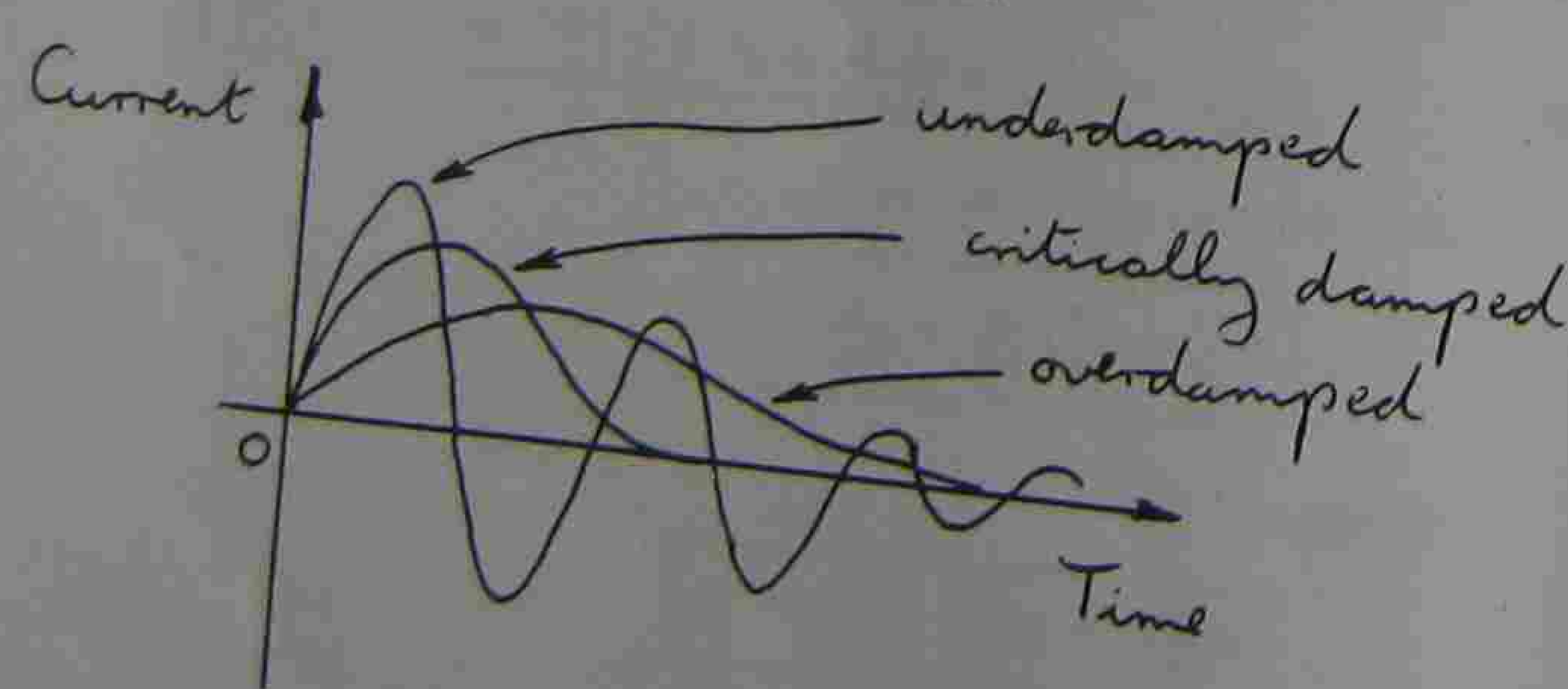


Figure 5.11

<u>Case 1 - Overdamped ($\alpha > w_o$)</u>

The solution to the second order differential equation is

$$i = A_1e^{(-\alpha + \beta)t} + A_2e^{(-\alpha - \beta)t}$$

$$\boxed{i = e^{-\alpha t}(A_1e^{\beta t} + A_2e^{-\beta t})}$$

To obtain the constants A_1 and A_2 we consider the initial values of current and rate of change of current.

<u>Example 5.3.1</u>

In figure 5.10 E is 100 volts, R is 500 ohms, L is 0.25 henrys and C is 100 microfarads. Determine the equation of the current if the initial charge on the capacitor is zero.

<u>Solution</u>

$$\alpha = \frac{R}{2L} = \frac{500}{2 \times 0.25} = 1000$$

$$w_o = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.25 \times 100 \times 10^{-6}}} = 200$$

$$\beta = \sqrt{|\alpha^2 - w_o^2|} = 979.8$$

$\alpha > w_o$ therefore overdamped.

$$i = A_1e^{(-\alpha + \beta)t} + A_2e^{(-\alpha - \beta)t}$$

$$= A_1e^{-20.2t} + A_2e^{-1979.8t} \text{ amperes.}$$

At $t = 0$; $i = 0$ and $L\dfrac{di}{dt} = E = 100$ volts.

$\therefore \dfrac{di}{dt} = \dfrac{E}{L} = \dfrac{100}{0.25}$

$$\therefore A_1 + A_2 = 0$$

$$A_1 = -A_2$$

Hence $i = -A_2e^{-20.2t} + A_2e^{-1979.8t}$

$$\frac{di}{dt} = 20.2A_2e^{-20.2t} - 1979.8A_2e^{-1979.8t}$$

At $t = 0$; $\frac{di}{dt} = 20.2A_2 - 1979.8A_2 = \frac{100}{0.25}$

$A_2 = -0.2041$

$$\boxed{i = 0.2041e^{-20.2t} - 0.2041e^{-1980t}} \text{ amperes}$$

Case 2 - Critically Damped ($\alpha = w_o$)

The solution to the second order differential equation is

$$\boxed{i = e^{-\alpha t}\,(A_1 + A_2 t)}$$

The constants A_1 and A_2 are obtained by consideration of the initial conditions.

Example 5.3.2

In figure 5.10 E is 200 volts, R is 100 ohms, L is 0.5 henrys and C is 200 microfarads. Determine the equation of the current if the initial charge on the capacitor is 20 millicoulombs.

$$\alpha = \frac{R}{2L} = \frac{100}{1} = 100$$

$$w_o = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.5 \times 200 \times 10^{-6}}} = 100$$

$\alpha = w_o$ therefore critically damped.

$$i = e^{-100t}\,(A_1 + A_2 t) \text{ amperes.}$$

At $t = 0$; $i = 0$; $v_c = \frac{q}{C} = \frac{20mC}{200\mu F} = 100$ volts

$L\frac{di}{dt} = E - v_c = 200 - 100 = 100$ volts

$1(A_1 + A_2 \times 0) = 0$

$\therefore \quad A_1 = 0$

$i = A_2 t\, e^{-100t}$ amperes.

$$\frac{di}{dt} = A_2 e^{-100t} - 100\,A_2 t e^{-100t}$$

At $t = 0$; $\frac{di}{dt} = A_2 = \frac{100}{0.5} = 200$

$$\therefore \quad \boxed{i = 200te^{-100t}} \text{ amperes.}$$

Case 3 - Underdamped ($\alpha < w_o$)

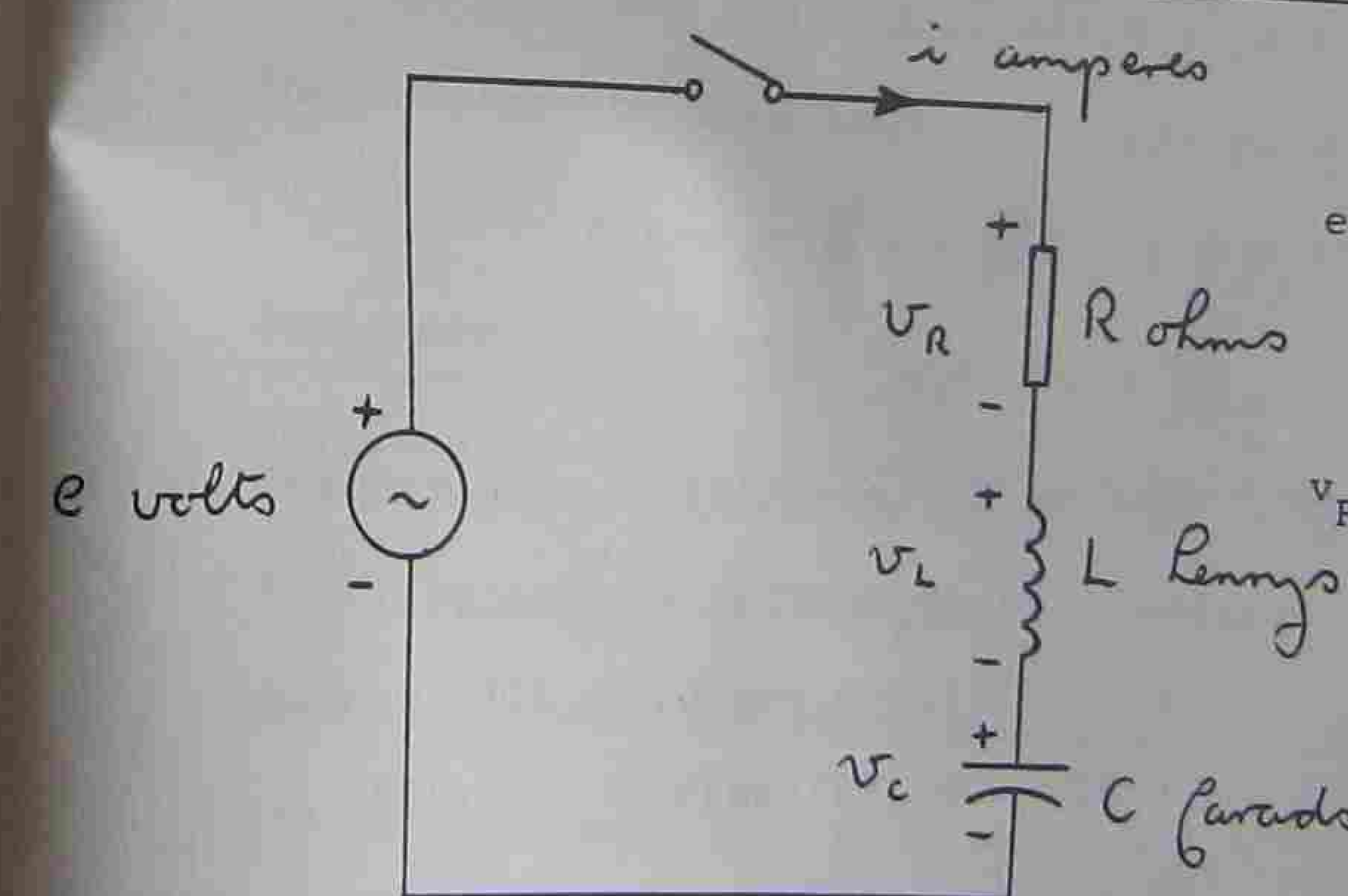
The solution to the second order differential equation is

$$\boxed{i = e^{-\alpha t}(A_1 \sin\beta t + A_2 \cos\beta t)}$$

Alternatively: $i = Ae^{-\alpha t}\sin(\beta t + \gamma)$

The constants A_1 and A_2 or A and γ are obtained by consideration of the initial conditions as shown in the overdamped and critically damped cases.

5.4 RESPONSE OF SERIES RLC CIRCUITS TO AC WAVEFORMS.



$e = E_m \sin(wt + \theta)$vol

$v_R + v_L + v_C = e$

Figure 5.12

When a sinusoidal voltage is applied to a series RLC circuit as shown above, the resulting current in the circuit consists of a transient current and a current that exists after the transient has diminished to zero.

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$$i_c = e^{-100t}(A_1\sin 100t + A_2\cos 100t) \text{ amperes.}$$

For the particular function:

$$\bar{Z} = R + j(X_L - X_c) = 100 + j(250 - 20)\ \Omega$$

$$= 100 + j230\ \Omega$$

$$= 250.8\underline{/1.161}\ \Omega$$

$$\bar{I}_p = \frac{100\underline{/1.571}}{250.8\underline{/1.161}} = 0.3987\underline{/0.4103} \text{ amperes.}$$

$$i_p = 0.3987\sin(500t + 0.4103) \text{ amperes.}$$

$$\therefore \quad i = e^{-100t}(A_1\sin 100t + A_2\cos 100t) + 0.3987\sin(500t + 0.4103) \text{ amperes.}$$

At the instant of closing the switch, $i = 0$ and $e = 100\sin 1.571$ i.e. $e = 100$ volts.

$$\therefore \quad 0 = A_2 + 0.3987\sin 0.4103$$

$$A_2 = -0.159$$

Also at $t = 0$; $L\frac{di}{dt} = e - v_c = 100$

$$\therefore \quad \frac{di}{dt} = 200 \text{ A/s}$$

In general, when we differentiate i with respect to time:

$$\frac{di}{dt} = -100e^{-100t}(A_1\sin 100t + A_2\cos 100t) + e^{-100t}(100A_1\cos 100t - 100A_2\sin 100t) + 500 \times 0.3987\cos(500t + 0.4103)$$

At $t = 0$; $\frac{di}{dt} = -100A_2 + 100A_1 + 182.8 = 200$

$$\therefore \quad A_1 = 0.013$$

$$\boxed{i = e^{-100t}(0.013\sin(100t) - 0.159\cos(100t)) + 0.3987\sin(500t + 0.4103) \text{ amperes.}}$$

The total current is given by the following:-

$$i = i_c + i_p$$

where i_c = complimentary function

i_p = particular function

For the above circuit we have

$$v_R + v_L + v_c = e$$

i.e. $$iR + L\frac{di}{dt} + \frac{1}{C}\int i\,dt = e = E_m\sin(wt + \theta)$$

Differentiate with respect to time.

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = w\ E_m\cos(wt + \theta)$$

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = \frac{w\ E_m}{L}\cos(wt + \theta)$$

The solution to this equation is

$$i = i_c + i_p$$

5.4.1 Complimentary Function (i_c)

The complimentary function is the solution for current if the voltage applied is considered to be constant.

i.e. $$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = 0 \quad \text{subst. } i_c = Ae^{mt}$$

As before, there are three possible solutions:

(1) Overdamped: $i_c = e^{-\alpha t}(A_1e^{\beta t} + A_2e^{-\beta t})$

$$= A_1e^{(-\alpha + \beta)t} + A_2e^{(-\alpha - \beta)t}$$

(2) Critically damped: $i_c = e^{-\alpha t}(A_1 + A_2t)$

(3) Underdamped: $i_c = e^{-\alpha t}(A_1\sin\beta t + A_2\cos\beta t)$

$$= Ae^{-\alpha t}\sin(\beta t + \gamma)$$

... is that current that flows after the transient subsides and may be obtained by considering current and voltage within the frequency domain.

Impedance of circuit $= \bar{Z} = Z\underline{/\varphi}$ $\qquad \tan\varphi = \frac{X}{R}$

Applied e.m.f. $= \bar{E}_m = E_m\underline{/\Theta}$

$$\bar{I}_p = \frac{\bar{E}_m}{\bar{Z}} = \frac{E_m\underline{/\Theta}}{Z\underline{/\varphi}}$$

$$= \frac{E_m}{Z}\ \underline{/\Theta - \varphi}$$

$$= I_p\underline{/\Theta - \varphi}$$

$$i_p = I_p\sin(wt + \Theta - \varphi)$$

Note: It is better to give Θ and φ in radians.

5.4.3 DETERMINATION OF CONSTANTS

$$\alpha = \frac{R}{2L}\ ; \quad w_o = \frac{1}{\sqrt{LC}}\ ; \quad \beta = \sqrt{|\alpha^2 - w_o^2|}$$

The constants A_1 and A_2 in the complimentary function may be obtained by consideration of the initial conditions for current (i) and rate of change of current $(\frac{di}{dt})$.

<u>Example 5.4.3.1</u>

In figure 5.12 R is 100 ohms, L is 0.5 henrys and C is 100 microfarads. If the switch is closed when $e = 100\sin(500t + 1.571)$ volts, determine the current in the circuit if the initial charge on the capacitor is zero.

<u>Solution</u>

$$\alpha = \frac{R}{2L} = 100$$

$$w_o = \frac{1}{\sqrt{LC}} = 141.4 \text{ rad/s}$$

$\alpha < w_o$ therefore underdamped. $\begin{cases} i_c = Ae^{mt} \\ m = -\alpha \pm j\beta \end{cases}$

$$\beta = \sqrt{|\alpha^2 - w_o^2|} = 100$$

DIFFERENTIATION

$$y = x^n$$

$$\frac{dy}{dx} = \frac{d}{dx} x^n$$

$$\boxed{\frac{dy}{dx} = n x^{n-1}}$$

$$y = a x^n$$

$$\frac{dy}{dx} = \frac{d}{dx} (a x^n)$$

$$= a \frac{d}{dx} x^n$$

$$\frac{dy}{dx} = a n x^{n-1}$$

Ex $x^5 \rightarrow \frac{dy}{dx} = ?$

$$\frac{d}{dx} x^5 = 5 x^{5-1} = 5 x^4$$

Ex $0.6 x^7$

$$\frac{d}{dx} (0.6 x^7) = 0.6 \frac{d}{dx} x^7$$

$$= 0.6 \times 7 \times x^{7-1}$$

$$= 4.2 x^6$$

EXERCISE Find $\frac{dy}{dx}$ for

$$2 x^{1.5}$$

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DIFFERENTIATION OF A SUM OF FUNCTIONS

THE DIFFERENTIATION OF A SUM OF FUNCTIONS IS EQUAL TO THE SUM OF THE INDIVIDUAL DIFFERENTIATIONS OF THE FUNCTIONS.

$$y = f_1(x) + f_2(x) + f_3(x)$$

$$\frac{dy}{dx} = \frac{d}{dx} f_1(x) + \frac{d}{dx} f_2(x) + \frac{d}{dx} f_3(x)$$

Ex $y = 5x^3 + 6x^2 + 7$

FIND $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{d}{dx} 5x^3 + \frac{d}{dx} 6x^2 + \frac{d}{dx} 7$$

$$= 5\frac{d}{dx}x^3 + 6\frac{d}{dx}x^2 + 0$$

$$= 5 \times 3\, x^{3-1} + 6 \times 2 \times x^{2-1}$$

$$= 15x^2 + 12x$$

Ex $y = \sin x + \cos x$

$\frac{dy}{dx} = ?$

$$\frac{dy}{dx} = \frac{d}{dx} \sin x + \frac{d}{dx} \cos x$$

$$= \cos x + (-\sin x)$$

$$= \cos x - \sin x$$

$$\boxed{\begin{aligned} \frac{d}{dx} \sin x &= \cos x \\ \frac{d}{dx} \cos x &= -\sin x \end{aligned}}$$

Ex

$$y = \sin 3x + \cos 5x$$

$$\frac{dy}{dx} = \frac{d}{dx}\sin 3x + \frac{d}{dx}\cos 5x$$

$$= \cos 3x \times \frac{d3x}{dx} + (-\sin 5x)\frac{d}{dx}5x$$

$$= \cos x \; 3\frac{\cancel{dx}}{\cancel{dx}} - \sin 5x \; 5\frac{\cancel{dx}}{\cancel{dx}}$$

$$= 3\cos x - 5\sin 5x$$

DIFFERENTIATION OF PRODUCT FUNCTIONS

$$y = u(x) \times v(x)$$

$$\frac{dy}{dx} = u(x)\frac{dv(x)}{dx} + v(x) \times \frac{du(x)}{dx}$$

Ex

$$y = \underbrace{(x+1)}_{u}\underbrace{(x+3)}_{v}$$

$$\frac{dy}{dx} = u\frac{dv(x)}{dx} + v\frac{du(x)}{dx}$$

$$= (x+1)\frac{d}{dx}(x+3) + (x+3)\frac{d}{dx}(x+1)$$

$$= (x+1)\left[\underbrace{\frac{dx}{dx}}_{1} + \underbrace{\frac{d3}{dx}}_{0}\right] + (x+3)\left[\underbrace{\frac{dx}{dx}}_{1} + \underbrace{\frac{d1}{dx}}_{0}\right]$$

$$= (x+1)(1) + (x+3)(1)$$

$$= x+1+x+3$$

$$= 2x+4$$

Ex

$$\frac{dy}{dx} =$$

$$=$$

$x)$

$+3)\ \frac{d}{dx}(x+1)$

$x+3)\left[\frac{dx}{dx} + \frac{d1}{dx}\right]$

$1 \qquad 0$

$)\ (1)$

Ex $\quad y = \underbrace{(x+1)^2}_{u}\ \underbrace{(x+3)^3}_{v}$

$$\frac{dy}{dx} = (x+1)^2 \frac{d}{dx}(x+3)^3 + (x+3)^3 \frac{d}{dx}(x+1)^2$$

$$= (x+1)^2 \times 3(x+3)^{3-1} \frac{d}{dx}(x+3) + (x+3)^3 \times 2(x+1)^{2-1} \frac{d}{dx}(x+1)$$

$$= (x+1)^2 \times 3(x+3)^2 \left[\frac{d}{dx}x + \underbrace{\frac{d3}{dx}}_{0}\right] + (x+3)^3 \times 2(x+1)\left[\frac{d}{dx}x + \underbrace{\frac{d}{dx}1}_{0}\right]$$

$$= 3(x+1)^2(x+3)^2(1) + (x+3)^3 \times 2(x+1)(1)$$

$$= 3(x+1)^2(x+3)^2 + 2(x+3)^3(x+1)$$

$$= (x+1)(x+3)^2\left[3(x+1) + 2(x+3)\right]$$

$$= (x+1)(x+3)^2\left[3x+3+2x+6\right]$$

$$= (x+1)(x+3)^2(5x+9)$$

DIFFERENTIATION OF QUOTIENT FUNCTIONS

$$y = \frac{u(x)}{v(x)}$$

$$\frac{dy}{dx} = \frac{v(x)\frac{du(x)}{dx} - u(x)\frac{dv(x)}{dx}}{(v(x))^2}$$

EX DIFFERENTIATE

$$y = \frac{x}{x+1}$$

$$\frac{dy}{dx} = \frac{(x+1)\frac{d}{dx}x - x\frac{d}{dx}(x+1)}{(x+1)^2}$$

$$\frac{dy}{dx} = \frac{(x+1)\times 1 - x\left[\frac{d}{dx}x + \frac{d}{dx}1\right]}{(x+1)^2}$$

$$= \frac{x+1 - x(1)}{(x+1)^2}$$

$$= \frac{x+1-x}{(x+1)^2}$$

$$= \frac{1}{(x+1)^2}$$

$$y = \frac{x^2+2x+1}{x^2-2x+1}$$

FIND $\frac{dy}{dx}$

ER $\longrightarrow$

=

ER

IE $\longrightarrow$

SUCCESSIVE DIFFERENTIATION

$$y = x^3 + 3x^2 + 4$$

$$\frac{dy}{dx} = \frac{d}{dx}\left[x^3 + 3x^2 + 4\right]$$

$$= \frac{d}{dx}x^3 + \frac{d}{dx}3x^2 + \frac{d}{dx}4]$$

$$= 3x^{3-1} + 3\times 2x^{2-1} + 0$$

$$= 3x^2 + 6x$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[3x^2 + 6x\right]$$

$$= \frac{d}{dx}3\left[x^2 + 2x\right]$$

$$\frac{d^2y}{dx^2} = 3\left[\frac{dx^2}{dx} + \frac{d2x}{dx}\right]$$

$$= 3\left[2x^{2-1} + 2\right]$$

$$= 3\left[2x + 2\right]$$

$$= 6(x+1)$$

pb FIND THE FIRS

$$y = \frac{1}{2x+1}$$

$$\frac{dy}{dx} = \frac{d}{dx}(2$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}[-$$

$$= -2x$$

$$= 4($$

$$= 8$$

pb FIND THE FIRST THREE DIFFERENTIAL COEFFICIENTS OF THE FUNCTION

$$y = \frac{1}{2x+1} = (2x+1)^{-1}$$

$$\frac{dy}{dx} = \frac{d}{dx}(2x+1)^{-1} = -1\,(2x+1)^{-1-1}\,\frac{d}{dx}(2x+1)$$

$$= -1\,(2x+1)^{-2} \times 2 = -2\,(2x+1)^{-2}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[-2\,(2x+1)^{-2}\right] = -2\,\frac{d}{dx}(2x+1)^{-2}$$

$$= -2 \times (-2)\,(2x+1)^{-2-1}\,\frac{d}{dx}(2x+1)$$

$$= 4\,(2x+1)^{-3} \times 2$$

$$= 8\,(2x+1)^{-3}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\,8\,(2x+1)^{-3}$$

$$= 8\,\frac{d}{dx}(2x+1)^{-3}$$

$$= 8\,(-3)\,(2x+1)^{-3-1}\,\frac{d}{dx}(2x+1)$$

$$= -24\,(2x+1)^{-4} \times 2$$

$$= -48\,(2x+1)^{-4}$$

$$= \frac{-48}{(2x+1)^4} \;//$$

EXERCISES

① DIFFERENTIATE

(i) $y = x^{10}$

(ii) $y = x^{3}$

(iii) $y = mx^{k}$

(iv) $y = (x-5)^{3}$

(v) $y = 15(x+4)^{7}$

(vi) $y = 4x^{-3.2}$

(vii) $y = 3.2(2x+3)^{1/5}$

② DIFFERENTIATE

(a) $y = 2x^{3/2} + x^{-3/2}$

(b) $y = 7x^{6} + 6x^{5} + 4x^{4} + 3x^{2} + 2x + 1$

(c) $y = \dfrac{1}{\sqrt{x}} + \dfrac{1}{\sqrt[3]{x^{2}}}$

(d) $y = -x^{-2} + \dfrac{1}{x} + x^{2}$

ARE NOT APPROPRIATE FOOTWEAR IN THIS BUILDING

$^2 + 2x + 1$

(3) THE LOSS IN ELECTRICAL MACHINE IS GIVEN BY

$$P = af + bf^2$$

WHERE P = POWER

f = FREQUENCY

a, b = CONSTANT

FIND $\frac{dP}{df}$

(4) THE INDUCED EMF OF A DC MACHINE IS GIVEN BY

$$E = 0.58 + 681.5\, I_f - 461.3\, I_f^2 + 46.3\, I_f^3$$

I_f = FIELD EXCITATION CURRENT

FIND $\frac{dE}{dI_f}$

(5) DIFFERENTIATE

(i) $(x+1)^{-\frac{1}{2}}(x-5)$

(ii) $(2x+9)^3(4x^2-5)^2$

(iii) $(5x^2+6x+3)(5x-1)$

(iv) $\frac{4-x}{x-x^2}$

(v) $\frac{x^{1/4}}{x^{1/2}-1}$

SUCCESSIVE DIFFEREN

$y = x^3 + 3$

$\frac{dy}{dx} = \frac{d}{dx}\left[x^3\right.$

$= \frac{d}{dx} x^3$

$= 3x^{3-1}$

$= 3x^2$

$\frac{d^2y}{dx^2} = \frac{d}{dx}$

$= \frac{d}{d}$

$$\frac{dy}{dx} = \frac{(x^2-2x+1)\frac{d}{dx}(x^2+2x+1) - (x^2+2x+1)\frac{d}{dx}(x^2-2x+1)}{[x^2-2x+1]^2}$$

$$= \frac{(x^2-2x+1)\left[\frac{d}{dx}x^2 + \frac{d}{dx}2x + \frac{d}{dx}1\right] - (x^2+2x+1)\left[\frac{d}{dx}x^2 - \frac{d}{dx}2x + \frac{d}{dx}1\right]}{[x^2-2x+1]^2}$$

$$= \frac{(x^2-2x+1)(2x^{2-1}+2+0) - (x^2+2x+1)\left[2x^{2-1} - 2 + 0\right]}{[x^2-2x+1]^2}$$

$$= \frac{(x^2-2x+1)(2x+2) - (x^2+2x+1)(2x-2)}{(x^2-2x+1)^2}$$

$$= \frac{(x^2-2x+1) \times 2(x+1) - (x^2+2x+1)\,2(x-1)}{(x^2-2x+1)^2}$$

$$\frac{2(x-1)^2(x+1) - 2(x+1)^2(x-1)}{[(x-1)^2]^2}$$

$$\frac{2(x-1)(x+1)[(x-1)-(x+1)]}{(x-1)^4}$$

$$\frac{2(x-1)(x+1)[\cancel{x}-1-\cancel{x}-1]}{(x-1)^4}$$

$$\frac{2(x-1)(x+1)[-2]}{(x-1)^4}$$

$$\frac{-4(x+1)}{(x-1)^3} \quad //$$

IN THIS BUILDING

DIFFERENTIATING TRIGONOMETRIC FUNCTIONS

$$\frac{d}{dx}\sin(x) = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\tan x = \sec^2 x$$

$$\frac{d}{dx}\cot x = -\mathrm{cosec}^2 x$$

$$\frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\mathrm{cosec}\, x = -\mathrm{cosec}\, x \cot x$$

DIFFERENTIATING EXPONENTIAL FUNCTION

$$y = e^x$$

$$\frac{dy}{dx} = \frac{de^x}{dx} = e^x$$

If $y = e^u$

$$\frac{dy}{dx} = \frac{d}{dx}e^u = e^u \frac{du}{dx}$$

pb DIFFERENTIATE (i) $y = e^{ax}$

(ii) $y = e^{\frac{1}{2}bx^2}$

(i)

$$y = e^{ax}$$

$$\frac{dy}{dx} = \frac{de^{ax}}{dx} = e^{ax}\frac{dax}{dx}$$

$$= e^{ax} \times a \frac{d}{}$$

$$= a e^{ax}$$

(ii) $y = e^{\frac{1}{2}bx^2+x}$

$$\frac{dy}{dx} = \frac{d}{dx}e^{\frac{1}{2}bx^2+}$$

$$= e^{(\frac{1}{2}bx^2+x}$$

...ENTIAL FUNCTION

e^{x}

$e^{u}\frac{du}{dx}$

Qb DIFFERENTIATE (i) $y = e^{ax}$

(ii) $y = e^{\frac{1}{2}bx^2 + x}$

(i) $y = e^{ax}$

$$\frac{dy}{dx} = \frac{de^{ax}}{dx} = e^{ax}\frac{dax}{dx}$$

$$= e^{ax} \times a\frac{\cancel{dx}}{\cancel{dx}}$$

$$= a\,e^{ax}$$

(ii) $y = e^{\frac{1}{2}bx^2 + x}$

$$\frac{dy}{dx} = \frac{d}{dx}e^{\frac{1}{2}bx^2 + x}$$

$$= e^{(\frac{1}{2}bx^2 + x)}\frac{d}{dx}\left(\frac{1}{2}bx^2 + x\right)$$

$$= e^{\frac{1}{2}bx^2 + x}\left[\frac{d}{dx}\frac{1}{2}bx^2 + \frac{d}{dx}x\right]$$

$$= e^{\frac{1}{2}bx^2 + x}\left[\frac{b}{2}\frac{d}{dx}x^2 + 1\right]$$

$$= e^{\frac{1}{2}bx^2 + x}\left[\frac{b}{2} \times 2x^{2-1} + 1\right]$$

$$= e^{\frac{1}{2}bx^2 + x}\left[bx + 1\right]$$

DIFFERENTIATING LOGARITHMIC FUNCTION

$$y = \log_e u(x) = \ln u(x)$$

$$\boxed{\frac{dy}{dx} = \frac{1}{u(x)} \frac{d}{dx} u(x)}$$

DIFFERENTIATE

$\log_e x^2$

$\log_e (x^2 - 1)$

$\log_e \sin x$

(i) $$\frac{d}{dx} \log_e x^2 = \frac{1}{x^2} \frac{d}{dx} x^2$$

$$= \frac{1}{x^2} \times 2x^{2-1}$$

$$= \frac{2x}{x^2}$$

$$= \frac{2}{x}$$

(ii) $$\frac{d}{dx} \log_e (x^2-1) = \frac{1}{(x^2-1)} \frac{d}{dx}(x^2-1)$$

$$= \frac{1}{(x^2-1)} \left[\frac{d}{dx} x^2 - \frac{d}{dx} 1\right]$$

$$= \frac{1}{(x^2-1)} \left[2x\right] = \frac{2x}{x^2-1}$$

(iii) $\frac{d}{dx} \log_e \sin x = \frac{1}{\sin x} \frac{d}{dx} \sin x$

$= \frac{1}{\sin x} \cos x$

$= \frac{\cos x}{\sin x} = \cot x$

$\log_e u = \frac{\log_{10} u}{\log_{10} e} = \frac{\log_{10} u}{\log_{10} 2.718} = \frac{\log_{10} u}{0.434}$

$= 2.3 \log_{10} u$

$\log_{10} u = \frac{\log_e u}{2.3}$

$\frac{d}{dx} \log_{10} u = \frac{d}{dx}\left(\frac{\log_e u}{2.3}\right)$

$$\boxed{\frac{d}{dx} \log_{10} u = \frac{1}{2.3} \frac{d}{dx} \log_e u = \frac{1}{2.3} \times \frac{1}{u} \frac{du}{dx}}$$

FIRST ORDER DERIVATIVE $\longrightarrow \frac{dy}{dx}$

SECOND ORDER DERIVATIVE $\longrightarrow \frac{d^2}{d}$

SUCCESS

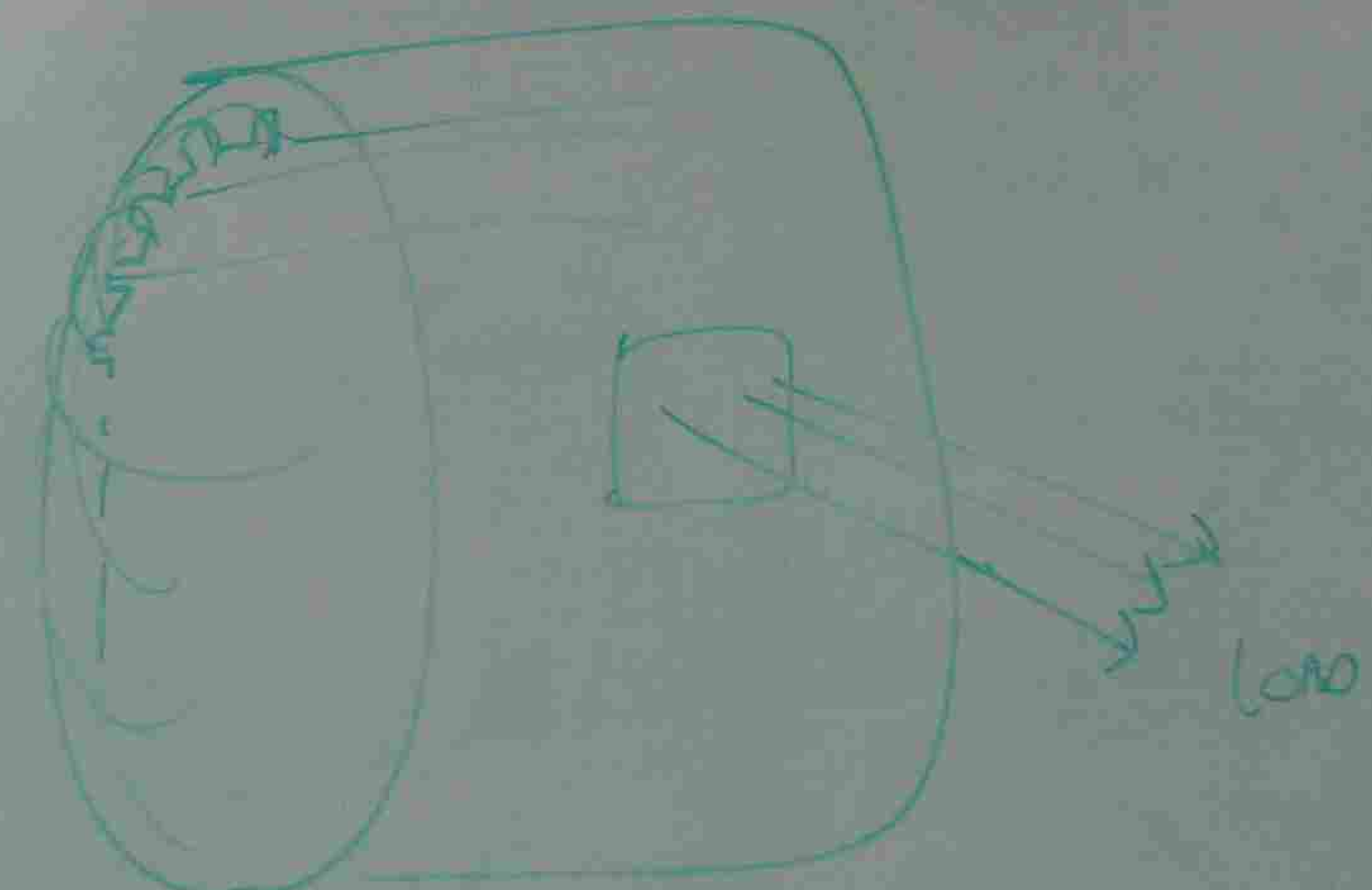
y

OF A SALIENT POLE

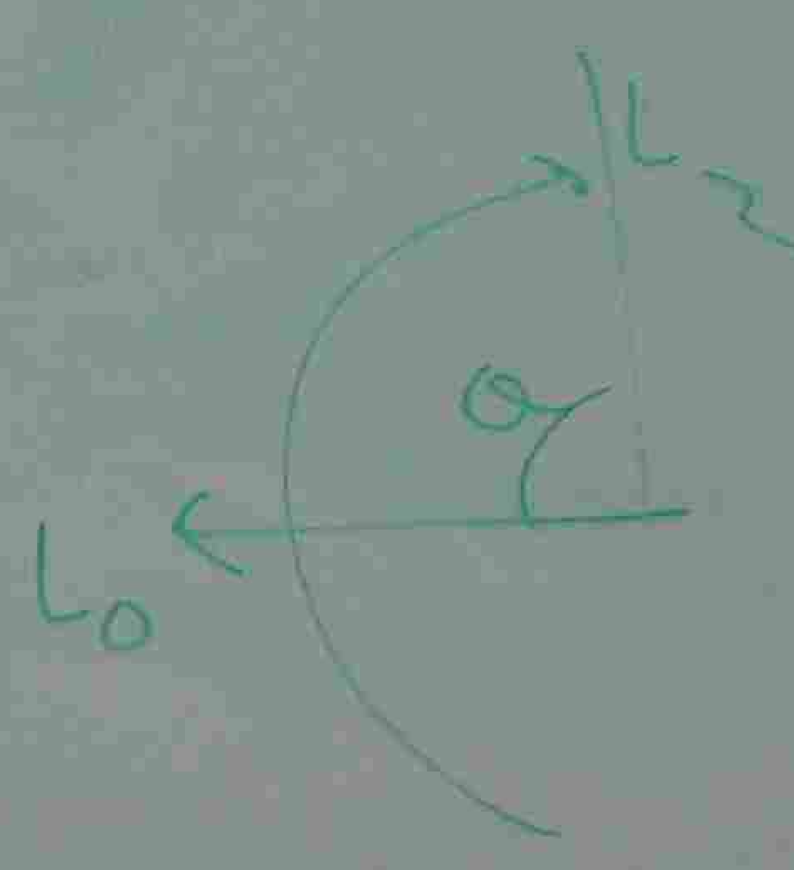
IS THE ANGULAR

OF CHANGE OF

$-2L_2 \sin 2\theta$



$$N = \frac{120 f}{P}$$



DIFFERENTI

LOGARITHMIC

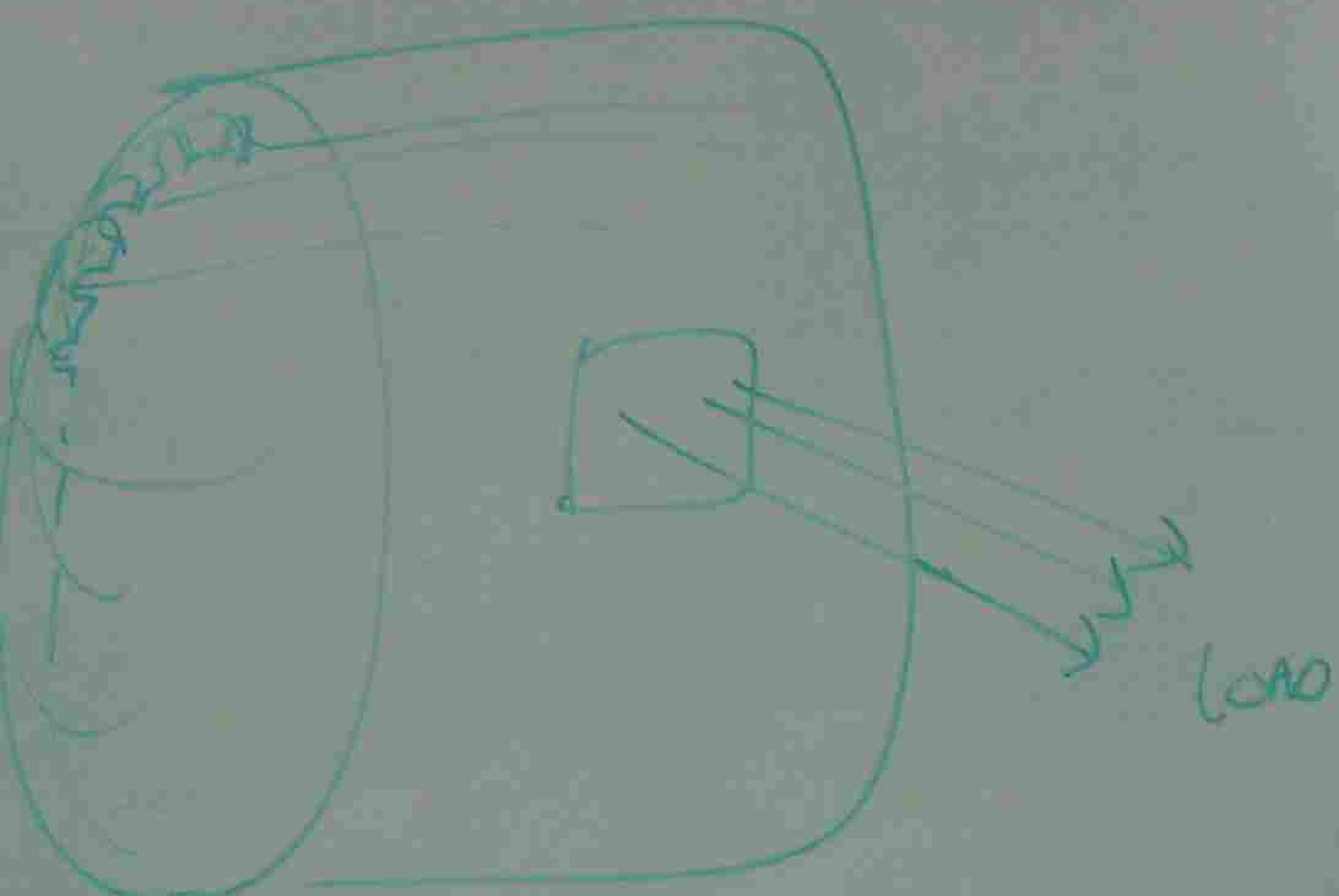
(i) e^{-ax}

(ii) $e^{(x^2}$

(iii) $\log_e ($

(iv) a^{x^2+}

(v) $x^{\sin}$



$= \frac{120 f}{p}$

pb

DIFFERENTIATE THE FOLLOWING EXPONENTIAL, LOGARITHMIC AND POWER FUNCTIONS.

(i) e^{-ax}

(ii) $e^{(x^2+2x)}$

(iii) $\log_e (x^2+2x+3)$

(iv) a^{x^2+1}

(v) $x^{\sin x}$

(i) $\frac{d}{dx} e^{-ax}$

(ii) $\frac{d}{dx} e^{x^2}$

(iii) $\frac{d}{dx}$

CANVAS SHOES ARE NOT APPROPRIATE FOOTWEAR IN THIS BUILDING

$$\frac{d}{dx}\frac{u}{v} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(x^2-1)\frac{d}{dx}2x - 2x\frac{d}{dx}(x^2-1)}{(x^2-1)^2}$$

$$= \frac{2(x^2-1) - 2x \times 2x}{(x^2-1)^2}$$

$$= \frac{2x^2-2-4x^2}{(x^2-1)^2}$$

$$= \frac{-2(x^2+1)}{(x^2-1)^2}$$

(ii) $y = \sin^2 x$

$$\frac{dy}{dx} = \frac{d}{dx}\sin^2 x = 2\sin^{2-1} x\,\frac{d\sin x}{dx} = 2\sin x\cos x$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \times 2\sin x\cos x$$

$$= 2\left[\sin x\frac{d}{dx}\cos x + \cos x\frac{d}{dx}\sin x\right]$$

$$= 2\left[\sin x(-\sin x) + \cos x \times \cos x\right]$$

$$= -2\left[\sin^2 x - \cos^2 x\right]$$

$$= 2\left[-\sin^2 x + \cos^2 x\right]$$

$$= 2\left[-\sin^2 x + 1 - \sin^2 x\right]$$

$$2\left[1-2\sin^2 x\right]$$

(iii

FOOTWEAR
N THIS BUILDING

(iii) $y = e^{\frac{x^2}{2}}$

$$\frac{dy}{dx} = \frac{d}{dx} e^{\frac{x^2}{2}} = e^{\frac{x^2}{2}} \frac{d}{dx} x^2/2$$

$$= e^{\frac{x^2}{2}} \times \frac{1}{2} \times 2x$$

$$= x e^{x^2/2}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} x e^{\frac{x^2}{2}}$$

$$= x \frac{d}{dx} e^{\frac{x^2}{2}} + e^{\frac{x^2}{2}} \frac{dx}{dx}$$

$$= x \times e^{\frac{x^2}{2}} \frac{d}{dx} x^2/2 + e^{\frac{x^2}{2}}$$

$$= x e^{x^2/2} \times \frac{\cancel{2}x}{\cancel{2}} + e^{x^2/2}$$

$$= e^{x^2/2} [x^2+1] \quad //$$

(iv) $y = a^x$

$$\frac{dy}{dx} = \frac{d}{dx} a^x = x a^{x-1}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} [x \; a^{x-1}]$$

$$= x \frac{d}{dx} a^{x-1} + a^{x-1} \frac{dx}{dx}$$

$$= x \times (x-1) a^{x-1-1} + a^{x-1}$$

$$= x(x-1) a^{x-2} + a^{x-1}$$

pb. THE CURRENT GROWTH IN A RESISTIVE - INDUCTIVE CIRCUIT FROM A SUDDENLY APPLIED BATTERY EMF IS

$$i = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$$

WHERE E, R, L ARE CONSTANTS, t IS TIME

FIND THE RATE OF CHANGE OF CURRENT WITH RESPECT TO TIME.

$$\frac{di}{dt} = \frac{d}{dt}\left[\frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right)\right]$$

$$= \frac{E}{R}\left[-\frac{d}{dt}e^{-\frac{Rt}{L}}\right]$$

$$= -\frac{E}{R} \times e^{-Rt/L} \times \frac{d}{dt}\left(-\frac{Rt}{L}\right)$$

$$= -\frac{E}{R} \times e^{-\frac{Rt}{L}} \times \left(-R/L\right)$$

$$\left\{ \frac{di}{dt} = \frac{E}{L} \times e^{-\frac{Rt}{L}} \right. \quad /\!/$$

DIFFERENTIATION OF A FUNCTION OF A FUNCTION

$y = \sin^2(x^2+1)$

Ex $y = \sin^3(2x^2-1)$ FIND y' $\left(\frac{dy}{dx} = ??\right)$

LET $u = 2x^2 - 1$

$\frac{du}{dx} = \frac{d}{dx}(2x^2-1)$

$= 2 \times 2\, x^{2-1}$

$= 4x$

$y = \sin^3 u$

$\frac{dy}{du} = 3 \sin^{3-1} u \, \frac{d \sin u}{du}$

$= 3 \sin^2 u \times \cos u$

BUT u

$\frac{dy}{dx}$

$$\boxed{\frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du}}$$

$$= 4x \times 3\sin^2 u \cos u$$

But $u = 2x^2 - 1$

$$= 4x \; 3\sin^2(2x^2-1) \; \cos 2x^2 - 1$$

$$= 12x \sin^2 2x^2 - 1 \; \cos 2x^2 - 1$$

#

DIFFERENTIATION OF IMPLICIT FUNCT

pb

$x^2 + y^2 = 9$ Find $\frac{dy}{dx}$

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx} 9$$

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = 0$$

$$2x^{2-1}\cancel{\frac{dx}{dx}} + 2y^{2-1}\frac{dy}{dx} = 0$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y \times \frac{dy}{dx} = -2x$$

...E NOT
...OPRIATE
...TWEAR
...BUILDING

$$\frac{dy}{dx} = \frac{-2x}{2y}$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

...b DIFFERENTIATE ...FOLLOWING FUNCTIONS WITH ...RESPECT TO X.

a) $x^2 + y^2 = 4$

b) $y \log_e x = 2$

(a) $\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx} 4$

$$\frac{d}{dx} x^2 + \frac{d}{dx} y^2 = 0$$

$$2x^{2-1} + 2y \frac{dy}{dx} = 0$$

$$2x + 2y \frac{dy}{dx} = 0$$

$$2y \frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

(b) $y \log_e x = 2$

$$\frac{d}{dx}\left[y \log_e x\right] = \frac{d}{dx} 2$$

$$\boxed{\frac{d}{dx} u.v = u \frac{dv}{dx} + v \frac{du}{dx}}$$

$$\frac{d}{dx} \log_e f(x) = \frac{f'(x)}{f(x)}$$

$$y \frac{d}{dx} \log_e x + \log_e x \frac{dy}{dx} = 0$$

$$y \times \frac{\frac{dx}{dx}}{x} + \log_e x \times \frac{dy}{dx} = 0$$

$$\frac{y}{x} + \log_e x \times \frac{dy}{dx} = 0$$

$\log_e x \times \frac{dy}{dx}$ ...

$\frac{dy}{dx}$ ...

$) = \frac{d}{dx} 4$

0

$= 0$

(b) $y \log_e x = 2$

$\frac{d}{dx}\left[y \log_e x\right] = \frac{d}{dx} 2$

$\boxed{\frac{d}{dx} u \cdot v = u \frac{dv}{dx} + v \frac{du}{dx}}$

$\boxed{\frac{d}{dx} \log_e f(x) = \frac{f'(x)}{f(x)}}$

$y \frac{d}{dx} \log_e x + \log_e x \frac{dy}{dx} = 0$

$y \times \frac{\frac{dx}{dx}}{x} + \log_e x \times \frac{dy}{dx} = 0$

$\frac{y}{x} + \log_e x \times \frac{dy}{dx} = 0$

$\log_e x \times \frac{dy}{dx} = -\frac{y}{x}$

$\frac{dy}{dx} = \frac{-y/x}{\log_e x}$

DIFFERENTIATE THE FOLLOWING TRIGONOMETRIC FUNCTIONS

(i) $\sin(10x+4) + \cos(7x+1)$

(ii) $\tan^2 3\theta$

(iii) $\sec x \tan x$

(iv) $\operatorname{cosec}^4 (x^2+1)$

(v) $\cot 5x \sin 6x$

(vi) $1 - 10 \sin 10t + 5 \sin 20t + 2.5 \sin 30t$

(i) $\frac{d}{dx}\left[\sin(10x+4) + \cos(7x+1)\right]$

$\frac{d}{dx}\sin(10x+4) + \frac{d}{dx}\cos(7x+1)$

$\cos(10x+4)\frac{d}{dx}(10x+4) + (-\sin(7x+1))\frac{d}{dx}(7x+1)$

$\cos(10x+4) \times 10 - \sin(7x+1) \times 7$

$10\cos(10x+4) - 7\sin(7x+1)$

(ii) $u = \tan 3\theta$

$u^2 = \tan^2 3\theta$

$\frac{d}{d\theta}u^2 = 2u\frac{du}{d\theta}$

$= 2u\frac{d}{d\theta}\tan 3\theta$

$2u \sec^2 3\theta$

$2\tan 3\theta \times \sec^2 3\theta$

$6\tan 3\theta \sec^2 3\theta$

NG TRIGONOMETRIC FUNCTIONS

$+1)$

$t + 2.5 \sin 30t$

(i) $\frac{d}{dx}\left[\sin(10x+4) + \cos(7x+1)\right]$

$\frac{d}{dx}\sin(10x+4) + \frac{d}{dx}\cos(7x+1)$

$\cos(10x+4)\frac{d}{dx}(10x+4) + \left(-\sin(7x+1)\right)\frac{d}{dx}(7x+1)$

$\cos(10x+4) \times 10 - \sin(7x+1) \times 7$

$10\cos(10x+4) - 7\sin(7x+1)$

(ii) $u = \tan 3\theta$

$u^2 = \tan^2 3\theta$

$\frac{d}{d\theta}u^2 = 2u\frac{du}{d\theta}$

$= 2u\frac{d}{d\theta}\tan 3\theta$

$2u \quad \sec^2 3\theta \times \frac{d\,3\theta}{d\theta}$

$2\tan 3\theta \times \sec^2 3\theta \times 3$

$6\tan 3\theta \sec^2 3\theta$ //

(iii) $\frac{d}{dx}\sec x \tan x = \sec x$

$= \sec$

$= s$

$= s$

(iv) $\frac{4}{\operatorname{cosec}(x^2+1)}$

$u = x^2+1 \longrightarrow$

cosec

4

$\frac{d}{dx}\operatorname{cosec}^4 u = 4\operatorname{cosec}$

$= 4\operatorname{cosec}^3 u$

$= 4\operatorname{cosec}^3$

$= 8x$

$+ \cos(7x+1)]$

$\cos(7x+1)$

$+ (-\sin(7x+1)) \frac{d}{dx}(7x+1)$

$\sin(7x+1) \times 7$

$\sin(7x+1)$ //

$\frac{d}{d\theta} \tan 3\theta$

$2u \quad \sec^2 3\theta \times \frac{d}{d\theta} 3\theta$

$2 \tan 3\theta \times \sec^2 3\theta \times 3$

$6 \tan 3\theta \sec^2 3\theta$ //

(iii) $\frac{d}{dx} \sec x \tan x = \sec x \frac{d}{dx} \tan x + \tan x \frac{d}{dx} \sec x$

$= \sec x \cdot \sec^2 x + \tan x \cdot \sec x \tan x$

$= \sec^3 x + \sec x \tan^2 x$

$= \sec x \left(\sec^2 x + \tan^2 x \right)$ //

(iv) $\operatorname{cosec}^4 (x^2+1)$

$u = x^2+1 \longrightarrow \operatorname{cosec}^4 u$

$\frac{d}{dx} \operatorname{cosec}^4 u = 4 \operatorname{cosec}^{4-1} u \frac{d}{dx} u$

$= 4 \operatorname{cosec}^3 u \times \frac{d}{dx}(x^2+1)$

$= 4 \operatorname{cosec}^3 (x^2+1) \times 2x$

$= 8x \operatorname{cosec}^3 (x^2+1)$

IN THIS BUILDING

(v) $\cot 5x \sin 6x$

$$\frac{d}{dx}\cot 5x.\sin 6x = \cot 5x \frac{d}{dx}\sin 6x + \sin 6x \frac{d}{dx}\cot 5x$$

$$= \cot 5x \cos 6x \frac{d}{dx}6x + \sin 6x \times (-\mathrm{cosec}^2 5x)\frac{d}{dx}5x$$

$$= 6\cot 5x \cos 6x - 5\sin 6x \,\mathrm{cosec}^2 5x$$

(vi) $\frac{d}{dt}\left[1 - 10\sin 10t + 5\sin 20t + 2.5\sin 30t\right]$

$$\frac{d}{dt}1 - \frac{d}{dt}10\sin 10t + \frac{d}{dt}5\sin 20t + \frac{d}{dt}2.5\sin 30t$$

$$0 - 10\cos 10t \frac{d}{dt}10t + 5\cos 20t \frac{d}{dt}20t + 2.5\cos 30t \frac{d}{dt}30t$$

$$-100\cos 10t + 100\cos 20t + 75\cos 30t$$

pb THE POTENTIAL DIFFE
OF SELF INDUCTANCE L

$$V_L = L\frac{di}{dt}$$

IF $i = 10\sin(314t +$

$$V_L = L\frac{d}{dt}[10\sin($$

$$= L \times 10 \cos(31$$

$$= 10L\cos(31$$

$$= 3140L\cos($$

$6x \frac{d}{dx} \cot 5x$

$\sin 6x \times (-\operatorname{cosec}^2 5x) \frac{d}{dx} 5x$

$-5 \sin 6x \operatorname{cosec}^2 5x$

$\sin 30t \,]$

$.5 \sin 30t$

$2.5 \cos 30t \frac{d}{dt} 30t$

t //

pb THE POTENTIAL DIFFERENCE ACROSS AN INDUC... OF SELF INDUCTANCE L IS

$$V_L = L \frac{di}{dt}$$

IF $i = 10 \sin(314t + 60)$, FIND THE POTENTIAL DIFFERENCE V_L.

$$V_L = L \frac{d}{dt} [10 \sin(314t + 60)]$$

$$= L \times 10 \cos(314t + 60) \times \frac{d}{dt}(314t + 60)$$

$$= 10L \cos(314t + 60) \times 314$$

$$= 3140L \cos(314t + 60)$$ //

Pb THE SELF INDUCTANCE OF A ROTOR WINDING OF A SALIENT POLE SYNCHRONOUS MACHINE IS

$$L = L_0 + L_2 \cos 2\theta$$

WHERE L_0 AND L_2 ARE CONSTANT AND θ IS THE ANGULAR POSITION OF THE ROTOR. FIND THE RATE OF CHANGE OF INDUCTANCE WITH ANGULAR POSITION.

$$\frac{dL}{d\theta} = \frac{d}{d\theta}\left(L_0 + L_2 \cos 2\theta\right)$$

$$= 0 + L_2 \frac{d}{d\theta}\cos 2\theta$$

$$= L_2(-\sin 2\theta) \times \frac{d}{d\theta} 2\theta$$

$$-2L_2 \sin 2\theta$$

ONENTIAL)

(i) $\frac{d}{dx} e^{-ax} = e^{-ax} \frac{d}{dx}(-ax)$

$= -a e^{-ax}$

(ii) $\frac{d}{dx} e^{x^2+2x} = e^{x^2+2x} \frac{d}{dx}(x^2+2x)$

$= e^{x^2+2x}\left[\frac{d}{dx}x^2 + \frac{d}{dx} \cdot 2x\right]$

$= e^{x^2+2x}\left[2x+2\right]$

$= 2\, e^{x^2+2x}\left[x+1\right]$

(iii) $\frac{d}{dx} \log_e (x^2+2x+3) = \frac{\frac{d}{dx}(x^2+2x+3)}{x^2+2x+3}$

$= \frac{2x+2}{x^2+2x+3} = \frac{2(x+1)}{x^2+2x+3}$

(iv) $\frac{d}{dx} a^{x^2+1} = a^{x^2+1}$

$= (x^2+1$

$= 2x$

(v) $\frac{d}{dx} x^{\sin x} =$

$=$

$=$

$2x \}$

$$\frac{\ldots x\,(x^2+2x+3)}{x^2+2x+3}$$

$$\frac{\ldots x+2}{\ldots+2x+3} = \frac{2(x+1)}{x^2+2x+3}$$

(iv) $\frac{d}{dx} a^{x^2+1} = x^2+1 \quad a^{x^2+1-1} \quad \frac{d}{dx} x^2+1$

$= (x^2+1) \quad a^{x^2} \quad 2x$

$= 2x(x^2+1) \, a^{x^2}$

(v) $\frac{d}{dx} x^{\sin x} = \sin x \quad x^{\sin x - 1} \quad \frac{d}{dx} \sin x$

$= \sin x \quad x^{\sin x - 1} \quad \cos x$

$= \sin x \cos x \quad x^{\sin x - 1}$

Ph FIND FIRST TWO DIFFERENTIAL COEFFICIENTS OF THE FOLLOWING FUNCTIONS

(i) $\log_e (x^2-1)$

(ii) $\sin^2 x$

(iii) $e^{\frac{x^2}{2}}$

(iv) a^x

(i) $y = \log_e (x^2-1)$

$$\frac{dy}{dx} = \frac{d}{dx} \log_e (x^2-1)$$

$$= \frac{\frac{d}{dx}(x^2-1)}{(x^2-1)}$$

$$= \frac{2x}{x^2-1}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \frac{2x}{x^2-1}$$

$$\frac{d}{dx} \frac{u}{v} =$$

$=$

$=$

$\cos x \quad \frac{d}{dx} \sin x$

$\cos x * \cos x$

$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$$

$$\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

pb $\int \sin 3x \cos 4x \, dx = ?$

$$\sin 3x \cos 4x = \frac{1}{2}\left[\sin(3x+4x) + \sin(3x-4x)\right]$$

$$= \frac{1}{2}\left[\sin 7x + \sin(-x)\right]$$

$$= \frac{1}{2}\left[\sin 7x - \sin x\right]$$

$$\int \frac{1}{2}\left[\sin 7x - \sin x\right] dx$$

$$\frac{1}{2}\left[\int \sin 7x \, dx - \int \sin x \, dx\right]$$

$$\frac{1}{2}\left[\int \frac{\sin 7x \, d7x}{7} - (-\cos x)\right]$$

$$\frac{1}{2}\left[\frac{(-\cos 7x)}{7} + \cos x\right] + c$$

$$\frac{1}{2}\left[-\frac{\cos 7x}{7} + \cos x\right] + c$$

$$-\frac{\cos 7x}{14} + \frac{\cos x}{2} + c$$

dx]

sx)]

+c

]+c

+c)

EXERCISE

INTEGRATE THE FOLLOWING TRIGONOMETRIC FUNCTIONS

(i) $\sec^2 x$

(ii) $\tan x$

(iii) $\cos 3x$

(iv) $\sin 6x$

(v) $\cot x$

(vi) $\operatorname{cosec} x \cot x$

(vii) $\sec x \tan x$

Pb $\int \frac{1}{5} \cos^2 5x \, dx = ?$

$\cos 2x = 2\cos^2 x - 1$

$\therefore \cos 10x = 2\cos^2 5x - 1$

$\cos^2 5x = \frac{\cos 10x + 1}{2}$

$\int \frac{1}{5} \left(\frac{\cos 10x + 1}{2} \right) dx$

$\frac{1}{10} \int (\cos 10x + 1) \, dx$

$\frac{1}{10} \left[\int \cos 10x \, dx + \int dx \right]$

$\frac{1}{10} \left[\int \right.$

$\frac{1}{10} \left[\right.$

$\frac{\sin}{10}$

Pb $\int$

$\int \sec^2$

$\sec^2$

pb $\int \sin x \sin 3x \, dx$

$$\sin x \sin 3x = \frac{1}{2}\left[\cos(x-3x) - \cos(x+3x)\right]$$
$$= \frac{1}{2}\left[\cos(-2x) - \cos 4x\right]$$
$$= \frac{1}{2}\left[\cos 2x - \cos 4x\right]$$

$$\int \frac{1}{2}\left[\cos 2x - \cos 4x\right] dx$$

$$\frac{1}{2}\left[\int \cos 2x \, dx - \int \cos 4x \, dx\right]$$

$$\frac{1}{2}\left[\int \frac{\cos 2x \, d2x}{2} - \int \frac{\cos 4x \, d4x}{4}\right]$$

$$\frac{\sin 2x}{4} - \frac{\sin 4x}{8} + C$$

$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$$

$$\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

$+\cos(3x-5x)$

$x \, dx$

$\cos 2x \, dx$

2

$+ C$

INTEGRATION OF EXPONENTIAL FUNCTION

$$\int e^u \, du = e^u + c$$

ph

$$\int e^{3x} \, dx$$

$$\int \frac{e^{3x} \, d3x}{3}$$

$$\frac{e^{3x}}{3} + c$$

ph

$$\int x e^{x^2} \, dx \longrightarrow \int e^{x^2} x \, dx$$

$$dx^2 = 2x^{2-1} dx$$

$$= 2x \, dx$$

$$\therefore x \, dx = \frac{dx^2}{2}$$

$$\int e^{x^2} \frac{dx^2}{2}$$

$$\frac{1}{2} \int e^{x^2} dx^2$$

$$\frac{1}{2} e^{x^2} + c$$

EXG...

IN...

$e^{x^2}\,dx \longrightarrow \int e^{x^2}\,x\,dx$

$= 2x^{2-1}\,dx$

$= 2x\,dx$

$dx = \frac{dx^2}{2}$

$\int e^{x^2}\,\frac{dx^2}{2}$

$\frac{1}{2}\int e^{x^2}\,dx^2$

$\frac{1}{2}e^{x^2} + c$

EXERCISE

INTEGRATE THE FOLLOWING EXPONENTIAL FUNCTION

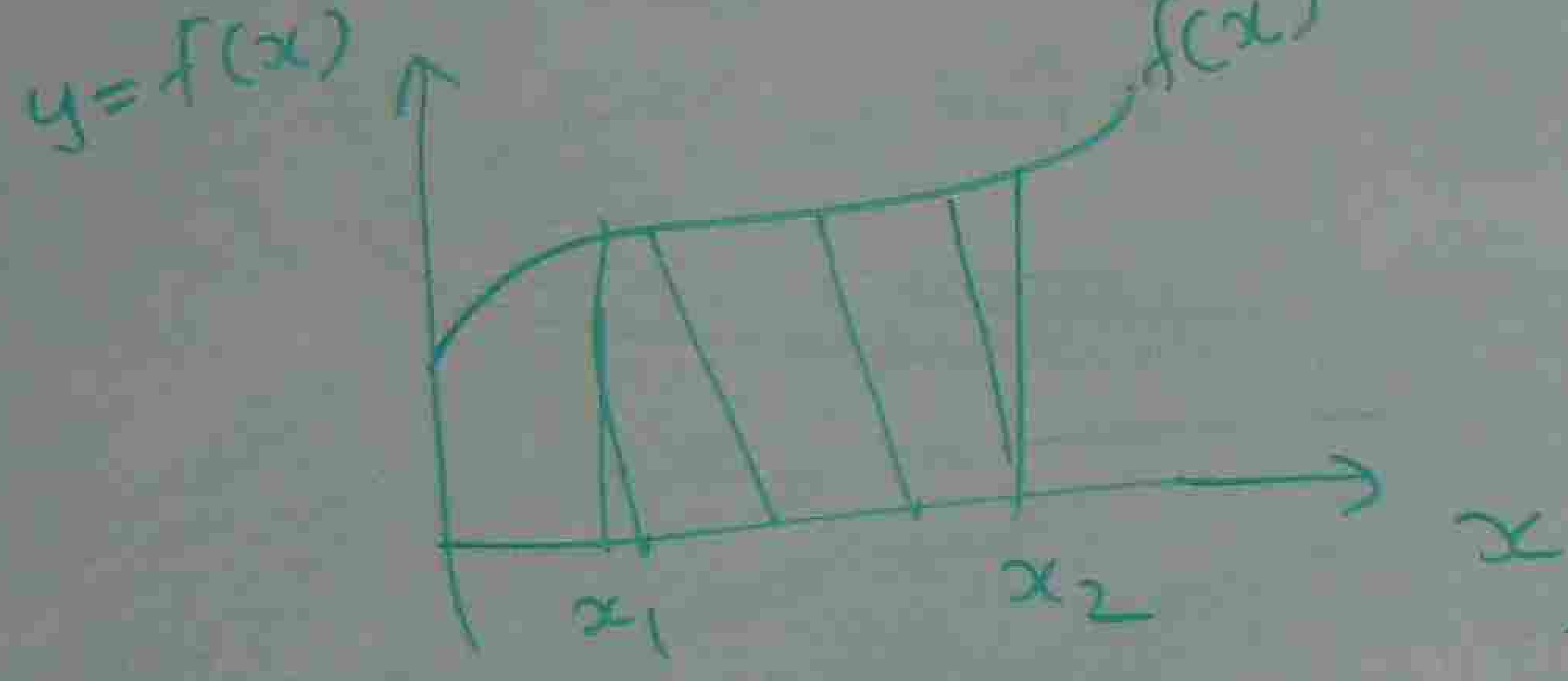
(i) e^{-3x}

(ii) $2e^{4x}$

(iii) $5e^{-\frac{1}{3}x}$

(iv) $3xe^{x^2}$

INTEGRATION



$$\text{AREA UNDER CURVE} = \int_{x_1}^{x_2} f(x)\,dx$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$$

Ex

$$\int x^7\,dx = \frac{x^{7+1}}{7+1} + c$$

$$= \frac{x^8}{8} + c$$

Ex

$$\int x^{\frac{1}{5}}\,dx = \frac{x^{\frac{1}{5}+1}}{\frac{1}{5}+1} + c$$

$$\frac{x^{6/5}}{6/5} + c$$

$$\frac{5}{6} x^{6/5} + c$$

Ex $\int x^{-3}\,dx = \dfrac{x^{-3+1}}{-3+1} + c$

$= \dfrac{x^{-2}}{-2} + c$

$= -\dfrac{1}{2}x^{-2} + c$

Ex $\int x^{-1/2}\,dx = \dfrac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$

$= \dfrac{x^{1/2}}{1/2} + c$

$= 2x^{1/2} + c$

$\int (x^n + x^m)\,dx = \int x^n\,dx + \int x^m\,dx$

$= \dfrac{x^{n+1}}{n+1} + \dfrac{x^{m+1}}{m+1} + c$

Ex $\int (x^4 + 2x^3)\,dx$

$\int x^4\,dx + \int 2x^3\,dx$

$\dfrac{x^{4+1}}{4+1} + 2\int x^3\,dx$

$\dfrac{x^5}{5} + 2 \times \dfrac{x^{3+1}}{3+1} + c$

$\dfrac{x^5}{5} + 2\,\dfrac{x^4}{4} + c$

$\frac{x^5}{5} + \frac{x^4}{2} + c$ //

$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$

$\int u^n \, du = \frac{u^{n+1}}{n+1} + c$

$\int (2x+3)^n \, d(2x+3) = \frac{(2x+3)^{n+1}}{n+1} + c$

$\int (2x+3)^n \, dx \neq \frac{(2x+3)^{n+1}}{n+1} + c$

<u>Ex</u> $\int (2x+3)^3 \, dx = ?$

$d(2x+3) = d2x + d3$
$= 2\,dx + 0$
$= 2\,dx$

$dx = \frac{d(2x+3)}{2}$

$\int (2x+3)^3 \, \frac{d(2x+3)}{2}$

$\frac{1}{2} \int (2x+3)^3 \, d(2x+3)$

$$\frac{1}{2}\,\frac{(2x+3)^{3+1}}{3+1}+c$$

$$\frac{1}{2}\times\frac{(2x+3)^{4}}{4}+c$$

$$\frac{1}{8}(2x+3)^{4}+c$$

Ex $\int(-3x+2)^{-1/3}\,dx$

$$d(-3x+2)= d(-3x)+d2$$

$$d(-3x+2) = -3dx$$

$$dx = -\frac{1}{3}\cdot d(-3x+2)$$

$$\int(-3x+2)^{-\frac{1}{3}}\left(-\frac{1}{3}d(-3x+2)\right)$$

$$-\frac{1}{3}\int(-3x+2)^{-\frac{1}{3}}d(-3x+2)$$

$$-\frac{1}{3}\times\frac{(-3x+2)^{-\frac{1}{3}+1}}{-\frac{1}{3}+1}+c$$

$$-\frac{1}{3}\times\frac{(-3x+2)^{2/3}}{2/3}+c$$

$$-\frac{1}{3}\times\frac{3}{2}(-3x+2)^{2/3}+c$$

$$-\frac{1}{2}(-3x+2)^{2/3}+c$$

EXERCISE

$$\int (5x+8)^{-2}\,dx$$

INTEGRATE THE FOLLOWINGS

(i) x^5

(ii) $\frac{1}{3} x^{-1/2}$

(iii) $6x^{-2}$

(iv) $3x^{1/5}$

(v) $\frac{1}{2} x^{-1/3} + x^2$

(vi) $(2x+3)^3$

(vii) $(1+x)^{-4}$

(viii) $(3-x)^{1/2}$

(ix) $(5x+6)^{-1/3}$

(x) $(3x-2)^{-3/2}$

INTEGRATION OF TR

$\frac{d}{dx} \sin x = \cos x$

$\frac{d}{dx} \cos x = -\sin$

$\frac{d}{dx} \tan x = \text{se}$

$\frac{d}{dx} \cot x = -\text{c}$

$\frac{d}{dx} \sec x = \text{se}$

$\frac{d}{dx} \text{cosec}\, x = -$

$+x)^{-4}$

$3-x)^{1/2}$

$x+6)^{-1/3}$

$x-2)^{-3/2}$

INTEGRATION OF TRIGONOMETRIC FUNCTIONS

$$\frac{d}{dx}\sin x = \cos x \longleftrightarrow \int \sin x\,dx = -\cos x + c$$

$$\frac{d}{dx}\cos x = -\sin x \longleftrightarrow \int \cos x\,dx = \sin x + c$$

$$\frac{d}{dx}\tan x = \sec^2 x \longleftrightarrow \int \sec^2 x\,dx = \tan x + c$$

$$\frac{d}{dx}\cot x = -\operatorname{cosec}^2 x \longleftrightarrow \int \operatorname{cosec}^2 x\,dx = -\cot x + c$$

$$\frac{d}{dx}\sec x = \sec x \tan x \longleftrightarrow \int \sec x \tan x = \sec x + c$$

$$\frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x \cot x \longleftrightarrow \int \operatorname{cosec} x \cot x = -\operatorname{cosec} x + c$$

$x + c$

$x + c$

$\tan x + c$

$= -\cot x + c$

$x = \sec x + c$

$\cot x = -\operatorname{cosec} x + c$

$$\frac{d}{dx}\sin^2 x = 2\sin^{2-1} x \frac{d}{dx}\sin x = 2\sin x\cos x$$

$$\int \sin^2 x \, dx$$

$$\boxed{\begin{aligned} \cos 2x &= 2\cos^2 x - 1 \\ \cos 2x &= 1 - 2\sin^2 x \end{aligned}}$$

$$\int \sin^2 x \, dx$$

$$\cos 2x = 1 - 2\sin^2 x$$

$$2\sin^2 x = 1 - \cos 2x \longrightarrow \sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\int \frac{1-\cos 2x}{2}\,dx = \int \frac{1}{2}\,dx - \int \frac{\cos 2x}{2}\,dx$$

$$= \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\,dx$$

$$= \frac{x}{2} - \frac{1}{2}\int \cos 2x\,\frac{d2x}{2}$$

$$= \frac{x}{2} - \frac{1}{4}\int \cos 2x\,d2x$$

$$= \frac{x}{2} - \frac{1}{4}\sin 2x + C$$

pb $\int \cos^2 x\,dx = ?$

$\cos 2x = 2\cos^2 x - 1$

$2\cos^2 x = 1 + \cos 2x$

$\cos^2 x = \frac{1+\cos 2x}{2}$

$$\int \frac{1+\cos 2x}{2}\,dx$$

$$\int \frac{1}{2}\,dx + \int \frac{\cos 2x\,dx}{2}$$

$$\frac{x}{2} + \int \frac{\cos 2x\,d2x}{2\times 2}$$

$\frac{x}{2}$ -

$\frac{x}{2}$

pb

Se

c

$\frac{x}{2} + \frac{1}{4} \int \cos 2x \, d2x$

$\frac{x}{2} + \frac{1}{4} \sin 2x + c$

pb

$\boxed{\int \tan^2 x \, dx = ?}$

$$\boxed{\begin{aligned} \sec^2 x &= \tan^2 x + 1 \\ \operatorname{cosec}^2 x &= \cot^2 x + 1 \end{aligned}}$$

$\sec^2 x = \tan^2 x + 1$

$\therefore \tan^2 x = \sec^2 x - 1$

$\int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx$

$= \int \sec^2 x \, dx - \int dx$

$= \tan x - x + c$

dx

$d2x$

$\sin 2x = 2\sin x \cos x$

ph

$$\int \sin x \cos x \, dx = ?$$

But

$\sin 2x = 2\sin x \cos x$

$\therefore \sin x \cos x = \frac{\sin 2x}{2}$

$$\int \frac{\sin 2x}{2} dx = \int \frac{\sin 2x \, d2x}{2 \times 2}$$

$$= \frac{1}{4} \int \sin 2x \, d2x$$

$$= -\frac{\cos 2x}{4} + C$$

$$\frac{d}{dx} \sin x \cos x = \sin x \frac{d}{dx} \cos x + \cos x \frac{d}{dx} \sin x$$

$$= \sin(-\sin x) + \cos x * \cos x$$

$$= -\sin^2 x + \cos^2 x$$

$$\boxed{\frac{d}{dx} uv = u\frac{dv}{dx} + v\frac{du}{dx}}$$

1

dx

dx

$\int dx \Big]$

$$\frac{1}{10}\left[\int \frac{\cos 10x \, d10x}{10} + x\right]$$

$$\frac{1}{10}\left[\frac{\sin 10x}{10} + x\right] + c$$

$$\frac{\sin 10x}{100} + \frac{x}{10} + c$$

pb $\int \tan^2 3x \, dx = ?$

$$\int \sec^2 x \, dx = \tan x + c$$

$$\sec^2 x = \tan^2 x + 1$$

$$\therefore \sec^2 3x = \tan^2 3x + 1$$

$$\tan^2 3x = \sec^2 3x - 1$$

$$\int (\sec^2 3x - 1)\, dx$$

$$\int \sec^2 3x \, dx - \int dx$$

$$\int \frac{\sec^2 3x \, d3x}{3} - x$$

$$\frac{\tan 3x}{3} - x + c$$

pb

$$\int \sin 6x \cos x \, dx$$

$$\sin 6x \cos x = \frac{1}{2}\left[\sin(6x + x) + \sin(6x - x)\right]$$

$$= \frac{1}{2}\left[\sin 7x + \sin 5x\right]$$

$$\int \frac{1}{2}\left[\sin 7x + \sin 5x\right] dx$$

$$\frac{1}{2}\left[\int \sin 7x \, dx + \int \sin 5x \, dx\right]$$

$$\frac{1}{2}\left[\int \frac{\sin 7x \, d7x}{7} + \int \frac{\sin 5x \, d5x}{5}\right]$$

$$\frac{1}{2}\left[-\frac{\cos 7x}{7} - \frac{\cos 5x}{5}\right] + C$$

$$-\frac{\cos 7x}{14} - \frac{\cos 5x}{10} + C$$

pb

$$\int \cos 3x \cos 5x \, dx = ?$$

$$\cos 3x \cos 5x = \frac{1}{2}\left[\cos(3x + 5x) + \cos(3x - 5x)\right]$$

$$= \frac{1}{2}\left[\cos 8x + \cos(-2x)\right]$$

$$= \frac{1}{2}\left[\cos 8x + \cos 2x\right]$$

$$\int \frac{1}{2}\left[\cos 8x + \cos 2x\right] dx$$

$$\frac{1}{2}\left[\int \cos 8x \, dx + \int \cos 2x \, dx\right]$$

$$\frac{1}{2}\left[\int \frac{\cos 8x \, d8x}{8} + \int \frac{\cos 2x \, d2x}{2}\right]$$

$$\frac{1}{2}\left[\frac{\sin 8x}{8} + \frac{\sin 2x}{2}\right] + C$$

$$\frac{\sin 8x}{16} + \frac{\sin 2x}{4} + C$$

pb

$\int$ s

$\sin x$

$\frac{1}{2}$

$\frac{1}{2}$

pb $\int \sin x \sin 3x \, dx$

$$\sin x \sin 3x = \frac{1}{2}\left[\cos(x-3x) - \cos(x+3x)\right]$$

$$= \frac{1}{2}\left[\cos(-2x) - \cos 4x\right]$$

$$= \frac{1}{2}\left[\cos 2x - \cos 4x\right]$$

$$\int \frac{1}{2}\left[\cos 2x - \cos 4x\right] dx$$

$$\frac{1}{2}\left[\int \cos 2x \, dx - \int \cos 4x \, dx\right]$$

$$\frac{1}{2}\left[\int \frac{\cos 2x \, d2x}{2} - \int \frac{\cos 4x \, d4x}{4}\right]$$

$$\frac{\sin 2x}{4} - \frac{\sin 4x}{8} + C$$

$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$$

$$\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

pb/ $\int \sin 6x \cos x \, dx$

$\sin 6x \cos x = \frac{1}{2}\left[\sin(6x+x) + \sin(6x-x)\right]$

$= \frac{1}{2}\left[\sin 7x + \sin 5x\right]$

$\int \frac{1}{2}\left[\sin 7x + \sin 5x\right] dx$

$\frac{1}{2}\left[\int \sin 7x \, dx + \int \sin 5x \, dx\right.$

$\frac{1}{2}\left[\int \frac{\sin 7x \, d7x}{7} + \int \frac{\sin 5x \, d5x}{5}\right]$

$\frac{1}{2}\left[-\frac{\cos 7x}{7} - \frac{\cos 5x}{5}\right] + C$

$-\frac{\cos 7x}{14} - \frac{\cos 5x}{10} + C$

pb/ $\int \cos 3x \cos 5x \, dx = ?$

$\cos 3x \cos 5x = \frac{1}{2}\left[\cos(3x+5x) + \cos(3x-5x)\right]$

$= \frac{1}{2}\left[\cos 8x + \cos(-2x)\right]$

$= \frac{1}{2}\left[\cos 8x + \cos 2x\right]$

$\int \frac{1}{2}\left[\cos 8x + \cos 2x\right] dx$

$\frac{1}{2}\left[\int \cos 8x \, dx + \int \cos 2x \, dx\right]$

$\frac{1}{2}\left\{\int \frac{\cos 8x \, d8x}{8} + \int \frac{\cos 2x \, d2x}{2}\right]$

$\frac{1}{2}\left[\frac{\sin 8x}{8} + \frac{\sin 2x}{2}\right] + C$

$\frac{\sin 8x}{16} + \frac{\sin 2x}{4} + C$

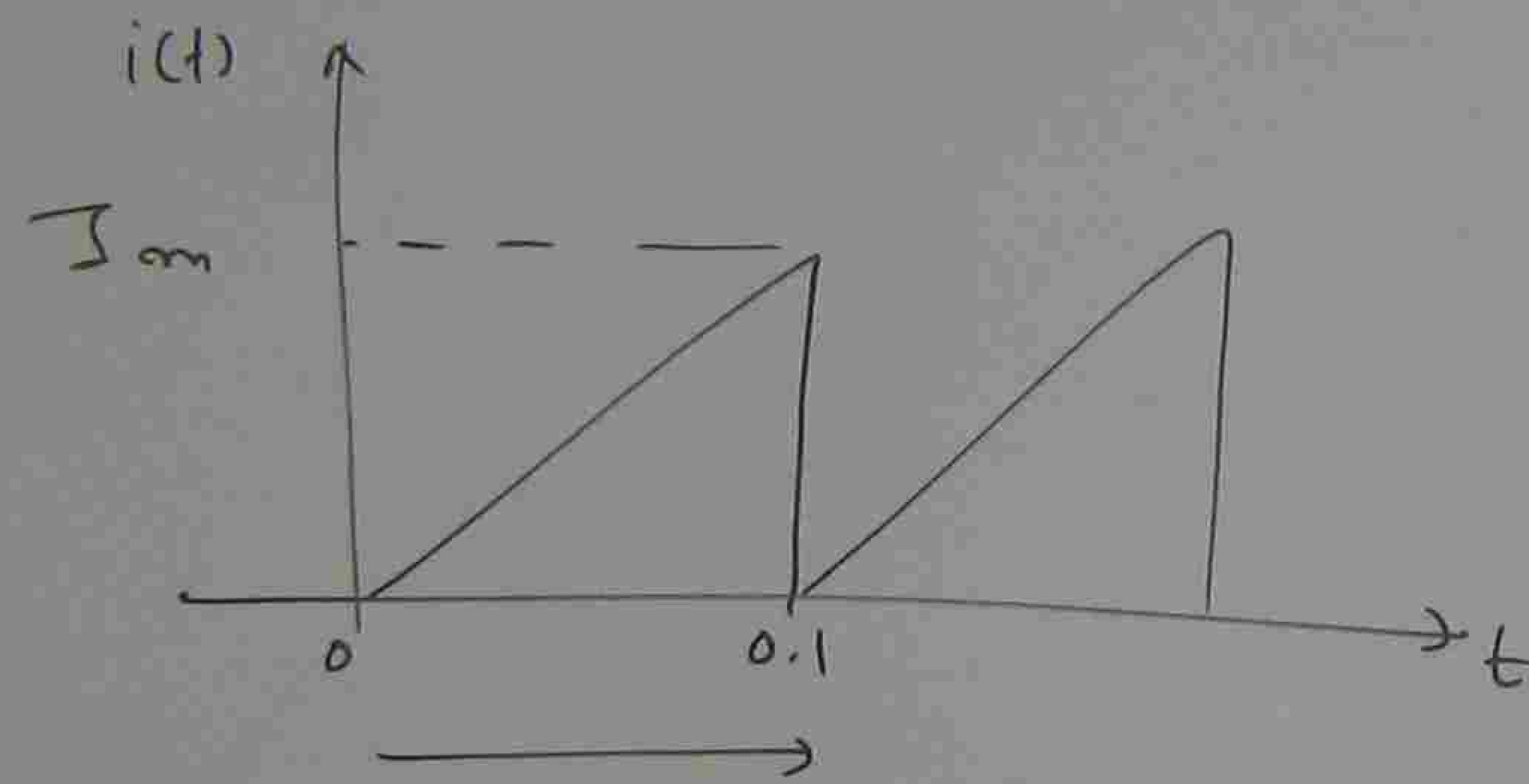
pb

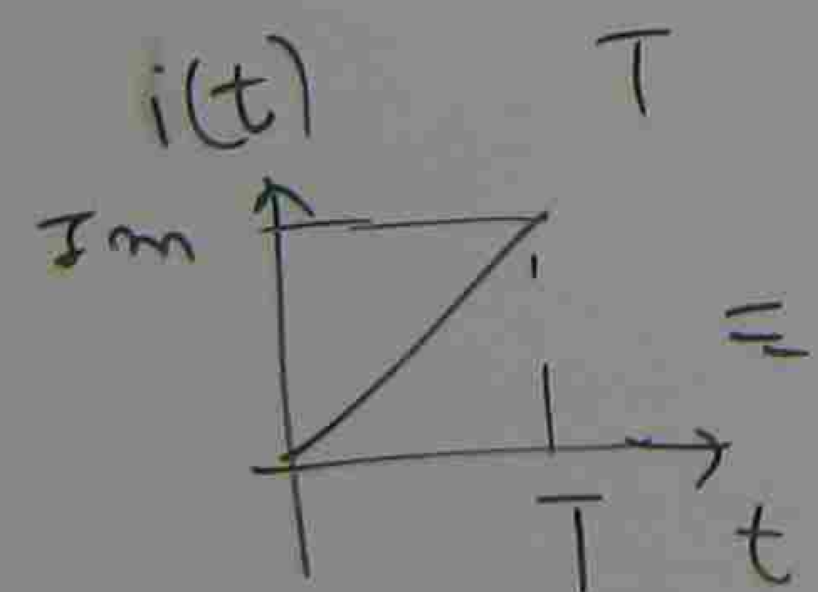
$\sin x$

$\frac{1}{2}$

INTEGRATION OF ELECTRICAL WAVE FORMS

AVERAGE VALUE





$$= i(t) = \frac{I_m}{T}\, t$$

$$I_{AVE} = \frac{1}{T}\int_0^T i(t)\,dt$$

$$I_{AVE} = \frac{1}{T}\int_0^T \frac{I_m t}{T}\,dt$$

$$= \frac{I_m}{T^2}\int_0^T t\,dt$$

$$= \frac{I_m}{T^2}\left[\frac{t^2}{2}\right]_0^T$$

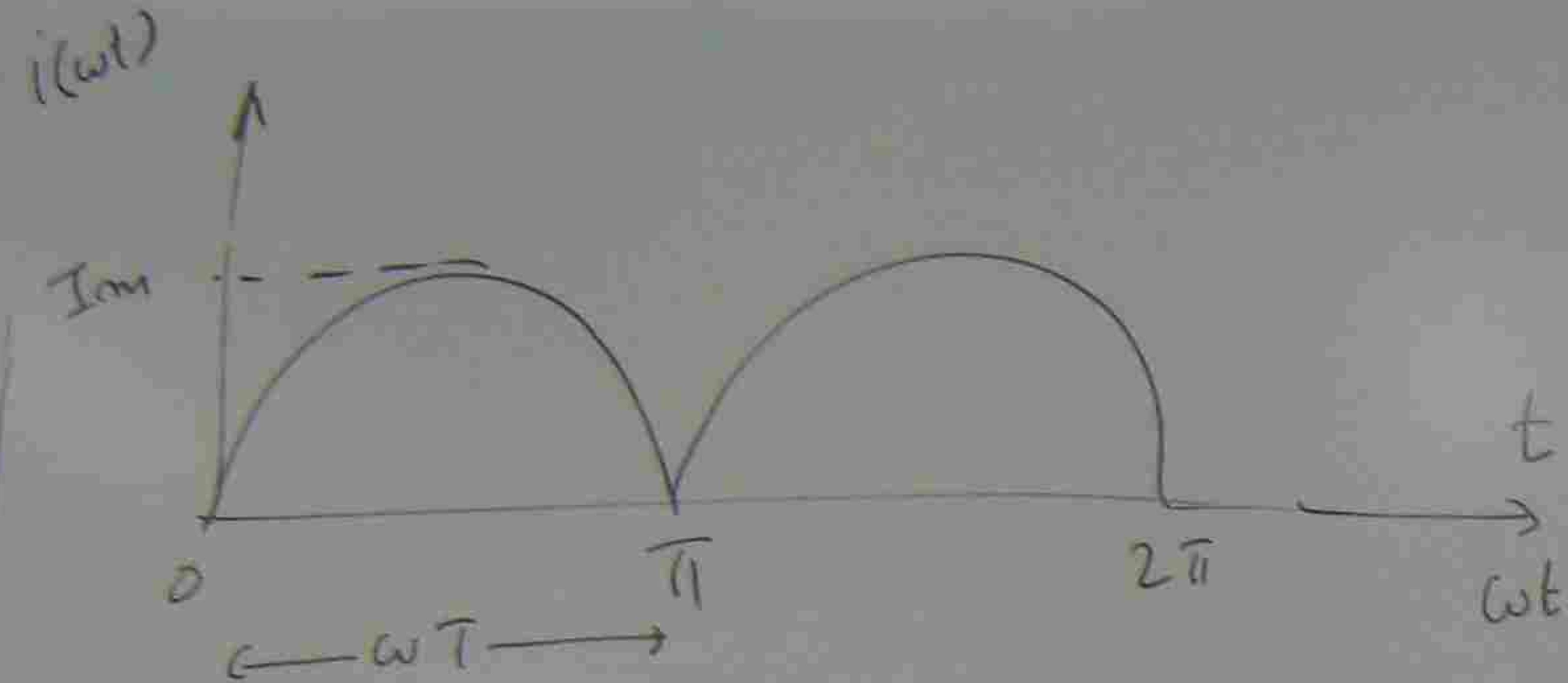
$$= \frac{I_m}{2T^2}\left[T^2 - 0\right]$$

$$= \frac{I_m}{2T^2} \times T^2$$

$$= \frac{I_m}{2}$$

i(ωt)

Im



$$i(\omega t) = I_m \sin \omega t \qquad 0 \leq \omega t \leq \pi$$

$$I_{AVE} = ?$$

$$I_{AVE} = \frac{1}{T} \int_0^T i(\omega t)\, dt$$

$$\boxed{T = \frac{\pi}{\omega}}$$

$$T = \pi$$

$$i(\omega t) = I_m \sin \omega t$$

$$I_{AVE} = \frac{1}{\pi/\omega} \int_0^{\pi} I_m \sin \omega t \, dt$$

$$I_{AVE} = \frac{\omega}{\pi} \cdot \int_0^{\pi} I_m \sin \omega t \, dt$$

$$= \frac{1}{\pi} \int_0^{\pi} I_m \sin \omega t \, d\omega t$$

$$= \frac{I_m}{\pi} \int_0^{\pi} \sin \omega t \, d\omega t$$

$$= \frac{I_m}{\pi} \Big[-\cos \omega t \Big]_0^{\pi}$$

$$= \frac{I_m}{\pi} (-) \Big[\cos \pi - \cos 0 \Big]$$

$$= \frac{I_m}{\pi} \times (-) \Big[-1 - 1 \Big]$$

$$= \frac{2 \cdot I_m}{\pi}$$

$$= 0.636\, I_m$$

$m \sin \omega t \, dt$

π

$I_m \sin \omega t \, d\omega t$

$$\int_0^{\pi} \sin \omega t \, d\omega t$$

$$\left[-\cos \omega t\right]_0^{\pi}$$

$$\left[\cos \pi - \cos 0\right]$$

$$\times(-)\left[-1-1\right]$$

$$\frac{I_m}{\pi}$$

$\cdot 636 \, I_m$

RMS VALUE

$$I_{Rms} = \sqrt{\frac{1}{T}\int_0^T [i(t)]^2 dt}$$

TRIANGULAR WAVE

$$i(t) = \frac{I_m}{T} t$$

$$I_{Rms} = \sqrt{\frac{1}{T}\int_0^T \left(\frac{I_m t}{T}\right)^2 dt}$$

$$= \sqrt{\frac{1}{T}\int_0^T \frac{I_m^2 t^2}{T^2} dt}$$

$$= \sqrt{\frac{1}{T} \times \frac{I_m^2}{T^2} \int_0^T t^2 dt}$$

$$= \sqrt{\frac{I_m^2}{T^3}\left[\frac{t^3}{3}\right]_0^T}$$

$$= \sqrt{\frac{I_m^2}{T^3} \times \frac{T^3 - 0}{3}}$$

$$= \sqrt{\frac{I_m^2}{T^3} \times \frac{T^3}{3}}$$

$$= \sqrt{\frac{I_m^2}{3}}$$

$$I_{rms} = \frac{I_m}{\sqrt{3}}$$

SINUSOIDAL WAVE

$$i(t) = I_m \sin \omega t$$

$$T = \frac{\pi}{\omega}$$

$$I_{rms} = \sqrt{\frac{1}{T}}$$

$$= \sqrt{\frac{1}{\frac{\pi}{\omega}}}$$

$$= \sqrt{\frac{\omega}{\pi}}$$

SINUSOIDAL WAVE

$$i(t) = I_m \sin \omega t$$

$$T = \frac{\pi}{\omega}$$

$$I_{rms} = \sqrt{\frac{1}{T} \int_0^T (I_m \sin \omega t)^2 \, dt}$$

$$= \sqrt{\frac{1}{\frac{\pi}{\omega}} \int_0^\pi I_m^2 \sin^2 \omega t \, dt}$$

$$= \sqrt{\frac{\omega}{\pi} \int_0^\pi I_m^2 \sin^2 \omega t \, dt}$$

$$= \sqrt{\frac{1}{\pi} \int_0^\pi I_m^2 \sin^2 \omega t \, d\omega t}$$

$$= \sqrt{\frac{I_m^2}{\pi} \int_0^\pi \sin^2 \omega t \, d\omega t}$$

$$\cos 2\omega t = 1 - 2 \sin^2 \omega t$$

$$\sin^2 \omega t = \frac{1 - \cos 2\omega t}{2}$$

$$= \sqrt{\frac{I_m^2}{\pi} \int_0^\pi \frac{1 - \cos 2\omega t}{2} \, d\omega t}$$

$m^2 \omega t \, d\omega t$

$^2 \omega t \, d\omega t$

$in^2 \omega t$

ωt

$\frac{- \cos 2\omega t}{2} \, d\omega t$

$$= \sqrt{\frac{I_m^2}{\pi \times 2} \int_0^{\pi} (1 - \cos 2\omega t) \, d\omega t}$$

$$= \sqrt{\frac{I_m^2}{2\pi} \left[\int_0^{\pi} d\omega t - \int_0^{\pi} \cos 2\omega t \, d\omega t \right.}$$

$$= \sqrt{\frac{I_m^2}{2\pi} \left[\omega t - \frac{1}{2} \sin \omega t \right]_0^{\pi}}$$

$$= \sqrt{\frac{I_m^2}{2\pi} \left[\left(\pi - \frac{1}{2} \underbrace{\sin \pi}_{0}\right) - \left(\underbrace{0 - \frac{1}{2} \sin 0}_{0}\right) \right.}$$

$$= \sqrt{\frac{I_m^2}{2\cancel{\pi}} \times \cancel{\pi}} = \frac{I_m}{\sqrt{2}} = 0.707 \, I_m$$

$\pi = 180^\circ$

$\sin 180 = 0$

Rms value of complex wave forms

period $T = \frac{2\pi}{\omega}$

$$i(t) = I_0 + A_1 \cos \omega t + A_2 \cos 2\omega t + A_3 \cos 3\omega t + \cdots + B_1 \sin \omega t + B_2 \sin 2\omega t + B_3 \sin 3\omega t +$$

$$I_{Rms} = \sqrt{I_0^2 + \frac{1}{2}(A_1^2 + A_2^2 + A_3^2 + \cdots) + \frac{1}{2}(B_1^2 + B_2^2 + B_3^2 + \cdots)}$$

$$= \sqrt{I_{dc}^2 + A_{1\,Rms}^2 + A_{2\,Rms}^2 + \cdots + B_{1\,Rms}^2 + B_{2\,Rms}^2 + \cdots}$$

$$= \sqrt{Dc^2 + (\text{sum of Rms})^2}$$

$\sin 3\omega t + - - -$

Q

$i(t) = 10 + 15 \sin 100t + 5 \sin 200t$ Amp, Find I_{Rms} = ?

$$I_{Rms} = \sqrt{10^2 + \frac{1}{2}(15^2 + 5^2)} = 15 \text{ Amp}$$

FORM FACTOR

$$\text{FORM FACTOR} = \frac{\text{RMS VALUE}}{\text{AVERAGE VALUE}} = \frac{I_{Rms}}{I_{AVE}}$$

WAVE SHAPE	FORM FACTOR
RECTIFIED SQUARE WAVE	1
RECTIFIED SINE WAVE	1.11
TRIANGULAR WAVE	1.15

Q

A VOLTAGE WAVE FORM IS REPRESENTED BY THE EQUATION

$v(t) = 10 - 10 \sin \omega t$ VOLT

DC COMPONENT + AC COMPONENT

DETERMINE THE FOLLOWINGS

(a) AVERAGE VALUE

(b) RMS VALUE

(c) FORM FACTOR

$$V_{AVE} = \frac{1}{T}\int_0^T v(t)\,dt$$

(OR)

$$V_{AVE} = \frac{1}{2\pi}\int_0^{2\pi} v(\theta)\,d\theta$$

$\boxed{\omega = 2\pi f}$ $\boxed{\theta = \omega t}$

$$V_{AVE} = \frac{1}{T}\int_0^T v(t)\,dt \qquad \boxed{\omega T = 2\pi}$$

$$= \frac{1}{T}\int_0^T (10 - 10\sin\omega t)\,dt$$

$$= \frac{1}{T}\left[\int_0^T 10\,dt\right] - \int_0^T 10\sin\omega t\,dt\Big]$$

$$= \frac{1}{T}\left[10(t)_0^T - \int_0^T \frac{10\sin\omega t\,d\omega t}{\omega}\right]$$

$$= \frac{1}{T}\left[10T - \frac{10}{\omega}\int_0^T \sin\omega t\,d\omega t\right]$$

$$= \frac{1}{T}\left[10T + \frac{10}{\omega}\left[\cos\omega t\right]_0^T\right.$$

$$= \frac{1}{T}\left[10T + \frac{10}{\omega}(\cos\omega T - \cos 0)\right.$$

$$= \frac{1}{T}\left[10T + \frac{10}{\omega}(\cos 2\pi - 1)\right.$$

$$\frac{1}{T}\Big(10T + \underbrace{\frac{10}{\omega}(1-1)}_{0}$$

$$= 10\,V$$

$$V_{AVE} = \frac{1}{2\pi}\int_0^{2\pi} v(\theta)\,d\theta$$

$$\frac{1}{2\pi}\int_0^{2\pi}(10 - 10\sin\theta)\,d\theta$$

$$\frac{1}{2\pi}\left[\int_0^{2\pi} 10\,d\theta - \int_0^{2\pi} 10\sin\theta\,d\theta\right]$$

$$\frac{1}{2\pi}\left[10(\theta)_0^{2\pi} + 10(\cos\theta)_0^{2\pi}\right]$$

$$\frac{1}{2\pi}\left[20\pi + 10\cancel{(\cos 2\pi - \cos 0)}\right]$$

$$= 10\,V$$

$V_{Rms} =$

$V_{Rms} =$

$=$

$=$

$=$

$$V_{Rms} = \sqrt{\frac{1}{T}\int_0^T (v(t))^2\,dt}$$

(OR)

$$V_{Rms} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} [v(\theta)]^2\,d\theta}$$

$$= \sqrt{\frac{1}{2\pi}\int_0^{2\pi} [10 - 10\sin\theta]^2\,d\theta}$$

$$= \sqrt{\frac{1}{2\pi}\int_0^{2\pi} 10^2(1-\sin\theta)^2\,d\theta}$$

$$= \sqrt{\frac{50}{\pi}\int_0^{2\pi} (1-\sin\theta)^2\,d\theta}$$

$$(1-x)^2 = 1 - 2x + x^2$$

$$= \sqrt{\frac{50}{\pi}\int_0^{2\pi} (1 - 2\sin\theta + \sin^2\theta)\,d\theta}$$

$$= \sqrt{\frac{50}{\pi}\left[\int_0^{2\pi} d\theta - \int_0^{2\pi} 2\sin\theta\,d\theta + \int_0^{2\pi}\sin^2\theta\,d\theta\right.}$$

$$= \sqrt{\frac{50}{\pi}\left[2\pi + 2(\cos\theta)_0^{2\pi} + \int_0^{2\pi}\left(\frac{1-\cos 2\theta}{2}\right)d\theta\right.}$$

$$= \sqrt{\frac{50}{\pi}\left[2\pi + 2\cancel{[\cos 2\pi - \cos 0]}\rightarrow 0 + \int_0^{2\pi}\frac{1}{2}d\theta - \int_0^{2\pi}\frac{\cos 2\theta}{2}\,d\theta\right.}$$

$$= \sqrt{\frac{50}{\pi}\left[2\pi + \frac{2\pi}{2} - \left[\frac{\sin 2\theta}{4}\right]_0^{2\pi}\right.}$$

$$= \sqrt{\frac{50}{\pi}\left[3\pi - \cancel{\frac{\sin 4\pi}{4}}\rightarrow 0\right.} = \sqrt{150} = 12.25\ V$$

$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\frac{\cos 2\theta\, d2}{4}$$

$$= \sqrt{\frac{50}{\pi}\int_0^{2\pi}(1-2\sin\theta+\sin^2\theta)\,d\theta}$$

$$= \sqrt{\frac{50}{\pi}\left[\int_0^{2\pi}d\theta-\int_0^{2\pi}2\sin\theta\,d\theta+\int_0^{2\pi}\sin^2\theta\,d\theta\right.}$$

$$= \sqrt{\frac{50}{\pi}\left[2\pi+2(\cos\theta)_0^{2\pi}+\int_0^{2\pi}\left(\frac{1-\cos2\theta}{2}\right)d\theta\right.}$$

$$= \sqrt{\frac{50}{\pi}\left[2\pi+2\left[\cancel{\cos2\pi-\cos0}\right]\to 0+\int_0^{2\pi}\frac{1}{2}d\theta-\int_0^{2\pi}\frac{\cos2\theta}{2}d\theta\right.}$$

$$= \sqrt{\frac{50}{\pi}\left[2\pi+\frac{2\pi}{2}-\left[\frac{\sin2\theta}{4}\right]_0^{2\pi}\right.}$$

$$= \sqrt{\frac{50}{\pi}\left[3\pi-\cancel{\frac{\sin4\pi}{4}}\to 0\right.} = \sqrt{150} = 12.25\ \mathrm{V}$$

$\cos2\theta = 1-2\sin^2\theta$

$\sin^2\theta = \frac{1-\cos2\theta}{2}$

$\frac{\cos2\theta\, d2\theta}{4}$

$(t))]^2\,dt$

$v(\theta)]^2\,d\theta$

$0 - 10\sin\theta]^2\,d\theta$

$(1-\sin\theta)^2\,d\theta$

$1-\sin\theta)^2\,d\theta$

$-2x+x^2$

$$= \sqrt{\frac{50}{\pi}\int_0^{2\pi}\left(1-2\sin\theta+\sin^2\theta\right)d\theta}$$

$$= \sqrt{\frac{50}{\pi}\left[\int_0^{2\pi}d\theta - \int_0^{2\pi}2\sin\theta\,d\theta + \int_0^{2\pi}\sin^2\theta\,d\theta\right.}$$

$$= \sqrt{\frac{50}{\pi}\left[2\pi + 2\left(\cos\theta\right)_0^{2\pi} + \int_0^{2\pi}\left(\frac{1-\cos 2\theta}{2}\right)d\theta\right.}$$

$$= \sqrt{\frac{50}{\pi}\left[2\pi + 2\left[\cos 2\pi - \cos 0\right]^{\to 0} + \int_0^{2\pi}\frac{1}{2}d\theta - \int_0^{2\pi}\frac{\cos 2\theta}{2}\,d\theta\right.}$$

$$= \sqrt{\frac{50}{\pi}\left[2\pi + \frac{2\pi}{2} - \left[\frac{\sin 2\theta}{4}\right]_0^{2\pi}\right.}$$

$$= \sqrt{\frac{50}{\pi}\left[3\pi - \frac{\sin 4\pi}{4}\right]^{\to 0}} = \sqrt{150} = 12.25\ V$$

$\cos 2\theta = 1 - 2\sin^2\theta$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$\frac{\cos 2\theta\, d2\theta}{4}$

(c) FORM FACTOR $= \frac{V_{rms}}{V_{AVE}} = \frac{12.25}{10}$

$= 1.225$

pb

DETERMINE THE AVERAGE VALUE, RMS VALUE AND FORM FACTOR FOR EACH WAVE FORM SHOWN IN FIGURE

USE INTEGRAL CALCULUS

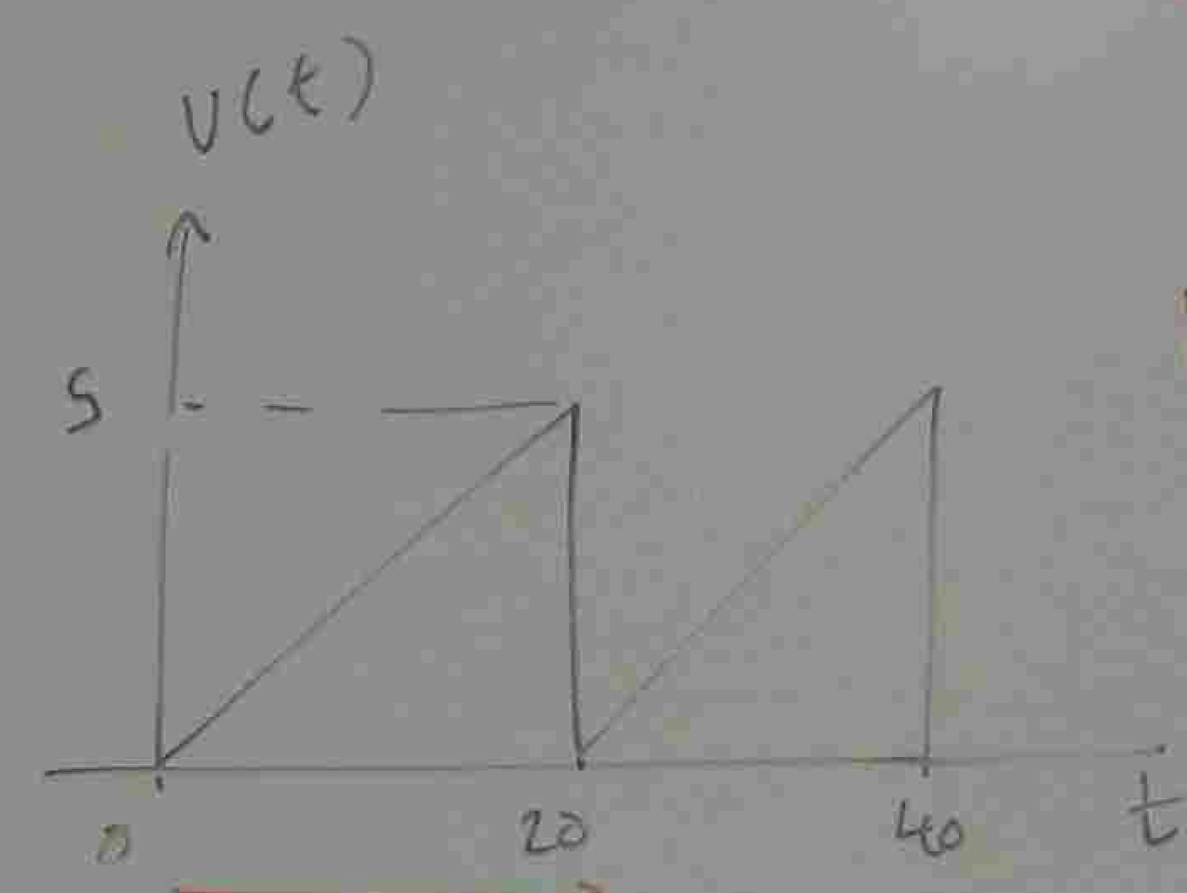
$$V_{AVE} = \frac{1}{T}\int_0^T v(t)\,dt$$

$$V_{rm} = \sqrt{\frac{1}{T}\int_0^T v(t)^2\,dt}$$

(a)

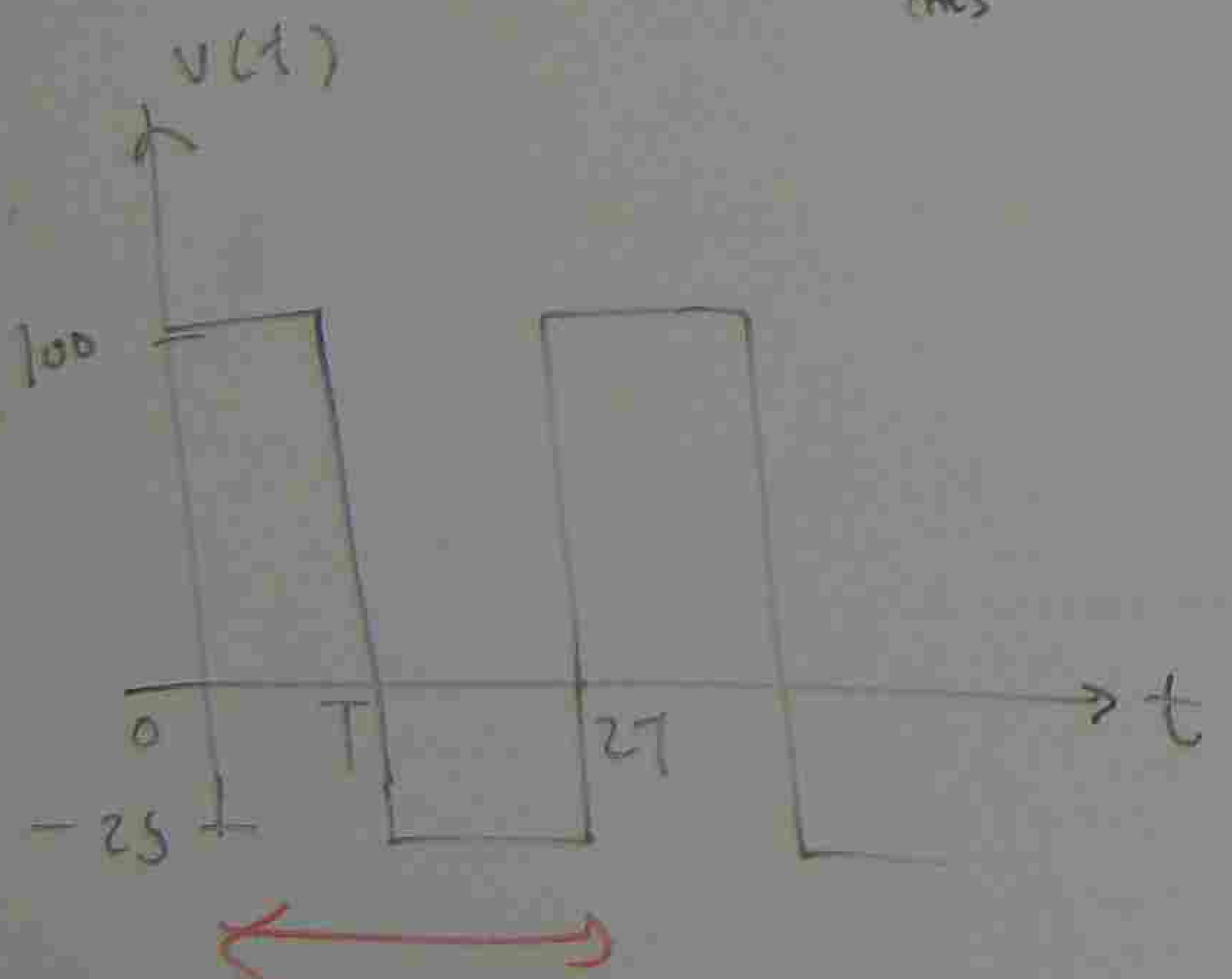
T = 20

(b)



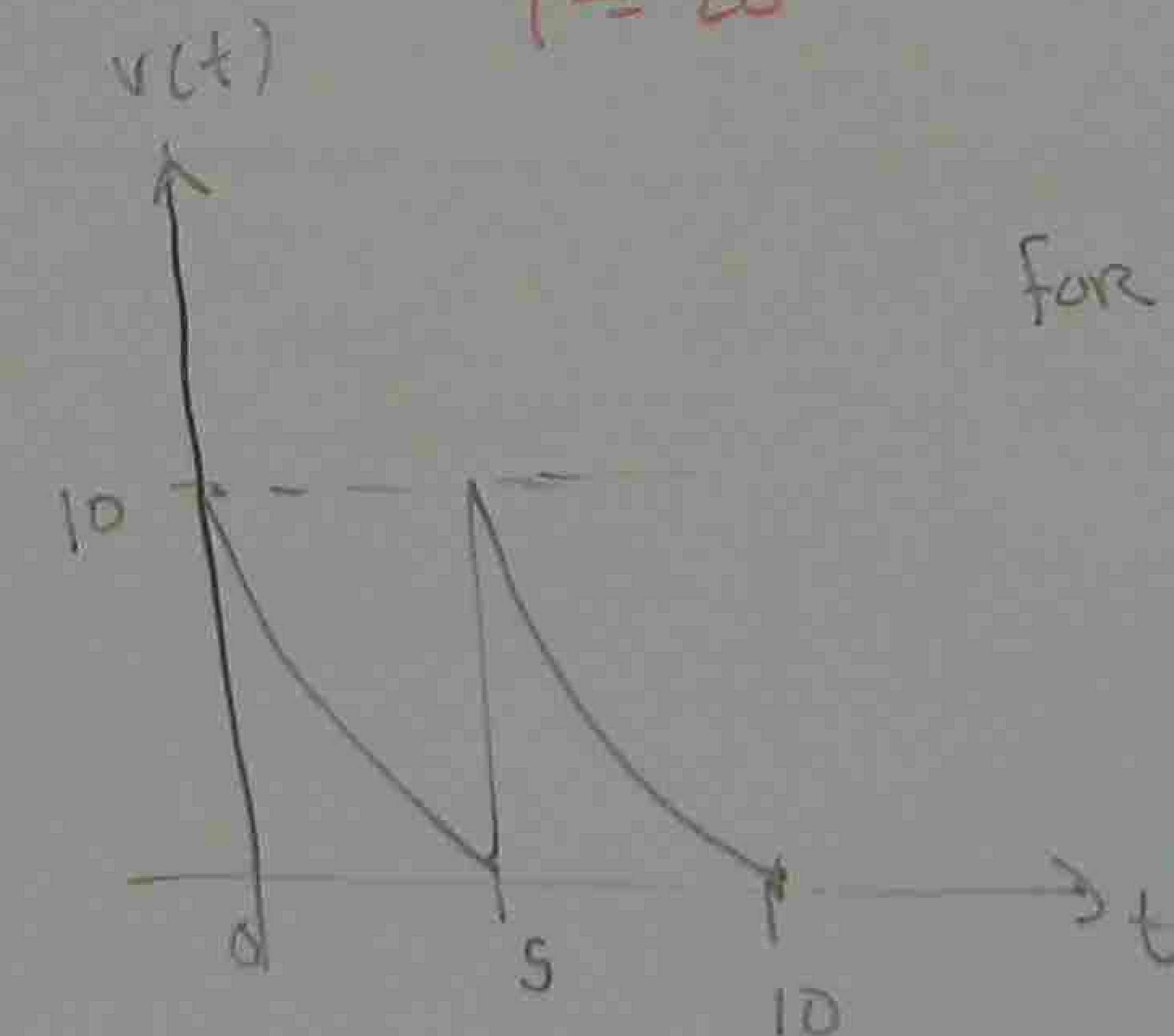
T = 20

(c)



T = 2T

(d)



T = 5

For $0 \leq t \leq 5\text{ms}$

$v(t) = 10e^{-1000t}$

(a)

V_{Rm}

(a) $V_{AVE} = \frac{1}{T}\int_0^T v(t)\,dt$

$= \frac{1}{20}\left[\int_0^{10} 100\,dt + \int_{10}^{20} 0\,dt \xrightarrow{} 0\right]$

$= \frac{1}{20}\left[100(t)_0^{10}\right]$

$= \frac{1}{20} \times 100 \times 10$

$= \frac{1000}{20} = 50 \quad V$

$V_{Rms} = \sqrt{\frac{1}{T}\int_0^T v(t)^2\,dt}$

$= \sqrt{\frac{1}{20}\left[\int_0^{10} 100^2\,dt + \cancel{\int_{10}^{20} 0^2\,dt} \to 0\right]}$

$= \sqrt{\frac{1}{20} \times 10000 (t)_0^{10}}$

$= \sqrt{\frac{1}{20} \times 100,00 \times 10}$

$= \sqrt{5000}$

$= 70.7\,V$

$\text{Form Factor} = \frac{V_{Rms}}{V_{AVG}}$

$= \frac{70.7}{50} = 1.4142$

(b) $v(t) = \frac{V_m}{T} \times t$

$= \frac{5}{20} t = 0.25t$

$V_{AVG} = \frac{1}{T}\int_0^T v(t)\,dt$

$V_{AVE} = \frac{1}{20}\int_0^{20} 0.25t\,dt$

$= \frac{1}{20} \times 0.25 \left[\frac{t^2}{2}\right]_0^{20}$

$= \frac{1}{20} \times 0.25 \times \frac{20^2}{2}$

$= \frac{0.25 \times 20}{2} = 2.5\,V$

$V_{rms} = \sqrt{\frac{1}{T}\int_0^T v(t)^2\,dt}$

$= \sqrt{\frac{1}{20}\int_0^{20} (0.25t)^2\,dt}$

$= \sqrt{\frac{1}{20} \times 0.0625 \times \left[\frac{t^3}{3}\right]_0^{20}}$

dt

$\Big]_0^{20}$

V

$(t)^2 dt$

dt

$\Big]_0^{20}$

$$= \sqrt{\frac{1}{20} \times 0.0625 \times \frac{20^3}{3}}$$

$$= \sqrt{\frac{20^2 \times 0.0625}{3}}$$

$$= 0.25 \times 20 \times \frac{1}{\sqrt{3}}$$

$$= \frac{5}{1.7321} = 2.887\ V$$

$$\text{Form Factor} = \frac{V_{Rms}}{V_{Avg}} = \frac{2.887}{2.5} = 1.155$$

(c) $$V_{AVE} = \frac{1}{T} \int_0^T v(t)\,dt$$

$$= \frac{1}{2T} \left[\int_0^T 100\,dt + \int_T^{2T} (-25)\,dt \right]$$

$$\frac{1}{2T} \left[\left[100\,(t)\right]_0^T - 25\,(t)\Big|_T^{2T} \right]$$

$$\frac{1}{2T} \left[100T - 25\left[2T - T\right] \right]$$

$$\frac{1}{2T} \left[100T - 25T \right]$$

$$\frac{75T}{2T} = 37.5\ V$$

$$V_{Rms} = \sqrt{\frac{1}{T} \int_0^T v(t)^2\,dt}$$

$$= \sqrt{\frac{1}{2T} \left[\int_0^T 100^2\,dt + \int_T^{2T} (-25)^2\,dt \right]}$$

$$= \sqrt{\frac{1}{2T} \left[100{,}00T + 625\,(2T - T) \right]}$$

$$= \sqrt{\frac{1}{2T} \left[10000T + 6 \right.}$$

$$= \sqrt{5312}$$

$$= 72.89\ V$$

$$\text{Form Factor} = \frac{72.89}{37.5}$$

(d) $$V_{AVE} = \frac{1}{T} \int_0$$

$$= \frac{1}{5/1000} \int_0$$

$$= 200 \int_0$$

$[10000T + 625T]$

12

$.89V$

$= \frac{72.89}{37.5} = 1.94$

$= \frac{1}{T}\int_0^T v(t)\,dt$

$= \frac{1}{5/1000}\int_0^{5\times10^{-3}} 10\times e^{-1000t}\,dt$

$= 200\int_0^{5\times10^{-3}} \frac{10\times e^{-1000t}}{-1000}\,d{-1000}$

$= \frac{2000}{-1000}\left[e^{-1000t}\right]_0^{5\times10^{-3}}$

$-2 \times \left[e^{-1000\times5\times10^{-3}} - e^{0}\right]$

$= -2 \times \left[e^{-5} - 1\right]$

$= -2 \times \left[\frac{1}{148} - 1\right]$

$= -2 \times -0.993$

$= 1.98\,V.$

$V_{Rms} = \sqrt{\frac{1}{T}\int_0^T v(t)^2\,dt}$

$V_{Rms} = \sqrt{\frac{1}{\frac{5}{1000}}\int_0^{5\times10^{-3}}\left[10 e^{-1000t}\right]^2 dt}$

$= \sqrt{\frac{1000}{5}\times10\int_0^{5\times10^{-3}} e^{-2000t}\,dt}$

$= \sqrt{2000\times\int_0^{5\times10^{-3}}\frac{e^{-2000t}\,d(-2000t)}{-2000}}$

$= \sqrt{-\int_0^{5\times10^{-3}} e^{-2000t}\,d(-2000t)}$

$\sqrt{-\left[e^{-2000t}\right]_0^{5\times10^{-3}}}$

$\sqrt{-\left[e^{-5}-1\right]}$

$\sqrt{1-e^{-5}}$

$3.996\,V$

$= \sqrt{\frac{1}{2T}\left[10000\,T + 625\,T\right]}$

$= \sqrt{5312}$

$= 72.89\,V$

$\text{Form Factor} = \frac{72.89}{37.5} = 1.94$

(d) $V_{AVE} = \frac{1}{T}\int_0^T v(t)\,dt$

$= \frac{1}{5/1000}\int_0^{5\times10^{-3}} 10\times e^{-1000t}\,dt$

$= 200\int_0^{5\times10^{-3}} 10\times e^{-1000t}\,\frac{d(-1000t)}{-1000}$

$= \frac{2000}{-1000}\left[e^{-1000t}\right]_0^{5\times10^{-3}}$

$= -2 \times \left[e^{-1000\times5\times10^{-3}} - e^{0}\right]$

$= -2 \times \left[e^{-5} - 1\right]$

$= -2 \times \left[\frac{1}{148} - 1\right]$

$= -2 \times -0.993$

$= 1.98\,V$

$V_{Rms} = \sqrt{\frac{1}{T}\int_0^T v(t)^2\,dt}$

$FF = \frac{3.1}{1.98} = 1.59$

$V_{Rms} = \sqrt{\frac{1}{\frac{5}{1000}}\int_0^{5\times10^{-3}}\left[10\,e^{-1000t}\right]^2 dt}$

$= \sqrt{\frac{1000}{5}\times100\int_0^{5\times10^{-3}} e^{-2000t}\,dt}$

$= \sqrt{20000\times\int_0^{5\times10^{-3}} e^{-2000t}\,\frac{d(-2000t)}{-2000}}$

$= \sqrt{-10\int_0^{5\times10^{-3}} e^{-2000t}\,d(-2000t)}$

$= \sqrt{-10\left[e^{-2000t}\right]_0^{5\times10^{-3}}}$

$\sqrt{\left[e^{-5} - 1\right]} \times 3.16$

$\sqrt{1 - e^{-5}} \times 3.16$

3.1

Q

IF A 100 Ω RESISTOR DISSIPATES AN AVERAGE POWER OF 1000 WATT, DETERMINE

(a) RMS VALUE OF CURRENT

(b) MAXIMUM VALUE OF CURRENT IF THE WAVE FORM IS SINUSOIDAL

(c) MAXIMUM VALUE OF CURRENT IF THE WAVE FORM IS TRIANGULAR.

(a) $POWER = I^2 R$

$I_{rms}^2 R = POWER$

$I_{rms}^2 \times 100 = 1000$

$I_{rms}^2 = 10$

$I_{rms} = \sqrt{10} = 3.16$ Amp.

I_{rm}

SI

I

$$I_{rms} = \sqrt{\frac{1}{T}\int_0^T I(t)^2\,dt}$$

SINUSOIDAL $\Rightarrow$ $$I_{rms} = \sqrt{\frac{1}{T}\int_0^T I(t)^2\,dt}$$

(OR)

$$= \sqrt{\frac{1}{2\pi}\int_0^{2\pi} I(\theta)^2\,d\theta}$$

$$I_{rms} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} (I_{max}\sin\theta)^2\,d\theta}$$

$$= \sqrt{\frac{1}{2\pi} I_m^2 \int_0^{2\pi} \sin^2\theta\,d\theta}$$

$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\sqrt{\frac{1}{2\pi} I_m^2 \int_0^{2\pi}\left[\frac{1-\cos 2\theta}{2}\right]d\theta}$$

$$I_m \sqrt{\frac{1}{2\pi}\left\{\left[\frac{1}{2}\theta\right]_0^{2\pi} - \left[\int_0^{2\pi}\frac{\cos 2\theta\, d2\theta}{4}\right]\right.}$$

$$\sqrt{\frac{1}{2\pi}\left[\frac{2\pi}{2} - \left[\frac{\sin 2\theta}{4}\right]_0^{2\pi}\right]}$$

$$I_m \sqrt{\frac{1}{2\pi}\left[\pi - \frac{\sin 4\pi - \sin 0}{4}\right]}$$

$$I_m \times \frac{1}{\sqrt{2}} = 3.16$$

$$I_m = \sqrt{2} \times 3.16 = 4.46 \text{ Amp.}$$

$- 2\sin^2 \theta$

$\theta = \frac{1 - \cos 2\theta}{2}$

TRIANGULAR

$$i(t) = \frac{I_m t}{T}$$

$$I_{rms} = \sqrt{\frac{1}{T} \int_0^T \left(\frac{I_m t}{T}\right)^2 dt}$$

$$= \sqrt{\frac{1}{T} \times \frac{I_m^2}{T^2} \int_0^T t^2 dt}$$

$$= \sqrt{\frac{I_m^2}{T^3} \left[\frac{t^3}{3}\right]_0^T}$$

$$= \sqrt{\frac{I_m^2}{T^3} \times \frac{T^3}{3}}$$

$$I_{rms} = \frac{I_m}{\sqrt{3}}$$

$$3.16 = \frac{I_m}{1.7321}$$

$$I_m = 3.16 \times 1.7321$$

$$= 5.47 \text{ Amp.}$$

$$= \sqrt{\frac{1}{2T} [10000\,T}$$

$$= \sqrt{5312}$$

$$= 72.89\,V$$

FORM FACTOR $= \frac{72.}{37}$

(d) $V_{AVE} = \frac{1}{}$

$= \frac{1}{5/1000}$

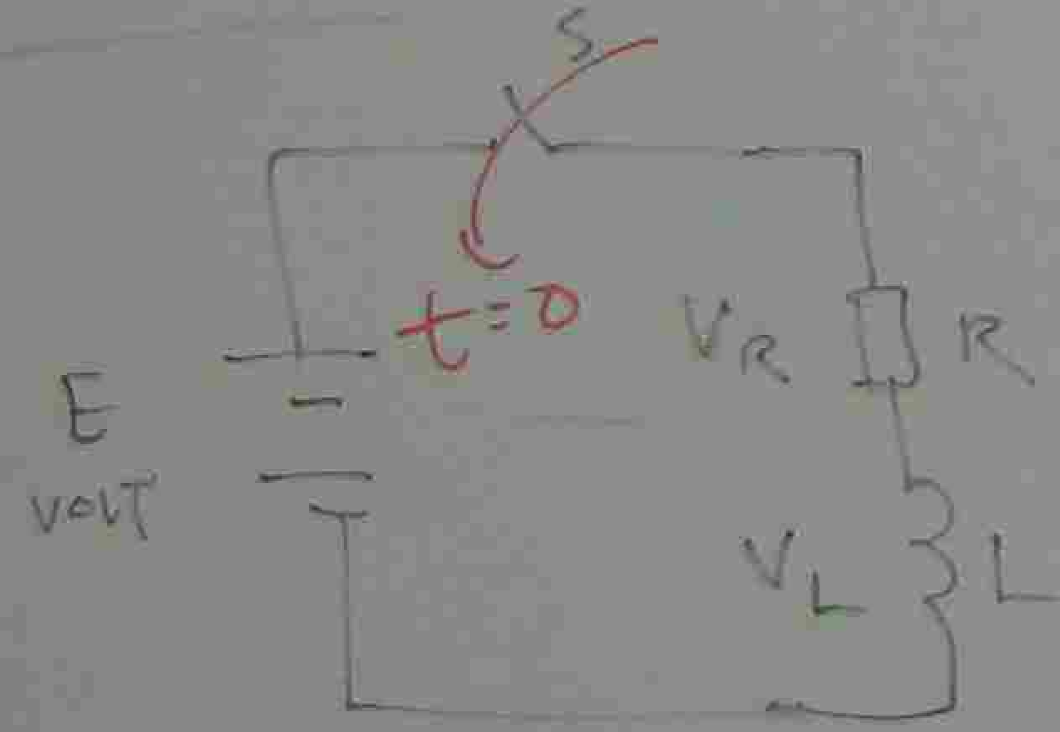
$= 200$

DIFFERENTIAL EQUATION & APPLICATION OF DIFFERENTIAL EQUATION IN ELECTRICAL CIRCUIT CALCULATION

$$\frac{dy}{dt} + 3y = 4 \quad \text{(FIRST ORDER)}$$

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 4 = 0 \quad \text{(SECOND ORDER)}$$

RL CIRCUIT



$V_R = IR$

$V_L = L\frac{di}{dt}$

$$E = V_R + V_L$$

$$E = IR + L\frac{di}{dt}$$

$$i = I\left(1 - e^{-\frac{R}{L}t}\right)$$

VALUE OF CURRENT AT $t=0$

SWITCHING ON $t=0$

$$i = I\left(1 - e^{-\frac{Rt}{L}}\right)$$

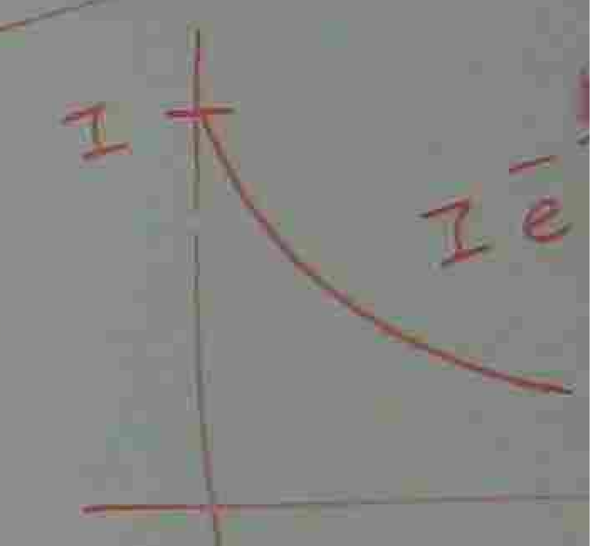
$$i = I\left(1 - e^{-\frac{Rt}{L}}\right)$$

$$= I - \underline{Ie^{-\frac{Rt}{L}}}$$

CONSTANT

VARIABLE WITH e FUNCTI

SWITCHING OFF



GENERALIZ

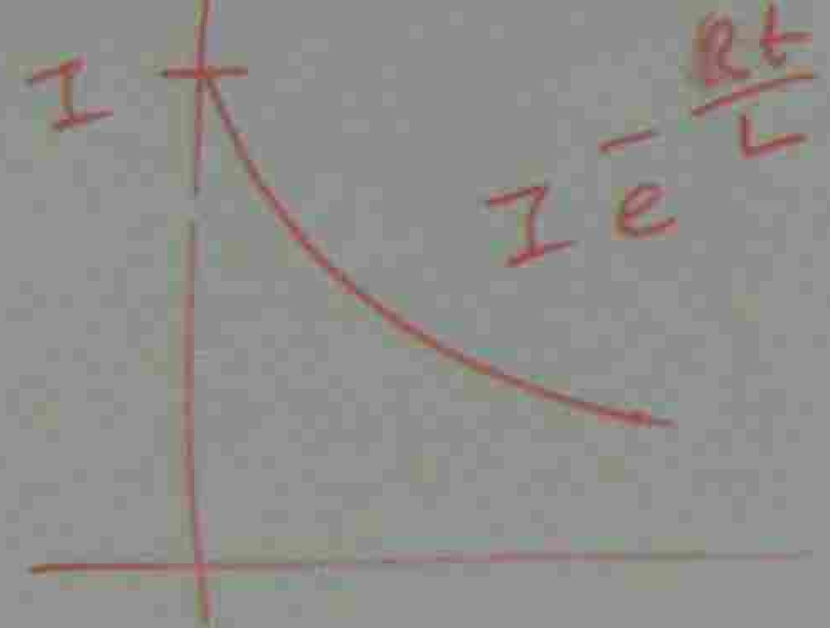
FOR ANS

THE EQU

L

i(t

SWITCHING OFF



GENERALIZED FORMAT FOR ANSWER OF THE EQUATION

$$L\frac{di}{dt} + IR = E$$

$$\boxed{i(t) = I_p + Ae^{-\frac{t}{\tau}} \qquad \tau = \frac{L}{R}}$$

)

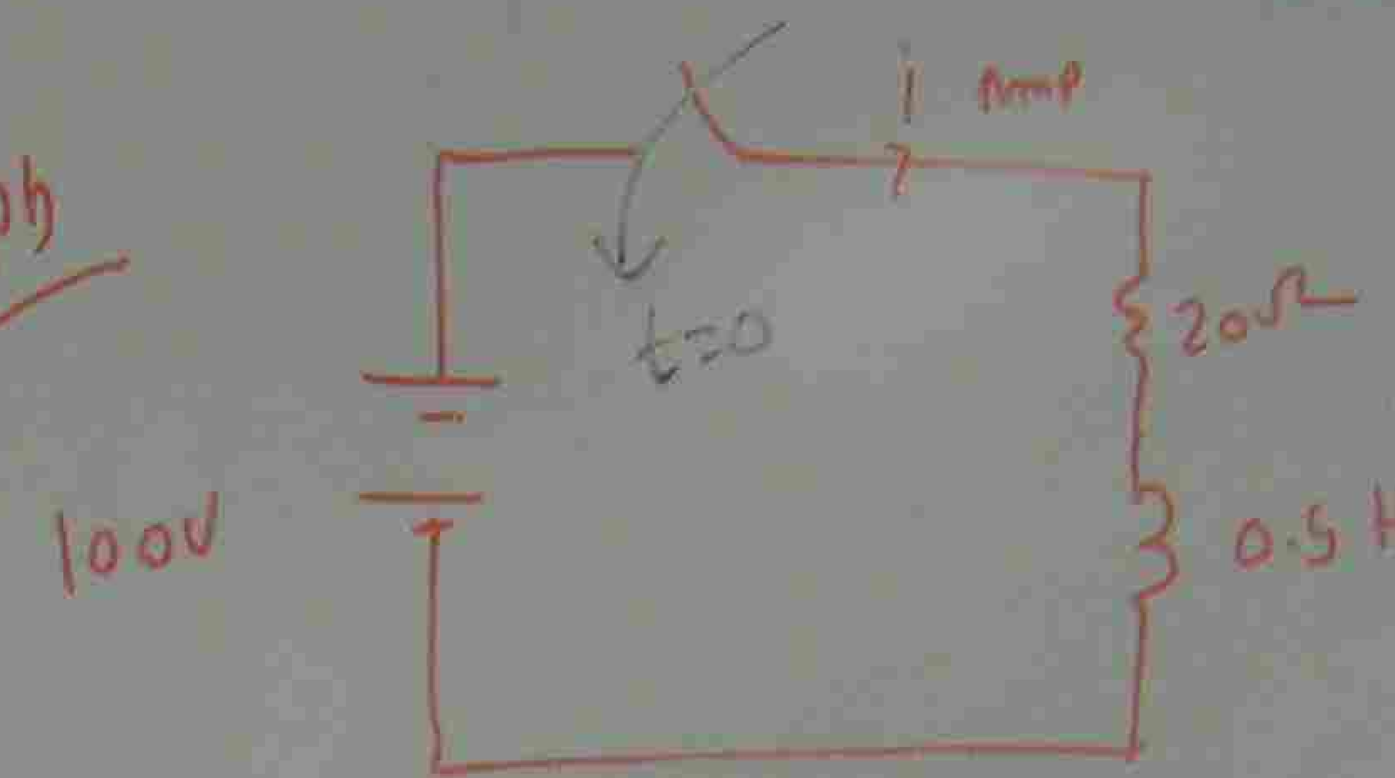
$(1 - e^{-\frac{Rt}{L}})$

$I - Ie^{-\frac{Rt}{L}}$

NSTANT

VARIABLE WITH e FUNCTION

ph



FOR THE CIRCUIT SHOWN ABOVE, DETERMINE THE following values AFTER THE SWITCH HAS BEEN CLOSED.

(a) THE FINAL VALUE OF CURRENT
(b) THE INITIAL VALUE OF CURRENT
(c) TIME CONSTANT OF THE CIRCUIT
(d) THE EQUATION OF THE CURRENT
(e) THE INITIAL RATE OF CHANGE OF CURRENT.

(a) SWITCH CLOSED, INDUCTOR IS SHORT CIRCUITED

$$I_{FINAL} = \frac{V}{R} = \frac{100}{20} = 5\,A$$

(b) $I_{INITIAL} = 0$

(c) TIME CONSTANT $(\tau) = \frac{L}{R} = \frac{0.5}{20} = 0.025$ sec $= 25$ ms

(d) $E = IR + L\frac{dI}{dt}$

$100 = 20I + 0.5\frac{dI}{dt}$ CIRCUIT EQUATION

$i(t) = I\left(1 - e^{-t/\tau}\right)$

FINAL CURRENT

$= 5\left(1 - e^{-\frac{t}{0.025}}\right)$

$i(t) = 5\left(1 - e^{-40t}\right)$

(e) $\frac{di(t)}{dt} = \frac{d}{dt}\,5\left(1 - e^{-40t}\right)$

$= \cancel{\frac{d}{dt}5}_{\to 0} - \frac{d}{dt}5e^{-40t}$

$= -5 \times (-40)\,e^{-40t}$

RATE OF CHANGE OF CURRENT $= 200\,e^{-40t}$ A/s

pb

DE T

CURR

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INITI

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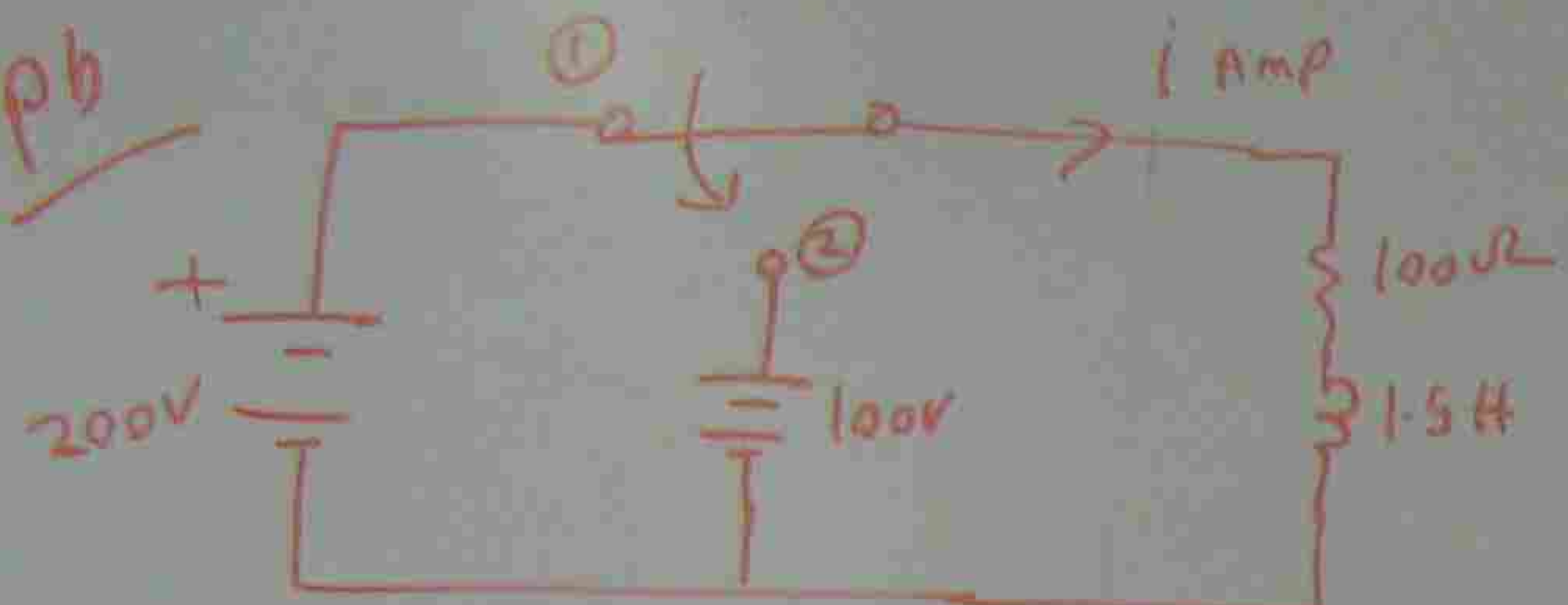
WHEN SWITCH

2

UGS

RGED

pb

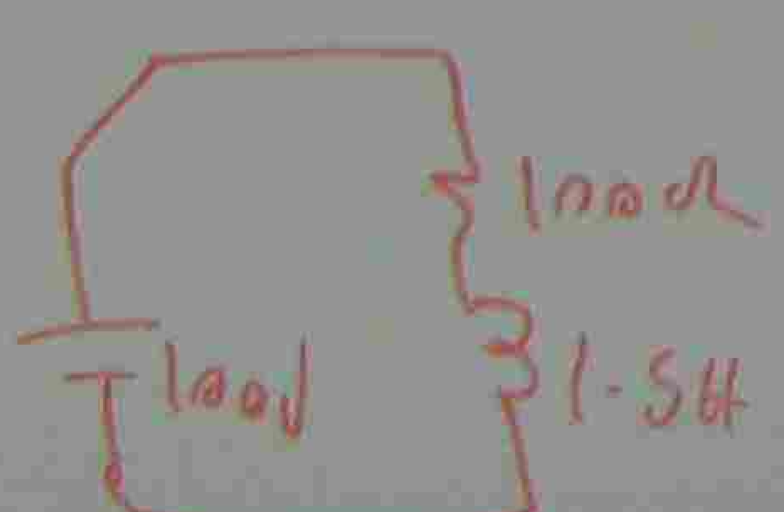


DETERMINE THE EQUATION OF THE CURRENT IN ABOVE FIGURE AFTER SWITCHING TO POSITION 2. ASSUME THAT STEADY STATE CURRENT HAD BEEN ATTAINED IN POSITION 1

INITIAL CURRENT $I(0) = \frac{200V}{100\Omega} = 2$ Amp.

(OR) $I(t)\Big|_{t=0} = 2$ Amp (NOT STEADY STATE)

N SWITCH AT 2



FINAL CURRENT $= \frac{100}{100} = 1$ Amp.

STEADY STATE CONSTANT VALUE

$$E = IR + L\frac{dI}{dt}$$

$$i(t) = I_p + A e^{-t/\tau}$$

FINAL

$$I = \frac{L}{R} = \frac{1.5}{100} = 0.015$$

$I_p = 1$ Amp ← FINAL

$$i(t) = 1 + A e^{-\frac{t}{0.015}}$$

WHEN TIME $(t = 0)$, THE INITIAL CURRENT IS 2 AMP.

$$i(t)\Big|_{t=0} = 2$$

$$2 = 1 + A e^{-\frac{0}{0.015}}$$

$$2 = 1 + A e^{0}$$

$2 = 1 + A$

$\therefore A = 2 - 1 = 1$

$$\therefore i(t) = 1 + 1 e^{-\frac{t}{0.015}}$$

$$= 1 + e^{-t/0.015} = 1 + e^{-6}$$

$e^{\sin x} dx$

$\cos x \, dx$

$\cos x \, dx$

$\sin x = e^{\sin x} + c$

$du = \ln u + c$

ph $\int \frac{4x \, dx}{2x^2+3}$

$d(2x^2+3) = d2x^2 + d3 = 2dx^2 + 0 = 2 \times 2 x^{2-1} dx$

$= \underline{\underline{4x \, dx}}$

$\int \frac{d(2x^2+3)}{(2x^2+3)} = \ln(2x^2+3) + c$

ph INTEGRATE THE FOLLOWING LOGARITHMIC FUNCTIONS

(i) $\dfrac{x}{3x^2+2}$ (ii) $\dfrac{e^{ax}}{e^{ax}+4}$ (iii) $\dfrac{\cos x}{\sin x+4}$ (iv) $\dfrac{\sec^2 x+1}{\tan x+x}$

(i) $\displaystyle\int \frac{x}{3x^2+2}\,dx$

$d(3x^2+2) = d\,3x^2 + d\,2 = 3\,dx^2 + 0 = 6x\,dx$

$\therefore\ x\,dx = \dfrac{d(3x^2+2)}{6}$

$$\int \frac{\frac{d(3x^2+2)}{6}}{(3x^2+2)} = \frac{1}{6}\int \frac{d(3x^2+2)}{(3x^2+2)}$$

$$= \frac{1}{6}\ln(3x^2+2)+c$$

(ii) $\displaystyle\int \frac{e^{ax}}{e^{ax}+4}\,dx$

$d(e^{ax}+4) = d\,e^{ax} + d4 = e^{ax}\,dax$

$= ae^{ax}\,dx$

$\therefore\ e^{ax}\,dx = \dfrac{d(e^{ax}+4)}{a}$

$$\int \frac{\frac{d(e^{ax}+4)}{a}}{e^{ax}+4} = \frac{1}{a}\int \frac{d(e^{ax}+4)}{e^{ax}+4}$$

$$= \frac{1}{a}\ln(e^{ax}+4)+c$$

(iii)

$d($

OF VARIABLE

dx

$+1$

$+1$

$x+1)$

$+d1$

dx

(i) $dx = u\,du$

(ii) $2x = u^2 - 1$

$x = \frac{u^2-1}{2}$

(iii) $\sqrt{2x+1} = u$

$$\int \frac{u^2-1}{2} \times u \times u\,du$$

$$\int \frac{(u^2-1)\,u^2\,du}{2}$$

$$\frac{1}{2}\int (u^4-u^2)\,du$$

$$\frac{1}{2}\left[\int u^4\,du - \int u^2\,du\right]$$

$$\frac{1}{2}\left[\frac{u^{4+1}}{4+1} - \frac{u^{2+1}}{2+1}\right] + c$$

$$\frac{1}{2}\left[\frac{u^5}{5} - \frac{u^3}{3}\right] + c$$

$$\frac{1}{2}\left[\frac{(\sqrt{2x+1})^5}{5} - \frac{(\sqrt{2x+1})^3}{3}\right] + c$$

$$\frac{1}{2}\left[\frac{(2x+1)^{5/2}}{5} - \frac{(2x+1)^{3/2}}{3}\right] + c \;//$$

$$\int \frac{x}{\sqrt{5-x}}\,dx$$

$u = \sqrt{5-x}$

$u^2 = 5-x$

$du^2 = d(5-x)$

$2u\,du = -dx$

$dx = -2u\,du$ — 1

$u^2 = 5-x$

$x = 5-u^2$ — 2

$$\int \frac{5-u^2}{u}\,(-2u\,du)$$

$$-\int \frac{(5-u^2)}{u}\cdot 2u\,du$$

$$-\left[\int (10-2u^2)\,du\right]$$

$$-\left[\int 10\,du - \int 2u^2\,du\right]$$

$$-\left[10u - 2\frac{u^3}{3}\right] + c$$

$$-\left[10u - \frac{2}{3}u^3\right] + c$$

$-\Big[10 \times \sqrt{5}$

$-\Big[10\,(5$

Exerc

$$-\int \frac{5-u^2}{u}(-2u\,du)$$

$$-\int \frac{(5-u^2)}{u} 2u\,du$$

$$-\left[\int (10-2u^2)\,du\right]$$

$$-\left[\int 10\,du - \int 2u^2\,du\right]$$

$$-\left[10u - 2\frac{u^3}{3}\right] + c$$

$$-\left[10u - \frac{2}{3}u^3\right] + c$$

$$-\left[10 \times \sqrt{5-x} - \frac{2}{3}\left(\sqrt{5-x}\right)^3\right] + c$$

$$-\left[10(5-x)^{1/2} - \frac{2}{3}(5-x)^{3/2}\right] + c$$

EXERCISE INTEGRATE

$$\int x\sqrt{5x+4}\,dx$$

$$\int \frac{x}{\sqrt{3x+4}}\,dx$$

REVISION (1)

(1) DIFFERENTIATE THE FOLLOWINGS

(a) $y = \dfrac{x^2 + 3x + 2}{x^2 - 3x + 2}$

(b) $y = \log_e (x^3 - 1)$

(c) $y = x^{\sin x}$

(d) $y = \operatorname{cosec}^4 (x^3 + 1)$

(e) $y = \tan \sin^3 \theta$

(2) THE CURRENT GROWTH IN A RESISTIVE INDUCTIVE CIRCUIT IS $i = \frac{E}{R}\left(1 - e^{-Rt/L}\right)$

FIND RATE OF CHANGE OF CURRENT WITH RESPECT TO TIME

(3) $V_L = L \frac{di}{dt}$

IF $i = 10 \sin (314t + 70)$

FIND V_L

(4) DIFFERENTIATE

(a) $y = 2x^{5/2} + x^{-1/2}$

(b) $y = e^{-3ax}$

(c) $y = e^{x^3 + x^2}$

(a)

$\frac{dy}{dx}$

x

x

IN A RESISTIVE

$= \frac{E}{R}\left(1 - e^{-Rt/L}\right)$

OF CURRENT

$+ 7i)$

$-\frac{1}{2}$

$t\ x$

(a) $y = \frac{u}{v} \Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{(x^2-3x+2)\frac{d}{dx}(x^2+3x+2) - (x^2+3x+2)\frac{d}{dx}(x^2-3x+2)}{\left[(x^2-3x+2)\right]^2}$$

$$= \frac{(x^2-3x+2)(2x+3) - (x^2+3x+2)[2x-3]}{(x^2-3x+2)^2}$$

$x-2$
$x-1$

$$= \frac{(x-2)(x-1)(2x+3) - (x+2)(x+1)(2x-3)}{\left[(x-2)^2(x-1)\right]^2}$$

(b) $\frac{d}{dx}\log_e f(x) = \frac{f(x)'}{f(x)}$

$y = \operatorname{cosec}^4 u$

$= 4\operatorname{cosec}$

$\frac{d}{dx}\log_e(x^3-1)$

(c) $y = x^{\sin}$

$\frac{dy}{dx} = \frac{d}{dx}$

(d) $y = c$

$\frac{dy}{dx} =$

$\frac{dv}{dx}$

$-(x^2+3x+2)\frac{d}{dx}(x^2-3x+2)$

$(x^2+3x+2)[2x-3]$

$+3)-(x+2)(x+1)(2x-3)$

$(x-1)]^2$

$\frac{f(x)'}{f(x)}$

$y = \operatorname{cosec}^4 u$

$= 4\operatorname{cosec}$

$\frac{d}{dx}\log_e(x^3-1) = \frac{\frac{d}{dx}(x^3-1)}{x^3-1} = \frac{3x^2}{x^3-1}$ //

(c) $y = x^{\sin x}$

$\frac{dy}{dx} = \frac{d}{dx}x^{\sin x} = \sin x \; x^{\sin x - 1}\frac{dx}{dx}$

$= \sin x \times x^{\sin x - 1}$ //

(d) $y = \operatorname{cosec}^4(x^3+1)$

$\frac{dy}{dx} = \frac{d}{dx}\operatorname{cosec}^4(x^3+1)$

$= 4\operatorname{cosec}^3(x^3+1)\frac{d}{dx}\operatorname{cosec}(x^3+1)$

$= 4\operatorname{cosec}^3(x^3+1)\times[-\operatorname{cosec}(x^3+1)\cot(x^3+1)\frac{d}{dx}(x^3$

$= 4\operatorname{cosec}^3(x^3+1)\operatorname{cosec}(x^3+1)\cot((x^3+1)\times 3x$

(e)

$y = t$

$\frac{dy}{d\theta} = \frac{d}{d}$

$=$

$=$

$=$

$=$

(a) $y = \tan \sin^3\theta$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\tan\sin^3\theta$$

$$= \sec \sin^3\theta \ \frac{d}{d\theta}\sin^3\theta$$

$$= \sec\underline{\underline{\sin^3\theta}} \times 3\ \underline{\underline{\sin^2\theta}} \times \frac{d}{d\theta}\sin\theta$$

$$= 3\ \sin^2\theta \ \sec\sin^3\theta \times \cos\theta$$

$$= 3\sin^2\theta\cos\theta\ \sec\sin^3\theta$$

(2) $i = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$

$$\frac{di}{dt} = \frac{d}{dt}\left[\frac{E}{R}\left(1-e^{-\frac{Rt}{L}}\right)\right]$$

$$= \frac{E}{R}\left[\frac{d}{dt}\left(1-e^{-\frac{Rt}{L}}\right)\right]$$

$$= \frac{E}{R}\left[0 - \frac{d}{dt}e^{-\frac{Rt}{L}}\right]$$

$$= -\frac{E}{R}\times e^{-\frac{Rt}{L}}\frac{d}{dt}\left(-\frac{Rt}{L}\right)$$

$$= -\frac{E}{R}\times e^{-\frac{Rt}{L}} \times -\frac{R}{L}$$

$$= \frac{E}{L}e^{-\frac{Rt}{L}}$$

(3) V_L

4 (i)

$\frac{du}{d}$

$$-12\ \operatorname{cosec}^4(x^3+1)\ \cot(x^3+1)\ x^2$$

$3+1)$

$) \cot(x^3+1)\ \frac{d}{dx}(x^3+1$

$+1)\cot(x^3+1)\ x\ 3x^2$

(3) $V_L = L\frac{di}{dt}$

$= L\frac{d}{dt}\left(10\sin(314t+70)\right]$

$= L\frac{d}{dt}10\sin(314t+70)$

$= L\times 10\cos(314t+70)\frac{d}{dt}(314t+70)$

$= 10L\cos(314t+70)\times 314$

$= 3140L\cos(314t+70)$ //

④ (i) $y = 2x^{5/2} + x^{-1/2}$

$\frac{dy}{dx} = 2\times\frac{5}{2}x^{5/2-1} + (-1/2)x^{-1/2-1}$

$= 5x^{1/2} - \frac{1}{2}x^{-3/2}$

(ii) $y = e^{-3ax}$

$\frac{dy}{dx} = \frac{d}{dx}e^{-3ax} = -3a$

$= -3$

$= 9$

(iii) $y = e^{x^3+x^2}$

$\frac{dy}{dx} = \frac{d}{dx}e^{x^3+x^2} =$

$=$

$=$

$.70)\,]$

$o)$

$\frac{d}{dt}(314t+70)$

314

$70)$

$(-1/2)\,x^{-1/2-1}$

$3/2$

(ii) $y = e^{-3ax}$

$$\frac{dy}{dx} = \frac{d}{dx} e^{-3ax} = -3a\, e^{-3ax-1} \frac{d}{dx}(-3ax)$$

$$= -3a\, e^{-3ax-1} \times -3a$$

$$= 9a^2 e^{-3ax-1}$$

(iii) $y = e^{x^3+x^2}$

$$\frac{dy}{dx} = \frac{d}{dx} e^{x^3+x^2} = e^{x^3+x^2}\left[\frac{d}{dx}(x^3+x^2)\right]$$

$$= e^{x^3+x^2}\left[3x^2+2x\right]$$

$$= x\left[3x+2\right] e^{x^3+x^2}$$

$$\int x^n \, dx = x^{n+1}$$

$$\int \sin x \, dx = -\cos x + c$$

$$\int \cos x \, dx = \sin x + c$$

$$\int \tan x \, dx = \ln \sec x + c$$

$$\int \cot x \, dx = \ln \sin x + c$$

$$\int \sec^2 x \, dx = \tan x + c$$

$$\int \operatorname{cosec}^2 x \, dx = -\cot x + c$$

$$\int \sec x \tan x \, dx = \sec x + c$$

$$\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c$$

$$\int e^u \, du = e^u + c$$

$$\int \sin^2 x \, dx = ?$$

$$\cos 2x = 1 - 2\sin^2 x$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\int \sin^2 x \, dx = \int \frac{1-\cos 2x}{2} \, dx$$

$$= \int \frac{dx}{2} - \int \frac{\cos 2x}{2} \, dx$$

$$= \frac{x}{2} - \frac{1}{2} \int \frac{\cos (2x) \, d(2x)}{2}$$

$$= \frac{x}{2} - \frac{1}{4} \sin 2x + c$$

$\sin A \cos B =$

$\int \sin 3x \cos$

$\sin 3x \cos 4x$

$\int \sin 3x$

$$\sin A \cos B = \frac{1}{2}\left(\sin(A+B) + \sin(A-B)\right)$$

$$\int \sin 3x \cos 4x \, dx = ?$$

$$\sin 3x \cos 4x = \frac{1}{2}\left[\sin(3x+4x) + \sin(3x-4x)\right]$$

$$= \frac{1}{2}\left[\sin 7x + \sin(-x)\right]$$

$$= \frac{1}{2}\left[\sin 7x - \sin x\right]$$

$$\int \sin 3x \cos 4x \, dx = \int \frac{1}{2}\left[\sin 7x - \sin x\right] dx$$

$$= \frac{1}{2}\left[\int \sin 7x \, dx - \int \sin x \, dx\right]$$

$$= \frac{1}{2}\left[\int \frac{\sin 7x \, d7x}{7} + \cos x\right]$$

$$= \frac{1}{2}\left[-\frac{\cos 7x}{7} + \cos x\right] + C$$

x

$$\frac{\cos 2x}{2} dx$$

$$\int \frac{\cos 2x \, d2x}{2}$$

$$\sin 2x + C$$

pb FIND THE AREA UNDER THE CURVE

$$y = 3\sin x + 10$$

BETWEEN π AND 2π

$$\int_{\pi}^{2\pi} y \, dx = \int_{\pi}^{2\pi} (3\sin x + 10)\, dx$$

$$= \int_{\pi}^{2\pi} 3\sin x \, dx + \int_{\pi}^{2\pi} 10 \, dx$$

$$= 3\left(-\cos x\right)_{\pi}^{2\pi} + 10 \left(\right.$$

$\pi = 180$

$= -3$

$2\pi - \pi)$

$\times \pi$

3.1416

6

EXERCISE

FIND THE AREA UNDER THE CURVE

$y = 4\sin x + 3$

BETWEEN π AND 2π

$\int e^u du = e^u + c$

INTEGRATE

$5e^{-\frac{1}{4}x}$

$\int 5e^{-\frac{1}{4}x} dx$

$$\int \frac{5e^{-\frac{1}{4}x} d(-\frac{1}{4}x)}{-\frac{1}{4}}$$

$-\frac{5}{\frac{1}{4}} \int e^{-\frac{1}{4}x} d(-\frac{1}{4}x)$

$-20 e^{-\frac{1}{4}x} + c$

INTEGRATE

① $-3x^2 e^{-x^3}$

② $\cos x \, e^{\sin x}$

① $\int -3x^2 e^{-x^3} dx$

$d(-x^3) = -3x^{3-1} dx$

$= -3x^2 dx$

$\int e^{-x^3} \times -3x^2 dx$

$\int e^{-x^3} d(-x^3)$

$e^{-x^3} + c$

② $\int \cos$

$d\sin x =$

$\int e^{\sin x}$

$\int e^{\sin x}$

$\int \frac{1}{4}$

① $\int -3x^2 e^{-x^3} dx$

$d(-x^3) = -3 x^{3-1} dx$

$= -3x^2 dx$

$\int e^{-x^3} \times (-3x^2 dx)$

$\int e^{-x^3} d(-x^3)$

$e^{-x^3} + c$

② $\int \cos x \, e^{\sin x} dx$

$d \sin x = \cos x \, dx$

$\int e^{\sin x} \cos x \, dx$

$\int e^{\sin x} d \sin x = e^{\sin x} + c$

$$\boxed{\int \frac{1}{u} du = \ln u + c}$$

ph $\int \frac{4x \, dx}{2x^2+3}$

$d(2x^2+3) = d2x^2 + d3 = 2dx^2 + 0 = 2 \times 2 x^{2-1} dx$

$= 4x \, dx$

$\int \frac{d(2x^2+3)}{(2x^2+3)} = \ln(2x^2+3) + c$

$+ d4 = e^{ax}\, d\,ax$

$= a e^{ax}\, dx$

$+4)$

$\frac{1}{a}\int \frac{d(e^{ax}+4)}{e^{ax}+4}$

$= \frac{1}{a}\ln(e^{ax}+4)+c$

(iii) $\int \frac{\cos x}{\sin x+4}\,dx$

$d(\sin x+4) = d\sin x + d4$

$= \cos x\, dx$

$\int \frac{d(\sin x+4)}{(\sin x+4}$

$= \ln(\sin x+4)+c$

(iv) $\int \frac{\sec^2 x+1}{\tan x+x}\,dx$

$d(\tan x+x) = d\tan x + dx$

$= \sec^2 x\, dx + dx$

$= (\sec^2 x+1)\,dx$

$\int \frac{d(\tan x+x)}{\tan x+x} = \ln(\tan x+x)+c$

INTEGRATION BY CHANGE OF VARIABLE

$$\int x\sqrt{2x+1}\;dx$$

SUBSTITUTE $u = \sqrt{2x+1}$

$$u^2 = 2x+1$$

$$du^2 = d(2x+1)$$

$$2u^{2-1}\,du = d2x + d1$$

$$2u\,du = 2dx$$

(i) $dx = u\,du$

(ii) $2x = u^2 - 1$

$$x = \frac{u^2-1}{2}$$

(iii) $\sqrt{2x+1} = u$

$$\int \frac{u^2-1}{2} \times u \times u\,du$$

$$\int \frac{(u^2-1)\,u^2\,du}{2}$$

$$\frac{1}{2}\int (u^4-u^2)\,du$$

$\frac{1}{2}\Big[\int u^4\,du - \int$

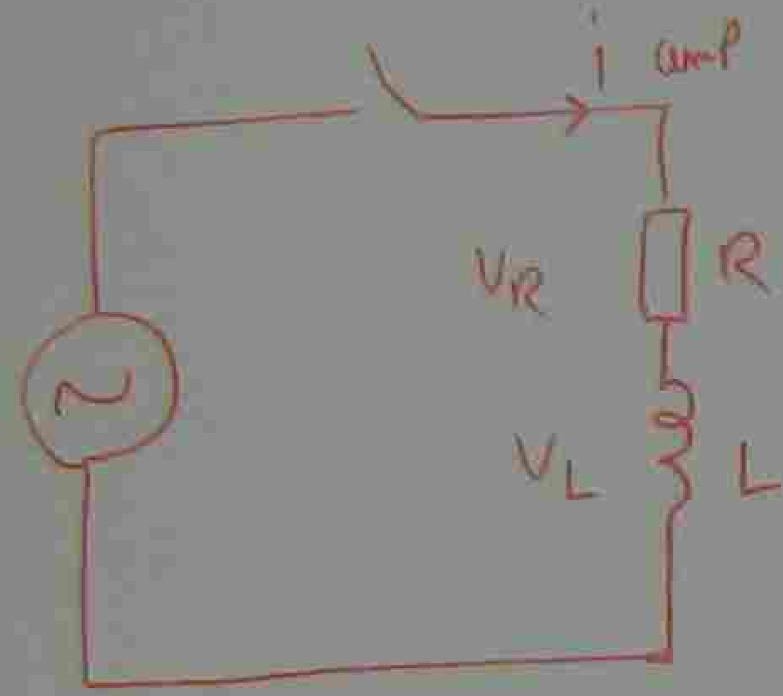
$\frac{1}{2}\Big[\frac{u^{4+1}}{4+1} -$

$\frac{1}{2}\Big[\frac{u^5}{5} -$

$\frac{1}{2}\Big[\frac{(\sqrt{2x+1})}{5}$

$\frac{1}{2}\Big[(2x$

RESPONSE OF RL AND RC CIRCUITS TO AC VOLTAGE



e volt

$e = E_m \sin(\omega t + \theta)$

$V_R = iR$

$V_L = L\frac{di}{dt}$

$$V_R + V_L = e$$

$$iR + L\frac{di}{dt} = E_m \sin(\omega t + \theta)$$

$$i = i_c + i_p$$

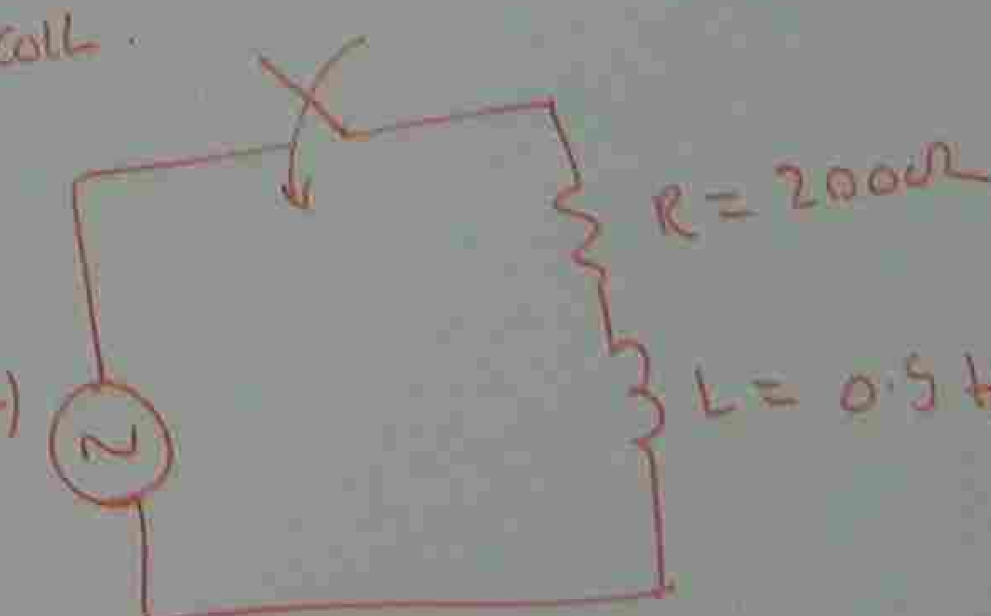
i_c: COMPLEMENTARY COMPONENT

i_p: PARTICULAR COMPONENT

GENERAL SOLUTION

$$i = A e^{-\frac{t}{\tau}} + I_p \sin(\omega t + \theta - \phi)$$

pb AN E.M.F $e = 100 \sin(314t + \theta)$ volt IS APPLIED TO A COIL OF RESISTANCE 200Ω AND INDUCTANCE 0.5 HENRY WHEN θ IS 30 DETERMINE THE EQUATION OF THE RESULTING CURRENT IN THE COIL.



$X_L = \omega L$

$= 314 \times 0$...

$= 157$

$Z = R + jX_L = 200 + j157$

$\phi = \tan^{-1}\frac{157}{200} = 38$...

$i =$

$\tau =$

$\theta =$

C VOLTAGE

GENERAL SOLUTION

$$i = A e^{-\frac{t}{\tau}} + I_p \sin(\omega t + \theta - \phi)$$

pb

AN E.M.F $e = 100 \sin(314t + \theta)$ VOLT IS APPLIED TO A COIL OF RESISTANCE $200\,\Omega$ AND INDUCTANCE 0.5 HENRY WHEN θ IS 30. DETERMINE THE EQUATION OF THE RESULTING CURRENT IN THE COIL

$+\theta)$

$$X_L = \omega L = 314 \times 0.5 = 157$$

$$Z = R + jX_L = 200 + j157$$

$$\phi = \tan^{-1} \frac{157}{200} = 38.13$$

$$i = A e^{-t/\tau} + I_p \sin(\omega t + \alpha - \phi)$$

$$\tau = \frac{L}{R} = \frac{0.5}{200} = 0.0025 = 2.5\,ms$$

$$\theta - \phi = 30 - 38.13 = -8.13$$

$$i = A e^{-\frac{t}{2.5 \times 10^{-3}}} + I_p \sin(314t - 8.13)$$

$$I_p = \frac{E}{Z} = \frac{100}{\sqrt{R^2 + X_L^2}} = \frac{100}{\sqrt{200^2 + 157^2}} = 0.$$

$$i = A e^{-400t} + 0.392 \sin(314t - 8.13)$$

$$t = 0 \rightarrow i = 0$$

R

NT

IN THIS BUILDING

$)$

ms

$(314t - 8.13)$

$$\frac{\ldots 0}{\ldots + X_L^2} = \frac{100}{\sqrt{200^2+157^2}} = 0.3932$$

$\ldots 2 \sin(314t - 8.13)$

$$0 = A e^{-400\times 0} + 0.392 \sin(314\times 0 - 8.13)$$

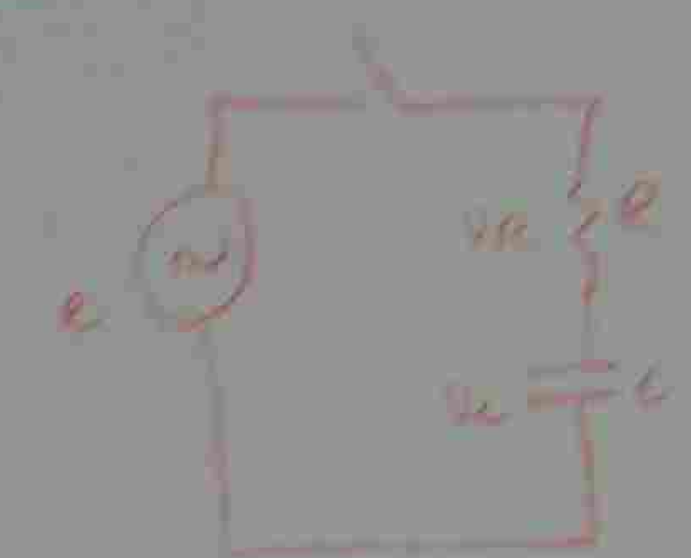
$$0 = A e^{0} + 0.392 \sin(-8.13)$$

$$0 = A - 0.392 \sin 8.13$$

$$A = 0.0556$$

$$i = 0.0556 e^{-400t} + 0.392 \sin(314t - 8.13)$$

RC circuit



$e = E_m \sin(\omega t + \alpha)$

$V_R = iR$

$V_c = \frac{1}{c}\int i\,dt$

$$V_R + V_c = e$$

$$iR + \frac{1}{c}\int i\,dt = e = E_m \sin(\omega t+\alpha)$$

$$\frac{d}{dt}(iR) + \frac{d}{dt}\left[\frac{1}{c}\int i\,dt\right] = \frac{d}{dt}\left[E_m \sin(\omega t+\alpha)\right]$$

$$R\frac{di}{dt} + \frac{1}{c}\frac{d}{dt}\int i\,dt = E_m \frac{d}{dt}\sin(\omega t+\alpha)$$

$$R\frac{di}{dt} + \frac{1}{c}\times i = E_m \omega \cos(\omega t+\alpha)$$

$Em\ sin(\omega t + \alpha)$

$\frac{d}{dt}\left[Em\ sin(\omega t + \alpha) \right]$

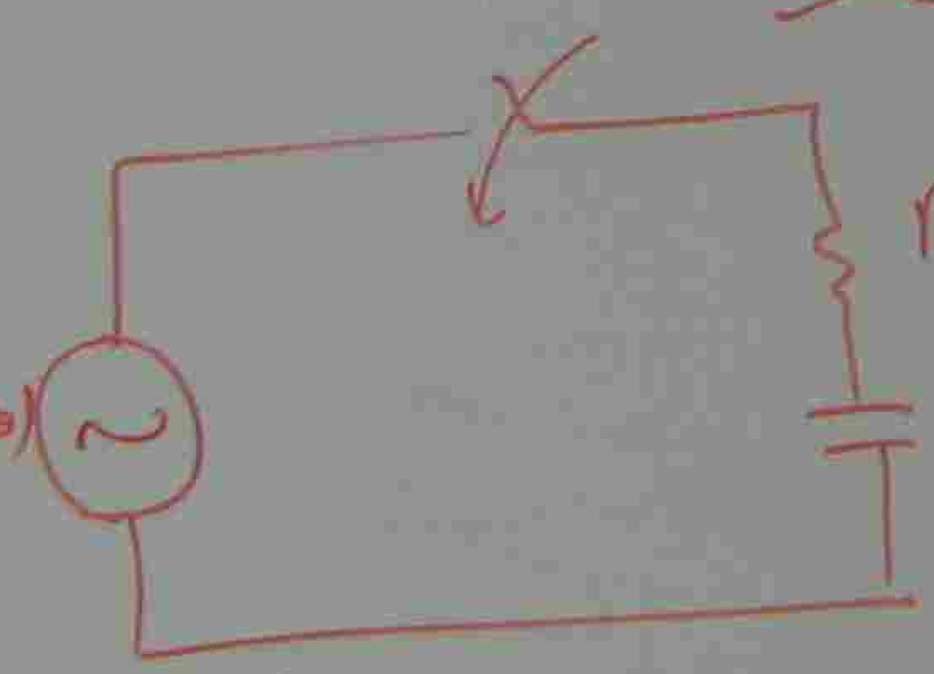
$= Em \frac{d}{dt} sin(\omega t + \alpha)$

$\omega\ cos(\omega t + \alpha)$

THE GENERAL SOLUTION

$$i = A e^{-t/\tau} + I_p \sin(\omega t + \theta - \phi)$$

pb / AN E.m.F OF $e = 200 \sin(500t + \theta)$ VOLT IS APPLIED TO AN RC CIRCUIT WHERE $R = 300\ \Omega$ AND $C = 5\ \mu F$ WHEN θ IS 60°. DETERMINE THE EQUATION OF THE CIRCUIT IF THE INITIAL CHARGE ON THE CAPACITOR IS 250 MICRO COULOMBS.



$$i = A e^{-t/\tau} + I_p \sin(\omega t + \theta - \phi)$$

$$\tau = R.C = 300 \times 5\times10^{-6} = 1.5\times10^{-3}$$

$$\theta = 60°$$

$Z = R - jX_c = 300 - j\frac{1}{2\pi fc}$

$= 300 - j\frac{1}{500 \times 5\times}$

$= 300 - j400\ \Omega$

$= \sqrt{300^2 + 400^2} \angle$

$Z = 500 \angle -53.1$

$\phi = -53.1$

$I_p = \frac{E}{Z} = \frac{200}{500} =$

$i = A e^{-\frac{t}{1.5\times10^{-3}}} + 0.4 \sin$

$i = A e^{-666.7t} + 0.4 \sin$

$t=0 \rightarrow i(0) = A e^{-666}$

$i(0) = A + 0.4$

OLLED

GE

3

$$Z = R - jX_c = 300 - j\frac{1}{2\pi fC}$$

$$= 300 - j\frac{1}{500\times 5\times10^{-6}}$$

$$= 300 - j400\ \Omega$$

$$= \sqrt{300^2+400^2}\ \angle -\tan^{-1}\frac{400}{300}$$

$$Z = 500\ \angle -53.1\ \Omega$$

$$\phi = -53.1$$

$$I_p = \frac{E}{Z} = \frac{200}{500} = 0.4$$

$$i = A e^{-\frac{t}{1.5\times10^{-3}}} + 0.4\sin(500t + 60 - (-53.1))$$

$$i = A e^{-666.7t} + 0.4\sin(500t + 113.1)\ \text{Amp}$$

$$t=0 \rightarrow i(0) = A e^{-666.7\times 0} + 0.4\sin(500\times 0 + 113.1)$$

$$i(0) = A + 0.4\sin 113.1 \quad \text{—— (1)}$$

$$q = \int i\,dt = 250\ \mu C$$

INITIAL

$$iR + \frac{1}{C}\int i\,dt = e$$

$$iR + \frac{1}{C}\times q = e$$

$$i\times 300 + \frac{250\times10^{-6}}{5\times10^{-6}} = 200\sin 60$$

$$i\times 300 + 50 = 200\sin 60$$

$$i(0) = \frac{200\sin 60 - 50}{300} = 0.4107$$

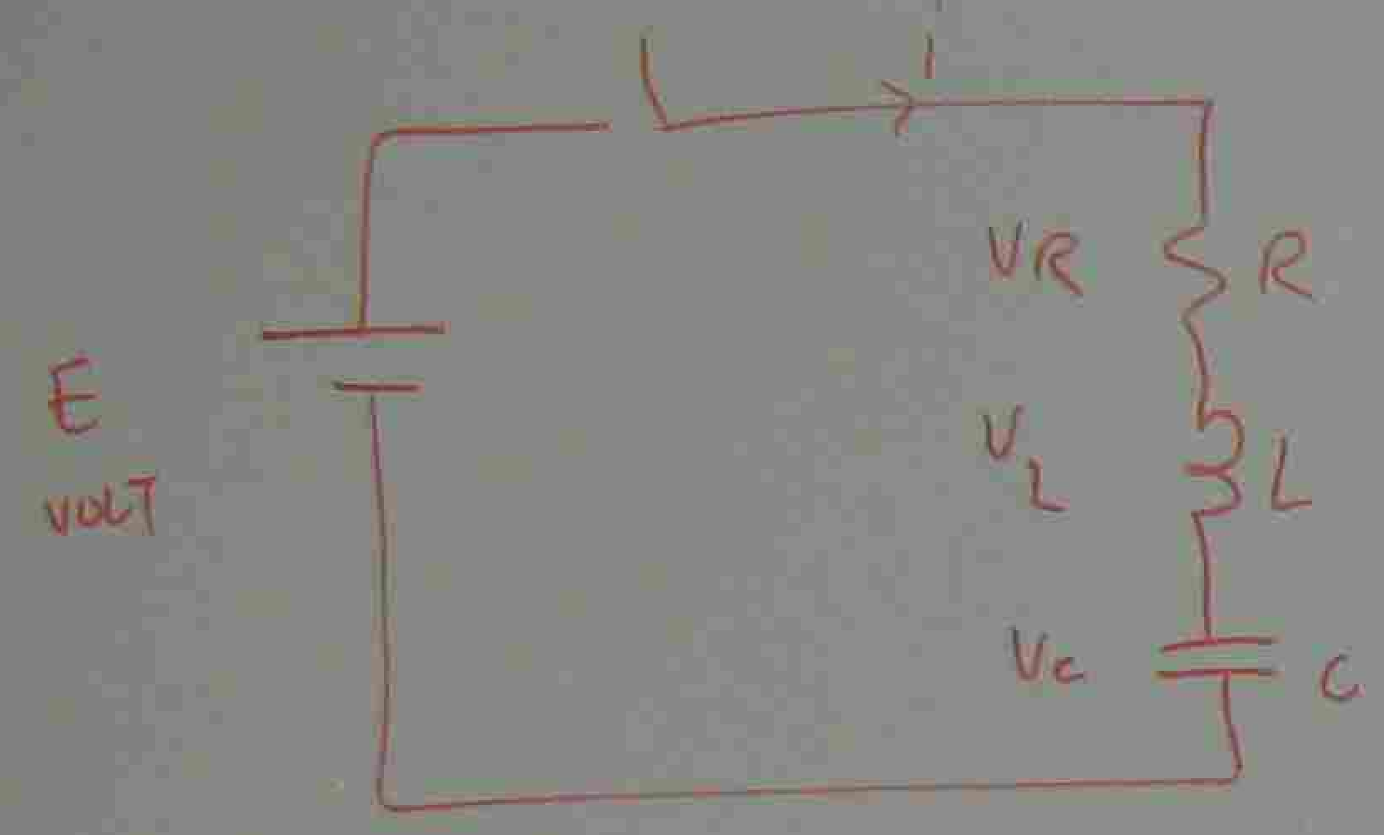
(1)

$$0.4107 = A + 0.4\sin 113.1$$

$$A = 0.0433$$

$$i(t) = 0.0433e^{-666.7t} + 0.4(500t + 113.1) \ldots$$

SOLVING THE SECOND ORDER DIFFERENTIAL EQUATION



$$V_R + V_L + V_c = E$$

$$iR + L\frac{di}{dt} + \frac{1}{c}\int i\,dt = E$$

$$\frac{d}{dt} iR + \frac{d}{dt} L\frac{di}{dt} + \frac{d}{dt}\frac{1}{c}\int i\,dt = \frac{d}{dt} E$$

$$R\frac{di}{dt} + L\frac{d^2 i}{dt^2} + \frac{1}{c} \times i = 0$$

$$\frac{R}{L}\frac{di}{dt} + \frac{d^2 i}{dt^2} + \frac{1}{Lc} i = 0$$

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{Lc} i = 0$$

COMPARE WITH

$$a\frac{d^2 i}{dt^2} + b\frac{di}{dt} + ci = 0$$

$$am^2 + bm + c = 0$$

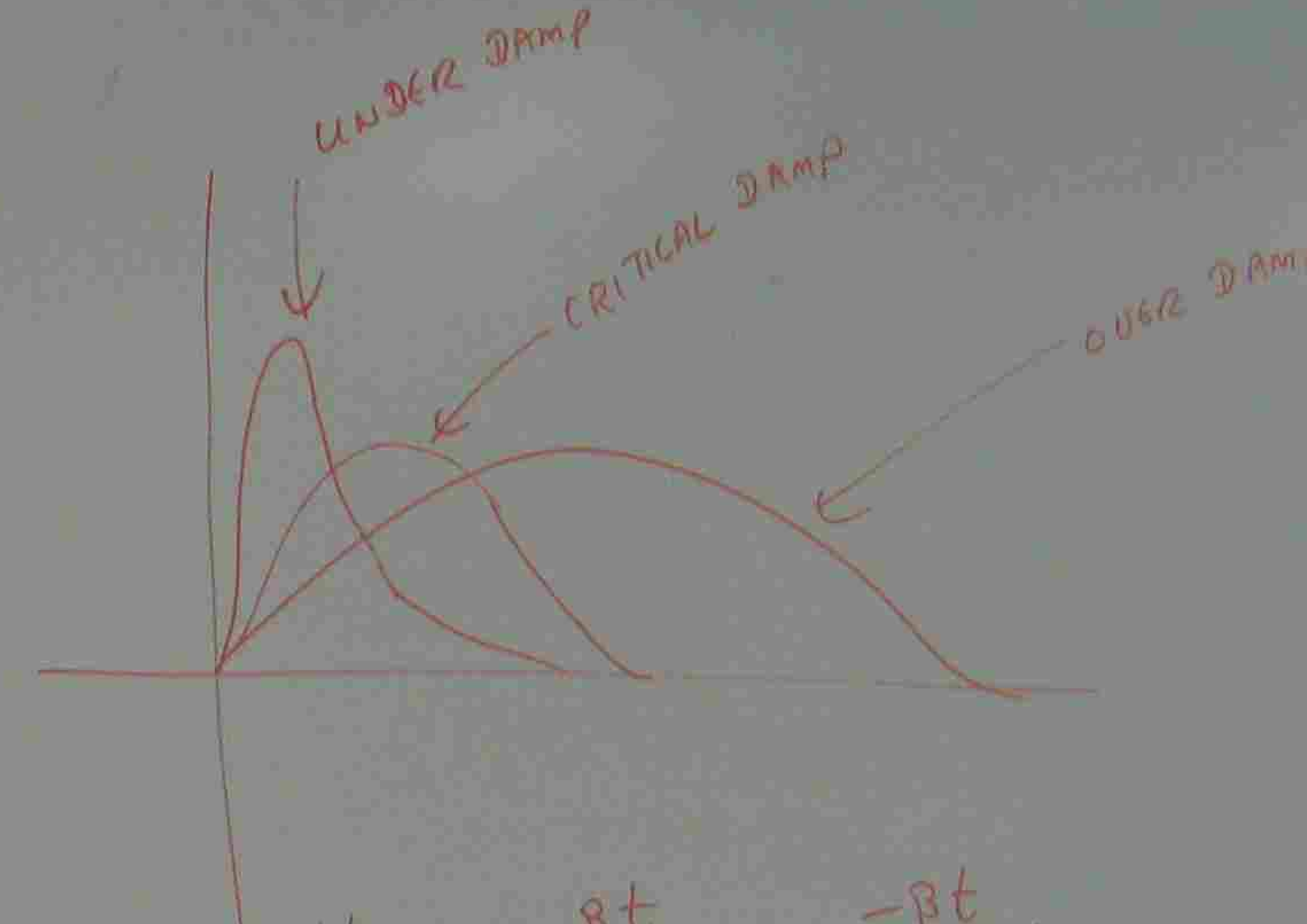
$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

LET

$$\alpha = \frac{R}{2L} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$\beta = \sqrt{\alpha^2 - \omega_0^2}$$



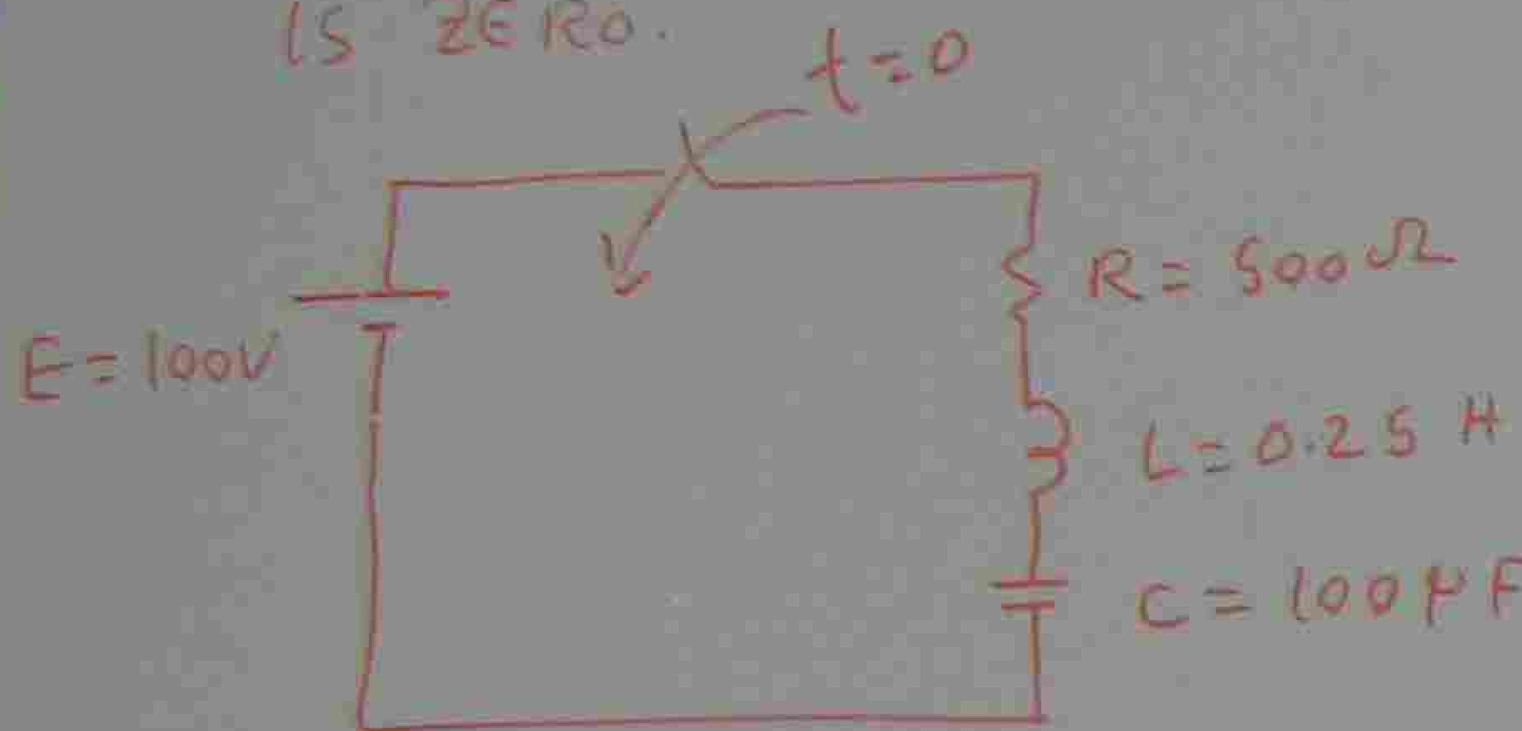
IF $\alpha > \omega_0 \Longrightarrow$ OVER DAMP $\longrightarrow$ $i = e^{-\alpha t}\left(A_1 e^{\beta t} + A_2 e^{-\beta t}\right)$

IF $\alpha = \omega_0 \Longrightarrow$ CRITICAL DAMP $\longrightarrow$ $i = e^{-\alpha t}\left(A_1 + A_2 t\right)$

IF $\alpha < \omega_0 \Longrightarrow$ UNDER DAMP $\longrightarrow$ $i = e^{-\alpha t}\left(A_1 \sin\beta t + A_2 \cos\beta t\right)$

Pb IN A GIVEN CIRCUIT, $E = 100V$ $R = 500\Omega$, $L = 0.25H$, $C = 100\mu F$.

DETERMINE THE EQUATION OF THE CURRENT IF THE INITIAL CHARGE ON CAPACITOR IS ZERO.



$$\alpha = \frac{R}{2L} = \frac{500}{2\times 0.25} = 1000$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.25\times 100\times 10^{-6}}} = 200$$

$$\beta = \sqrt{\alpha^2 - \omega_0^2}$$

$\alpha > \omega_0 \Rightarrow$ OVER DAMP

$$= \sqrt{1000^2 - 200^2}$$

$$= 979.8$$

$$i = e^{-\alpha t}\left[A_1 e^{\beta t} + A_2 e^{-\beta t}\right]$$

$$= A_1 e^{(-\alpha+\beta)t} + A_2 e^{(-\alpha-\beta)t}$$

$$= A_1 e^{(-1000+979.8)t} + A_2 e^{(-1000-979.8)t}$$

$$i = A_1 e^{-20.2t} + A_2 e^{-1979.8t}$$

I $t = 0 \rightarrow i = 0$

$$0 = A_1 e^{-20.2\times 0} + A_2 e^{-1979.8\times 0}$$

$$A_1 + A_2 = 0 \quad (1) \Rightarrow A_1 = -A_2$$

$$i = A_1 e^{-20.2t} + A_2 e^{-1979.8t}$$

$$\frac{di}{dt} = A_1 \frac{d}{dt} e^{-20.2t} + A_2 \frac{d}{dt} e^{-1979.8t}$$

$$\frac{di}{dt} = -20.2 A_1 e^{-20.2t} + (-1979.8) A_2 e^{-1979.8t}$$

II

$L \frac{di}{dt}$ = INDUCTOR VOLTAGE

$t=0 \rightarrow L\frac{di}{dt} = E$

$$\frac{di}{dt} = \frac{E}{L} = \frac{100}{0.25} = 400$$

$$400 = -20.2 A_1 e^{-20.2\times 0} - 1979.8 A_2 e^{-1979.8\times 0}$$

$$-20.2 A_1 - 1979.8 A_2 = 400$$

$$-20.2(-A_2) - 1979.8 A_2 = 400$$

$$A_2 = -0.2041$$

$$\therefore A_1 = 0.02041$$

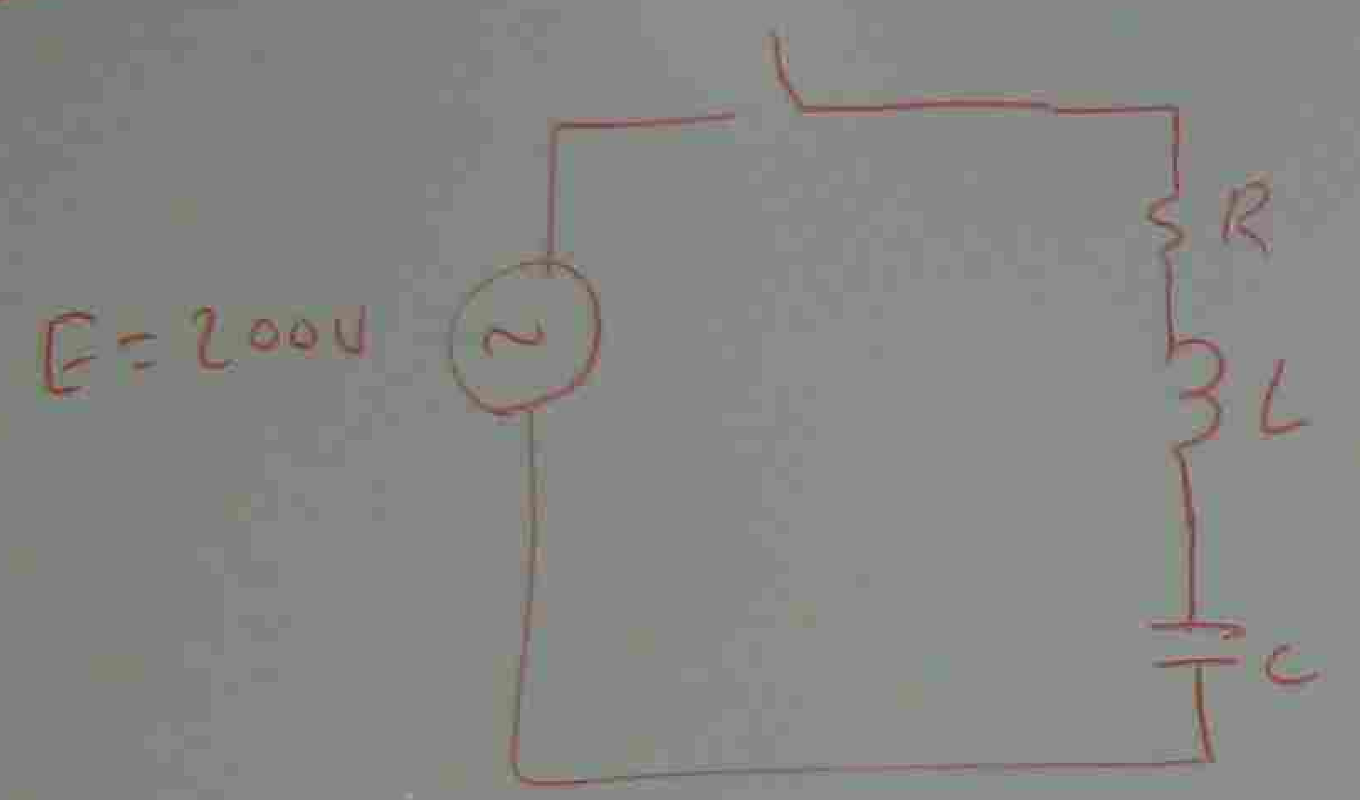
$$i = 0.02041 e^{-20.2t} - 0.204 e^{-1979.8t}$$

$0 - 979.8)t$

$\times 0$

$\Rightarrow A_1 = -A_2$

EXERCISE



E = 200V, R = 100Ω L = 0.5 HENRY

C = 200 μF

FIND THE EQUATION OF THE CURRENT

IF THE INITIAL CHARGE ON CAPACITOR IS

20 MILLICOULOMBS

$\alpha = \frac{R}{2L}$ $\omega_0 = \frac{1}{\sqrt{LC}}$

COMPARE α & ω_0 SELECT THE EQUATION

CONDITION (1) $t = 0 \longrightarrow i = 0$

CONDITION 2

$q = 20 \text{ mC}$

$V_c = \frac{q}{C} = \frac{20 \times 10^{-3}}{200 \times 10^{-6}}$

$= 100V$

$V_c = \frac{1}{C} \int i\,dt$

$t=0 \Big| \frac{di}{dt} = \frac{E}{L} = \frac{100V}{0.5}$

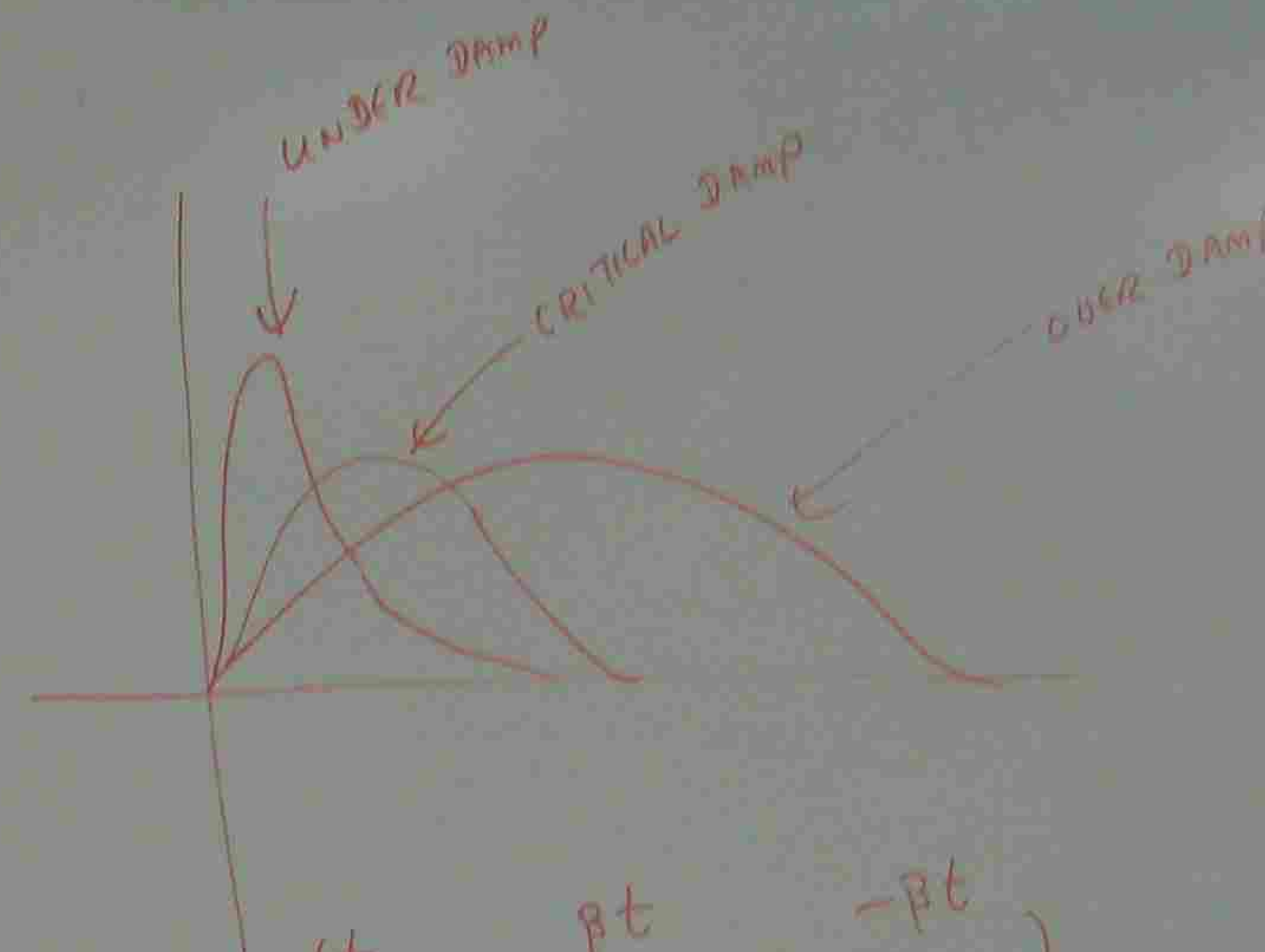
$= 200$

IF

IF

IF

$$m = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$



LET

$$\alpha = \frac{R}{2L} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$\beta = \sqrt{\alpha^2 - \omega_0^2}$$

IF $\alpha > \omega_0 \Longrightarrow$ OVER DAMP $\longrightarrow$ $i = e^{-\alpha t}\left(A_1 e^{\beta t} + A_2 e^{-\beta t}\right)$

IF $\alpha = \omega_0 \Longrightarrow$ CRITICAL DAMP $\longrightarrow$ $i = e^{-\alpha t}\left(A_1 + A_2 t\right)$

IF $\alpha < \omega_0 \Longrightarrow$ UNDER DAMP $\longrightarrow$ $i = e^{-\alpha t}\left(A_1 \sin \beta t + A_2 \cos \beta t\right)$

-3

0

-6

0

$= \frac{100V}{0.5}$

00

$\pi \;-\; 20\times - \left[\cos\theta\right]_0^{2\pi} \Big]$

$\pi + 20\left[\cos 2\pi - \cos 0\right]$

$20\left[1 - 1\right]$

$\to 0$

$20\,\omega t$

dt (or) $V_{Rms} = \sqrt{\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} V(\theta)^2\, d\theta}$

2π

$(20 - 20\sin\theta)^2\, d\theta$

$$V_{Rms} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} (400 - 800\sin\theta + 400\sin^2\theta)\, d\theta}$$

$$\cos 2\theta = 1 - 2\sin^2\theta \longrightarrow \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$= \sqrt{\frac{1}{2\pi}\left[\int_0^{2\pi} 400\, d\theta - \int_0^{2\pi} 800\sin\theta\, d\theta + 400\int_0^{2\pi}\left(\frac{1-\cos 2\theta}{2}\right) d\theta\right]}$$

$$= \sqrt{\frac{1}{2\pi}\left[400\times 2\pi + 800(\cos\theta)_0^{2\pi} + \frac{400}{2}(\theta)_0^{2\pi} - \frac{400}{4}(\sin 2\theta)_0^{2\pi}\right]}$$

$$= \sqrt{\frac{1}{2\pi}\left[\left(800\pi + 800\underbrace{\cancel{(\cos 2\pi - \cos 0)}}_{0}\right) + 200\times 2\pi - \frac{400}{4}\underbrace{\cancel{(\sin 4\pi - \sin 0)}}_{0}\right]}$$

$$= \sqrt{\frac{1}{2\pi}\times(800\pi + 400\pi)}$$

$$V_{Rms} = \sqrt{600} = 24.49$$

(c)

$$\text{Form Factor} = \frac{V_{Rms}}{V_{AVG}} = \frac{24.49}{20} = 1.225$$

REVISION

① $$\int_{\pi}^{2\pi} (3\sin x + 10)\,dx$$

$$\int_{\pi}^{2\pi} 3\sin x\,dx + \int_{\pi}^{2\pi} 10\,dx$$

$$3\int_{\pi}^{2\pi} \sin x\,dx + 10\int_{\pi}^{2\pi} dx$$

$$3 \times \left[-\cos x\right]_{\pi}^{2\pi} + 10\left[x\right]_{\pi}^{2\pi}$$

$$-3\left[\cos 2\pi - \cos \pi\right] + 10\left[2\pi - \pi\right]$$

$$-3\left[\cos 360 - \cos 180\right] + 10 \times \pi$$

$$-3\left[1 - (-1)\right] + 10 \times 3.1416$$

$$-3 \times 2 + 31.416$$

$$25.416$$

② $\int (5x+7)^3 \, dx$

$d(5x+7) = d5x + d7$

$= 5dx$

$\therefore dx = \dfrac{d(5x+7)}{5}$

$\int (5x+7)^3 \, \dfrac{d(5x+7)}{5}$

$\dfrac{1}{5} \int (5x+7)^3 \, d(5x+7)$

$\dfrac{1}{5} \times \dfrac{(5x+7)^{3+1}}{3+1} + c$

$\dfrac{1}{5} \dfrac{(5x+7)^4}{4} + c = \dfrac{1}{20}(5x+7)^4 + c$

③ $\int \sin^2 2x \, dx$

$\cos 2x = 1 - 2\sin^2 x$

$\cos 4x = 1 - 2\sin^2 2x$

$2\sin^2 2x = 1 - \cos 4x$

$\sin^2 2x = \dfrac{1 - \cos 4x}{2}$

$\int \dfrac{1 - \cos 4x}{2} \, dx$

$\dfrac{1}{2} \int [1 - \cos 4x] \, dx$

$\dfrac{1}{2} \left[\int dx - \int \cos 4x \, dx \right]$

$d4x = 4dx$

$\therefore dx = \dfrac{d4x}{4}$

$\dfrac{1}{2} \left[x - \int \cos 4x \times \dfrac{d4x}{4} \right]$

$\dfrac{1}{2} \left[x - \dfrac{\sin 4x}{4} \right] + c$

$\dfrac{x}{2} - \dfrac{\sin 4x}{8} + c$

(4) $\int \sin 3x \cos 4x \, dx$

$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$

$\sin 3x \cos 4x = \frac{1}{2}\left[\sin(3x+4x) + \sin(3x-4x)\right]$

$= \frac{1}{2}\left[\sin 7x + \sin(-x)\right]$

$= \frac{1}{2}\left[\sin 7x - \sin x\right]$

$\int \frac{1}{2}\left[\sin 7x - \sin x\right] dx$

$\frac{1}{2}\left[\int \sin 7x \, dx - \int \sin x \, dx\right]$

$\frac{1}{2}\left[\int \frac{\sin 7x \, d7x}{7} - (-\cos x)\right]$

$\frac{1}{2}\left[-\frac{\cos 7x}{7} + \cos x\right] + C$

$\frac{\cos x}{2} - \frac{\cos 7x}{14} + C$

(5) $\int x\sqrt{2x+1} \, dx$

LET

$\sqrt{2x+1} = u$

$2x+1 = u^2$

$2x = u^2 - 1$

$x = \frac{u^2-1}{2}$

$dx = d\left(\frac{u^2-1}{2}\right)$

$dx = \frac{du^2}{2} - d\frac{1}{2}$

$= \frac{2u\,du}{2}$

$dx = u\,du$

$$\int \frac{u^2-1}{2} \times u \times u\,du$$

$$\frac{1}{2}\left[\int (u^2-1)\,u^2\,du\right]$$

$$\frac{1}{2}\left[\int (u^4-u^2)\,du\right]$$

$$=\frac{1}{2}\left[\int u^4du-\int u^2du\right]$$

$$=\frac{1}{2}\left[\frac{u^5}{5}-\frac{u^3}{3}\right]+c$$

$$=\frac{u^5}{10}-\frac{u^3}{6}+c$$

$$=\frac{(\sqrt{2x+1})^5}{10}-\frac{(\sqrt{2x+1})^3}{6}+c$$

$$\frac{(2x+1)^{5/2}}{10}-\frac{(2x+1)^{3/2}}{6}+c$$

(6) $$\int \frac{x}{\sqrt{7-x}}\,dx$$

$$u=\frac{1}{\sqrt{7-x}}$$

$$u=(7-x)^{-1/2}$$

$$u^2=(7-x)^{-1}$$

$$u^2=\frac{1}{7-x}$$

$$7-x=\frac{1}{u^2}$$

$$x=7-\frac{1}{u^2}$$

$$dx=d\left(7-\frac{1}{u^2}\right)$$

$$dx=-d\,u^{-2}$$

$$=-(-2)\,u^{-2-1}\,du$$

$$dx=2u^{-3}du$$

$$=\frac{2du}{u^3}$$

$$\int\left(14-\frac{2}{u^2}\right)\times u\times\frac{2du}{u^3}$$

$$\int\left(14-\frac{2}{u^2}\right)\times\frac{1}{u^2}du$$

$$\int\frac{14}{u^2}du-\frac{2}{u^4}du$$

$$14\int u^{-2}du-2\int u^{-4}du$$

$\left.\right]$

$-du \left.\right]$

$\left.\right] + c$

$+ c$

$\frac{\left(\sqrt{2x+1}\right)^3}{6} + c$

$$\frac{(2x+1)^{5/2}}{10} - \frac{(2x+1)^{3/2}}{6} + c$$

(6) $\int \frac{x}{\sqrt{7-x}}\,dx$

$$u = \frac{1}{\sqrt{7-x}}$$

$$u = (7-x)^{-1/2}$$

$$u^2 = (7-x)^{-1}$$

$$u^2 = \frac{1}{7-x}$$

$$7 - x = \frac{1}{u^2}$$

$$x = 7 - \frac{1}{u^2}$$

$$dx = d\left(7 - \frac{1}{u^2}\right)$$

$$dx = -d\,u^{-2}$$

$$= -(-2)\,u^{-2-1}\,du$$

$$dx = 2\,u^{-3}\,du$$

$$= \frac{2\,du}{u^3}$$

$$\int \left(14 - \frac{2}{u^2}\right) \times u \times \frac{2\,du}{u^3}$$

$$\int \left(14 - \frac{2}{u^2}\right) \times \frac{1}{u^2}\,du$$

$$\int \frac{14}{u^2}\,du - \frac{2}{u^4}\,du$$

$$14\int u^{-2}\,du - 2\int u^{-4}\,du$$

$$14 \times \frac{u^{-2+1}}{-2+1} - 2\frac{u^{-4+1}}{-4+1} + c$$

$$14\,\frac{u^{-1}}{-1} - \frac{2u^{-3}}{-3} + c$$

$$-\frac{14}{u} + \frac{2}{3u^3} + c$$

$$-\frac{14}{\frac{1}{\sqrt{7-x}}} + \frac{2}{3\times\left(\frac{1}{\sqrt{7-x}}\right)^3} + c$$

$$-14\sqrt{7-x} + \frac{2(7-x)^{3/2}}{3} + c$$

$$-14(7-x)^{1/2} + \frac{2}{3}(7-x)^{3/2} + c$$

7 $$\int \frac{e^{ax}}{e^{ax}+4}\,dx$$

$$u = e^{ax}+4$$

$$du = d\,e^{ax} + d4$$

$$du = e^{ax}\,d\,ax$$

$$du = e^{ax} \times a\,dx$$

$$\therefore e^{ax} = u-4$$

$$du = (u-4)\times a \times dx$$

$$dx = \frac{du}{(u-4)\times a}$$

$$\int \frac{u-4}{u} \times \frac{du}{(u-4)\times a}$$

$$\int \frac{du}{u \times a}$$

$$\frac{1}{a}\int \frac{du}{u}$$

$$\frac{1}{a}\int \frac{1}{u}\,du$$

$$\frac{1}{a}\ln u + c$$

$$\frac{1}{a}\ln(e^{ax}+4)+c$$

8

d sinx

(9) $\displaystyle\int_2^5 \frac{1}{x}\,dx = \left[\ln x\right]_2^5$

$= \ln 5 - \ln 2 = 1.609 - 0.693 = 0.915$

(10) $\displaystyle\int_1^2 e^{-3x}\,dx$

$d(-3x) = -3\,dx$

$dx = -\frac{1}{3}\,d(-3x)$

$\displaystyle\int_1^2 e^{-3x}\left(-\frac{1}{3}\,d(-3x)\right)$

$\displaystyle -\frac{1}{3}\int_1^2 e^{-3x}\,d(-3x)$

$\displaystyle -\frac{1}{3}\left[e^{-3x}\right]_1^2$

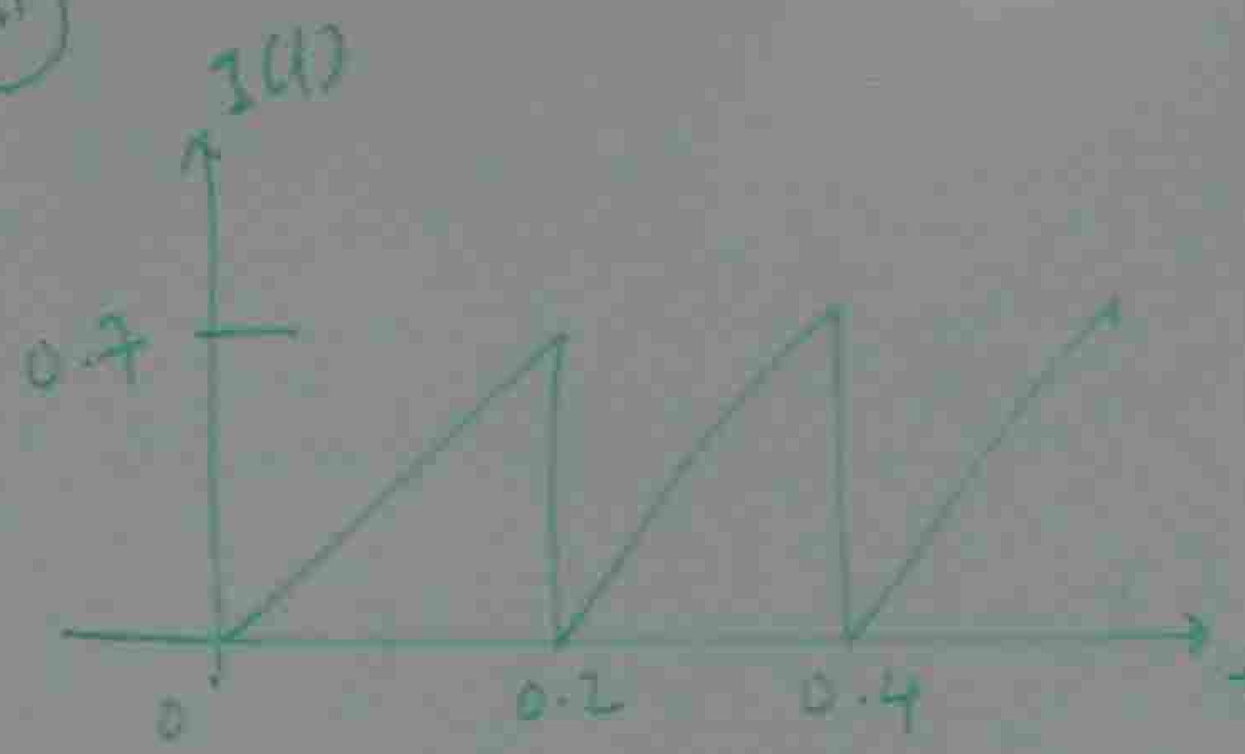
$\displaystyle -\frac{1}{3}\left[e^{-6} - e^{-3}\right]$

$\displaystyle -\frac{1}{3}\left[2.478\times10^{-3} - 0.049\right]$

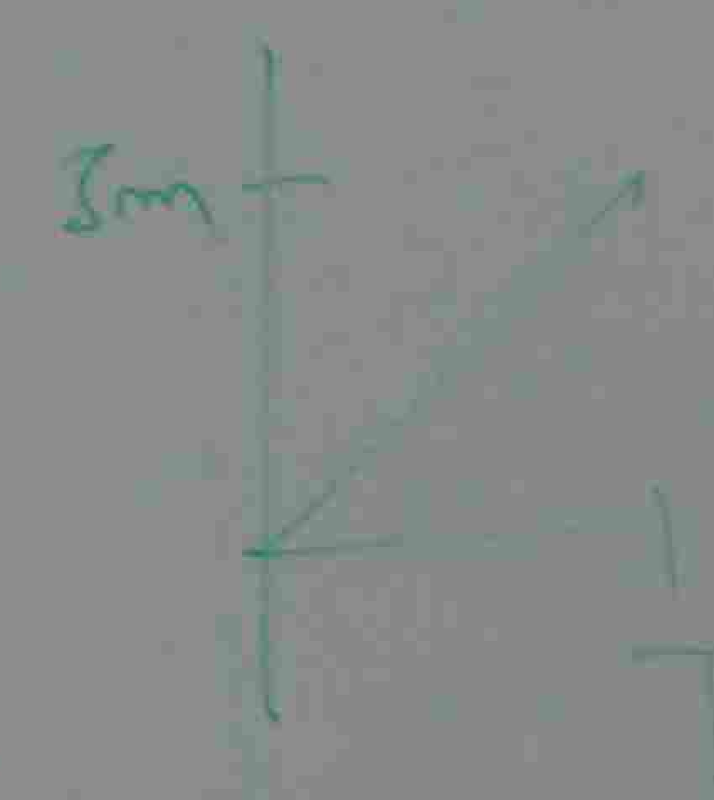
$\displaystyle -\frac{1}{3}\left[-0.046\right]$

$\therefore 0.0157$

(11)



FIND I_{AVG} FOR GIVEN WAVE.



$f(t) = \frac{I_m}{T}\times t$

$I_{AV} = \frac{1}{T}\int_0^T f(t)\,dt$

$I_{AVE} = \frac{1}{T}\int_0^T \frac{I_m}{T}\, t \; dt$

$= \frac{I_m}{T^2}\int_0^T t\,dt$

$= \frac{I_m}{T^2}\left[\frac{t^2}{2}\right]_0^T$

$= \frac{I_m}{T^2} \times \frac{T^2}{2}$

$= \frac{I_m}{2}$

$= \frac{0.7}{2}$

$= 0.35$

(12) A VOLTAGE WAVE FORM IS REPRESENTED BY THE EQUATION

$V(t) = 20 - 20 \sin \omega t$

DETERMINE THE FOLLOWINGS

(a) AVERAGE VALUE (b) RMS VALUE (c) FORM FACTOR

$\omega t = \theta$

$V_{AVE} = \frac{1}{T}\int_0^T f(t)\,dt$ (or) $V_{AVE} = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta$

$v(t) = 20 - 20\sin\omega t$ (or) $v(\theta) = 20 - 20\sin\theta$

$V_{AVE} = \frac{1}{2\pi}\int_0^{2\pi}(20 - 20\sin\theta)\,d\theta$

$= \frac{1}{2\pi}\left[\int_0^{2\pi} 20\,d\theta - \int_0^{2\pi} 20\sin\theta\,d\theta\right]$

$= \frac{1}{2\pi}\left[20 (\ldots\right.$

$= \frac{1}{2\pi}\left[20 \ldots\right.$

$= \frac{1}{2\pi}\left(40 \ldots\right.$

$V_{AVG} = \frac{1}{2\pi} \times 40\pi$

$V_{rms} = \sqrt{\frac{1}{T}\int_0 \ldots}$

$V_{rms} = \sqrt{\frac{1}{2}\ldots}$

$$= \frac{1}{2\pi}\left[20\,(\theta)_0^{2\pi} - 20\times -\left[\cos\theta\right]_0^{2\pi}\right]$$

$$= \frac{1}{2\pi}\left[20\times 2\pi + 20\left[\cos 2\pi - \cos 0\right]\right]$$

$$= \frac{1}{2\pi}\left[40\pi + 20\left[1 - 1\right]\right]$$

$$V_{AVG} = \frac{1}{2\pi}\times 40\pi = 20\ \text{unit}$$

$$V_{Rms} = \sqrt{\frac{1}{T}\int_0^T f(t)^2\,dt}\quad (\text{OR})\quad V_{Rms} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} v(\theta)^2\,d\theta}$$

$$V_{Rms} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi}(20-20\sin\theta)^2\,d\theta}$$

$$V_{Rms} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi}(400 - 800\sin\theta + 400\sin^2\theta)\,d\theta}$$

$$\cos 2\theta = 1 - 2\sin^2\theta \longrightarrow \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$= \sqrt{\frac{1}{2\pi}\left[\int_0^{2\pi}400\,d\theta - \int_0^{2\pi}800\sin\theta\,d\theta + 400\int_0^{2\pi}\left(\frac{1-\cos 2\theta}{2}\right)d\theta\right]}$$

$$= \sqrt{\frac{1}{2\pi}\left[400\times 2\pi + 800(\cos\theta)_0^{2\pi} + \frac{400}{2}(\theta)_0^{2\pi} - \frac{1}{4}(\sin 2\theta)_0^{2\pi}\right]}$$

$$= \sqrt{\frac{1}{2\pi}\left[\left(800\pi + \underbrace{800(\cos 2\pi - \cos 0)}_{0} + 200\times 2\pi - \frac{1}{4}\underbrace{(\sin 4\pi - \sin 0)}_{0}\right)\right]}$$

$$= \sqrt{\frac{1}{2\pi}\times(800\pi + 400\pi)}$$

$$V_{Rms} = \sqrt{600} = 24.49$$

(c)

$$\text{FORM FACTOR} = \frac{V_{Rms}}{V_{AVG}} = \frac{24.49}{20} = 1.225$$

STATISTICS

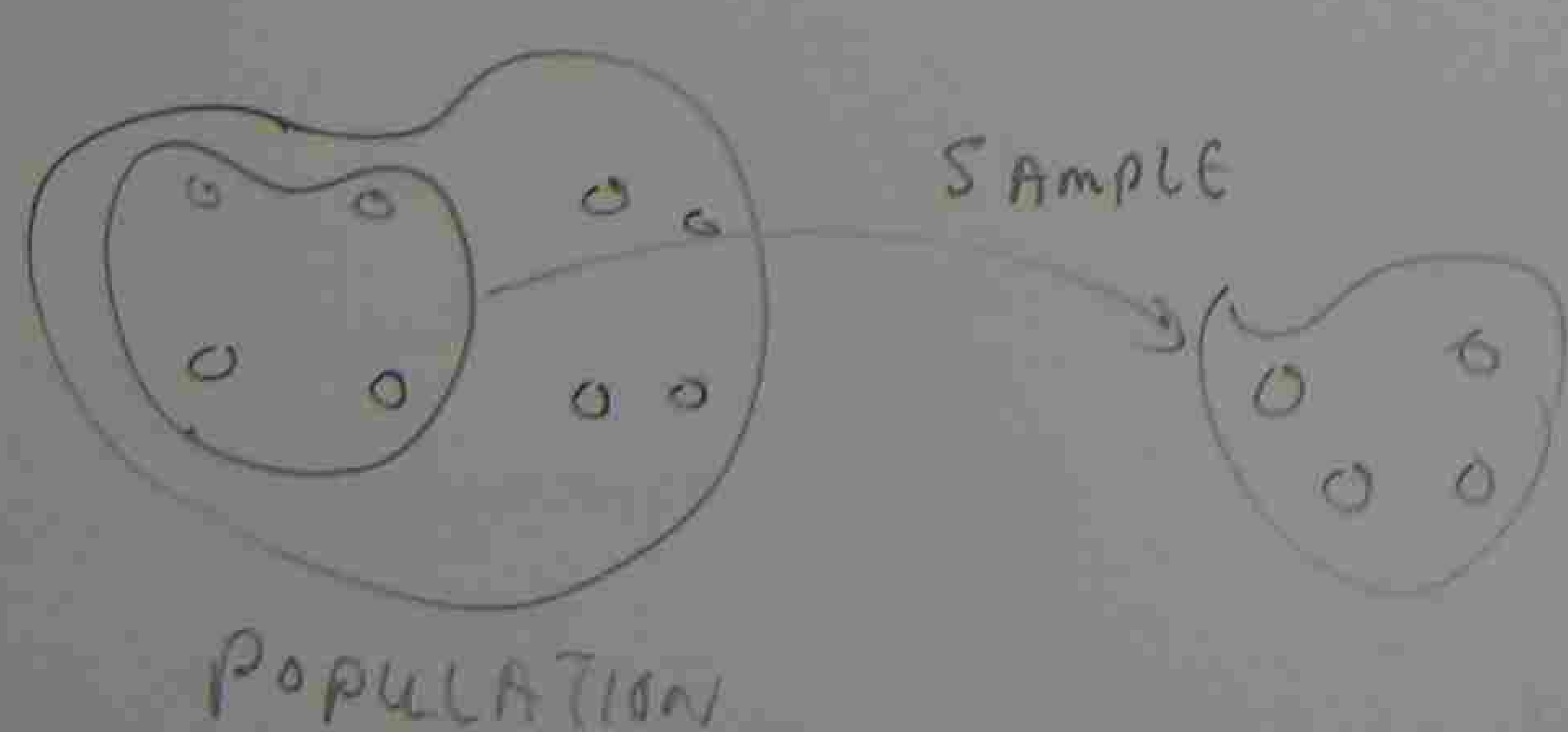
STATISTICS INVOLVES COLLECTING, SUMMARISING ANALYZING AND INTERPRETING DATA.

TYPES OF DATA

CATEGORICAL DATA

NUMERICAL DATA

DISCRETE — CONTINUOUS



DISCRETE – COUNTABLE SUCH AS NUMBER OF DAYS.

CONTINUOUS – INFINITE SUCH AS DAILY TEMPERATURE

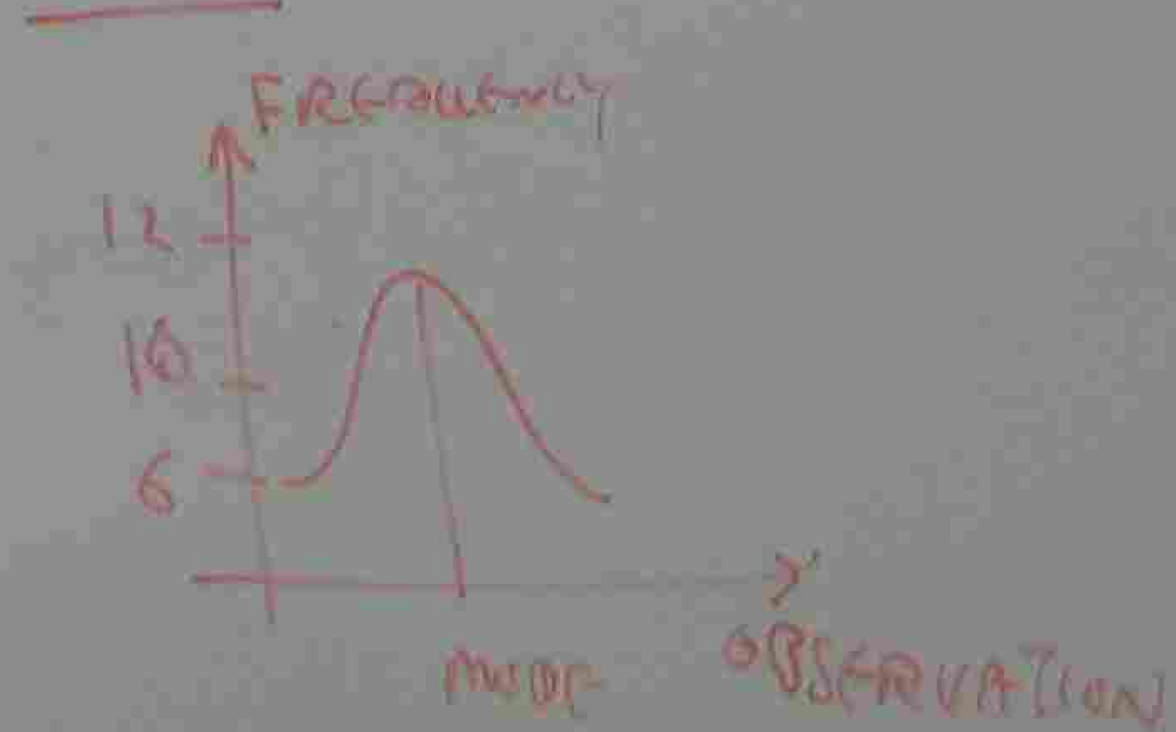
PARAMETER

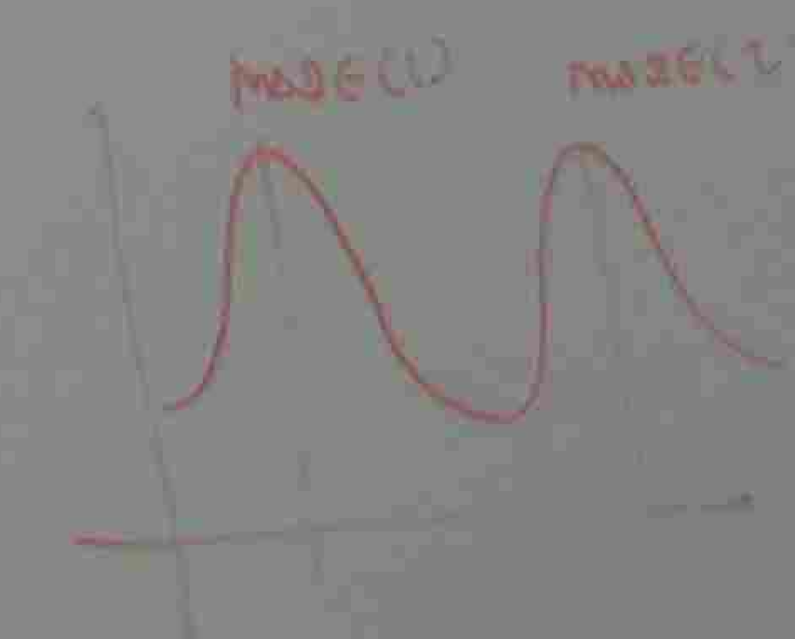
A NUMERICAL MEASURE THAT DESCRIBES SOME CHARACTERISTICS OF POPULATION

STATISTICS

- NUMERICAL MEASURE THE DESCRIBES SOME CHARACTERISTICS OF SAMPLE

MODE THE DATA SET THAT OCCURS MOST OFTEN





OF DAYS.

PERATURE

SOME

SOME

OFTEN

MODE(2)

NO MODE

Pb (1) FIND THE MODE OF THE FOLLOWING SAMPLES OF DATA

(a) 18, 19, 18, 20, 18, 18, 20, 21, 37, 18

(b) 27, 28, 29, 36, 14, 24

(c) 19, 18, 20, 39, 18, 18, 19, 46, 19

(a) MODE = 18

(b) NO MODE

(c) 19, 18 = MODES

18 = MODE 1

19 = MODE 2

MEDIAN

THE MIDDLE NUMBER IN AN ORDERED SET.

Pb (2)

FIND THE MEDIAN OF EACH OF THE FOLLOWING SAMPLES OF DATA

(a) X: 7, 3, 2, 8, 11

(b) X: 18, 19, 18, 20, 18, 18, 20, 21, 37, 18

(a) 2, 3, 7, 8, 11

MEDIAN = 7

(b) X = 18, 18, 18, 18, 18, 19, 20, 20, 21, 37

MEDIAN = $\frac{18+19}{2}$

= 18.

MEAN THE AR...

MEASUREMENTS

BY ADDING TH...

DIVIDING THIS...

THE DATA S...

$\bar{X} = $

MEDIAN = $\frac{18+19}{2}$

= 18.5

MEAN: THE ARITHMETIC AVERAGE OF ALL MEASUREMENTS IN THE DATA SET OBTAINED BY ADDING THEM ALL UP AND DIVIDING THIS SUM BY THE SIZE OF THE DATA SET.

$$\bar{X} = \frac{\Sigma X}{n}$$

21, 37

pb 3 FIND THE MEAN OF THE FOLLOWING SAMPLES OF DATA

(i) X: 11, 14, 10, 5

(ii) X: 2, 2, 4, 5, 5, 5, 6, 7, 8, 9, 11

(i) $\text{MEAN} = \bar{x} = \frac{11+14+10+5}{4} = \frac{40}{4} = 10$

(ii) $\text{MEAN} = \bar{x} = \frac{2+2+4+5+5+5+6+7+8+9+11}{11}$

$= \frac{64}{11} = 5.818$

pb 4 FIND THE MEAN OF THE FOLLOWING GROUP FREQUENCY DISTRIBUTION (GFD)

INTERVAL	FREQUENCY	INTERVAL	FREQUENCY
$0 \to 9$	1	$50 \to 59$	1
$10 \to 19$	2		
$20 \to 29$	4		
$30 \to 39$	8		
$40 \to 49$	4		

+11

P FREQUENCY

INTERVAL	MEAN	f	fx
$0 \longrightarrow 9$	$\frac{0+9}{2} = 4.5$	1	$4.5 \times 1 = 4.5$
$10 \longrightarrow 19$	$\frac{10+19}{2} = 14.5$	2	$14.5 \times 2 = 29$
$20 \longrightarrow 29$	$\frac{20+29}{2} = 24.5$	4	$24.5 \times 4 = 98$
$30 \longrightarrow 39$	$\frac{30+39}{2} = 34.5$	8	$34.5 \times 8 = 276$
$40 \longrightarrow 49$	$\frac{40+49}{2} = 44.5$	4	$44.5 \times 4 = 178$
$50 \longrightarrow 59$	$\frac{50+59}{2} = 54.5$	1	$54.5 \times 1 = 54.5$
		$\sum f = 20$	$\sum fx = 640$

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{640}{20} = 32$$

RANGE

THE DIFFERENCE BETWEEN THE LARGEST AND SMALLEST VALUES IN THE SAMPLE.

Pb 5 FIND THE RANGE OF THE FOLLOWING SAMPLE OF DATA.

(i) 18, 19, 18, 20, 18, 37, 21, 20, 18, 18

(ii) 18, 19, 18, 20, 18, 21, 20, 18, 18

(i) RANGE = LARGEST − SMALLEST

$= 37 - 18 = 19$

(ii) RANGE = LARGEST − SMALLEST

$= 21 - 18 = 3$

RESIDUAL

THE DIFFERENCE BETWEEN EACH SAMPLE x AND THE MEAN $\bar{x}$ IS CALLED THE DEVIATION FROM THE MEAN OR RESIDUAL.

$$\text{RESIDUAL} = x - \bar{x}$$

Pb 6 10, 12, 15, 17, 21

CALCULATE (i) MEAN (ii) RESIDUAL (DEVIATION)

(i) $\bar{x} = \dfrac{10+12+15+17+21}{5} = \dfrac{75}{5} = 15$

(ii) RESIDUAL: $(x - \bar{x}) = (10-15), (12-15), (15-15), (17-15)$

$= -5, -3, 0, 2, 6$

VARIANCE

$$\text{VARIANCE} = \frac{\sum (x - \bar{x})^2}{n-1}$$

...CE BETWEEN EACH SAMPLE x AND THE MEAN $\bar{x}$

...HE DEVIATION FROM THE MEAN OR RESIDUAL.

...IDUAL $= x - \bar{x}$

..., 12, 15, 17, 21

(i) MEAN (ii) RESIDUAL (DEVIATION)

$$\frac{10+12+15+17+21}{5} = \frac{75}{5} = 15$$

...L $= (x - \bar{x}) = (10-15), (12-15), (15-15), (17-15), (21-15)$

$= -5, -3, 0, 2, 6$

$$\text{VARIANCE} = \frac{\sum (x-\bar{x})^2}{n-1}$$

pb 7 CALCULATE VARIANCE OF 10, 12, 15, 17, 21

$$\bar{x} = \text{MEAN} = \frac{\sum x}{n} = \frac{10+12+15+17+21}{5} = 15$$

$$\text{VARIANCE} = s^2 = \frac{\sum(x-\bar{x})^2}{n-1} = \frac{(10-15)^2 + (12-15)^2 + (15-15)^2 + (17-15)^2 + (21-\ldots}{5-1}$$

$$= \frac{(-5)^2 + (-3)^2 + (0)^2 + (2)^2 + (6)^2}{4}$$

$$= \frac{25+9+4+36}{4} = \frac{74}{4} = 18.5$$

STANDARD DEVIATION

THE STANDARD DEVIATION OF A SAMPLE OF DATA IS DEFINED TO THE POSITIVE SQUARE ROOT OF THE VARIANCE

$$s = \sqrt{\frac{\sum(x-\bar{x})^2}{n-1}}$$

APPROPRIATE
FOOTWEAR
THIS BUILDING

$(21-15)^2$

pb ⑧ CALCULATE STANDARD DEVIATION OF 10, 12, 15, 17, 21

$$s = \sqrt{\frac{\sum (x-\bar{x})^2}{n-1}} = \sqrt{\frac{(10-15)^2 + (12-15)^2 + (15-15)^2 + (17-15)^2 + (21-15)^2}{5-1}}$$

$$\bar{x} = \frac{10+12+15+17+21}{5} = 15$$

$$= \sqrt{18.5} = 4.3$$

CALCULATION OF s^2 (VARIANCE) AND s (STANDARD DEVIATION) FOR A SAMPLE OF RAW DATA

$$s^2 = \frac{\sum (x-\bar{x})^2}{n-1} = \frac{\sum x^2 - n(\bar{x})^2}{n-1}$$

pb ⑨

CALCULATE THE VARIANCE AND STANDARD DEVIATION FOR THE FO[…] SAMPLE OF DATA.

X: 2, 3, 5, 5, 9

$$\bar{x} = \frac{\sum x}{n} = \frac{2+3+5+5+9}{5} = \frac{24}{5} = 4.8$$

$$\sum x^2 = 2^2 + 3^2 + 5^2 + 5^2 + 9^2 = 144$$

$$n(\bar{x})^2 = 5 \times (4.8)^2 = 115.2$$

$$s^2 = \frac{\sum (x-\bar{x})^2}{n-1} = \frac{\sum x^2 - n(\bar{x})^2}{n-1}$$

$$s^2 = \frac{144 - 115.2}{5-1} = 7.2$$

$$s = \sqrt{7.2} = 2.68$$

VARIAN[…]

STAND[…] DEVIA[…]

pb (1[…]

$\bar{x} =$

$s^2 = \frac{\sum f x^2 - n(\ldots}{n-1}$

$s = \sqrt{32.89} = 5.[…]$

ANCE AND STANDARD DEVIATION FOR THE FOLLOWING

5, 5, 9

$\frac{5+5+9}{} = \frac{24}{5} = 4.8$

$+5^2+9^2 = 144$

$)^2 = 115.2$

$= \frac{\sum x^2 - n(\bar{x})^2}{n-1}$

$2 = 7.2$

.68

CALCULATION OF S^2 AND S FOR SAMPLE DATA PRESENTED AS GFD (GROUP FREQUENCY DISTRIBUTION).

VARIANCE: $S^2 = \frac{\sum f(x-\bar{x})^2}{n-1} = \frac{\sum f(x-\bar{x})^2}{\sum f - 1} = \frac{\sum fx^2 - n(\bar{x})^2}{n-1}$

STANDARD DEVIATION $= S = \sqrt{\frac{\sum fx^2 - n(\bar{x})^2}{n-1}}$

Pb 10 FIND THE VARIANCE AND STANDARD DEVIATION OF THE FOLLOWING SAMPLE OF GROUPED DATA.

INTERVAL	FREQUENCY
10 — 14	4
15 — 19	7
20 — 24	5
25 — 29	3
30 — 34	1

INTERVAL	x (MID POINT)	f	fx	fx^2
10 — 14	$\frac{10+14}{2} = \frac{24}{2} = 12$	4	4x12 = 48	$4\times12^2 = 576$
15—19	$\frac{15+19}{2} = 17$	7	17x7 = 119	$7\times17^2 = 2023$
20—24	$\frac{20+24}{2} = 22$	5	22x5 = 110	$5\times22^2 = 2420$
25—29	$\frac{25+29}{2} = 27$	3	27x3 = 81	$3\times27^2 = 2187$
30—34	$\frac{30+34}{2} = 32$	1	32x1 = 32	$1\times32^2 = 1024$
		$\sum f = 20$	$\sum fx = 390$	$\sum fx^2 = 8230$

$\bar{x} = \frac{\sum fx}{\sum f} = \frac{390}{20} = 19.5$

$S^2 = \frac{\sum fx^2 - n(\bar{x})^2}{n-1} = \frac{8230 - 20\times(19.5)^2}{20-1}$

$= \frac{8230 - 7605}{19} = 32.89$

$S = \sqrt{32.89} = 5.739$

SET THEORY AND ELEMENTARY PROBABILITY

DEFINITION of PROBABILITY

$$P(A) = \frac{\text{Number of Times A occur}}{\text{Total number of Repetitions of Experiment}} = \frac{n}{N}$$

pb 12

pb 11 A BAG CONTAINS 7 RED BALLS, 5 BLUE AND 4 GREEN BALLS. FIND THE PROBABILITY THAT IF ONE BALL ONLY IS DRAWN RANDOMLY, IT WILL BE GREEN.

7 RED

5 BLUE

4 GREEN

N = TOTAL NUMBER OF BALLS = 7 + 5 + 4 = 16

n = NO. of GREEN BALL = 4

$$P(\text{GREEN}) = \frac{n}{N} = \frac{4}{16} = \frac{1}{4}$$

FOOTWEAR
IN THIS BUILDING

pb 12 IF A SINGLE DRAW IS MADE FROM A STANDARD PACK OF CARDS. FIND THE PROBABILITY OF DRAWING A JACK (OR) A QUEEN (OR) A KING.

1 PACK = 52 CARDS.

JACK = 4 CARDS

QUEEN = 4 CARDS

KING = 4 CARDS.

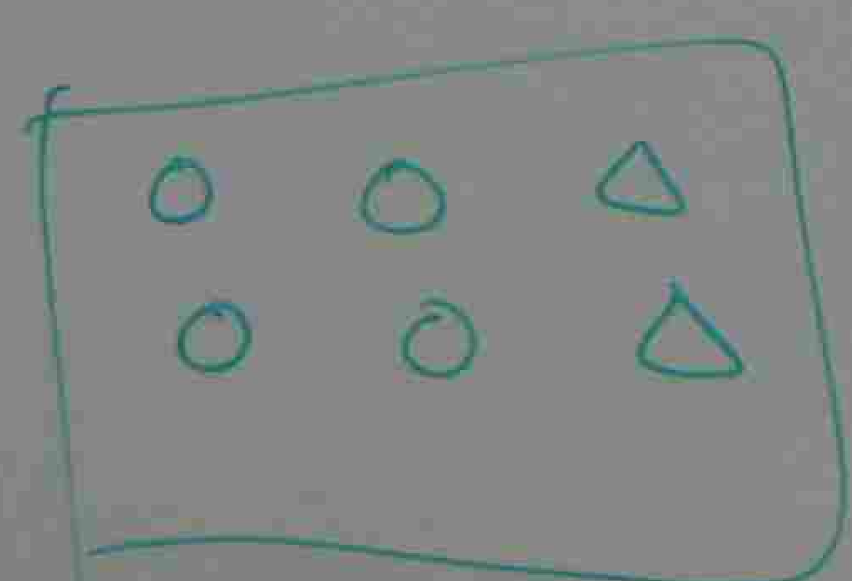
$P(A) = P(\text{JACK}) = \frac{4}{52} = \frac{1}{13}$

$P(B) = P(\text{QUEEN}) = \frac{4}{52} = \frac{1}{13}$

$P(C) = P(\text{KING}) = \frac{4}{52} = \frac{1}{13}$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$
$$= \frac{1}{13} + \frac{1}{13} + \frac{1}{13}$$
$$= \frac{3}{13}$$

$4 = 16$



$\frac{2}{6}$ $\qquad 1 - \frac{2}{6} = \frac{4}{6}$

$P(\triangle)$ $\qquad 1 - P(\triangle)$

$$\boxed{P(\bar{A}) = 1 - P(A)}$$

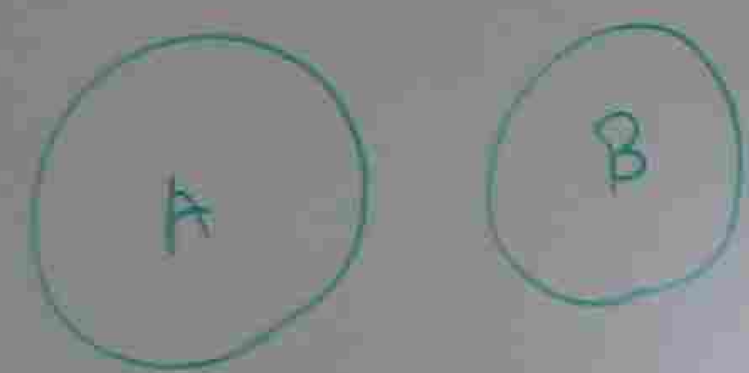
pb 14 THERE ARE 52 CARDS IN A STANDARD PACK IN WHICH 39 CARDS CONTAIN HEART.

CALCULATE THE PROBABILITY THAT THE CARD CONTAINING HEART IS DRAWN AND PROBABILITY THAT THE CARD NOT CONTAINING HEART IS DRAWN.

$P(A) = P(\text{HEART}) = \frac{39}{52} = \frac{3}{4}$

$$P(\bar{A}) = 1 - P(A)$$
$$= 1 - \frac{3}{4}$$
$$= \frac{1}{4}$$

UNION, INTERSECT, SUBSET



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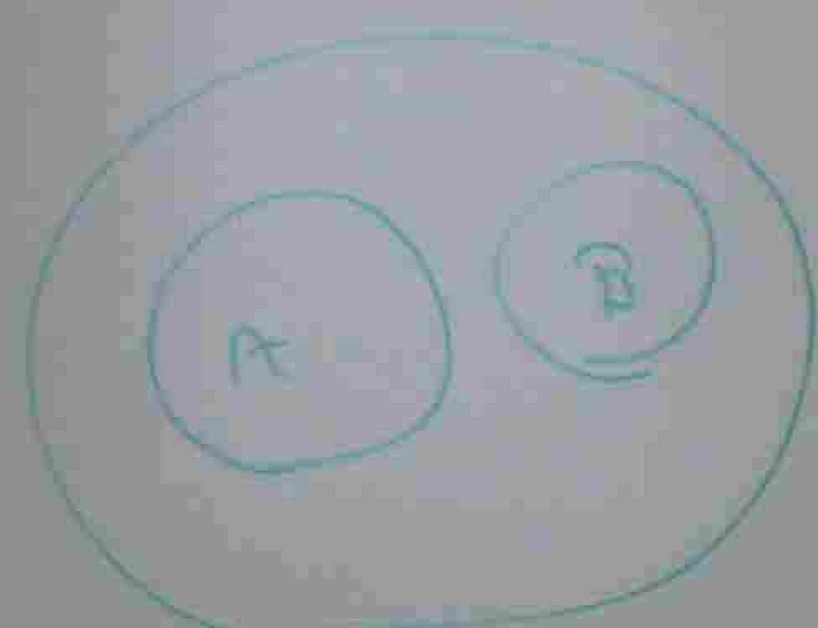
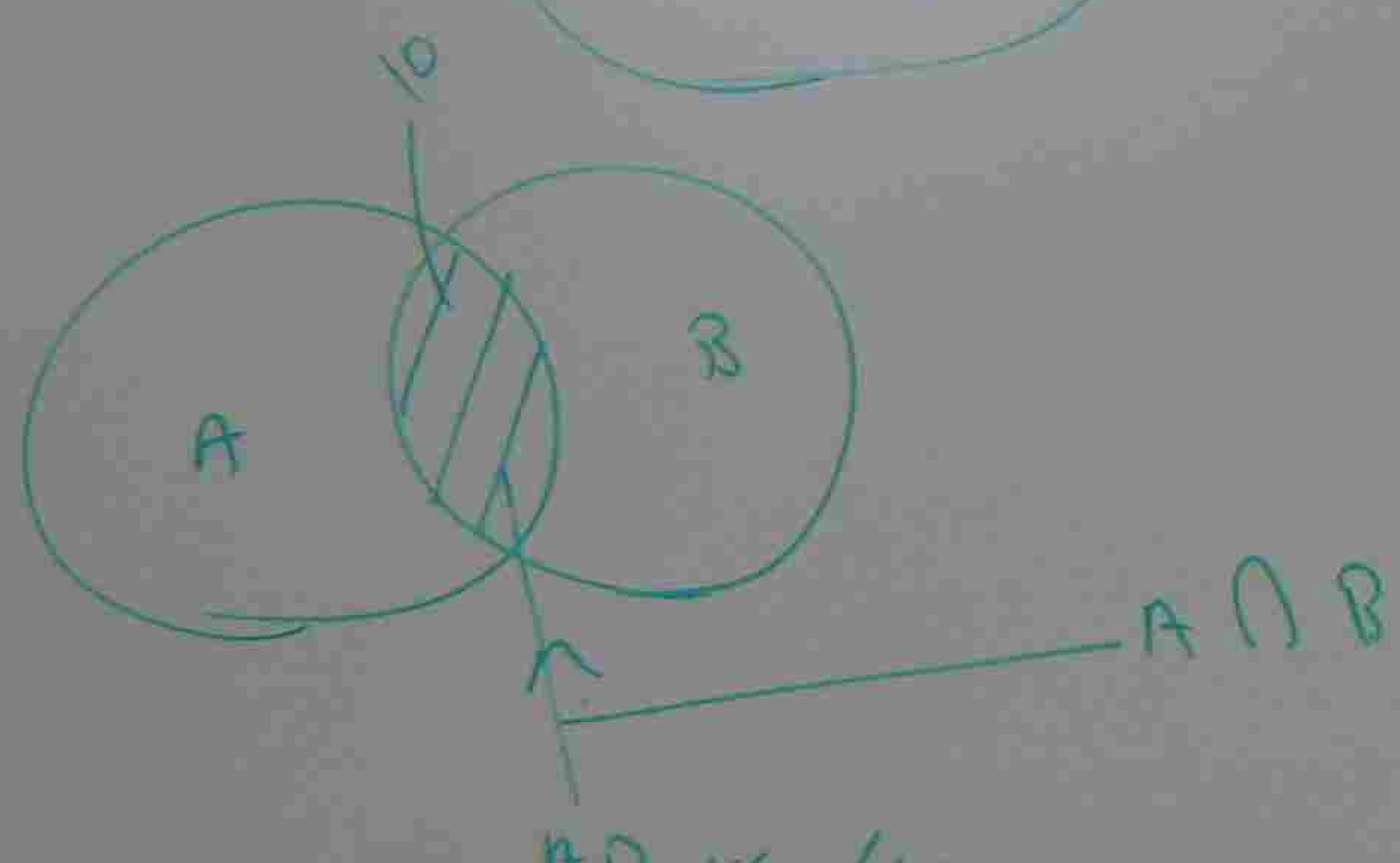
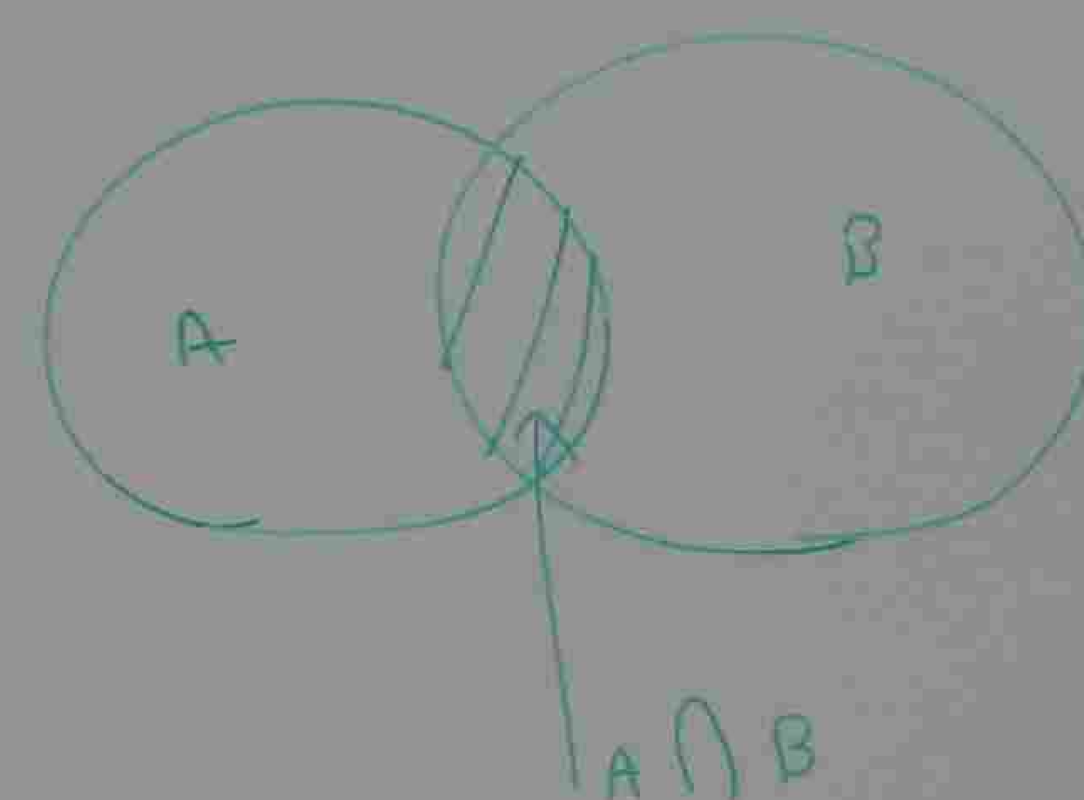




A 60 STUDENTS, 10 ABOVE AGE 40

B 50 STUDENTS, 10 ABOVE AGE 40

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$



$P(A) = \frac{60}{110}$

$P(B) = \frac{50}{110}$

$P(A \cap B) = \frac{10}{110}$

$P(A \cup B) = \frac{60}{110} + \frac{50}{110} - \frac{10}{110}$

$= \frac{100}{110}$

Pb 15 P(S) = 0.4

FIND P(

P(S ∪ T) = P(S)

= 0.4

=

PROBABILITY OF SU

Pb 16 A SIN

IS THE P

ON THE UP

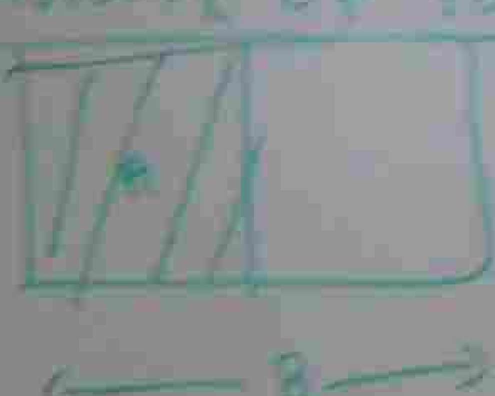
NUMBER

Pb (15) $P(S) = 0.4$, $P(T) = 0.5$, $P(S \cap T) = 0.15$

FIND $P(S \cup T)$

$$P(S \cup T) = P(S) + P(T) - P(S \cap T)$$
$$= 0.4 + 0.5 - 0.15$$
$$= 0.75$$

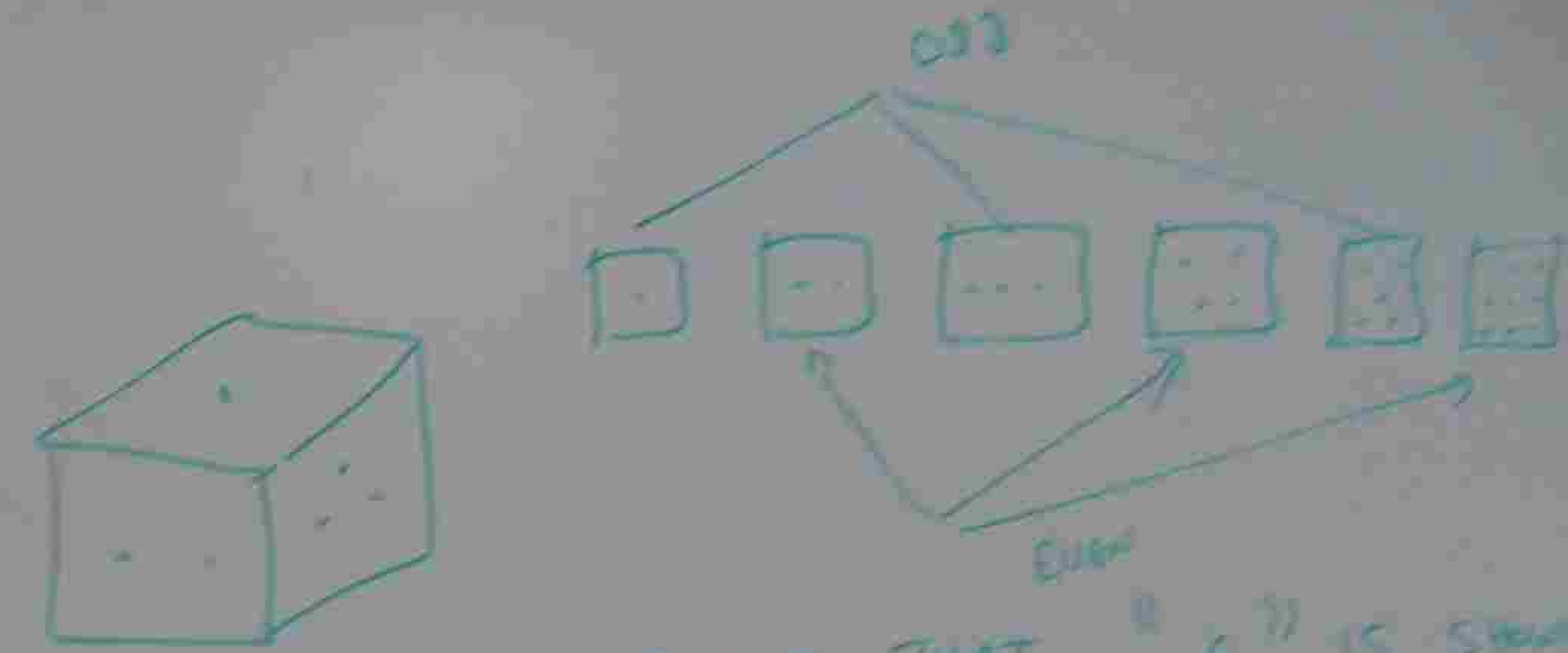
PROBABILITY OF SUBSET



$A \subset B$, A IS SUB SET OF B

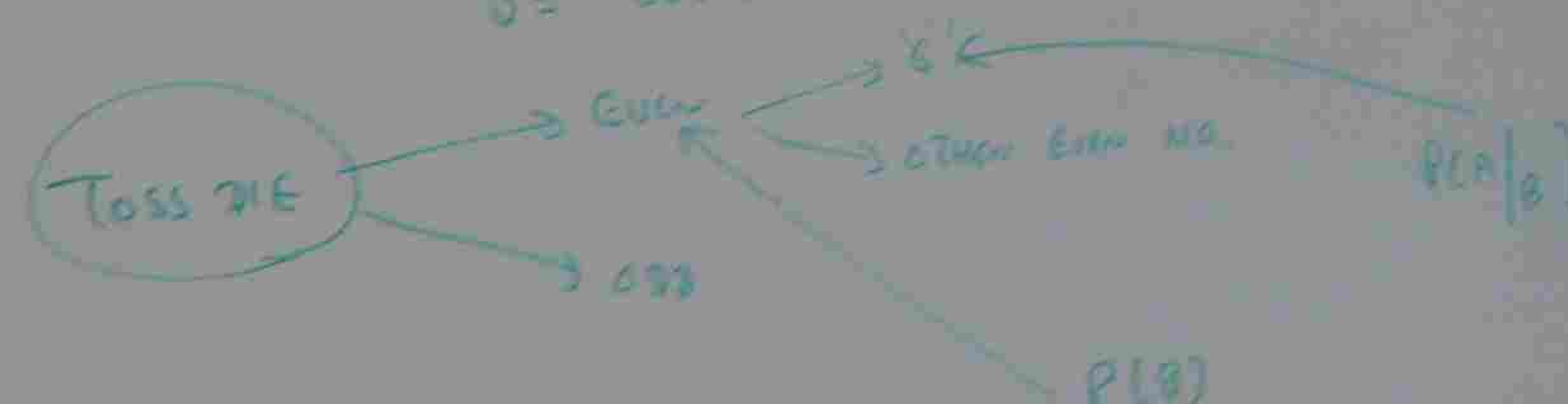
$$\boxed{P(A/B) = \frac{P(A \cap B)}{P(B)}}$$

Pb (16) A SINGLE DIE IS TOSSED. WHAT IS THE PROBABILITY OF A 6 APPEARING ON THE UPPER MOST FACE GIVEN THAT NUMBER SHOWING IS EVEN.



A = EVENT THAT "6" IS SHOWING

B = EVENT THAT EVEN NUMBER IS SHOWING



$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{6}}{\frac{3}{6}} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$$

$$\boxed{P(A \cap B) = P(A) \times P(B)}$$

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

$$\boxed{P(\bar{A}) = 1 - P(A)}$$

pb 17 SUPPOSE WE HAVE 3 EVENTS X, Y, Z

$P(X) = \frac{1}{2}$, $P(Y) = \frac{1}{4}$, $P(Z) = \frac{1}{8}$

FIND (a) $P(Z \cap X)$, (b) $P(Z \cup X)$

(c) $P(X \cup Y)$, (d) $P(X \cup \bar{Y})$

(a) $P(Z \cap X) = P(Z) \times P(X)$

$= \frac{1}{8} \times \frac{1}{2} = \frac{1}{16}$

(b) $P(Z \cup X) = P(Z) + P(X) - P(Z \cap X)$

$= \frac{1}{8} + \frac{1}{2} - \frac{1}{16}$

$= \frac{2+8}{16} - \frac{1}{16}$

$= \frac{10}{16} - \frac{1}{16}$

$= \frac{10-1}{16} = \frac{9}{16}$

(c) $P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$

$= P(X) + P(Y) - P(X) \times P(Y)$

$= \frac{1}{2} + \frac{1}{4} - \frac{1}{2} \times \frac{1}{4}$

$= \frac{4+2}{8} - \frac{1}{8} = \frac{6}{8} - \frac{1}{8} = \frac{5}{8}$

(d)

(d) $P(X \cup \bar{Y}) = P(X) + P(\bar{Y}) - P(X \cap \bar{Y})$

$= P(X) + [1 - P(Y)] - P(X) \times P(\bar{Y})$

$= P(X) + [1 - P(Y)] - P(X) \times [1 - P(Y)]$

$= \frac{1}{2} + [1 - \frac{1}{4}] - \frac{1}{2} \times [1 - \frac{1}{4}]$

$= \frac{1}{2} + \frac{3}{4} - \frac{1}{2} \times \frac{3}{4}$

$= \frac{4+6}{8} - \frac{3}{8}$

$= \frac{10}{8} - \frac{3}{8} = \frac{7}{8}$ //

Pb(13) SUPPOSE THAT WE KNOW FROM EXPERIMENT THAT 28% OF PEOPLE READ THE WOMAN'S DAY ARTICLE AND 45% OF PEOPLE READ THE NEW IDEA.

READING WOMEN'S DAY AND NEW IDEA ARE INDEPENDENT OF EACH OTHER.

TWO PEOPLE ARE SELECTED AT RANDOM

FIND THE PROBABILITY THAT

(a) NEITHER READ NEW IDEA

(b) BOTH READ WOMEN'S DAY

(c) BOTH DO NOT READ WOMEN'S DAY

$P(W) = P(\text{WOMEN'S DAY}) = 0.28$

$P(\bar{W}) = P(\text{DO NOT READ WOMEN DAY}) = 1 - P(W)$

$= 1 - 0.28$

$= 0.72$

$P(N) = 0.45$

$P(\bar{N}) = 1 - P(N) = 1 - 0.45 = 0.55$

(a) NEITHER READS NEW IDEA (FROM TWO PEOPLE)

$P(\bar{N} \cap \bar{N}) = P(\bar{N}) \times P(\bar{N})$

$= 0.55 \times 0.55 = 0.3025$

$\frac{1}{8} = \frac{5}{8}$

(b) P(

(c) P

Pb (14

6 A

OF

OF

FIND

(i) F

40

(i

(b) $P(w \cap w) = P(w) \times P(w)$

$= 0.28 \times 0.28 = 0.0784$

(c) $P(\bar{w} \cap \bar{w}) = P(\bar{w}) \times P(\bar{w})$

$= 0.72 \times 0.72 = 0.5184$

Pb (19) THERE ARE 10 PEOPLE TO BE CLASSIFIED ACCORDING TO SEX & AGE

6 ARE FEMALES AND 4 ARE MALES.

OF 6 FEMALES, 4 ARE UNDER 40, 2 ARE 40 (OR) MORE

OF 4 MALES. 3 ARE UNDER 40, 1 IS 40 (OR) MORE

FIND THE PROBABILITY THAT PERSON IS

(i) FEMALE (ii) UNDER 40, (iii) FEMALE UNDER 40 (iv) UNDER 40 GIVEN THAT SHE IS FEMALE ⟷ AMONG THE FEMALES, FIND UNDER 40

(v) MALE AND 40 (OR) MORE, (vi) MALE GIVEN THAT HE IS UNDER 40.

(i) $P(F) = \frac{6}{10}$

P(A) UNDER 40

(ii) $P(A) = \frac{4+3}{10} = \frac{7}{10}$

$\underset{\text{FEMALE}}{P(F)}, \underset{\text{UNDER 40}}{P(A)}$

$\boxed{P(A \cap F) = P(F \cap A)}$

$P(F \cap A) = \text{FEMALE UNDER 40} = \frac{4}{10}$

(iv) $P(A/F) = \frac{P(A \cap F)}{P(F)} = \frac{P(F \cap A)}{P(F)} = \frac{\frac{4}{10}}{\frac{6}{10}}$

$= \frac{4}{10} \times \frac{10}{6}$

$= \frac{4}{6}$

(v)

P(M) ...

(vi)

(v) $P(A)$ = UNDER 40

40 OR MORE = $P(\bar{A})$

$P(m)$ = MALE

MALE AND 40 (OR) MORE = $P(m \cap \bar{A}) = P(m) \times P(\bar{A})$

$= P(m) \times (1 - P(A))$

$= \frac{4}{10} \times \left(1 - \frac{7}{10}\right)$

$= \frac{4}{10} \times \frac{3}{10} = \frac{12}{100} \approx \frac{1}{10}$

(vi) $P(A/m) = \dfrac{P(m \cap A)}{P(m)}$

$= \dfrac{3/10}{4/10} = \frac{3}{4}$

$) = P(F \cap A)$

$\frac{A)}{)} = \dfrac{\frac{4}{10}}{\frac{6}{10}}$

$= \frac{4}{10} \times \frac{10}{6}$

$= \frac{4}{6}$

REVISION (STATISTICS)

① FIND THE MEAN OF THE FOLLOWING GFD

INTERVAL	FREQUENCY
0 → 9	1
10 → 19	2
20 → 29	4
30 → 39	8
40 → 49	4
50 → 59	1

② FIND THE VARIANCE AND STANDARD DEVIATION OF THE FOLLOWING SAMPLE OF GROUPED DATA

INTERVAL	FREQUENCY
10 — 14	4
15 — 19	7
20 — 24	5
25 — 29	3
30 — 34	1

VARIANCE

$$s^2 = \frac{\sum f (x-\bar{x})^2}{n-1} = \frac{\sum f x^2 - n\bar{x}^2}{n-1}$$

STANDARD DEVIATION

$$s = \sqrt{\frac{\sum f x^2 - n\bar{x}^2}{n-1}}$$

①

10

20

30

40 →

50 →

...THE VARIANCE AND STANDARD DEVIATION OF
...NG SAMPLE OF GROUPED DATA

	FREQUENCY
	4
	7
	5
	3
	1

$$\frac{\ldots f(x-\bar{x})^2}{n-1} = \frac{\sum f x^2 - n\bar{x}^2}{n-1}$$

$$\sqrt{\frac{\sum f x^2 - n\bar{x}^2}{n-1}}$$

(1)

INTERVAL	MEAN = $\frac{\text{SMALLEST} + \text{HIGHEST}}{2}$ (x)	f	fx
0 → 9	$\frac{0+9}{2} = 4.5$	1	$4.5 \times 1 = 4.5$
10 → 19	$\frac{10+19}{2} = 14.5$	2	$14.5 \times 2 = 29$
20 → 29	$\frac{20+29}{2} = 24.5$	4	$24.5 \times 4 = 98$
30 → 39	$\frac{30+39}{2} = 34.5$	8	$34.5 \times 8 = 276$
40 → 49	$\frac{40+49}{2} = 44.5$	4	$44.5 \times 4 = 178$
50 → 59	$\frac{50+59}{2} = 54.5$	1	$54.5 \times 1 = 54.5$

$\sum f = 20$ $\quad$ $\sum fx = 640$

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{640}{20}$$

$$= 32$$

(2)

INTERVAL	
10 — 14	
15 → 19	
20 — 24	
25 — 29	
30 — 34	

$\bar{x} = \sum \ldots$

VARIANCE = ...

STANDARD DEVIATION

...AN = SMALLEST + HIGHEST / 2 (...s)	f	fx
= 4.5	1	4.5×1 = 4.5
= 14.5	2	14.5×2 = 29
= 24.5	4	24.5×4 = 98
= 34.5	8	34.5×8 = 276
= 44.5	4	44.5×4 = 178
= 54.5	1	54.5×1 = 54.5

$\Sigma f = 20$ $\quad \Sigma fx = 640$

$$\frac{\ldots x}{\ldots f} = \frac{640}{20} = 32$$

(2)

INTERVAL	$\bar{x}$ = MID POINT	f	fx	fx^2
10 − 14	$\frac{10+14}{2} = 12$	4	12×4 = 48	$4\times12^2 = 576$
15 → 19	$\frac{15+19}{2} = 17$	7	17×7 = 119	$7\times17^2 = 2023$
20 − 24	$\frac{20+24}{2} = 22$	5	22×5 = 110	$5\times22^2 = 2420$
25 − 29	$\frac{25+29}{2} = 27$	3	27×3 = 81	$3\times27^2 = 2187$
30 − 34	$\frac{30+34}{2} = 32$	1	32×1 = 32	$1\times32^2 = 1024$

$\Sigma f = 20$ $\quad \Sigma fx = 390$ $\quad \Sigma fx^2 = 8230$

$$\bar{x} = \frac{\Sigma fx}{\Sigma f} = \frac{390}{20} = 19.5 \quad , n = \Sigma f = 20$$

$$\text{VARIANCE} = s^2 = \frac{\Sigma fx^2 - n(\bar{x})^2}{n-1} = \frac{8230 - 20\times(19.5)^2}{20-1} = \frac{8230 - 7605}{19} = 32.89$$

$$\text{STANDARD DEVIATION} = s = \sqrt{32.89} = 5.735$$

76

023

420

187

024

230

= 20

$(19.5)^2$

89 = 32.89

MATHS A + B REVISION

1. FACTORISE $8x^2 + 73x + 9$
2. $(x-7)(x+2) = 4x - 1$
   FIND x
3. $\log_{10} \frac{k}{k-x} = t$, FIND x
4. PLOT GRAPH FOR $y = 2x^2 - 12x + 13$
5. THE FOLLOWING VALUES OF X AND Y ARE BELIEVED TO SATISFY THE EQUATION $y = Ax^2 + Bx$
   FIND A LINEAR EQUATION THAT SUITS THE INFORMATION AND EVALUATE A & B
6. CALCULATE THE AMOUNT THAT \$5000 GROWS TO WHEN IT IS INVESTED FOR 3 WEEKS AT INTEREST RATE 13% PER YEAR INTEREST IS COMPOUNDED DAILY
7. PLOT $y = 7 \sin 2\theta$

---

1. $8x^2 + 73x + 9$

$8x \quad 1$

$x \quad 9$

$72x + x = 73x$

$(8x+1)(x+9)$

2. $(x-7)(x+2) = 4x - 1$

$x^2 - 7x + 2x - 14 = 4x - 1$

$x^2 - 5x - 14 = 4x - 1$

$x^2 - 5x - 14 - 4x + 1 = 0$

$x^2 - 9x - 13 = 0$

$Ax^2 + Bx + C = 0$

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$= \frac{-(-9) \pm \sqrt{(-9)^2 - 4 \times 1 \times (-13)}}{2 \times 1}$$

$x = \frac{9 \pm \sqrt{}}{2}$

$= \frac{9 \pm \sqrt{}}{}$

$= \frac{9 \pm 11.5}{2}$

$= \frac{9 + 11.53}{2}$

$= 10.265$

3. $\log_{10} \frac{k}{k-x}$

$\frac{k}{k-x}$

$\frac{k}{10^t}$

$x =$

$x =$

$x =$

$x + 9$

1

9

$= 73x$

$x + 9\,]$

$x+2) = 4x - 1$

$2x - 14 = 4x - 1$

$-14 = 4x - 1$

$4 - 4x + 1 = 0$

$-13 = 0$

$+ c = 0$

$\sqrt{B^2 - 4AC}$

A

$) \pm \sqrt{(-9)^2 - 4 \times 1 \times (-13)}$

2×1

$x = \dfrac{9 \pm \sqrt{81 + 52}}{2}$

$= \dfrac{9 \pm \sqrt{133}}{2}$

$= \dfrac{9 \pm 11.53}{2}$

$= \dfrac{9 + 11.53}{2}$ (or) $\dfrac{9 - 11.53}{2}$

$= 10.265$ (OR) -1.265

③ $\log_{10} \dfrac{k}{k-x} = t$

$\dfrac{k}{k-x} = 10^t$

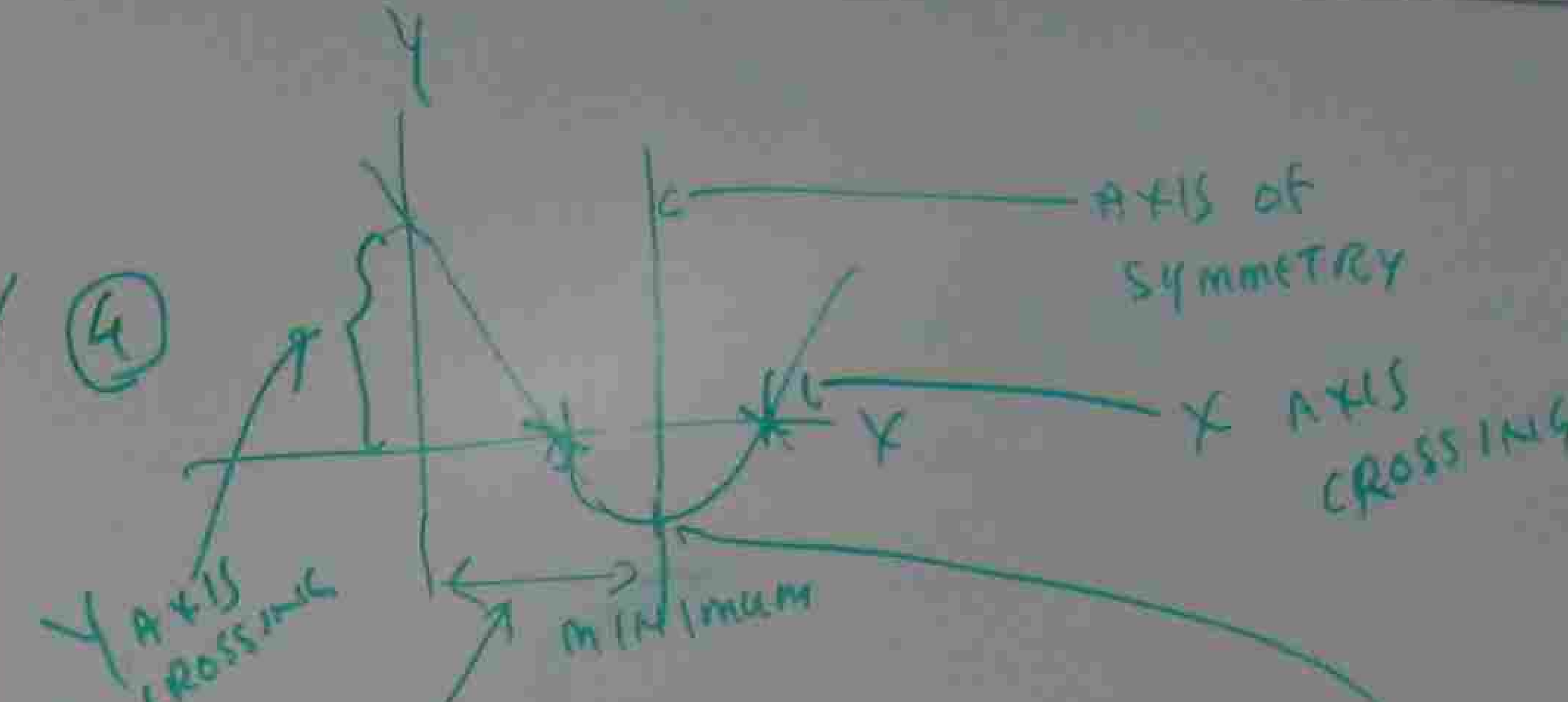
$\dfrac{k}{10^t} = k - x$

$x = k - \dfrac{k}{10^t}$

$x = k\left(1 - \dfrac{1}{10^t}\right)$

$x = k\left(\dfrac{10^t - 1}{10^t}\right)$

④



① COMPARE WITH $Y = Ax^2 + Bx + c$

THEN $\dfrac{-B}{2A}$ = AXIS OF SYMMETRY

② FIND Y VALUE AT AXIS OF SYMMETRY

③ FIND Y AXIS CROSSING BY PUTTING $x = 0$

④ FIND X AXIS CROSSING BY PUTTING $Y = 0$

$Y = 2x^2 - 12x + 13$

$Y = Ax^2 + Bx + c$

$A = 2,\ B = -12,\ C = 13$

① AXIS OF SY

② $Y = 2x$

Y AT AXIS

$= 2$

③ Y AX

$Y = 2$

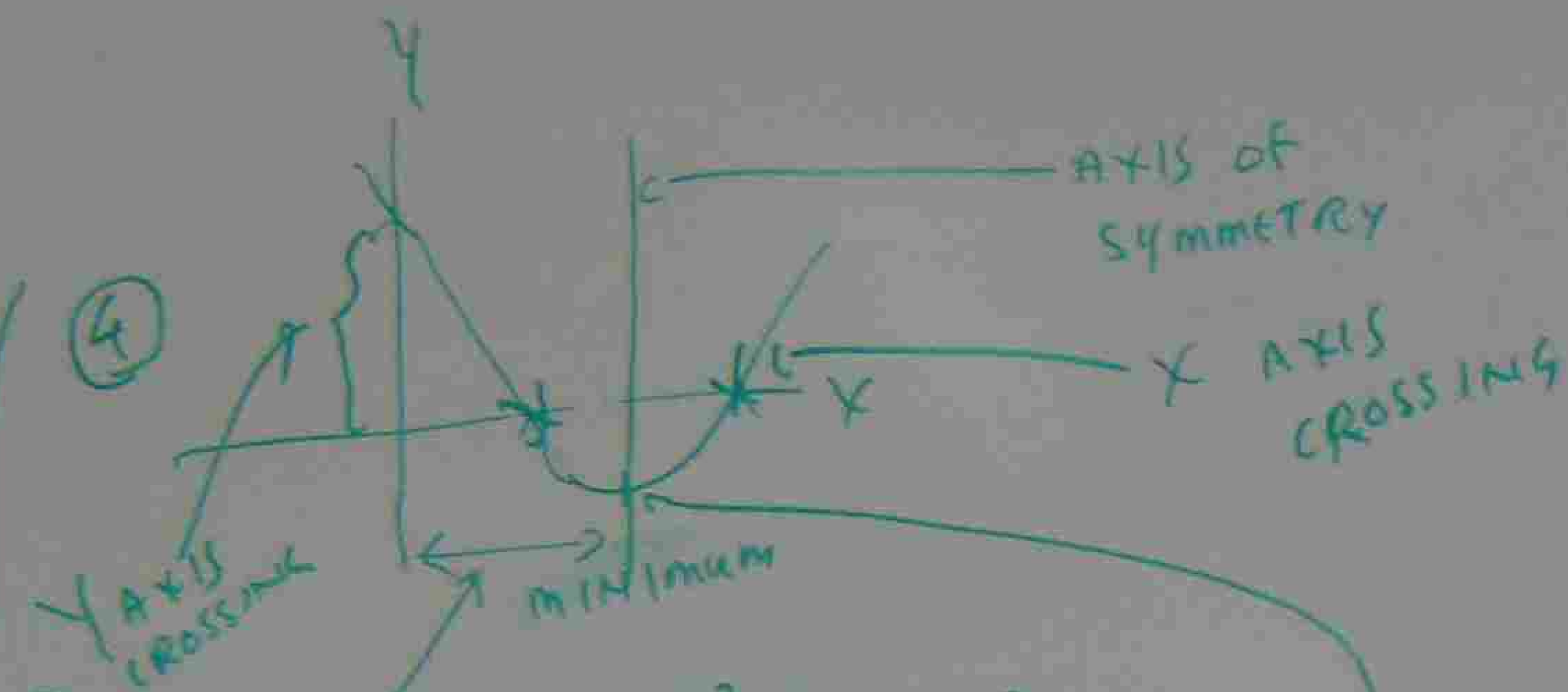
④ X A

$0 =$

$x =$

13

-5



1 COMPARE WITH $Y = Ax^2 + Bx + C$

THEN $\frac{-B}{2A}$ = AXIS OF SYMMETRY

2 FIND Y VALUE AT AXIS OF SYMMETRY

3 FIND Y AXIS CROSSING BY PUTTING $X = 0$

4 FIND X AXIS CROSSING BY PUTTING $Y = 0$

$$Y = 2x^2 - 12x + 13$$

$$Y = Ax^2 + Bx + C$$

$A = 2$, $B = -12$, $C = 13$

1 AXIS OF SYMMETRY $= \frac{-B}{2A} = \frac{-(-12)}{2\times 2} = \frac{12}{4} = 3$

X VALUE

2 $Y = 2x^2 - 12x + 13$

Y AT AXIS OF SYMMETRY

$= 2\times 3^2 - 12\times 3 + 13 = 2\times 9 - 36 + 13 = -5$

3 Y AXIS CROSSING, $X = 0$

$Y = 2x^2 - 12x + 13 = 2\times 0^2 - 12\times 0 + 13 = 13$

4 X AXIS CROSSING $Y = 0$

$0 = 2x^2 - 12x + 13$

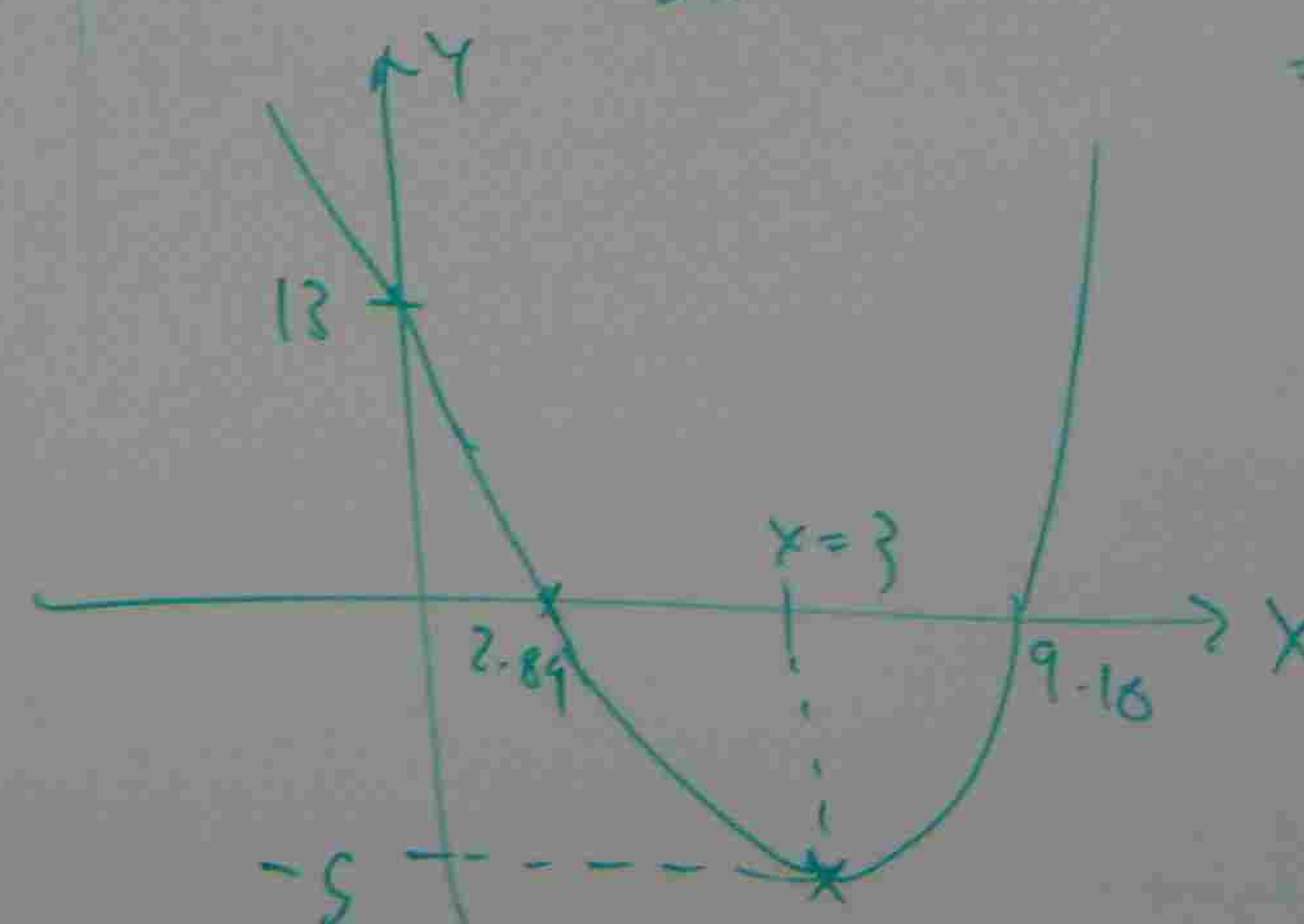
$Ax^2 + Bx + C = 0$

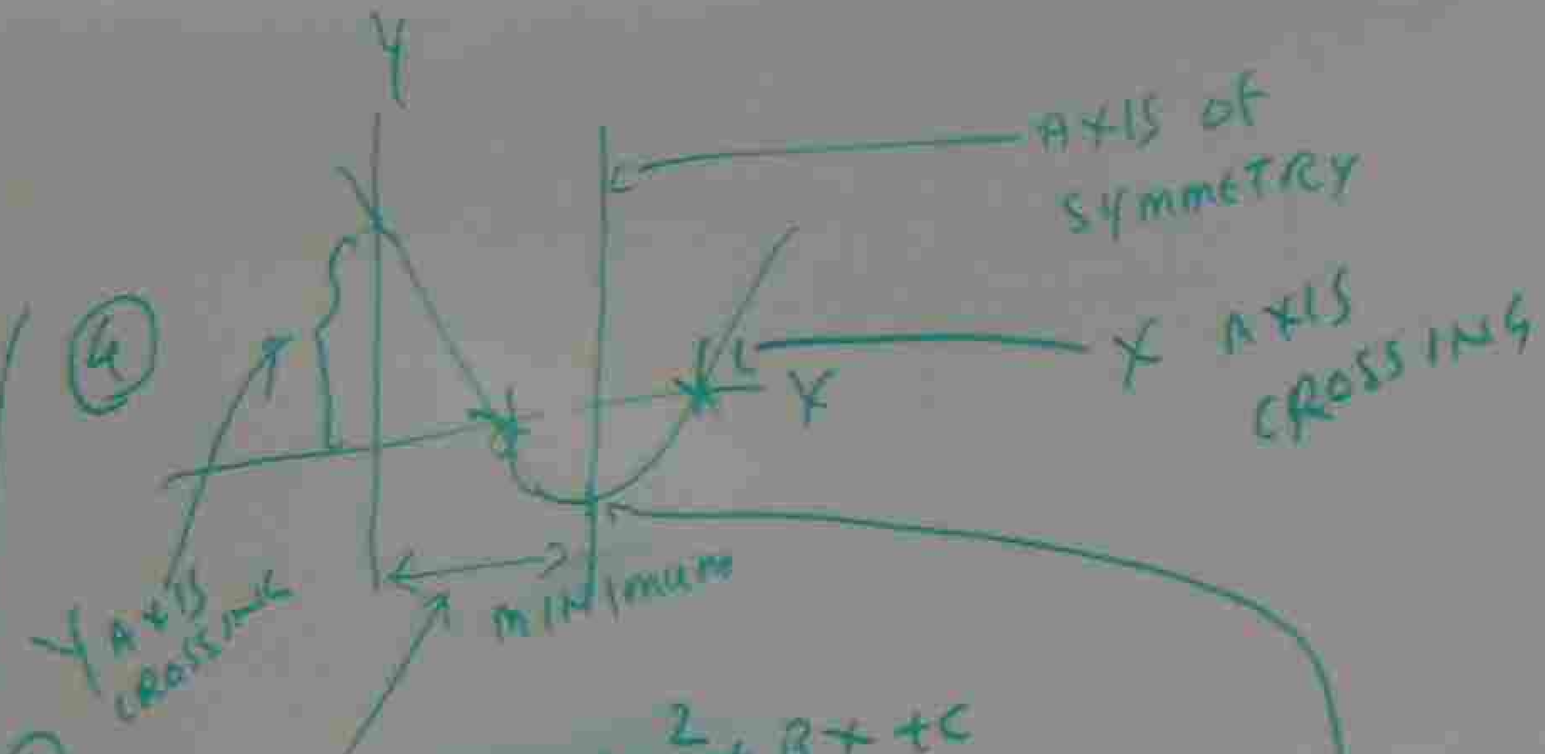
$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = -\frac{(-12) \pm \sqrt{(-12)^2 - 4\times 2\times 13}}{2\times 2}$$

$$= \frac{12 \pm \sqrt{144 - 104}}{4}$$

$$= \frac{12 \pm \sqrt{40}}{4} = \frac{12 \pm 6.32}{4}$$

$= 9.16$ (OR) 2.84





1 COMPARE WITH $Y = AX^2 + BX + C$

THEN $\frac{-B}{2A}$ = AXIS OF SYMMETRY

2 FIND Y VALUE AT AXIS OF SYMMETRY

3 FIND Y AXIS CROSSING BY PUTTING $X = 0$

4 FIND X AXIS CROSSING BY PUTTING $Y = 0$

$Y = 2X^2 - 12X + 13$

$Y = AX^2 + BX + C$

$A = 2, \; B = -12, \; C = 13$

1 AXIS OF SYMMETRY $= \frac{-B}{2A} = \frac{-(-12)}{2 \times 2} = \frac{12}{4} = 3$

X VALUE

2 $Y = 2X^2 - 12X + 13$

Y AT AXIS OF SYMMETRY

$= 2 \times 3^2 - 12 \times 3 + 13 = 2 \times 9 - 36 + 13 = -5$

3 Y AXIS CROSSING, $X = 0$

$Y = 2X^2 - 12X + 13 = 2 \times 0^2 - 12 \times 0 + 13 = 13$

4 X AXIS CROSSING $Y = 0$

$0 = 2X^2 - 12X + 13$

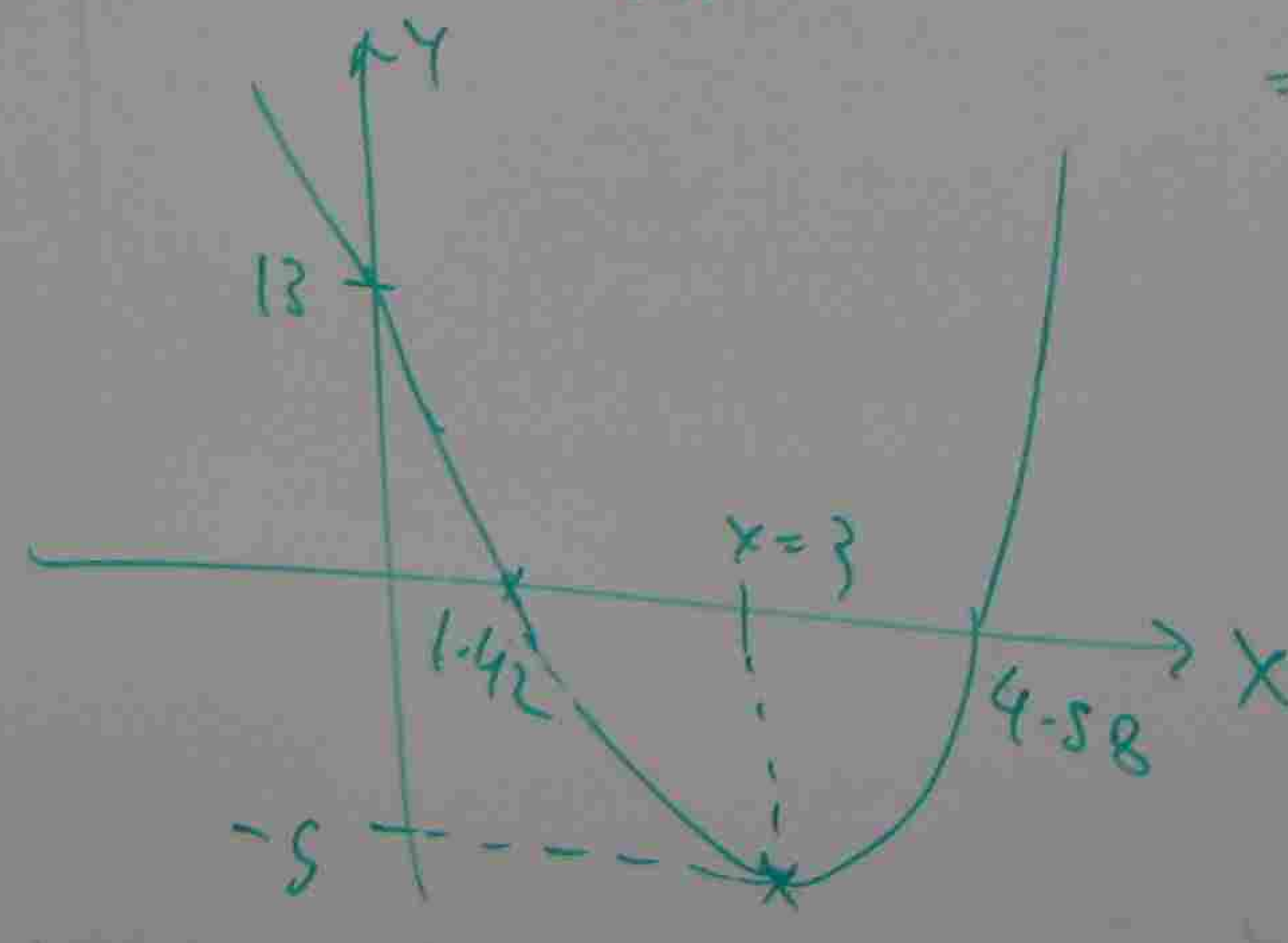
$AX^2 + BX + C = 0$

$$X = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{-(-12) \pm \sqrt{(-12)^2 - 4 \times 2 \times 13}}{2 \times 2}$$

$$= \frac{12 \pm \sqrt{144 - 104}}{4}$$

$$= \frac{12 \pm \sqrt{40}}{4} = \frac{12 \pm 6.32}{4}$$

$= 4.58$ (OR) 1.42



MATHS A+B REVISION

1. FACTORISE $8x^2 + 73x + 9$
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5. THE FOLLOWING VALUES OF x AND y ARE BELIEVED TO SATISFY THE EQUATION $y = Ax^2 + B\cancel{x}$ FIND A LINEAR EQUATION THAT SUITS THE INFORMATION AND EVALUATE A & B

x = 1	2	3	4	5
y = 5	16	34	57	84

6. CALCULATE THE AMOUNT THAT $5000 GROWS TO WHEN IT IS INVESTED FOR 3 WEEKS AT INTEREST RATE 13% PER YEAR INTEREST IS COMPOUNDED DAILY
7. PLOT $y = 7 \sin 2\theta$

1. $8x^2 + 73$
$8x$
x
$72x + x$
$(8x+1)($

2. $(x-7)$
$x^2 - 7x$
$x^2 - 5$
$x^2 - 5x -$
$x^2 - 9x$
$Ax^2 + Bx$
$x = \frac{-B}{}$
$= -($

(5)

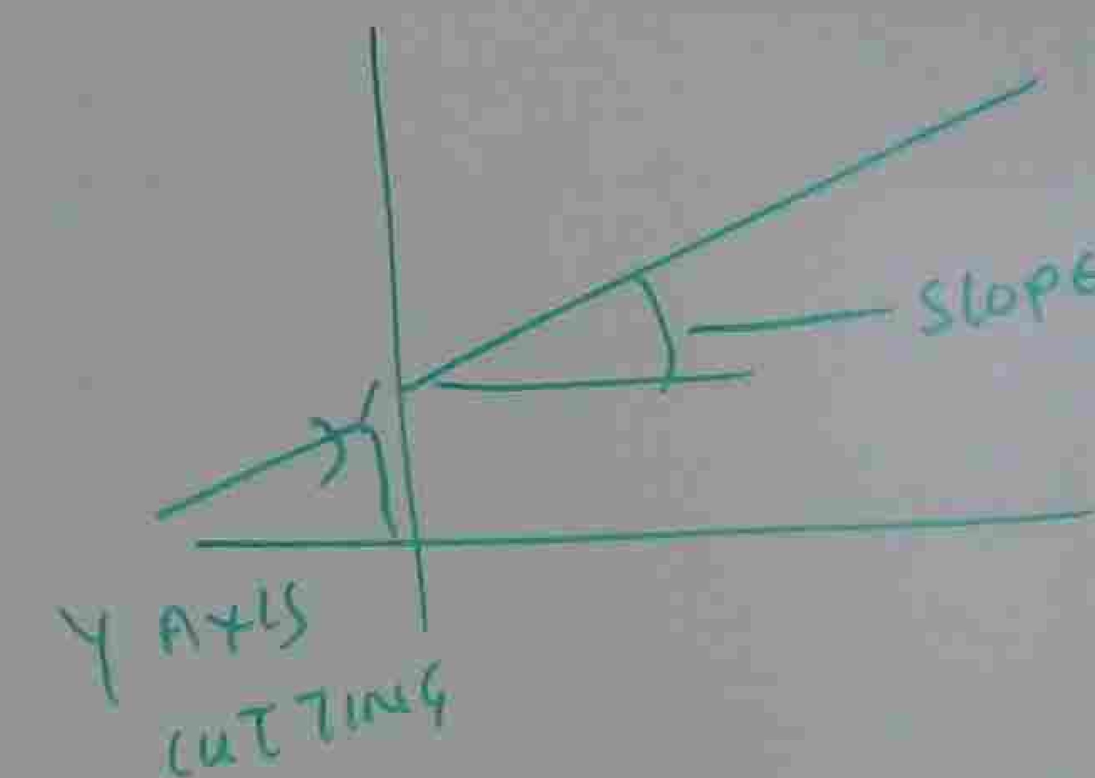
X	1	2	3	4	5
Y	5	16	34	57	84

X	1	2	3	4	5
$\frac{y}{x} = Y$	$\frac{5}{1} = 5$	$\frac{16}{2} = 8$	$\frac{34}{3} = 11.3$	$\frac{57}{4} = 14.3$	$\frac{84}{5} = 16.8$

$$y = Ax^2 + Bx$$

$$\frac{y}{x} = Ax + B$$

$$Y = Ax + B$$



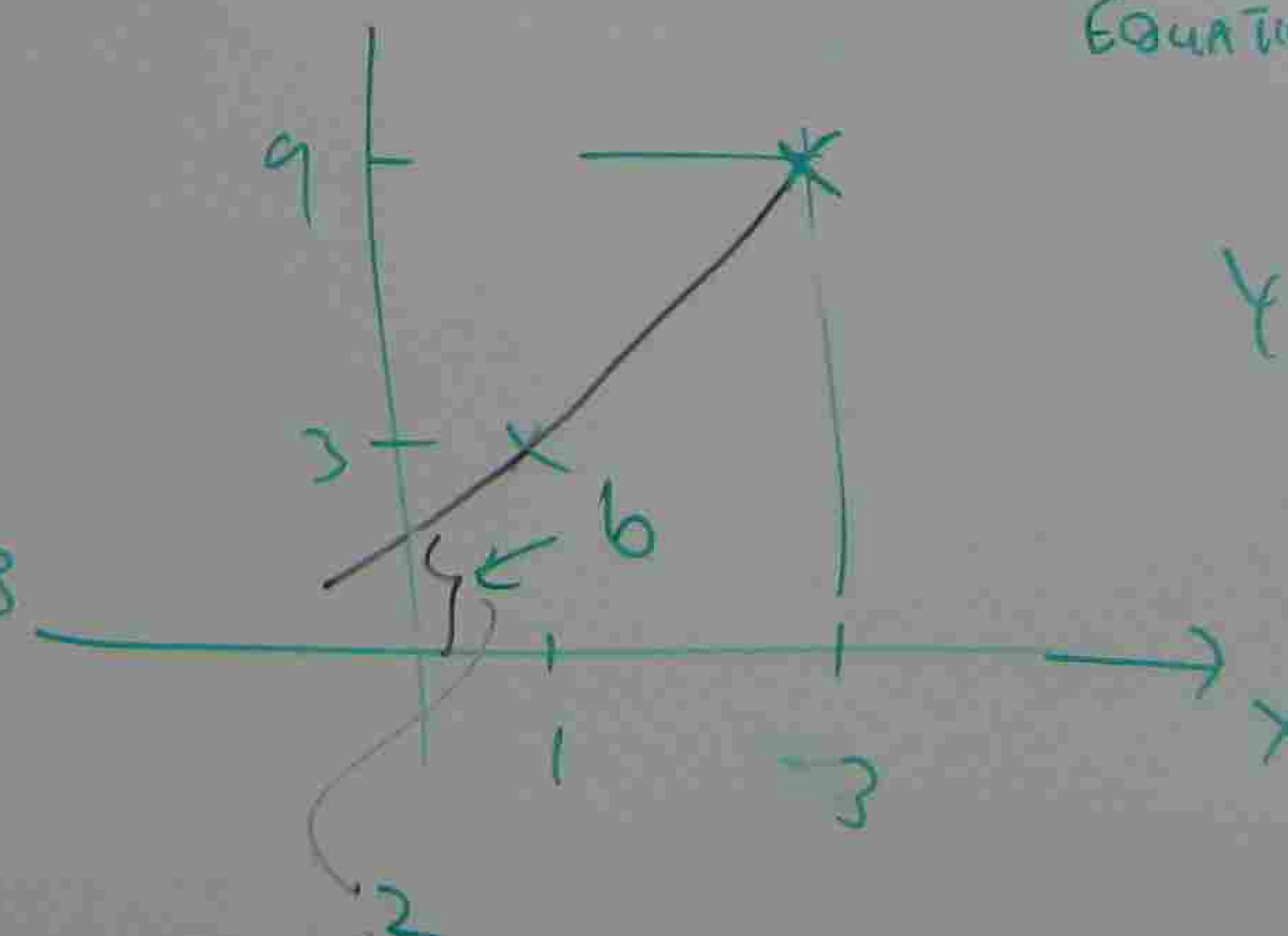
$$\text{Slope} = \frac{Y_2 - Y_1}{x_2 - x_1}$$

$$= \frac{8 - 5}{2 - 1} = 3$$

Slope = 3

$x = 1 \longrightarrow y = 3x = 3 \times 1 = 3$

$x = 3 \longrightarrow y = 3 \times 3 = 9$



EQUATION = SLOPE × X + Y CUTTING POINT

$$Y = 3X + 2$$

$$\frac{y}{x} = 3x + 2$$

$$y = 3x^2 + 2X$$

(6)

A

(7)

— SLOPE

...TION = SLOPE × X + Y CUTTING POINT

$y = 3x + 2$

$\frac{y}{x} = 3x + 2$

x

$y = 3x^2 + 2x$

⑥

$$A = \text{INVEST AMOUNT} \times \left(1 + \frac{\frac{\text{ANNUAL INTEREST RATE}}{365}}{100}\right)^{\text{NO. OF DAYS INVESTED}}$$

$$= 5000 \times \left(1 + \frac{\frac{13}{365}}{100}\right)^{3 \times 7}$$

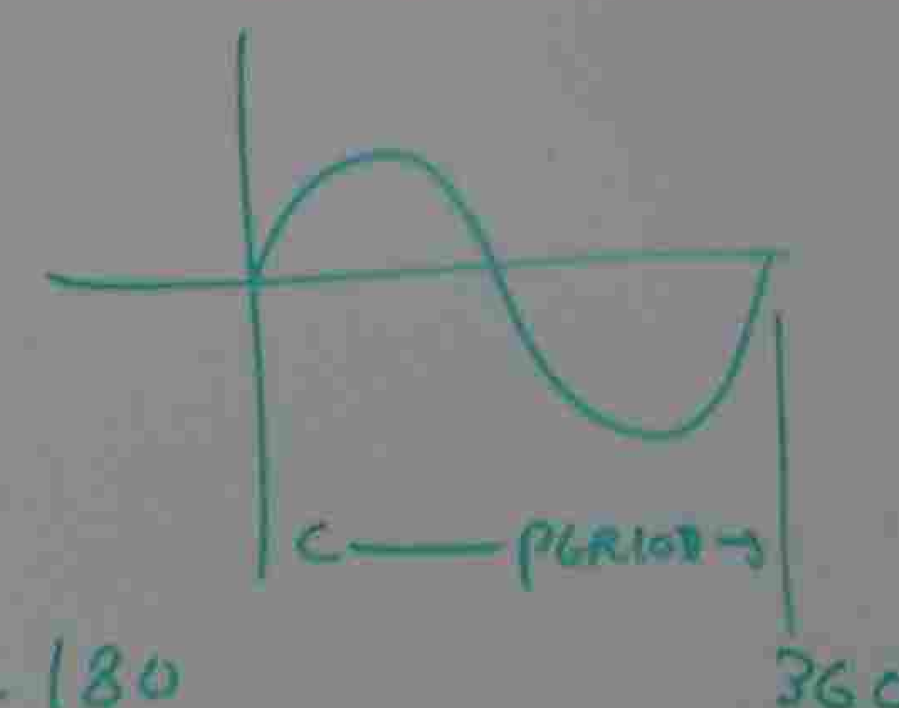
$$= 5000 \left(1 + \frac{\frac{13}{365}}{100}\right)^{21} = \$5037.53$$

⑦

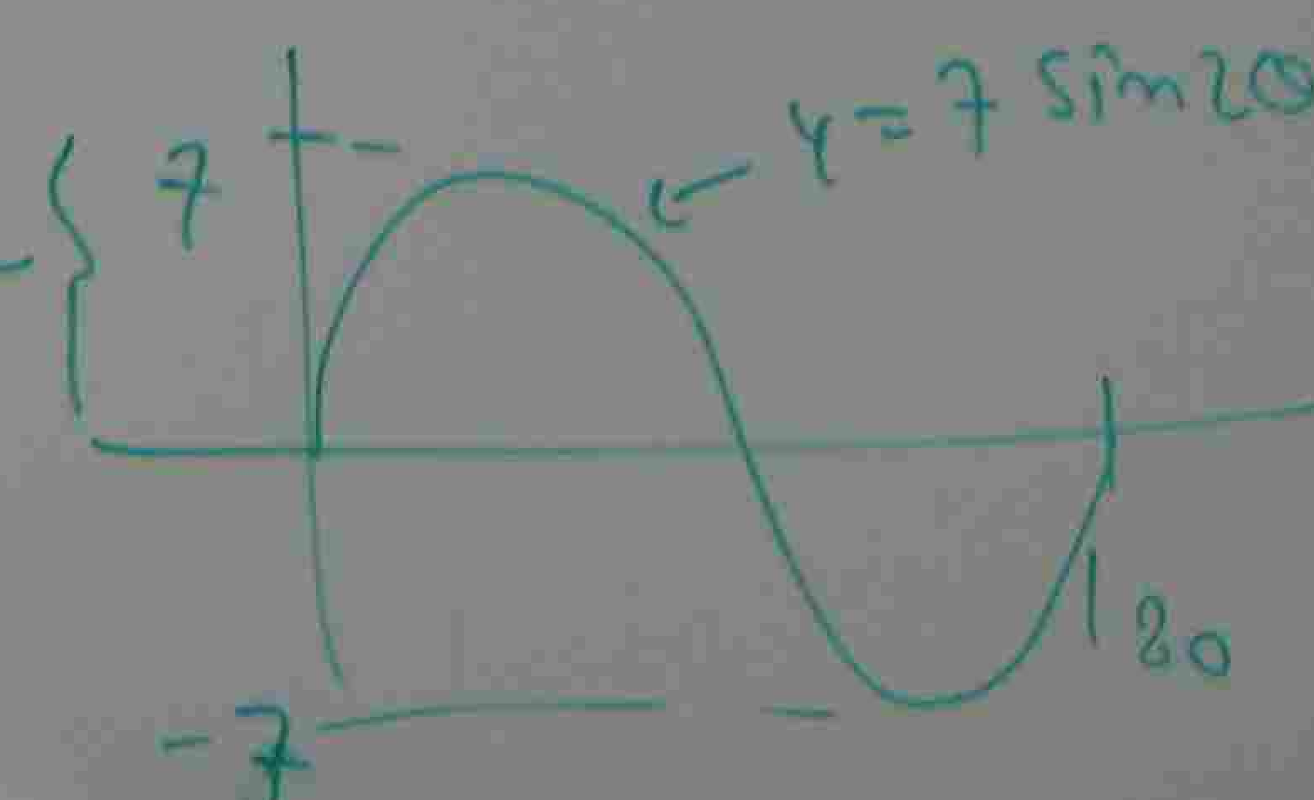
$$y = 7 \sin \underline{2\theta}$$

HALF OF AMPLITUDE

PERIOD = 360°



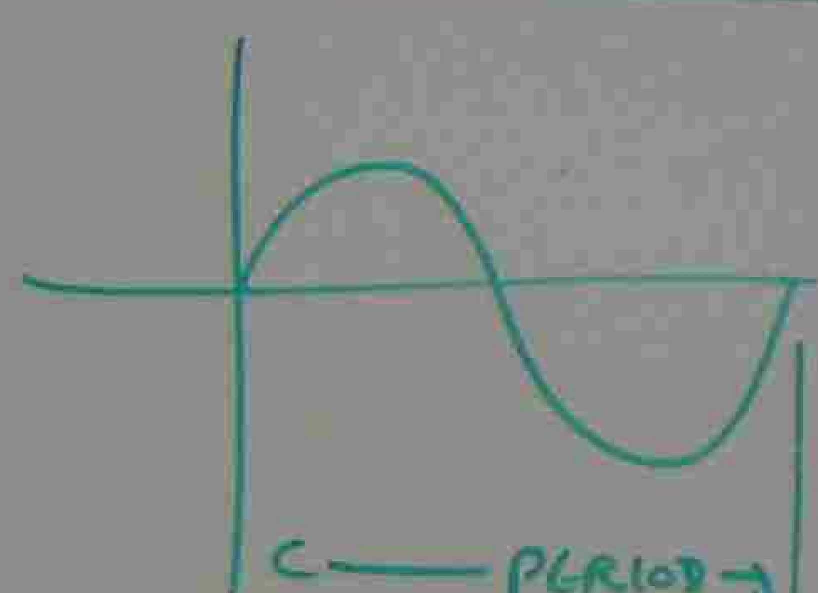
$2\theta = 360 \rightarrow \theta = 180$



$$\left(1 + \frac{\frac{\text{ANNUAL INTEREST RATE}}{365}}{100}\right)^{\text{NO. OF DAYS INVESTED}}$$

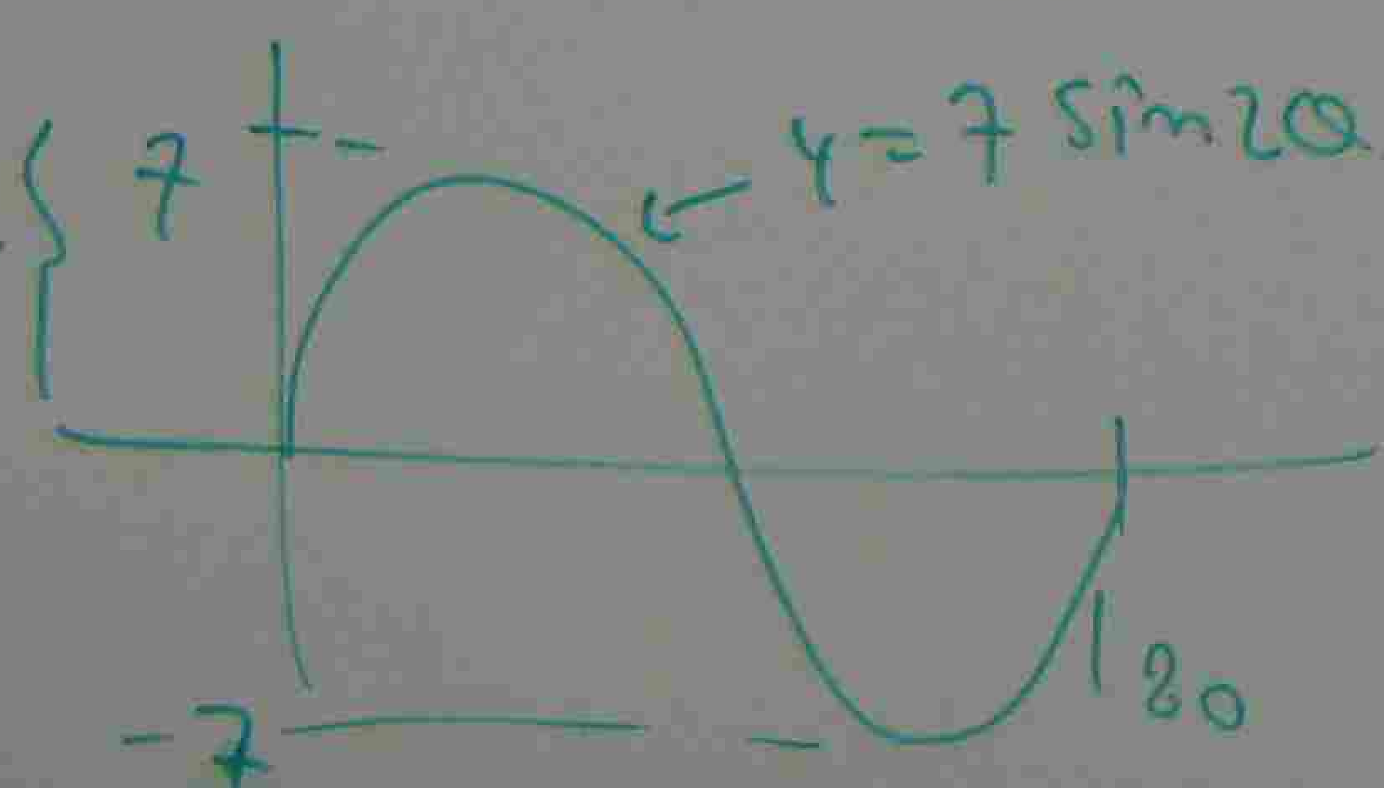
$$\left(\frac{\frac{13}{365}}{100}\right)^{3\times 7}$$

$$\left(\;\right)^{21} = \$\,5037.53$$

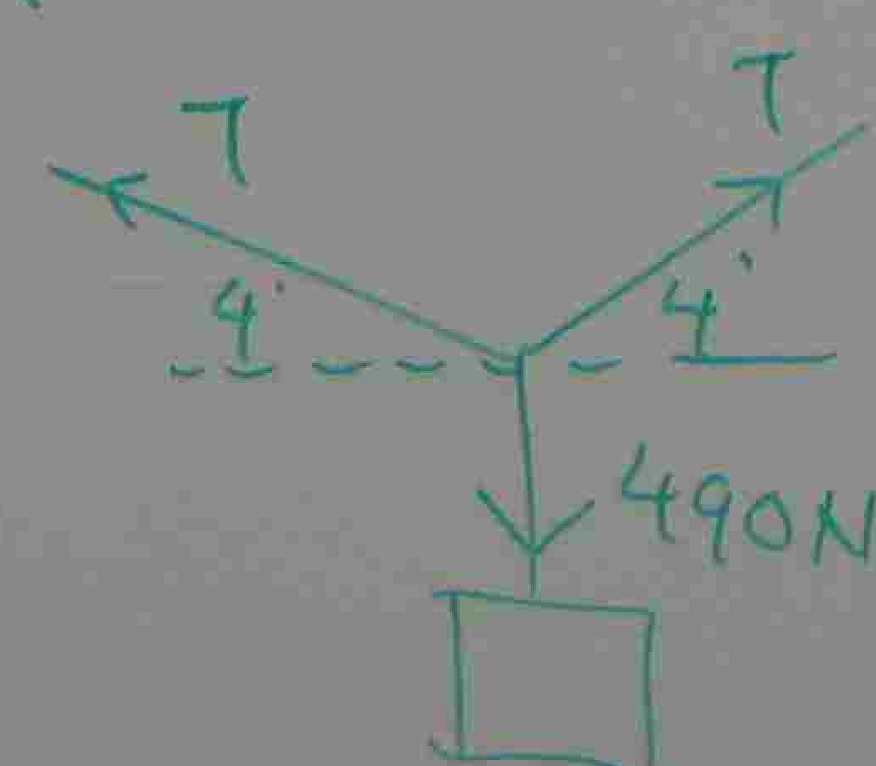


PERIOD = 360°

$= 360 \rightarrow \theta = 180$



MATHS (3)

1. SKETCH THE CURVE $5.37 \sin(136 \times 10^3 t - 0.955)$ mA
   PERIOD
   PHASE SHIFT
2. PROVE THAT $\dfrac{\sec^2 A - 1}{1 + \tan^2 A} = \sin^2 A$
3. FIND THE PHASE DIFFERENCE $\sin(\theta - 30)$ & $\sin(\theta - 100)$ GRAPHICALLY
4. IN $\triangle ABC$, $\angle B = 65°$, $\angle C = 40°$, $AB = 5.4$ m, FIND AC, BC, $\angle A$
5. SIMPLIFY $\sin 5x \cos 4x - \cos 5x \sin 4x$
6. FIND EXACT VALUE OF $\sin 15°$
7. 

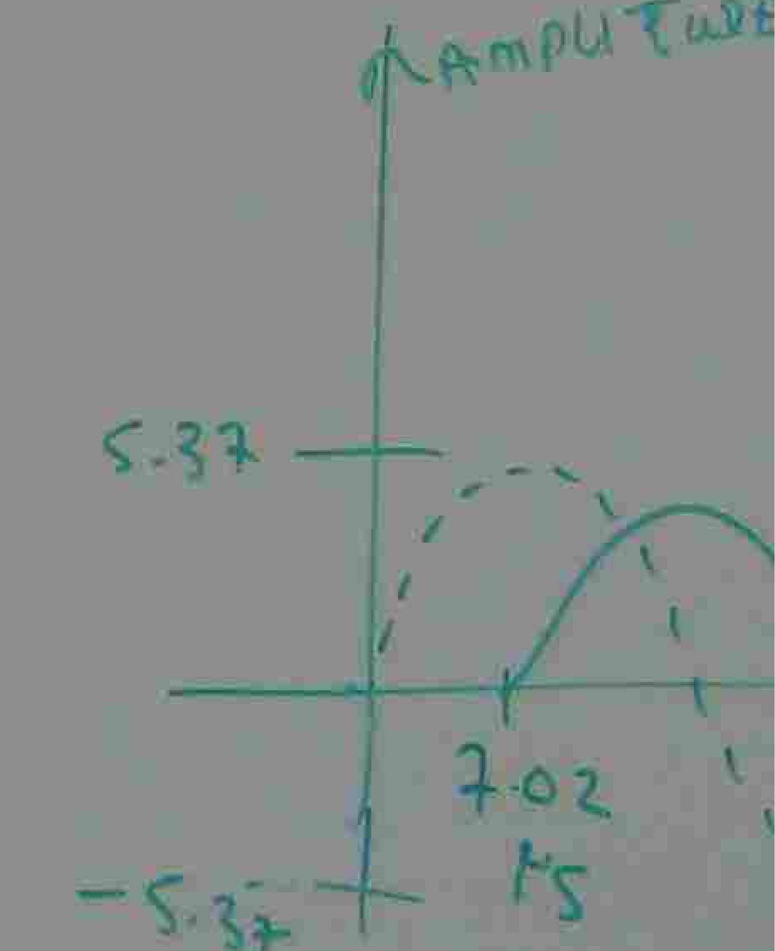
   FIND (i) TENSION IN CABLE
   (ii) RATIO OF TENSION TO LOAD SUPPORTED

1. AMPLITUDE, P

5.37 si

HALF OF AMPLITUDE

PERI



ss) mA

PHASE SHIFT

① AMPLITUDE, PERIOD, PHASE SHIFT ANGLE

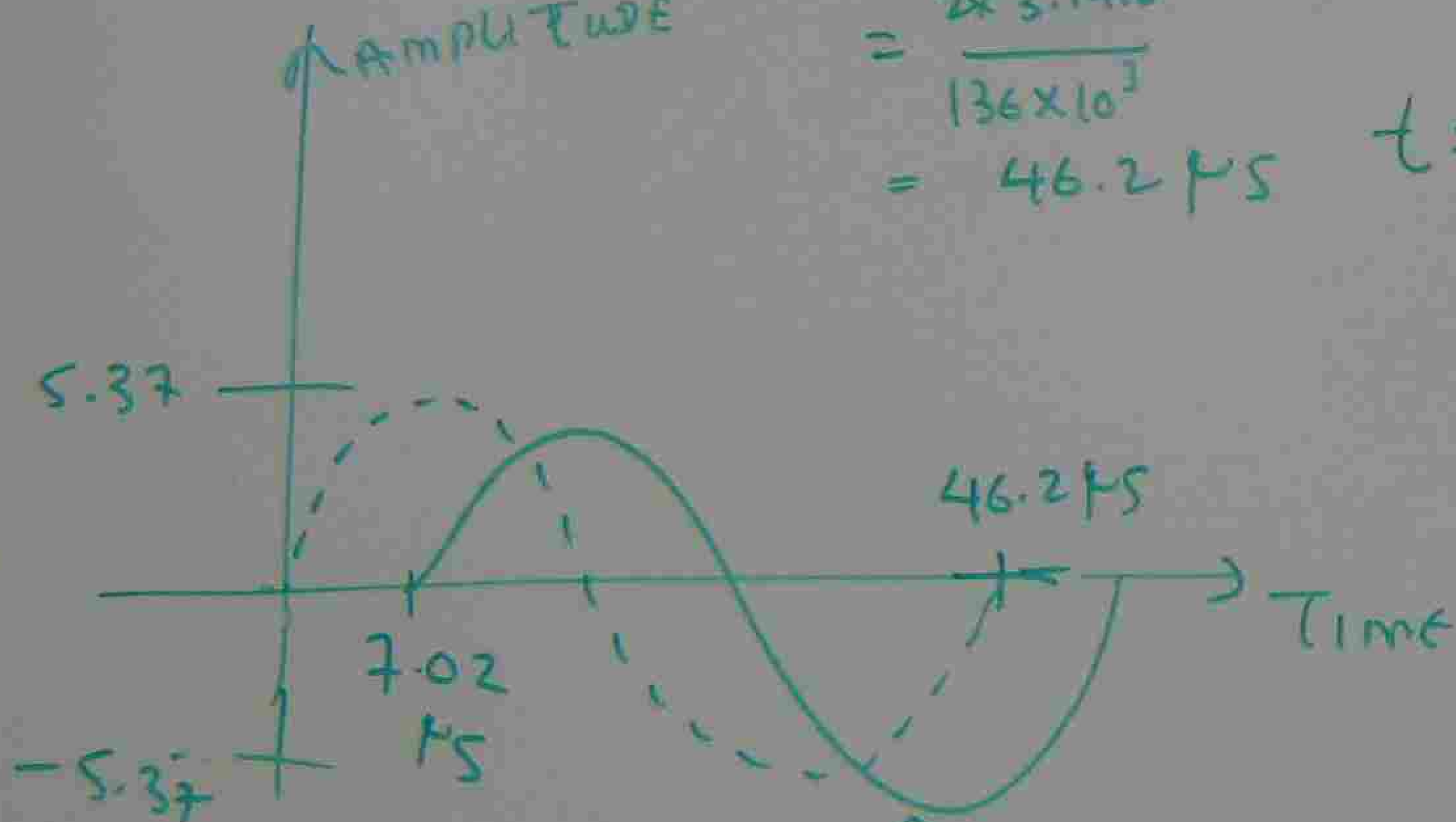
$$5.37 \sin\left(\underbrace{136 \times 10^3}_{\omega} t - \underbrace{0.955}_{\phi}\right) \text{ mA}$$

5.37 ← HALF OF AMPLITUDE

ϕ ← $\text{Time} = \dfrac{\phi}{\omega}$

$$\text{PERIOD} = T = \frac{2\pi}{\omega} = \frac{2 \times 3.1416}{136 \times 10^3} = 46.2\ \mu s$$

$$t = \frac{0.955}{136 \times 10^3} = 7.02\ \mu s$$



$$5.37 \sin\left(136 \times 10^3 t - 0.955\right)$$

mA

② LHS

$$\frac{\sec^2 A - 1}{1 + \tan^2 A}$$

$$\boxed{\sec A = \frac{1}{\cos A} \qquad \tan A = \frac{\sin A}{\cos A}}$$

$$\boxed{\sin^2 A + \cos^2 A = 1}$$

$$\frac{\dfrac{1}{\cos^2 A} - 1}{1 + \dfrac{\sin^2 A}{\cos^2 A}}$$

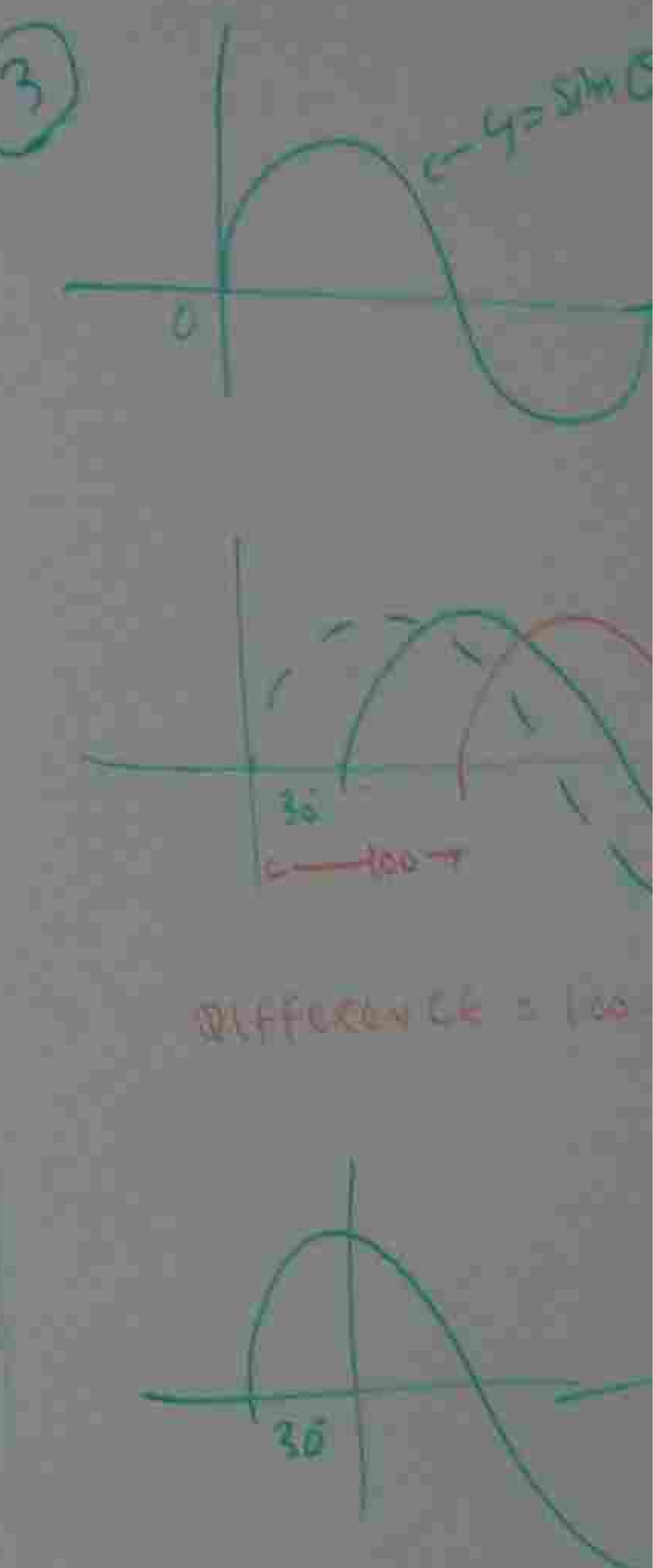
$$\frac{\dfrac{1 - \cos^2 A}{\cancel{\cos^2 A}}}{\dfrac{\cos^2 A + \sin^2 A}{\cancel{\cos^2 A}}} = \frac{1 - \cos^2 A}{1}$$

$$= 1 - \cos^2 A$$

$$= \sin^2 A$$

RHS

③



4x

to

orted

$\boxed{\sin^2 A + \cos^2 A = 1}$

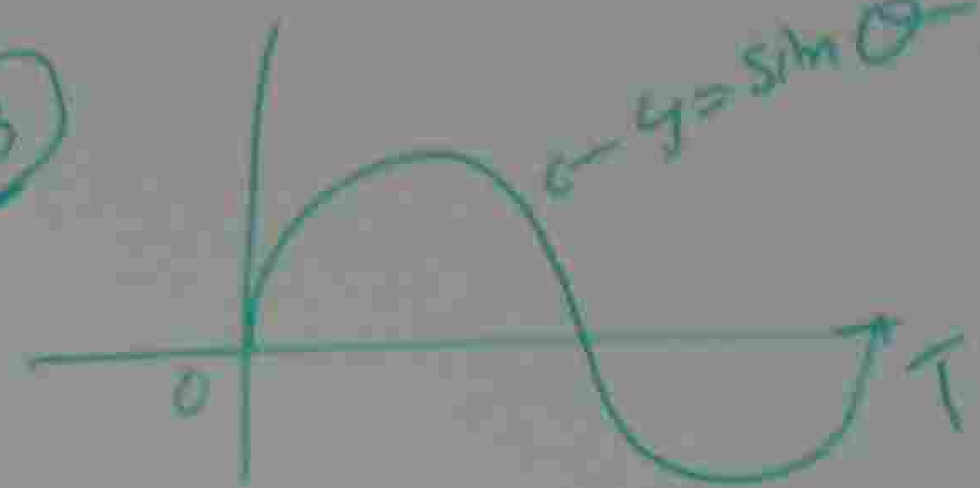
$= \dfrac{1-\cos^2 A}{1}$

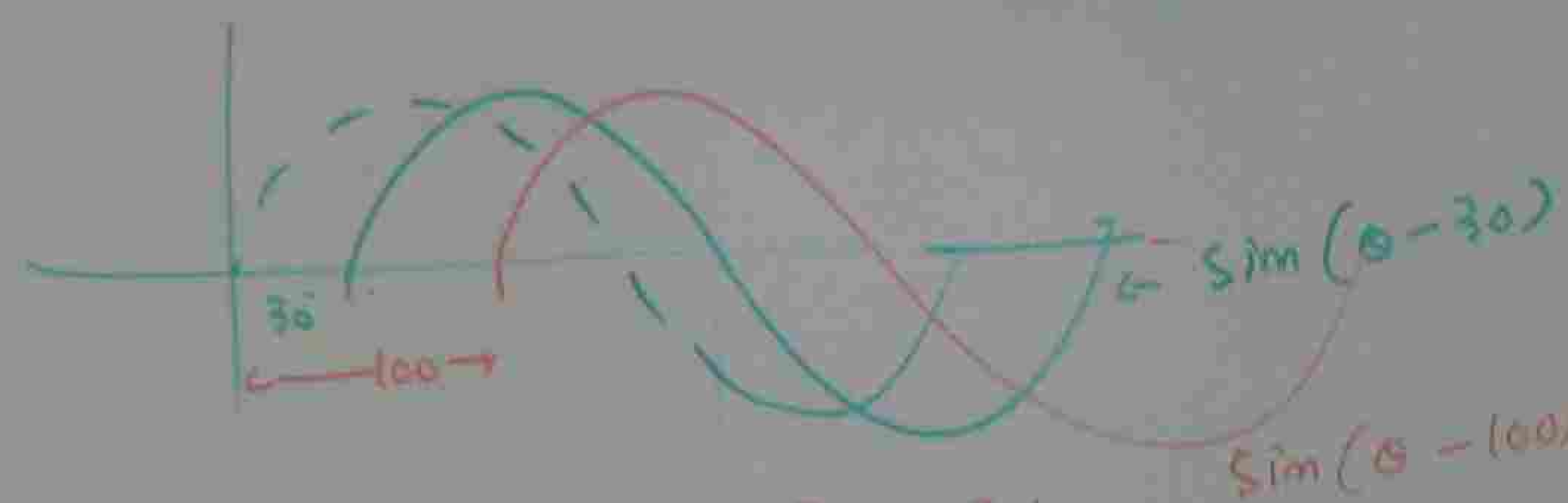
$= 1 - \cos^2 A$

$= \sin^2 A$

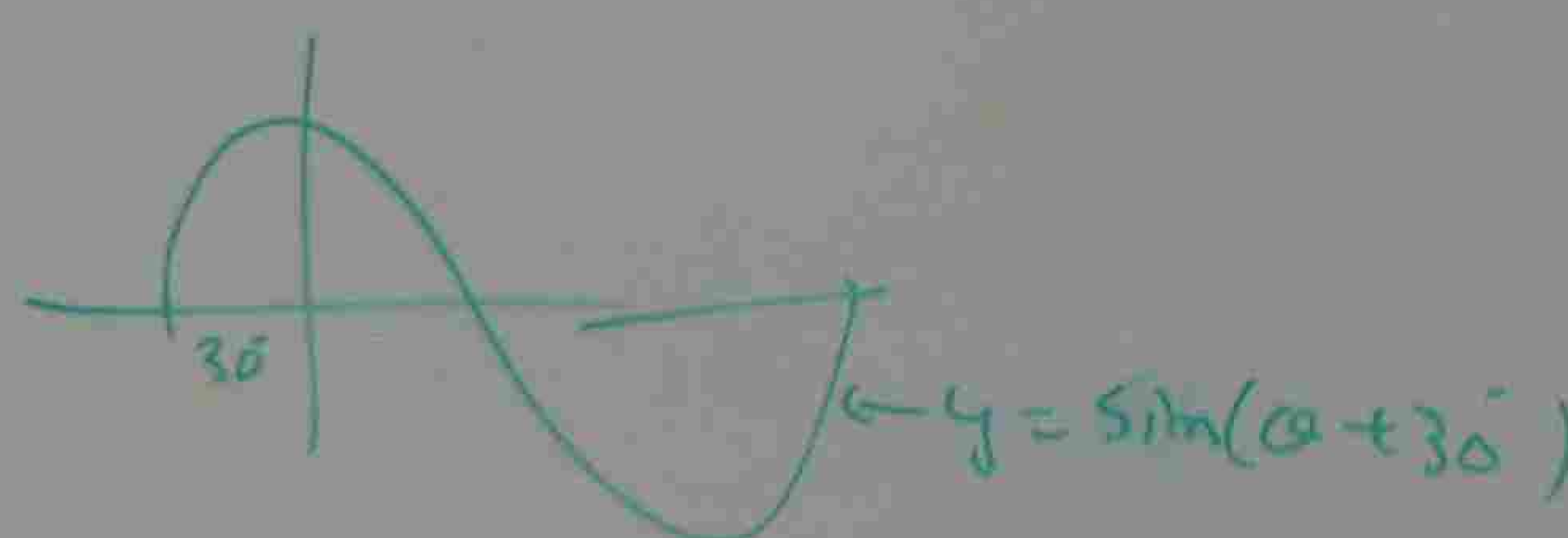
RHS

3

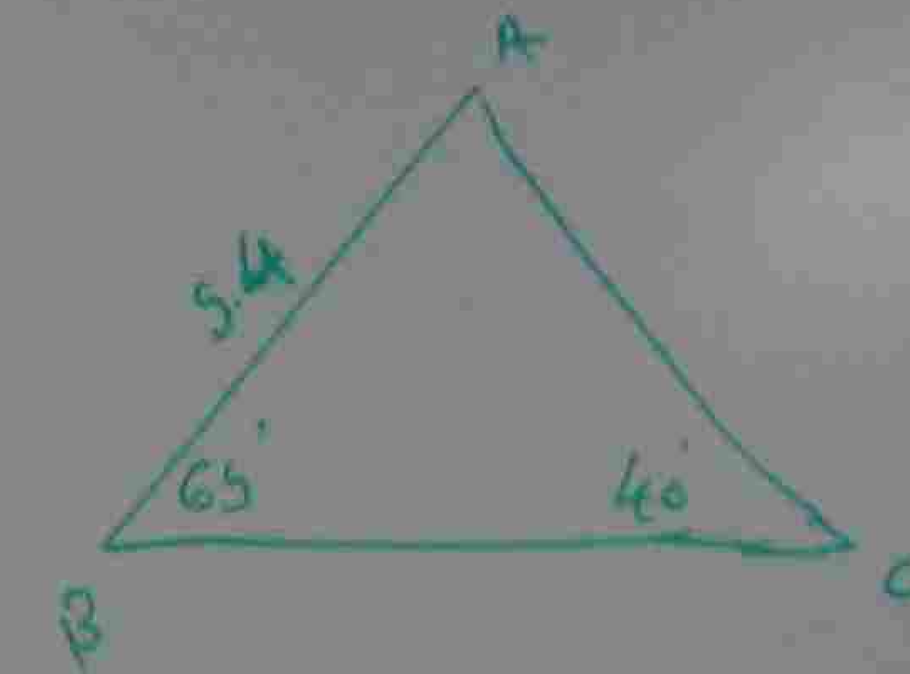




DIFFERENCE = 100 − 30 = 70°



4



$\hat{A} = 180 - (\hat{B} + \hat{C})$

$= 180 - (65 + 40)$

$= 180 - 105 = 75°$

$\dfrac{AB}{\sin C} = \dfrac{AC}{\sin B}$

$\dfrac{5.4}{\sin 40} = \dfrac{AC}{\sin 65}$

$AC = \dfrac{5.4 \sin 65}{\sin 40}$

$= \dfrac{5.4 \times 0.906}{0.642}$

$= 7.62$

$\dfrac{BC}{\sin A} = \dfrac{A\ldots}{s\ldots}$

$\dfrac{BC}{\sin 75} = \dfrac{s\ldots}{s\ldots}$

$BC = \dfrac{5.4}{\sin\ldots}$

$= \dfrac{5.4\ldots}{0.6\ldots}$

$= 8.1\ldots$

5 sin(A+B)

sin(A−B)

sin 5x cos…

= sin(…

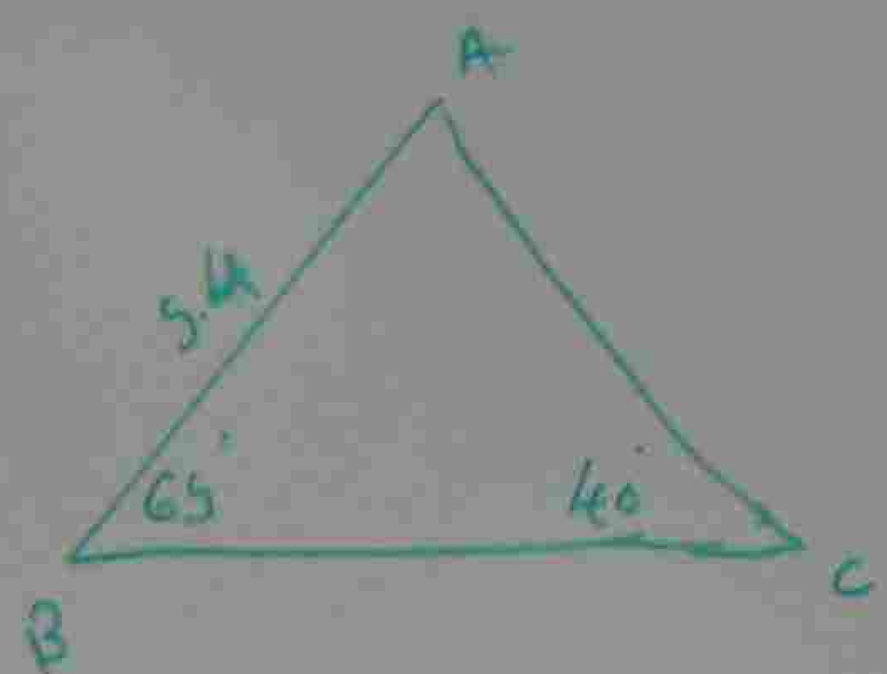
= sin…

$= \sin(\theta - 30)$

$\sin(\theta - 100)$

$= \sin(\theta + 30)$

④



$$\hat{A} = 180 - (\hat{B} + \hat{C})$$
$$= 180 - (65 + 40)$$
$$= 180 - 105 = 75$$

$$\frac{AB}{\sin C} = \frac{AC}{\sin B}$$

$$\frac{5.4}{\sin 40} = \frac{AC}{\sin 65}$$

$$AC = \frac{5.4 \sin 65}{\sin 40}$$
$$= \frac{5.4 \times 0.906}{0.642}$$
$$= 7.62$$

$$\frac{BC}{\sin A} = \frac{AB}{\sin C}$$

$$\frac{BC}{\sin 75} = \frac{5.4}{\sin 40}$$

$$BC = \frac{5.4 \sin 75}{\sin 40}$$
$$= \frac{5.4 \times 0.965}{0.642}$$
$$= 8.11$$

⑤ $\sin(A+B) = \sin A \cos B + \cos A \sin B$

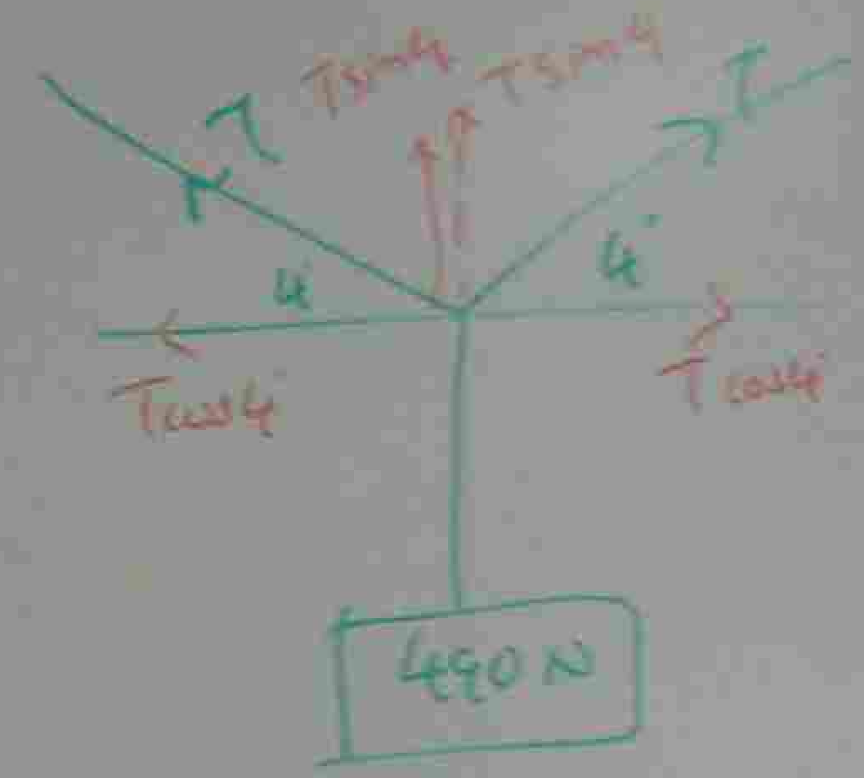
$\sin(A-B) = \sin A \cos B - \cos A \sin B$

$$\sin 5x \cos 4x - \cos 5x \sin 4x$$
$$= \sin(5x - 4x)$$
$$= \sin x$$

⑥ $\sin 15 = ?$

$$\sin 15 = \sin(45 - 30)$$
$$= \sin 45 \cos 30 - \cos 45 \sin 30$$
$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}$$
$$= \frac{\sqrt{3} - 1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$
$$= \frac{\sqrt{6} - \sqrt{2}}{2 \times 2}$$
$$= \frac{\sqrt{6} - \sqrt{2}}{4}$$

7



$$2T \sin 4 = 490$$

$$T = \frac{490}{2 \sin 4}$$

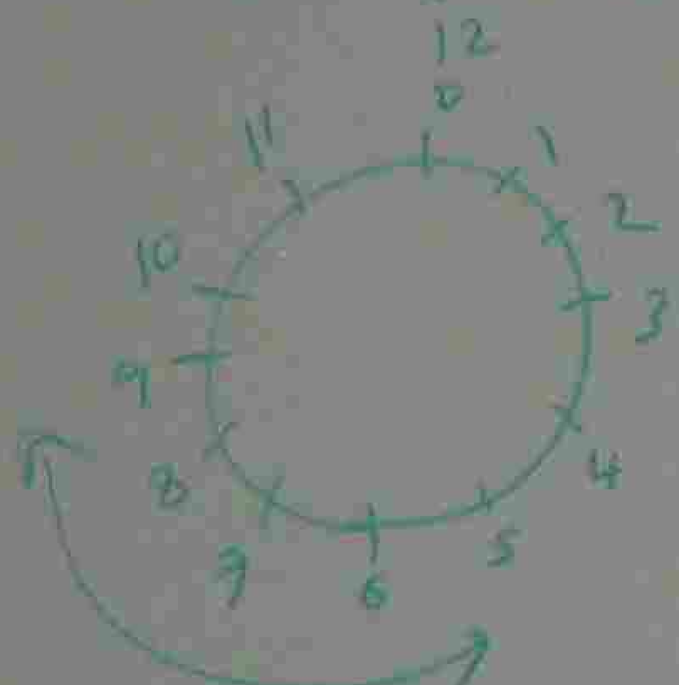
$$= \frac{490}{2 \times 0.069}$$

$$= 3550$$

$$\frac{T}{W} = \frac{3550}{490} = 7.24 : 1$$

NORMAL PROBABILITY DISTRIBUTION

PROBABILITY – CHANCE (OR) LIKELIHOOD

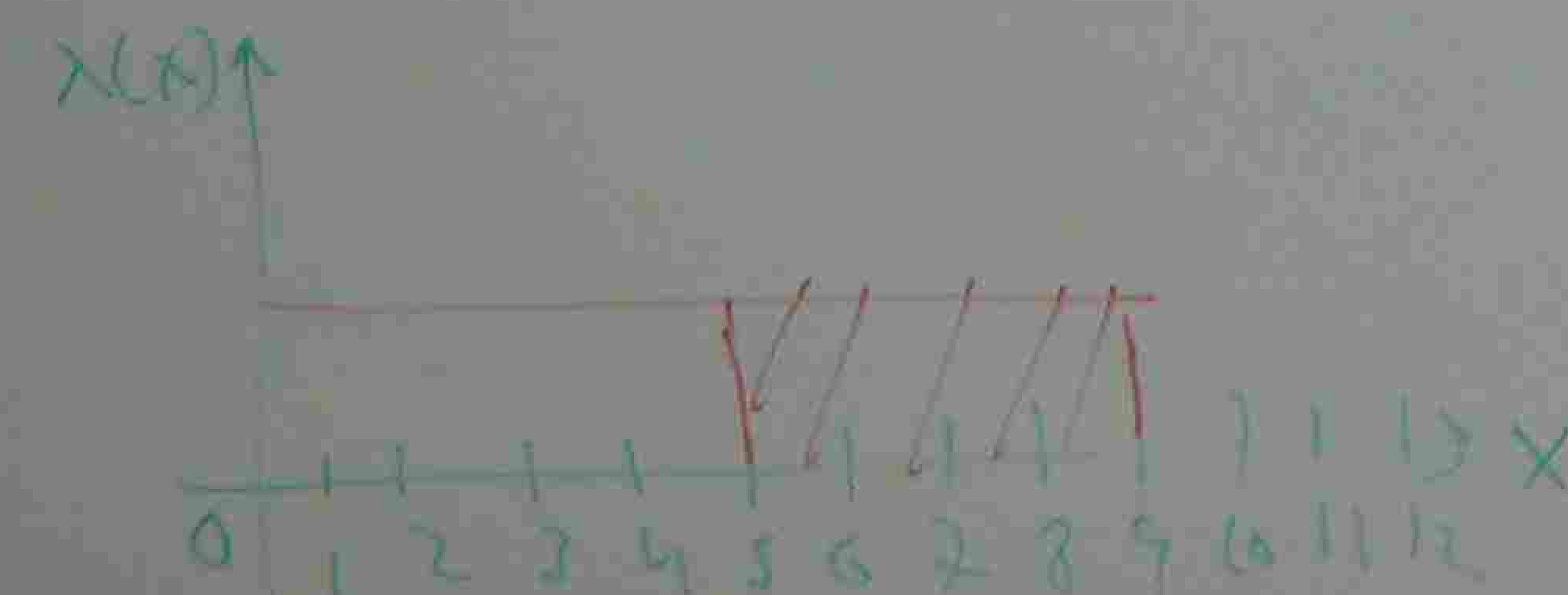


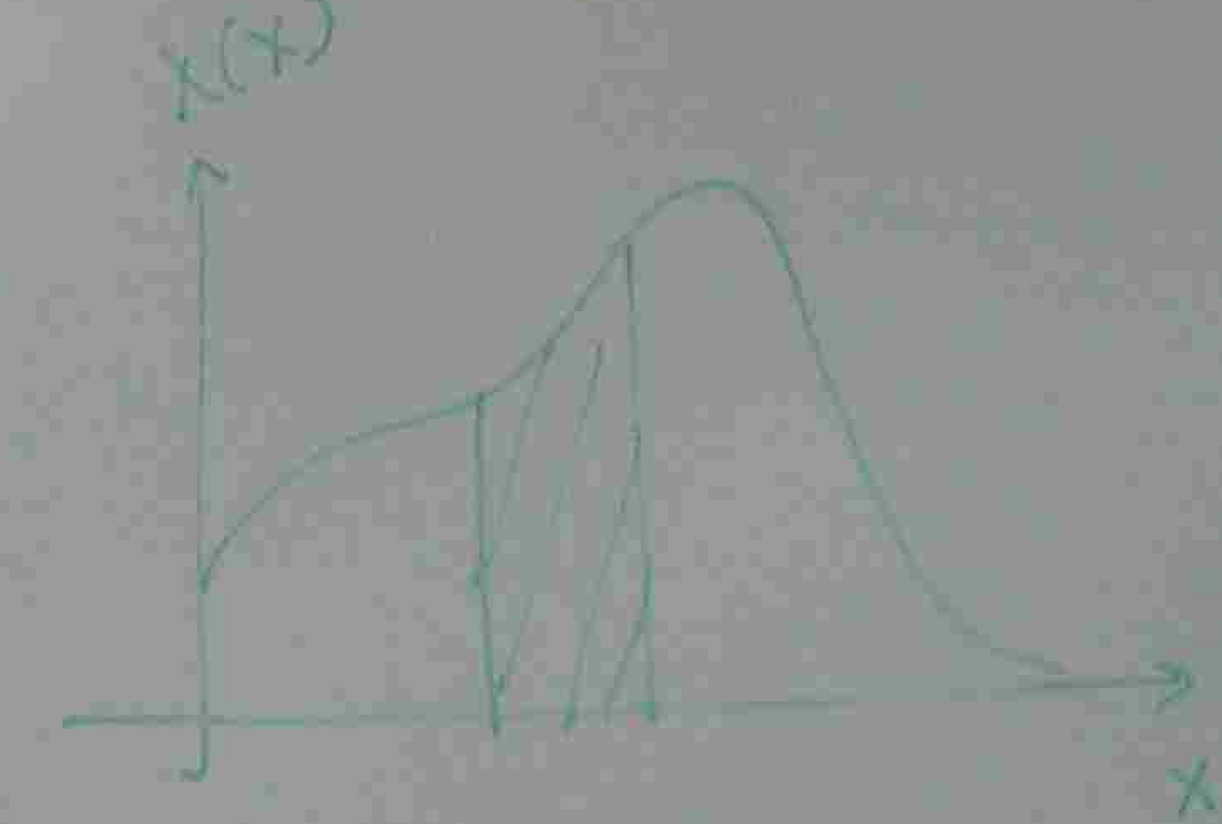
CHANCE TO STOP THE WATCH BETWEEN 5 → 9

TOTAL = 12 DIVISIONS

5 → 9 = 4 DIVISIONS.

$$P(5 < x < 9) = \frac{4}{12} = \frac{1}{3}$$





PRACTICAL PROBABILITY GRAPH

NORMAL DISTRIBUTION

$$\text{STANDARD DEVIATION} = \delta = \sqrt{\frac{\sum f x^2 - n(\bar{x})^2}{n-1}}$$

HOW MUCH THE NUMBER DIFFERS FROM AVERAGE VALUE

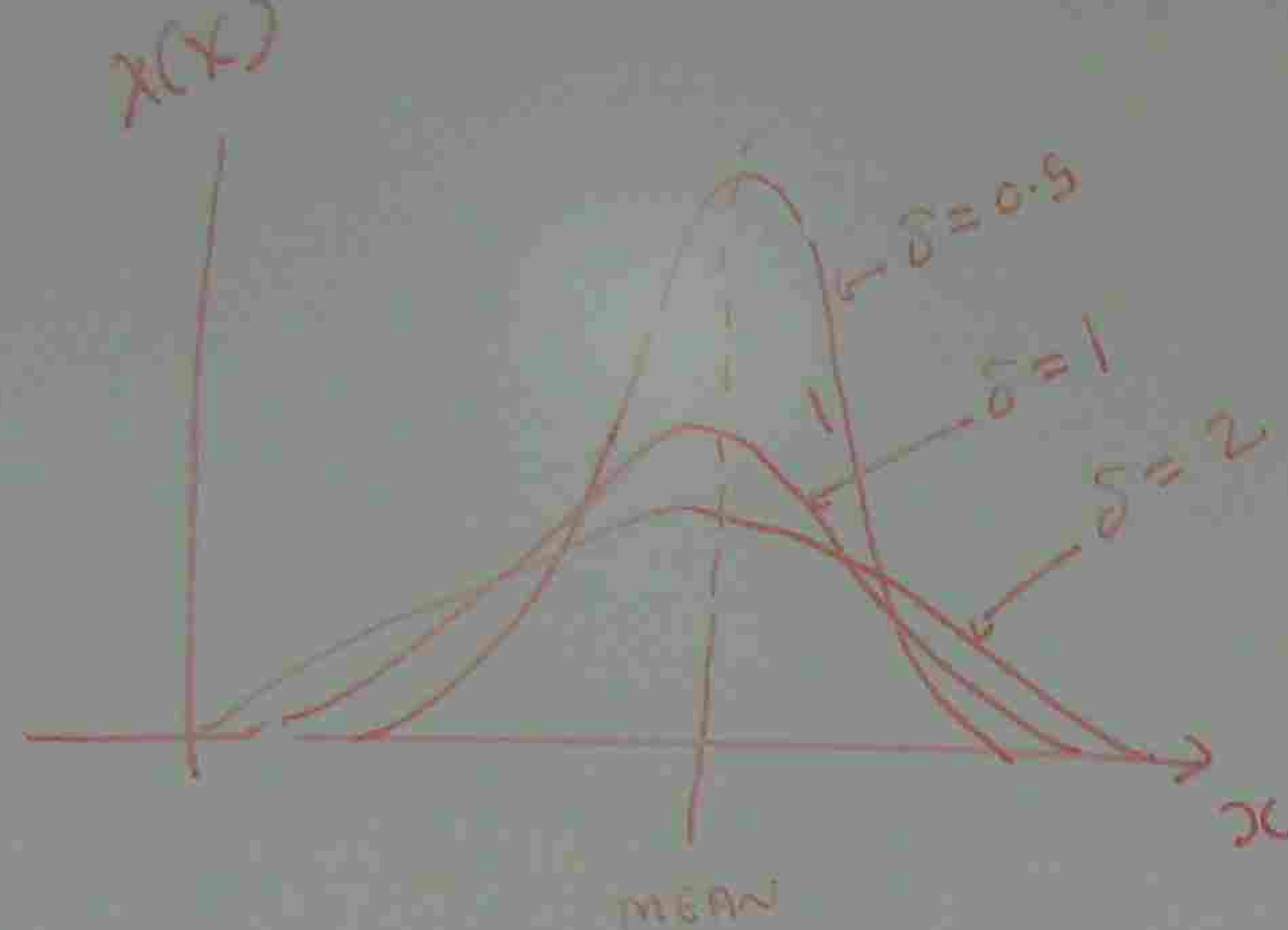
X	f
EVENT 1 →	5 Times
2 →	3 Times
3 →	4 Times.

$\bar{X}$ = MEAN = AVERAGE

n = TOTAL NUMBER OF TIMES.

$(\overline{x})^2$

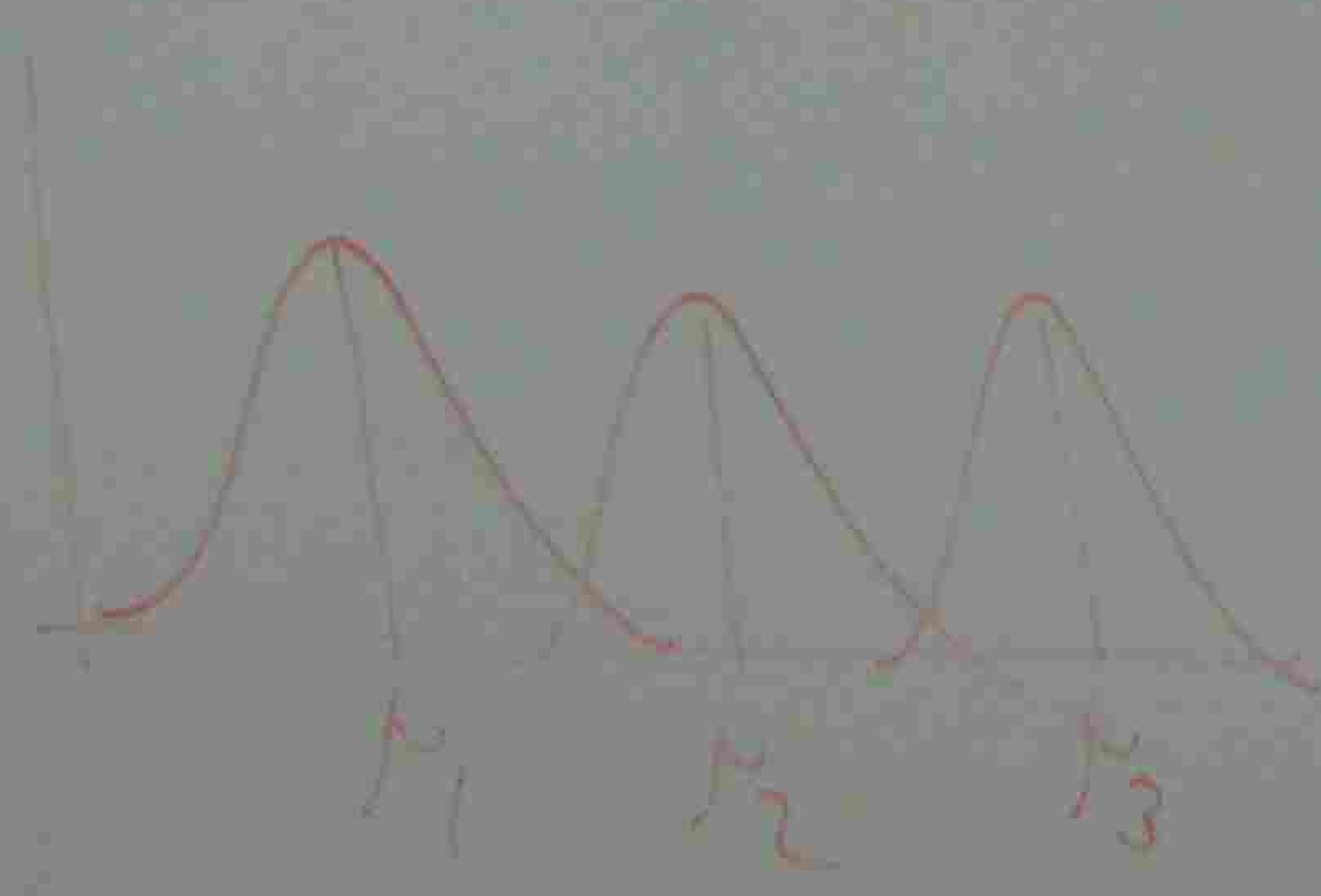
THE
DIFFERS
ERAGE



SAME MEN $= \mu = \frac{\sum x_1}{n_1} = \frac{\sum x_2}{n_2} = \frac{\sum x_3}{n_3}$

NORMAL DISTRIBUTION $\left\{\right.$ SAME MEAN
DIFFERENT
STANDARD DEVIATION

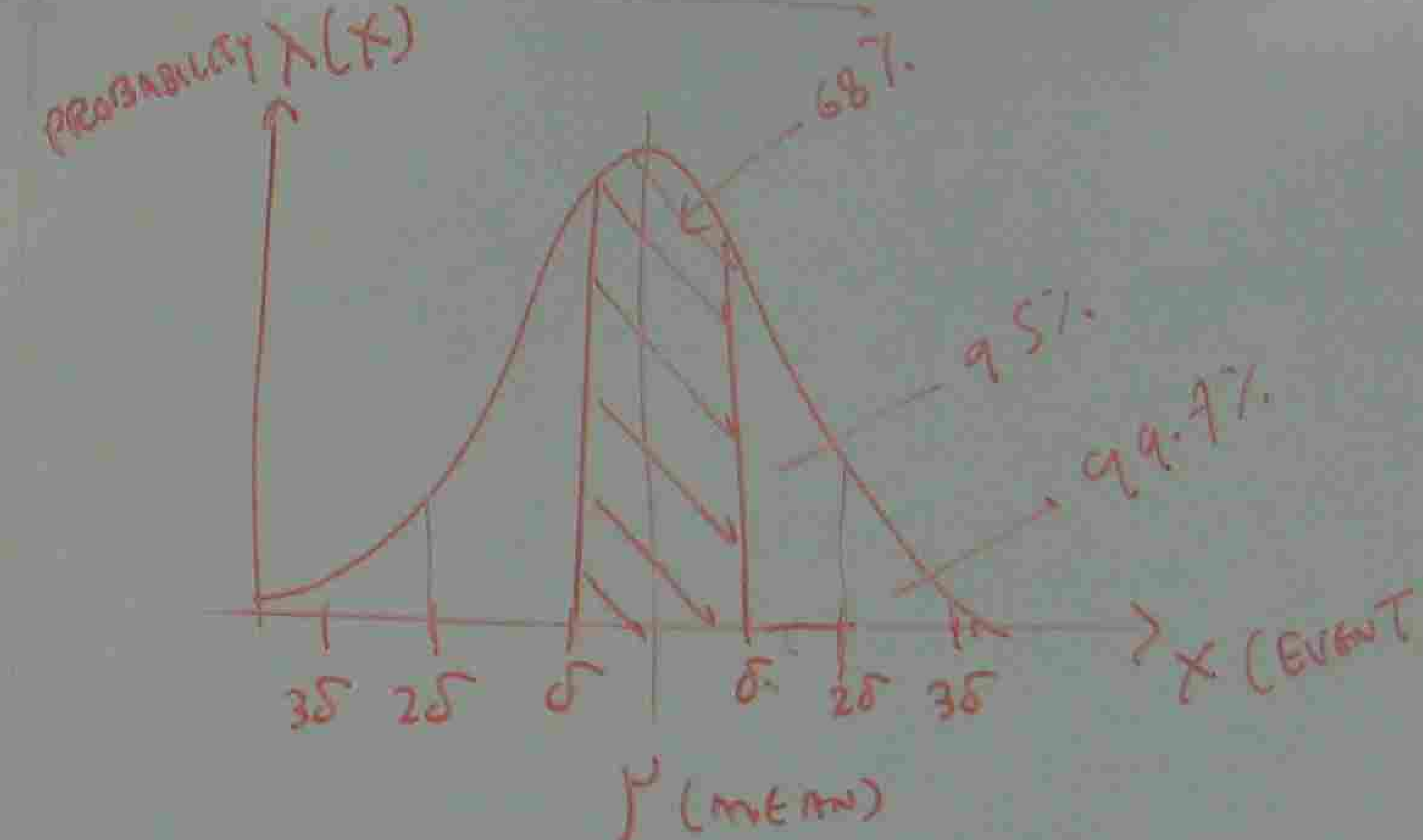
DIFFERENT MEAN, SAME STANDARD DEVIATION



S, δ

μ

NORMAL DISTRIBUTION



$P(\mu - \delta < x < \mu + \delta) = 68\% = 0.68$

$P(\mu - 2\delta < x < \mu + 2\delta) \rightarrow 0.95$

$P(\mu - 3\delta < x < \mu + 3\delta) \longrightarrow 0.997$

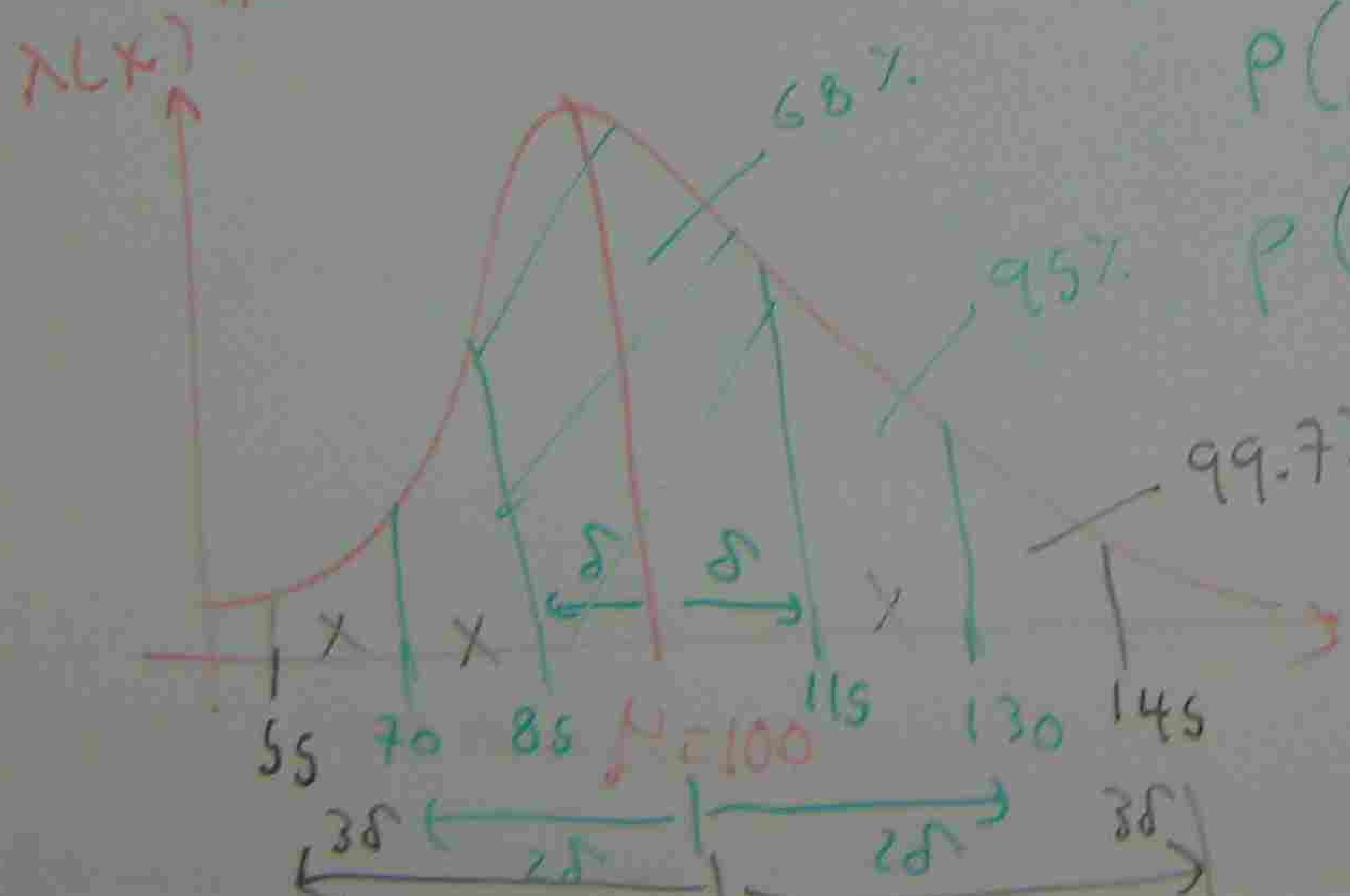
Ex 20 Suppose that IQ scores are normally distributed with a mean $\mu = 100$ and a standard deviation $\sigma = 15$.

Indicate the followings on the graph.

(a) Approximately 68% of people have IQ between 85 and 115

(b) Approximately 95% of people have IQ between 70 and 130

(c) Almost every one (approximately 99.7%) have IQ between 55 and 145.



$P(\mu - \sigma < x < \mu + \sigma) = 0.68$

$P(\mu - 2\sigma < x < \mu + 2\sigma) = 0.95$

$P(\mu - 3\sigma < x < \mu + 3\sigma) = 0.997$

Ex (20) Suppose that IQ scores are normally distributed with a mean $\mu = 100$ and a standard deviation $\delta = 15$.

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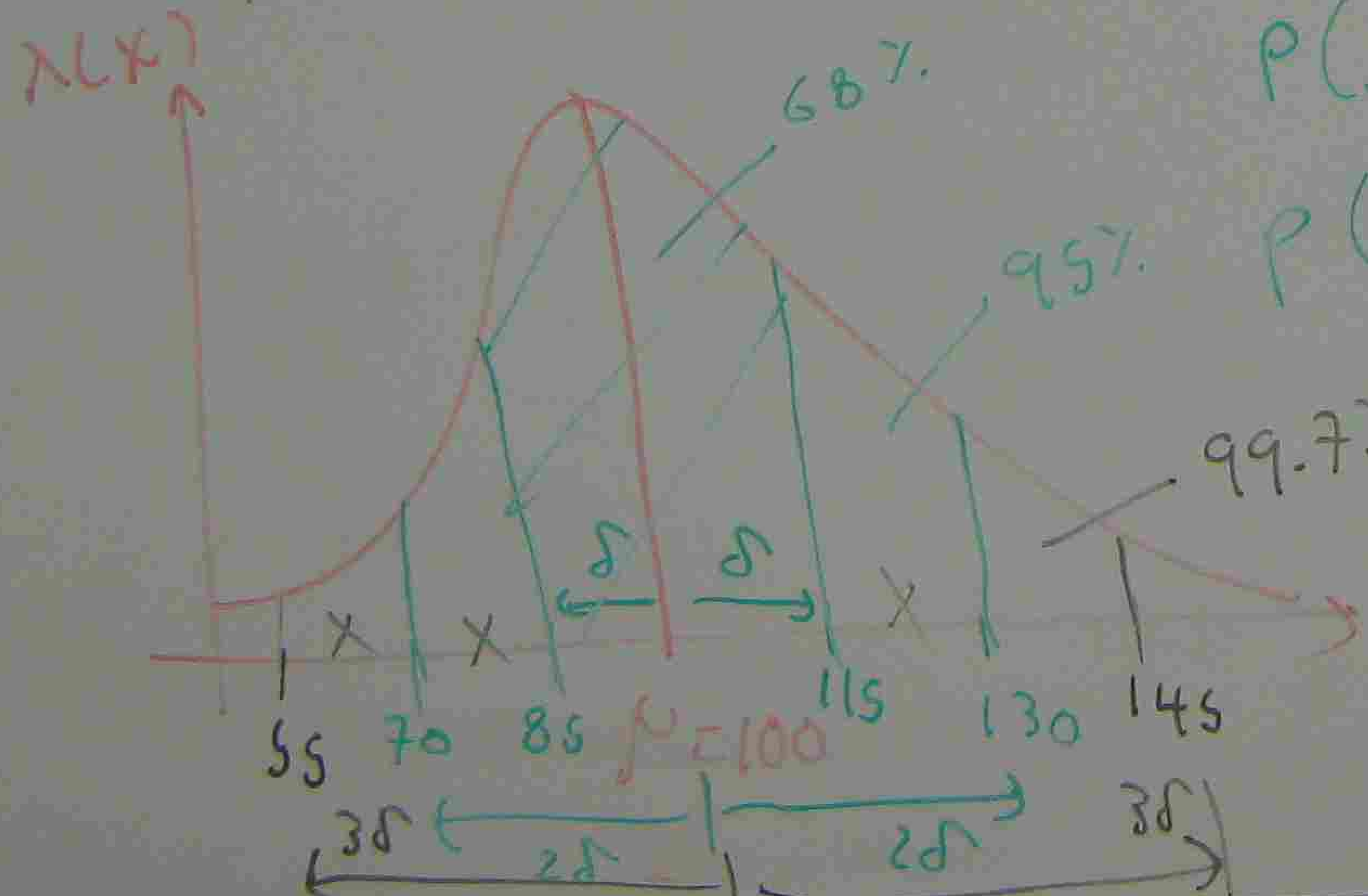


$P(\mu - \delta < x < \mu + \delta) = 0.68$

$P(\mu - 2\delta < x < \mu + 2\delta) = 0.95$

$P(\mu - 3\delta < x < \mu + 3\delta) = 0.997$

.7%

x (event)

68% = 0.68

0.95

→ 0.997

STANDARD SCORE (

STANDARD SCORE (Z-SCORE)

$z =$

If $x > \mu$ $z =$

If $x < \mu$ $z =$

If $x = \mu$ z

EX (21) THE

HEIGHT HAS

STANDARD

FIND THE

HEIGHT TO

FEMALE HE

STANDARD SCORE (Z SCORE)

STANDARD SCORE (Z-SCORE) = $\frac{\text{RAW DATA} - \text{MEAN}}{\text{STANDARD DEVIATION}}$

$$Z = \frac{x - \mu}{\delta}$$

IF $x > \mu$ $Z = +$

IF $x < \mu$ $Z = -$

IF $x = \mu$ $Z = 0$ NO ERROR

EX 21 THE DISTRIBUTION OF ADULT FEMALE HEIGHT HAS A MEAN $\mu = 165.5$ cm, STANDARD DEVIATION $\delta = 8$ cm.

FIND THE STANDARD SCORE (RELATIVE HEIGHT TO MEAN) CORRESPONDING TO A FEMALE HEIGHT OF (a) 164 cm (b) 178 cm

(a) $Z = \frac{x - \mu}{\delta} = \frac{164 - 165.5}{8} = -0.3$

(b) $Z = \frac{x - \mu}{\delta} = \frac{178 - 165.5}{8} = 1.44$

EX 22 THE WEIGHT OF A CERTAIN SPECIES OF ELEPHANT ARE DISTRIBUTED WITH A MEAN OF 20 TONNES AND A STANDARD DEVIATION OF 2 TONNES. SUPPOSE ALSO THAT WEIGHTS OF GARDEN SNAILS ARE DISTRIBUTED WITH A MEAN OF 30 GRAM AND STANDARD DEVIATION 3 GRAM.

WHICH IS HEAVIER, AN ELEPHANT WEIGHING 23 TON (OR) A SNAIL WEIGHING 34.5 GRAMS.

ELEPHANT

μ_1

$X_1 =$

$Z = \frac{X_1}{}$

SNAIL

$\mu_2 = 3$

$X_2 =$

$Z = \frac{X}{}$

TWO AN

SAME R

IS

$= 0.997$

MEAN

EVIATION

ADULT FEMALE

55.5 cm ,

n.

(RELATIVE

DING TO A

m (b) 178 cm

(a) $z = \frac{x-\mu}{\delta} = \frac{164-165.5}{8} = -0.3$

(b) $z = \frac{x-\mu}{\delta} = \frac{178-165.5}{8} = 1.44$

Ex 22 THE WEIGHT OF A CERTAIN SPECIES OF ELEPHANT ARE DISTRIBUTED WITH A MEAN OF 20 TONNES AND A STANDARD DEVIATION OF 2 TONNES. SUPPOSE ALSO THAT WEIGHTS OF GARDEN SNAILS ARE DISTRIBUTED WITH A MEAN OF 30 GRAM AND STANDARD DEVIATION 3 GRAM. WHICH IS HEAVIER, AN ELEPHANT WEIGHING 23 TON (OR) A SNAIL WEIGHING 34.5 GRAMS.

ELEPHANT

$\mu_1 = 20 \quad \delta_1 = 2$

$x_1 = 23$

$z = \frac{x_1 - \mu_1}{\delta_1} = \frac{23-20}{2} = 1.5$

SNAIL

$\mu_2 = 30 , \quad \delta_2 = 3$

$x_2 = 34.5$

$z = \frac{x_2 - \mu_2}{\delta_2} = \frac{34.5-30}{3}$

$z = 1.5$

TWO ANIMALS HAVE SAME RELATIVE WEIGHT.

Ex 23

IF A STUDENT SHOWS THE FOLLOWING RESULTS IN A SERIES OF TESTS. COMPARE THE OVERALL PERFORMANCE.

SUBJECT	STUDENT SCORE X	MEAN μ	STANDARD DEVIATION δ
MATHS	170	150	15
ENGLISH	120	120	10
HISTORY	188	200	12
SCIENCE	120	100	8

MATHS $z = \frac{X-\mu}{\delta} = \frac{170-150}{15} = 1.33$

ENGLISH $z = \frac{X-\mu}{\delta} = \frac{120-120}{10} = 0$

HISTORY $z = \frac{X-\mu}{\delta} = \frac{188-200}{12} = -1.0$

SCIENCE $z = \frac{X-\mu}{\delta} = \frac{120-100}{8} = 2.5$

PERFORMANCE

SCIENCE → BEST (2.5)

MATHS ⇒ 2nd BEST (1.33)

ENGLISH - NORMAL (0)

HISTORY : MOST POOR (-1.0)

READING Z TA

Z TABLE

z_0	0.00
0.0	
0.1	
0.2	
1.1	
1.[illegible]	
3.0	

1.12 = 1.1

0.3686

0.3690

...S THE FOLLOWING RESULTS IN A
...ARE THE OVERALL PERFORMANCE.

...	MEAN μ	STANDARD DEVIATION σ
	150	15
	120	10
	200	12
	100	8

$\frac{...-150}{...} = 1.33$

$\frac{...-120}{...} = 0$

$\frac{...-200}{12} = -1.0$

$\frac{...0-100}{8} = 2.5$

PERFORMANCE

SCIENCE ⇒ BEST (2.5)

MATHS ⇒ 2nd BEST (1.33)

ENGLISH - NORMAL (0)

HISTORY = MOST POOR (−1.0)

READING Z TABLE

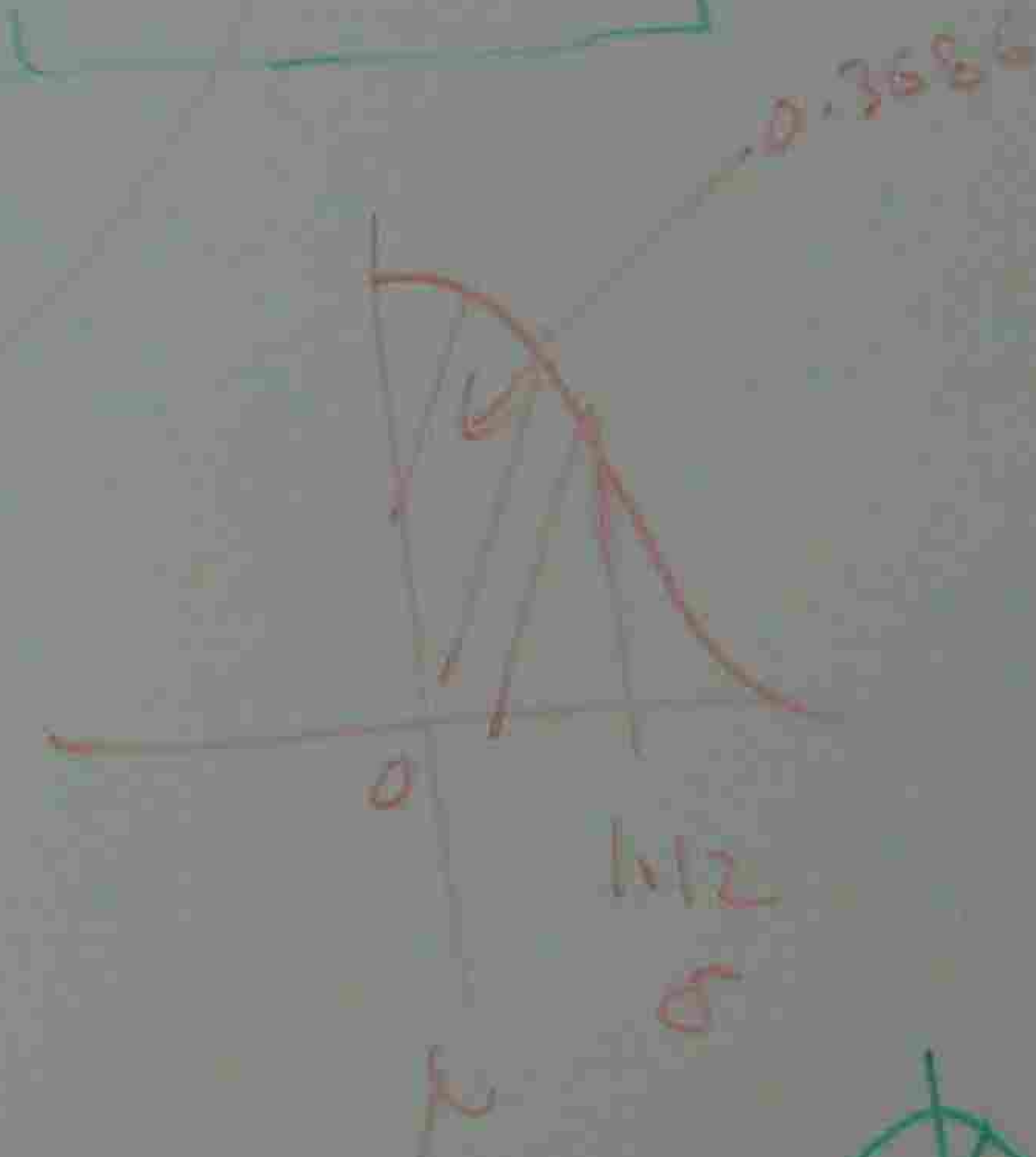
Z TABLE

z_0	0.00	0.01	0.02	0.03 - - - 0.04
0.0				
0.1				
0.2				
1.1			0.3686	
3.0				

$1.12 = 1.1 + .02$

$0.3686 \rightarrow 1.12$

$0.3690 \rightarrow 1.12$



Ex (24) USE ...

NORMA...

(a) BET...

(b) BET...

(c) BET...

(d) TO...

(e) TO...

(f) BE...

(a) P...

(b) P...

(c) P...

(e)

Ex 24 USE Z TABLE, CALCULATE THE AREA UNDER THE STANDARD NORMAL CURVE FOR THE FOLLOWING INTERVALS.

(a) BETWEEN $Z=0$ AND $Z=1.50$

(b) BETWEEN $Z=0$ AND $Z=-2.10$

(c) BETWEEN $Z=-0.30$ AND $Z=+2.25$

(d) TO THE RIGHT OF $Z=1.95$

(e) TO THE LEFT OF $Z=1.64$

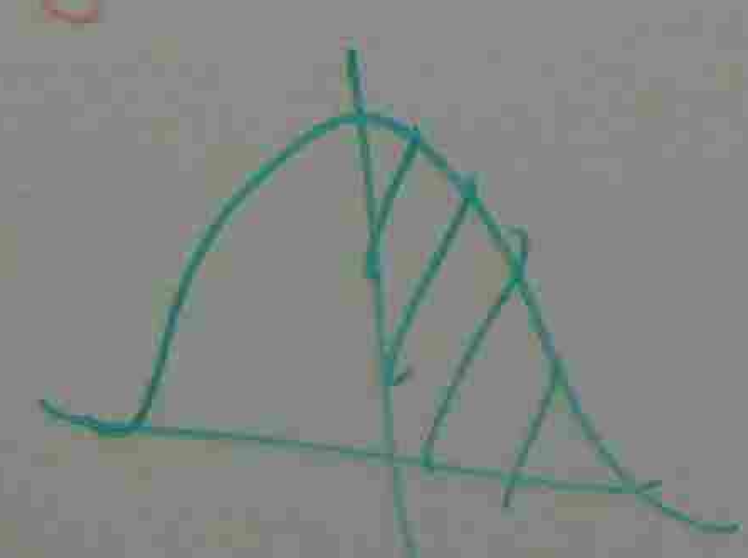
(f) BETWEEN 0.60 AND 1.80

(a) $P(0<Z<1.5) = 0.4332 - 0.0000$
$= 0.4332$

(b) $P(-2.1<Z<0) = 0 - (-0.4821)$
$= 0.4821$

(c) $P(-0.30<Z<2.25) = 0.4878 - (-0.1179)$
$= 0.4878 + 0.1179$
$= 0.6057$

(e) $P(Z>1.95) =$ THE WHOLE AREA $- P(0<Z<1.95)$
$= 0.5 - [0.4744 - 0]$
$= 0.5 - 0.4744 = 0.0256$



(e) $P(Z<1.64)$

$=$ THE WHOLE AREA $+ (P(0<Z<1.64))$

$= 0.5 + (0.4495 - 0)$

$= 0.5 + 0.4495$

$= 0.9495$

(f) $P(0.6<Z<1.8)$

$= P(1.8) - P(0.6)$

$= 0.4641 - 0.2257$

$= 0.2384$

EX 25

(a) IF 37.7% OF POPULATION HAS A STANDARD SCORE BETWEEN THE MEAN (z=1) AND SOME POSITIVE z VALUE FIND THAT z VALUE

(b) FIND THE z VALUE CUTTING OFF TOP 5% OF POPULATION

(c) FIND THE z SCORE CUTTING OFF BOTTOM 10% OF POPULATION

(d) FIND THE VALUE OF z SO THAT 95% OF POPULATION HAS A SCORE BETWEEN − z AND + z

(a) AREA = 37.7% ⇒ 0.377 ⟶ z = ?

1.1 — 0.3770 (column 0.06)

z = 1.16

(b) CUTTING OFF TOP 5% = THE WHOLE ARE − 5%

= 0.5 − 0.05 = 0.45

1.6 —— 0.4495 (column 0.04) − 0.4505 (column 0.05)

0.4495 = 1.64

0.4505 = 1.65

$0.4500 = \frac{1.64+1.65}{2}$

= 1.645

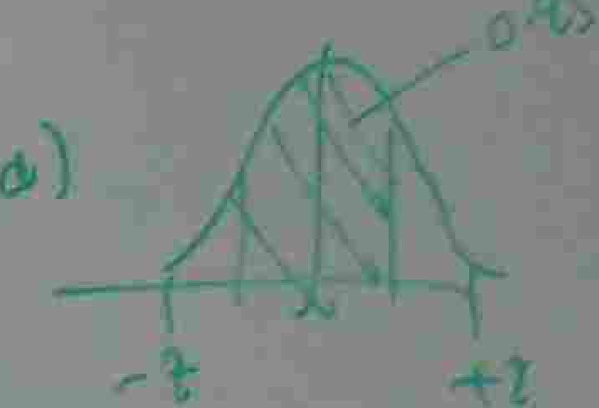
(c) CUTTING OFF BOTTOM 10% = THE WHOLE AREA − 10%

= 0.5 − 0.1 = 0.4

1.2 —— 0.3997 (column 0.08)

0.3997 ≃ 0.4 ⇒ 1.28

(d)



2 AREA = 0.95

$1 \text{ AREA} = \frac{0.95}{2} = 0.475$

0.475 ⟶ z = ?

1.9 —— 0.475 (column 0.06) ⇒ 1.96

EX 26 THE HEIGHT OF ADULT MALES IN AUSTRALIA IS APPROXIMATELY NORMALLY DISTRIBUTED WITH MEAN μ = 170cm AND STANDARD DEVIATION σ = 7cm.

(a) WHAT IS THE PROBABILITY THAT A MALE RANDOMLY SELECTED HAS HEIGHT (?) BETWEEN 160cm & 175 cm

(ii) GREATER THAN 188CM

(b) WHAT IS THE GREATEST HEIGHT EXCEEDED BY 22% OF POPULATION.

(a) $P(\underset{x_1}{160} < z < \underset{x_2}{175})$ = THE AREA BETWEEN

$z_1 = \frac{x_1 - \mu}{\delta}$ AND $z_2 = \frac{x_2 - \mu}{\delta}$

$z_1 = \frac{160 - 170}{7}$ AND $z_2 = \frac{175 - 170}{7}$

$z_1 = -1.43$ AND $z_2 = 0.71$

TOTAL = AREA (1.43) + AREA (0.71)

	0.03
1.4	0.4236

	0.01
0.7	0.2611

= 0.4236 + 0.2611

= 0.6847

(ii) $P(X > 188)$

= TOTAL AREA − THE AREA $z = \frac{x - \mu}{\delta}$

$= 0.5 - \left(z = \frac{188 - 170}{7} = 2.57\right)$

	0.07
2.5	0.4949

= 0.5 − 0.4949 = 0.0051

(b) GREATEST HEIGHT EXCEED BY 22% OF POPULATION

= THE WHOLE AREA − 0.22

= 0.5 − 0.22 = 0.28 → FIND (Z)

	0.07
0.7	0.2794 ≃ 0.28

∴ z = 0.77

MALES
ATELY
ARD
THAT
HAS
& 175

CATEGORY	OBSERVED FREQUENCY (O)	EXPECTED FREQUENCY (E) CUSTOMER X %SHARE	O-E	$\frac{(O-E)^2}{E}$
A	48	$200 \times \frac{30}{100} = 60$	O-E = 48-60 = -12	$\frac{(-12)^2}{60} = 2.4$
B	98	$200 \times \frac{50}{100} = 100$	O-E = 98-100 = -2	$\frac{(-2)^2}{100} = 0.04$
C	54	$200 \times \frac{20}{100} = 40$	54-40 = 14	$\frac{(14)^2}{40} = 4.9$

$\Sigma O-E = -12 + (-2) + 14 = 0$

$\Sigma \frac{(O-E)^2}{E} = 7.34$

$X^2_{\alpha, V} = X^2_{0.05, (\text{NO. OF PRODUCTS} - 1)}$

$= X^2_{0.05, (3-1)} = X^2_{0.05, 2}$

K

V_2 $V_1 = 2$

4 — 6.94

REJECTIONS POINT

FIND FIRST NUMBER LESS THAN 7.34 IN X^2 TABLE

pb 32

A CER

THERE IS A

PER MONTH AN

TO EXAMINE

EXPENDITURES

SALE VOLUME

MONTH
1
2
3
4
5
6
7
8
9
10

SAMPLING DISTRIBUTION | CORRELATION & REGRESSION ANALYSIS

$(\bar{x})$ MEAN = $\dfrac{\text{DATA RANGE (MAXIMUM NUMBER + MINIMUM NUMBER)}}{2}$

STANDARD DEVIATION = $S = \sqrt{\dfrac{\sum x^2 - n(\bar{x})^2}{n-1}}$

SAMPLE ERROR = $x - \mu$

= MAXIMUM NUMBER AT DATA RANGE − MEAN

MEAN OF DISTRIBUTION = $\dfrac{\sum \text{MEAN}}{\text{TOTAL SAMPLE SIZE}}$

SAMPLE MEAN = $\dfrac{\sum \text{TOTAL}}{\text{NO. OF EVENTS}}$

pb 23

A P

TO BOT

ONE

IS SHO

(a) LIST A

(b) FOR EA

SAMPLE

(c) CALCUL

(d) GRAPH

...ION ANALYSIS

...NIMUM NUMBER)

pb 23

A POPULATION OF 5 SALES PERSONS SELLING CAR TELEPHONES TO BOTH PRIVATE AND COMMERCIAL CUSTOMERS OVER A PERIOD OF ONE MONTH. THE NUMBER OF TELEPHONES SOLD FOR EACH PERSON IS SHOWN IN FOLLOWING TABLE

SALES PERSON	PHONES SOLD
ADAM (A)	14 ✓
BAKER (B)	20 ✓
COLLING (C)	12
DAVID (D)	8
EDWARDS (E)	16

(a) LIST ALL POSSIBLE SAMPLES OF SIZE 2

(b) FOR EACH SAMPLE, CALCULATE DATA RANGE, SAMPLE MEAN SAMPLE STANDARD DEVIATION

(c) CALCULATE MEAN, FREQUENCY, RELATIVE FREQUENCY

(d) GRAPH.

A B

SAMPLE COMBINATION
AB
AC
AD
AE
BC
BD
BE
CD
CE
DE

... CAR TELEPHONES
... OVER A PERIOD OF
... SOLD FOR EACH PERSON

A B C D E → AB, AC, AD, AE, BC, BD, BE
CD, CE, DE

...LE MEAN

...ENCY

LAST − MEAN

SAMPLE COMBINATION	DATA RANGE	MEAN = $\frac{MAX + MIN}{2}$ $\bar{X}$	STANDARD DEVIATION	SAMPLE ERROR $\bar{X} - \mu$
AB	14 − 20	$\bar{X} = \frac{14+20}{2} = 17$	$\sqrt{\frac{14^2 + 20^2 - 2 \times 17^2}{2-1}} = 4.24$	20 − 17 = 3
AC	14 − 12	$\bar{X} = \frac{14+12}{2} = 13$	$\sqrt{\frac{14^2 + 12^2 - 2 \times 13^2}{2-1}} = 1.414$	12 − 13 = −1
AD	14 − 8	11	4.243	−3
AE	14 − 16	15	1.414	+1
BC	20 − 12	16	5.657	+2
BD	20 − 8	14 ←	8.485	0
BE	20 − 16	18	2.828	4
CD	12 − 8	10	2.828	−4
CE	12 − 16	14 ←	2.828	0
DE	8 − 16	12	5.657	−2

B C(...
B+...

AE, BC, BD, BE

DE

LAST − MEAN

SAMPLE ERROR

$\bar{x} - \mu$

$17^2 = 4.24$ | $20 - 17 = 3$

$13^2 = 1.414$ | $12 - 13 = -1$

3 | −3

+1

+2

85 | 0

828 | 4

MEAN	FREQUENCY	RELATIVE FREQUENCY
10	1	$\frac{1}{\Sigma f} = \frac{1}{10} = 0.1$
11	1	$\frac{1}{10} = 0.1$
12	1	$\frac{1}{10} = 0.1$
13	1	$\frac{1}{10} = 0.1$
14	2	$\frac{2}{10} = 0.2$
15	1	$\frac{1}{10} = 0.1$
16	1	$\frac{1}{10} = 0.1$
17	1	$\frac{1}{10} = 0.1$
18	1	$\frac{1}{10} = 0.1$

$$\Sigma f = 10$$

ACCORDING TO FREQUENCY DISTRIBUTION, B+D, C+E CAN SELL MORE PRODUCTS.

Pb 29 IN ABOVE P
DISTRIBUTIO

(i) MEAN OF DISTRIBUTION =

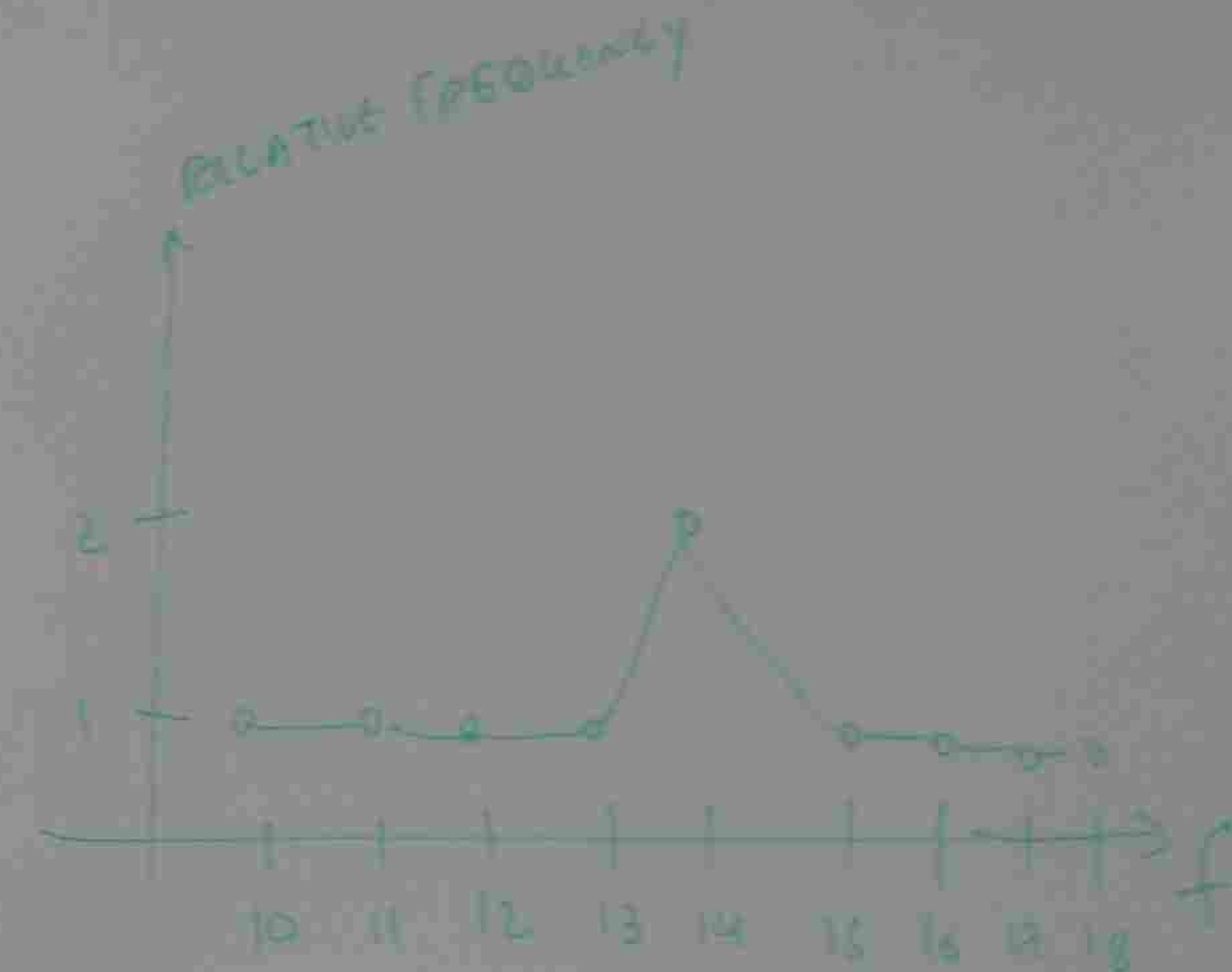
(ii) SAMPLE MEAN = Σ

FREQUENCY	RELATIVE FREQUENCY
1	$\frac{1}{\Sigma f} = \frac{1}{10} = 0.1$
1	$\frac{1}{10} = 0.1$
1	$\frac{1}{10} = 0.1$
1	$\frac{1}{10} = 0.1$
2	$\frac{2}{10} = 0.2$
1	$\frac{1}{10} = 0.1$
1	$\frac{1}{10} = 0.1$
1	$\frac{1}{10} = 0.1$
1	$\frac{1}{10} = 0.1$

$\Sigma f = 10$

... TO FREQUENCY DISTRIBUTION,

...E CAN SELL MORE PRODUCTS.



Pb 29 IN ABOVE PROBLEM, CALCULATE (i) MEAN OF DISTRIBUTION μ_x, (ii) SAMPLE MEAN

(i) MEAN OF DISTRIBUTION $= \frac{\Sigma \text{MEAN}}{\text{TOTAL SAMPLE}} = \frac{17+13+11+15+16+14+18+10+14+12}{10}$

$= 14$

(ii) SAMPLE MEAN $= \frac{\Sigma \text{TOTAL EVENTS (SALES)}}{\text{TOTAL PERSON}} = \frac{14+20+12+8+16}{5}$

$= 14$

STANDARD DISTRIBUTION OF SELECTED SAMPLE

STANDARD DISTRIBUTION OF SELECTED SAMPLE $\delta x = \dfrac{\text{PARENT STANDARD DISTRIBUTION}}{\sqrt{\text{Number of sample}}}$

pb 30 A CERTAIN BRAND OF ROPE IS KNOWN TO HAVE A MEAN BREAKING STRENGTH 25 kg WITH A STANDARD DEVIATION OF 0.5 kg. A RANDOM SAMPLE OF ROPE IS TESTED FOR BREAKING STRENGTH. WHAT IS THE POSSIBILITY THAT SAMPLE MEAN BREAKING STRENGTH OF 50 PIECES OF ROPE WILL BE (i) BETWEEN 24.9 AND 25.1 kg (ii) LESS THAN 24.8 kg.

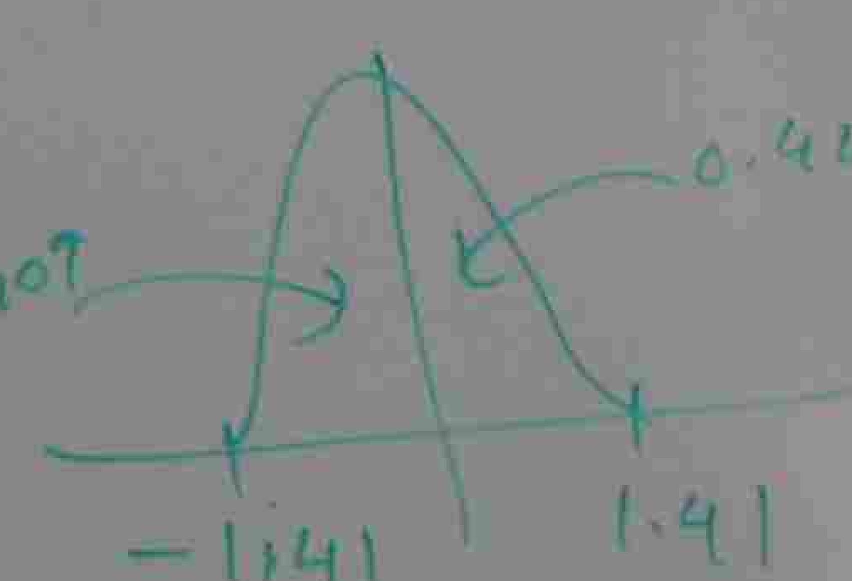
MEAN = 25 kg (μ) $\delta = 0.5$ kg ← PARENT DISTRIBUTION

STANDARD DEVIATION OF SELECTED SAMPLE $(\delta_x) = \dfrac{\text{PARENT STANDARD DEVIATION}}{\sqrt{\text{No. of sample}}} = \dfrac{0.5}{\sqrt{50}} = 0.071$

(a) BETWEEN 24.9 AND 25.1 kg

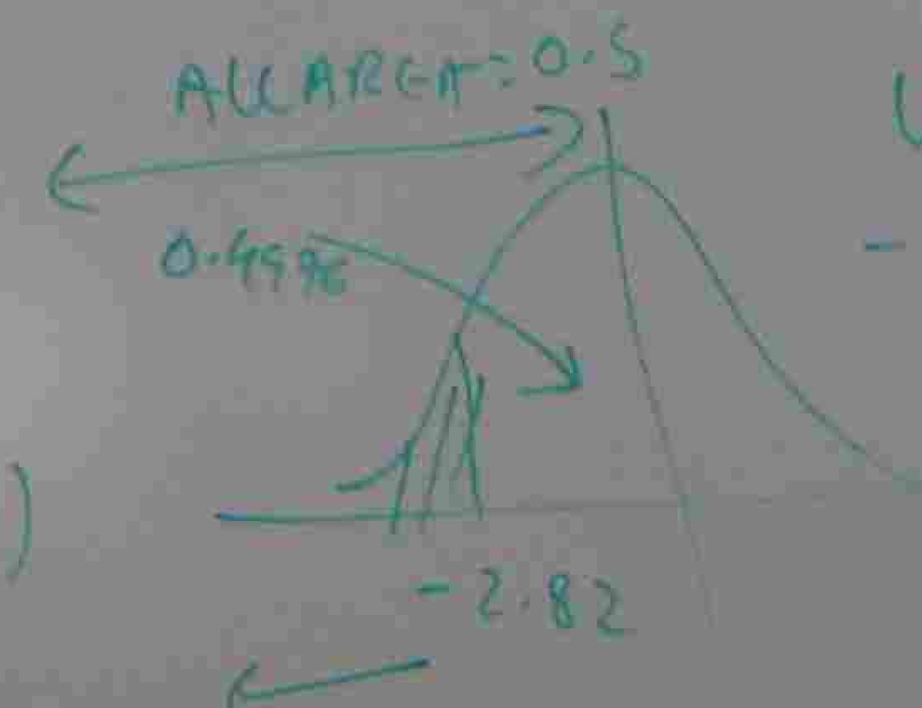
$$P(24.9 < x < 25.1) \longrightarrow P\left(\frac{x_1-\mu}{\delta x} < x < \frac{x_2-\mu}{\delta x}\right) = P\left(\frac{24.9-25}{0.071} < x < \frac{25.1-25}{0.071}\right)$$

$$P(-1.41 < x < 1.41)$$



TOTAL AREA = 0.4207 + 0.4207 = 0.8414

$$(b)\ P\left(\frac{x-\mu}{\delta x}\right) = P\left(\frac{24.8-25}{0.071}\right) = P(-2.82)$$



LESS THAN -2.82 = ALL − 2.82 AREA

= 0.5 − 0.4976

= 0.0024

MEAN OF

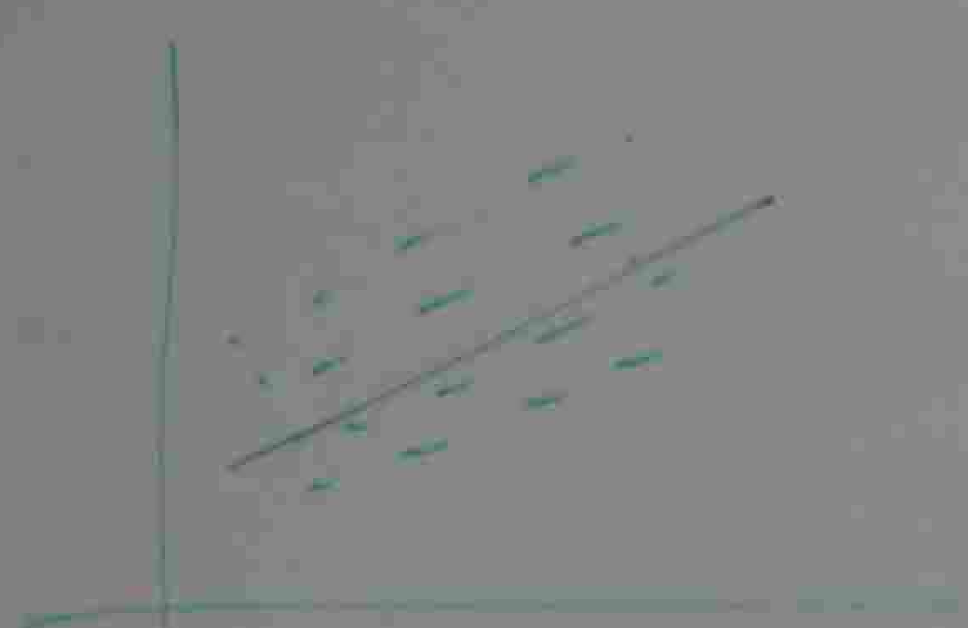
10+14

+12

8+16

CORRELATION AND REGRESSION ANALYSIS

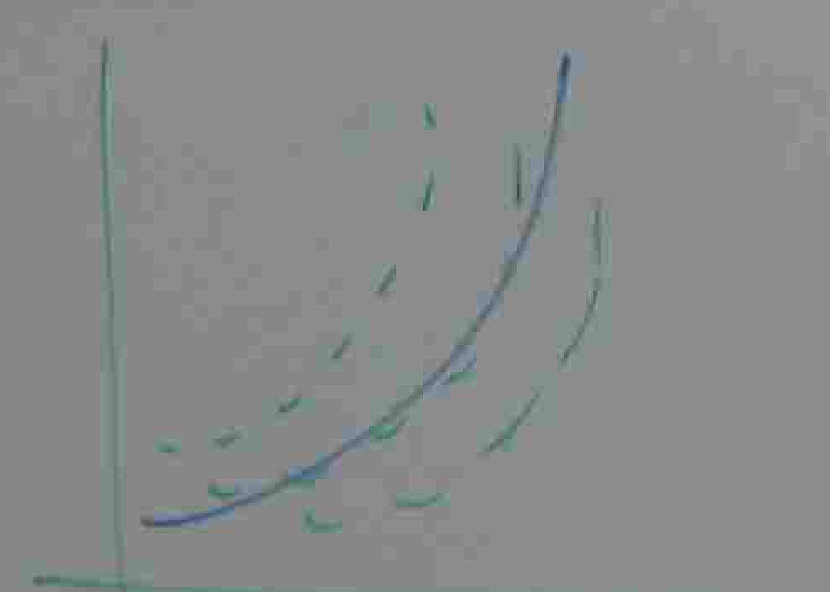
POSSIBLE RELATIONSHIPS BETWEEN X AND Y SCATTER PLOT



DIRECT LINEAR

INVERSE LINEAR

INVERSE CURVILINEAR



DIRECT CURVILINEAR

NON RELATIONSHIP

CORRELATION ANALYSIS $\longrightarrow$ ACTIVITY & EFFECTIVENESS

INPUT

$$\bar{X} = \frac{X_1 + X_2 + \cdots}{n}$$

AVERAGE

STANDARD DEVIATION $S_X = \sqrt{\frac{X_1^2 + X_2^2 + \cdots - n(\bar{X})^2}{n-1}}$

EFFECT

OUTPUT

$$\bar{Y}_{\text{AVERAGE}} = \frac{Y_1 + Y_2 + Y_3 + \cdots}{n}$$

$$S_y = \sqrt{\frac{y_1^2 + y_2^2 + \cdots - n(\bar{y})^2}{n-1}}$$

ΣX

S

S

S

CORRE

Σxy

$SS_{xy} = \Sigma xy - n\bar{x}\bar{y}$

$SS_x = (n-1)s_x^2$

$SS_y = (n-1)s_y^2$

CORRELATION $r = \dfrac{SS_{xy}}{\sqrt{SS_x \times SS_y}}$

IF r IS POSITIVE

CLOSE TO +1

THEN INPUT IS STRONGLY RELATED TO OUT PUT

IF r IS NEGATIVE, IN PUT DOES NOT EFFECTED ON OUT PUT

pb(32) A CERTAIN LARGE COMPANY IS INTERESTED IN WHETHER OR NOT THERE IS A RELATIONSHIP BETWEEN THE AMOUNT SPENT ON ADVERTISING PER MONTH AND THE GROSS MONTHLY SALE VOLUME.
TO EXAMINE THIS PROBLEM, THE FOLLOWING SAMPLE OF ADVERTISING EXPENDITURES AND ASSOCIATED SALE VOLUMES.
[illegible] VOLUMES ARE RANDAMLY SELECTED FOR MONTHLY DATA

MONTH	ADVERTISING EXPENDITURE (X) \$ 10000	SALE (Y) (X \$ 10000)
1	1.2	101
2	0.8	92
3	1.0	110
4	1.3	120
5	0.7	90
6	0.8	82
7	1.0	93
8	0.6	75
9	0.9	91
10	1.1	108

ADVERTISING

MEAN $\bar{x} = \dfrac{1.2+0.8+1.0+1.3+0.7+0.8+1.0+0.6+0.9+1.1}{10} = 0.94$

$S_x = \sqrt{\dfrac{1.2^2+0.8^2+1.0^2+1.3^2+0.7^2+0.8^2+1.0^2+0.6^2+0.9^2+1.1^2-10\,(0.94)^2}{10-1}} = 0.2221$

SALE

MEAN $= \bar{y} = \dfrac{101+92+110+120+90+82+93+75+91+108}{10} = 95.9$

$S_y = \sqrt{\dfrac{101^2+92^2+110^2+120^2+90^2+82^2+93^2+75^2+91^2+108^2-10\times 95.9^2}{10-1}} = 13.37$

$\sum xy = 1.2\times 101+0.8\times 92+1.0\times 110+1.3\times 120+0.7\times 90+.8\times 82+1.0\times 93+0.6\times 75+0.9\times 91+1.1\times 108$
$= 924.8$

$SS_{xy} = \sum xy - n\,\bar{x}\,\bar{y} = 924.8 - 10\times 0.94\times 95.9 = 23.34$

$SS_x = (n-1)\,s_x^2 = (10-1)\times(0.2221)^2 = 0.444$

$SS_y = (n-1)\,s_y^2 = (10-1)\times(13.37)^2 = 1600.9$

$r = \dfrac{SS_{xy}}{\sqrt{SS_x \times SS_y}} = \dfrac{23.34}{\sqrt{0.444\times 1600.9}} = 0.875$

POSITIVE
CLOSE TO +1
THUS ADVERTISEMENT
AFFECTS THE SALE

$\alpha = 0.05$

v_2 \ v_1	1	2	3	4	5	6	7	8
1								
2								
3								
4								
5								
6								
7								

CATEGORY	OBSERVED FREQUENCY (O)	E
		CUST
A	48	200
B	98	200
C	54	20

$\chi^2_{\alpha, v} =$