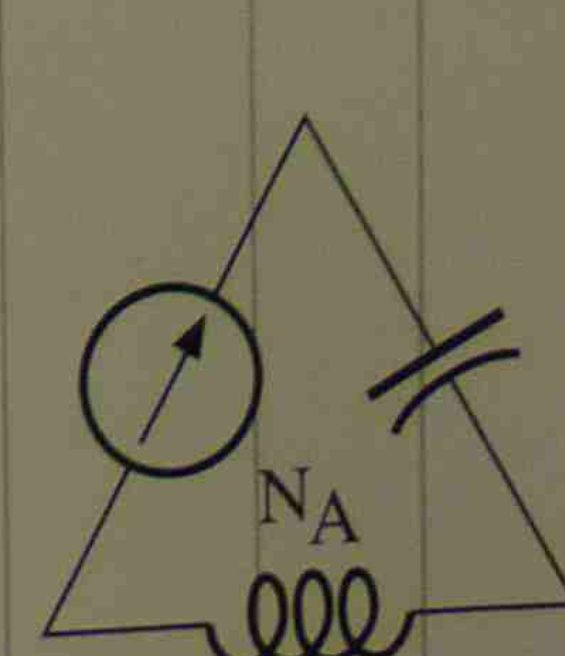
17

Methods of Analysis and Selected Topics (ac)



17.1 INTRODUCTION

For networks with two or more sources that are not in series or parallel, the methods described in the last two chapters cannot be applied. Rather, such methods as mesh analysis or nodal analysis must be employed. Since these methods were discussed in detail for dc circuits in Chapter 8, this chapter will consider the variations required to apply these methods to ac circuits.

The branch-current method will not be discussed again, since it falls within the framework of mesh analysis. In addition to the methods mentioned above, the bridge network and Δ-Y, Y-Δ conversions will also be discussed for ac circuits.

Before we examine these topics, however, we must consider the subject of independent and controlled sources.

17.2 INDEPENDENT VERSUS DEPENDENT (CONTROLLED) SOURCES

In the previous chapters, each source appearing in the analysis of dc or ac networks was an *independent source* such as E and I (or **E** and **I**) in Fig. 17.1.

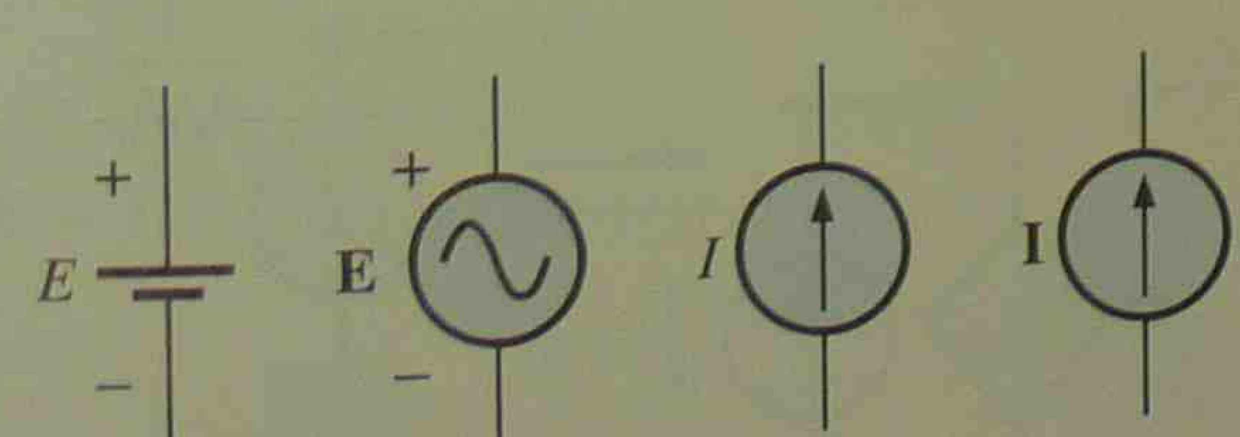


FIG. 17.1
Independent sources.

The term independent specifies that the magnitude of the source is independent of the network to which it is applied and that it displays its terminal characteristics even if completely isolated.

A dependent or controlled source is one whose magnitude is determined (or controlled) by a current or voltage of the system in which it appears.

There are currently two symbols used for controlled sources. One simply uses the independent symbol with an indication of the controlling element, as shown in Fig. 17.2. In Fig. 17.2(a), the magnitude and

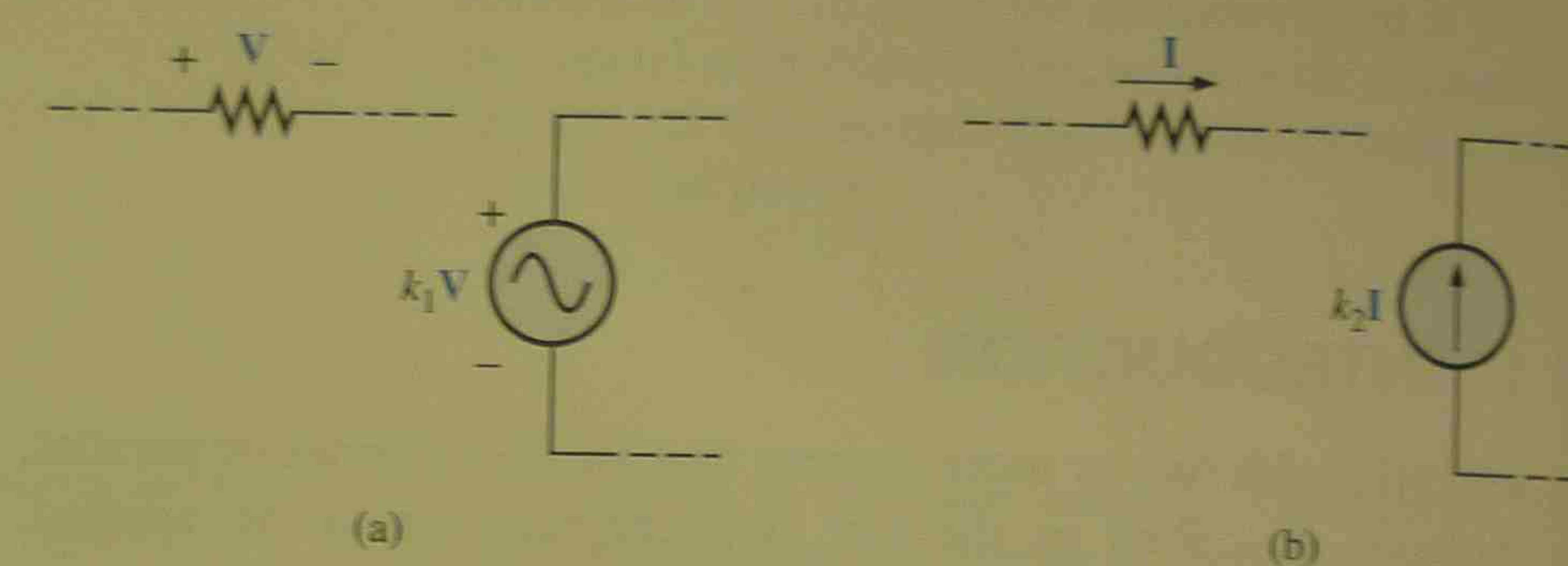


FIG. 17.2
Controlled or dependent sources.

phase of the voltage are controlled by a voltage **V** elsewhere in the system with the magnitude further controlled by the constant k_1. In Fig. 17.2(b), the magnitude and phase of the current source are controlled by a current **I** elsewhere in the system with the magnitude further controlled by the constant k_2. To distinguish between the dependent and independent sources, the notation of Fig. 17.3 was introduced. In recent years a number of respected publications on circuit analysis have accepted the notation of Fig. 17.3, although a number of excellent publications in the area of electronics continue to use the symbol of Fig. 17.2, especially in the circuit modeling for a variety of electronic devices such as the transistor and FET. This text will use both with an effort toward using the symbol most commonly applied to the area of investigation and to ensure that when the student encounters either symbol, he or she will be aware of its characteristics.

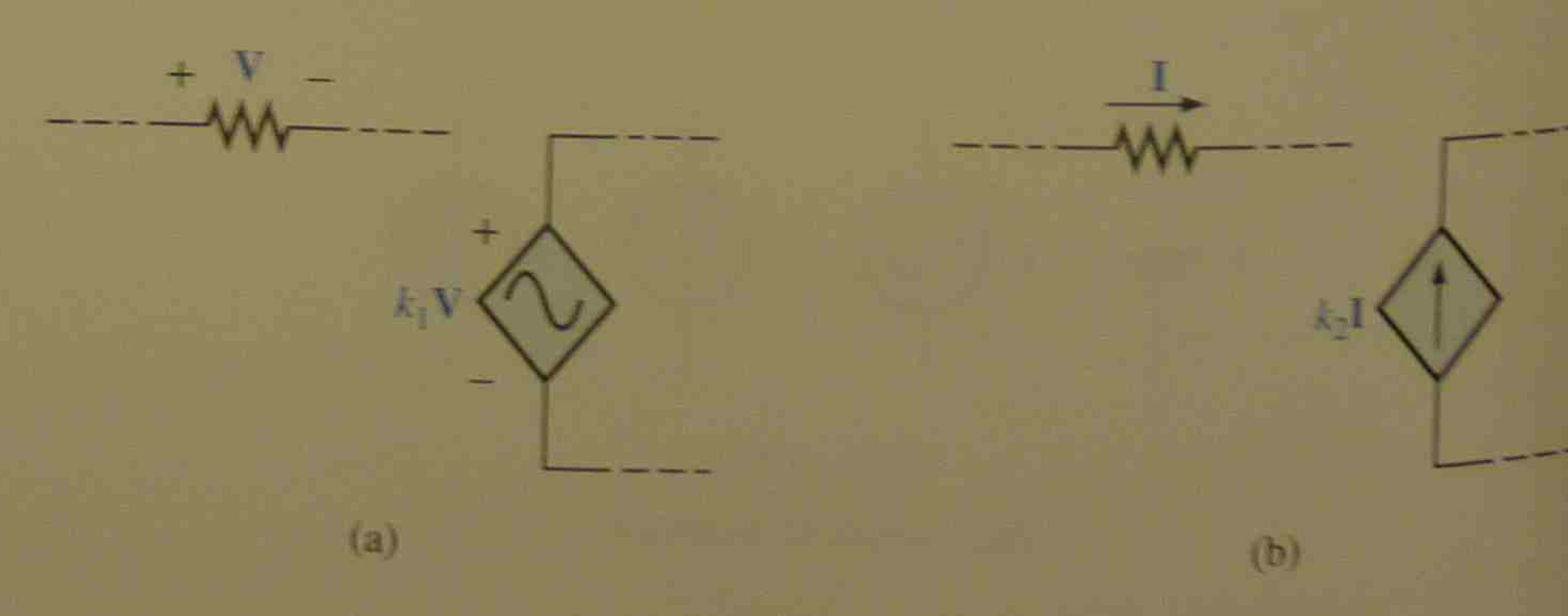


FIG. 17.3

Possible combinations for controlled sources are indicated in Fig. 17.4. Note that the magnitude of current sources or voltage sources can be controlled by a voltage and a current, respectively. Unlike with the independent source, isolation such that **V** or **I** = 0 in Fig. 17.4(a) will result in the short-circuit or open-circuit equivalent as indicated in Fig. 17.4(b). Note that the type of representation under these conditions is controlled by whether it is a current source or a voltage source, not by the controlling agent (**V** or **I**).

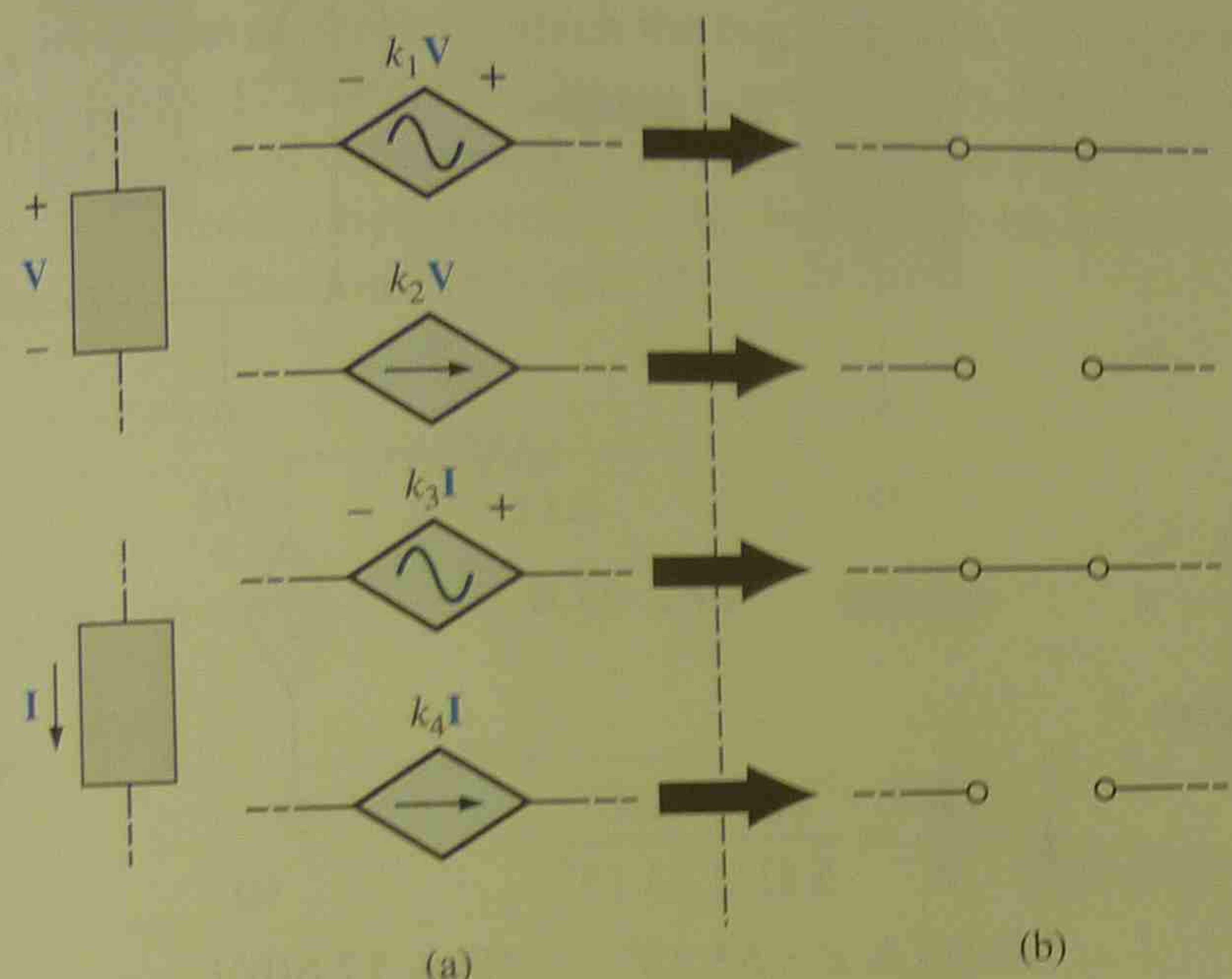


FIG. 17.4

Conditions of V = 0 V and I = 0 A for a controlled source.

17.3 SOURCE CONVERSIONS

When applying the methods to be discussed, it may be necessary to convert a current source to a voltage source, or a voltage source to a current source. This can be accomplished in much the same manner as for dc circuits, except now we shall be dealing with phasors and impedances instead of just real numbers and resistors.

In general, the format for converting from one to the other is as shown in Fig. 17.5.

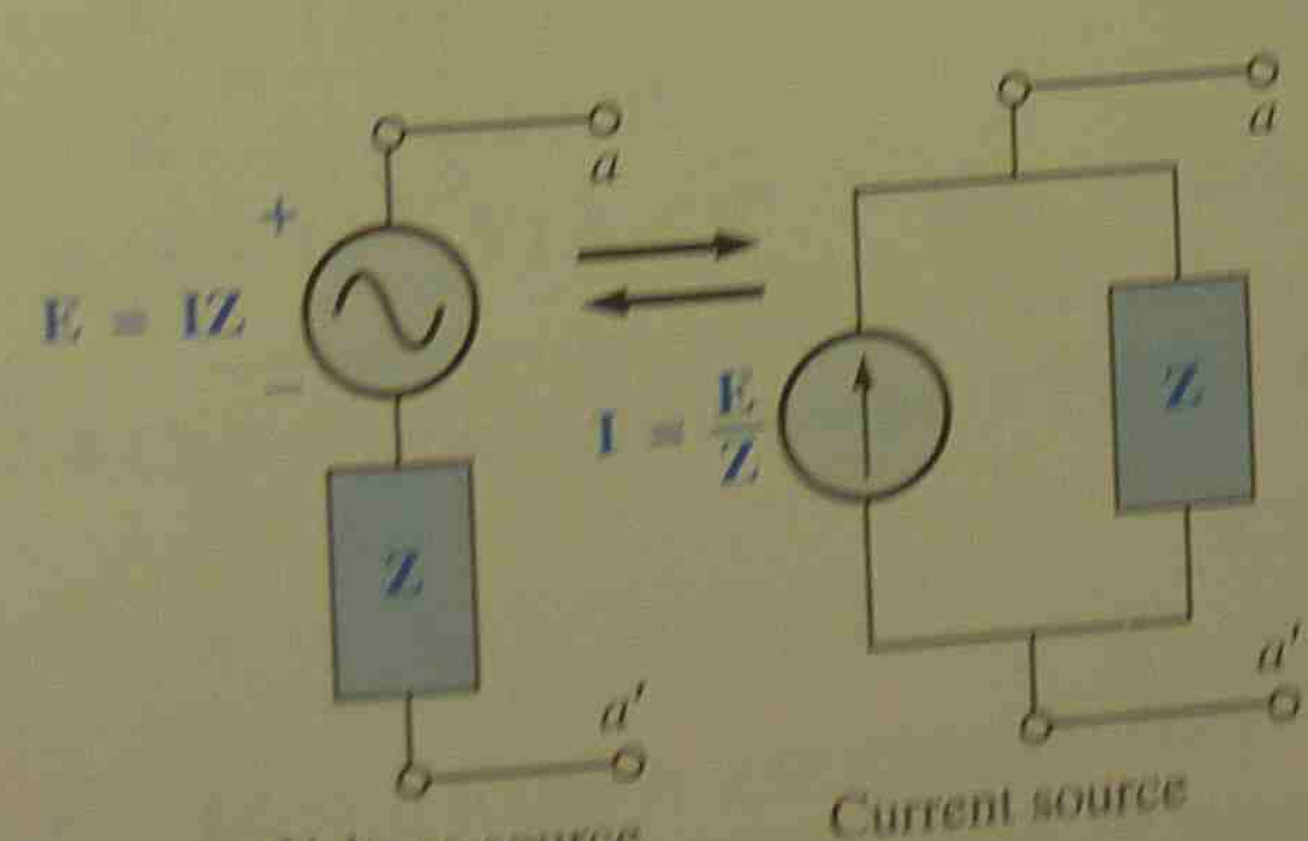


FIG. 17.5

EXAMPLE 17.1 Convert the voltage source of Fig. 17.6(a) to a current source.

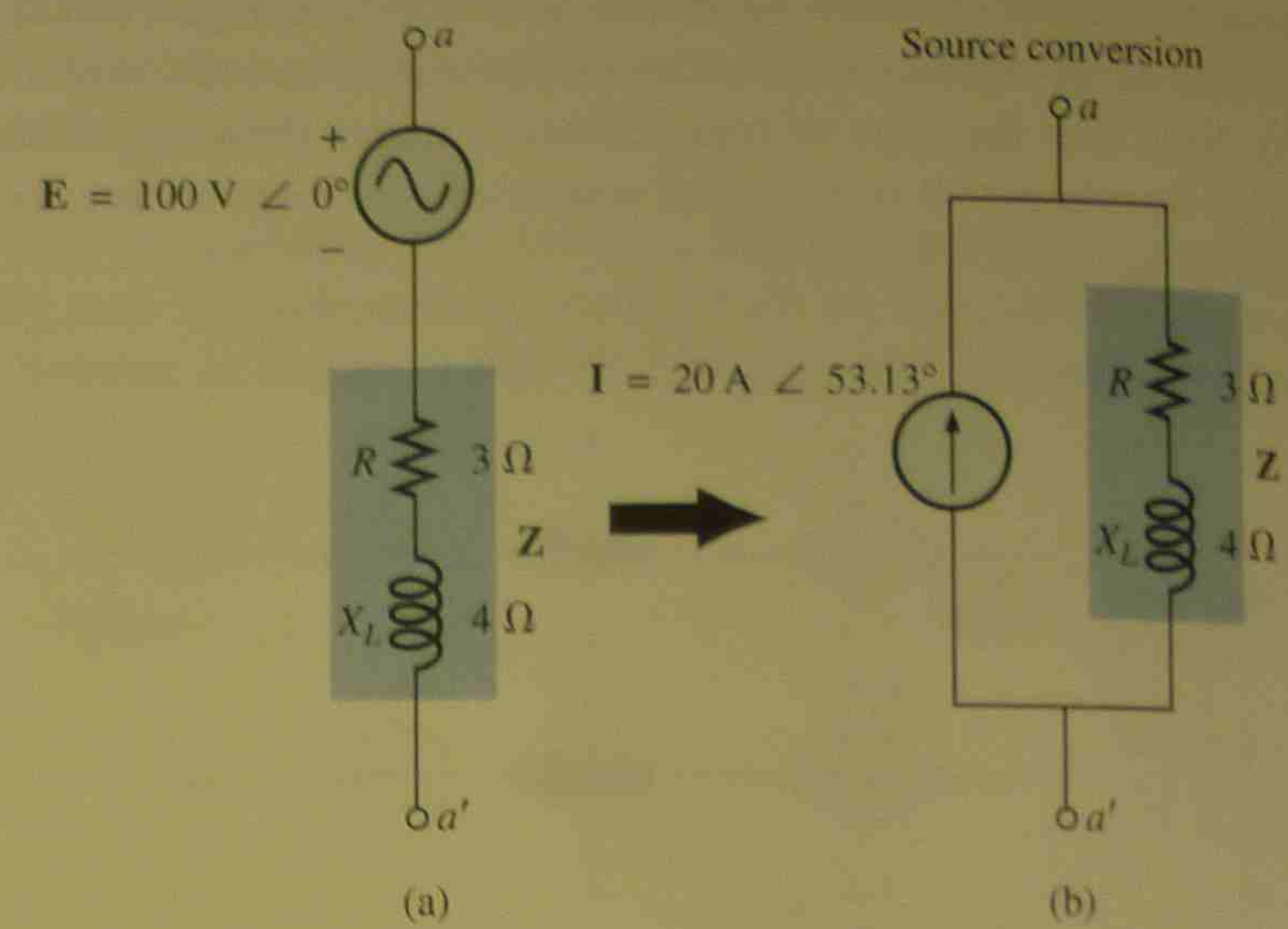


FIG. 17.6

Solution:

$$\mathbf{I} = \frac{\mathbf{E}}{\mathbf{Z}} = \frac{100\text{ V}\angle 0^\circ}{5\ \Omega\angle 53.13^\circ}$$
$$= \mathbf{20\ A\angle -53.13^\circ} \qquad \text{[Fig. 17.6(b)]}$$

EXAMPLE 17.2 Convert the current source of Fig. 17.7(a) to a voltage source.

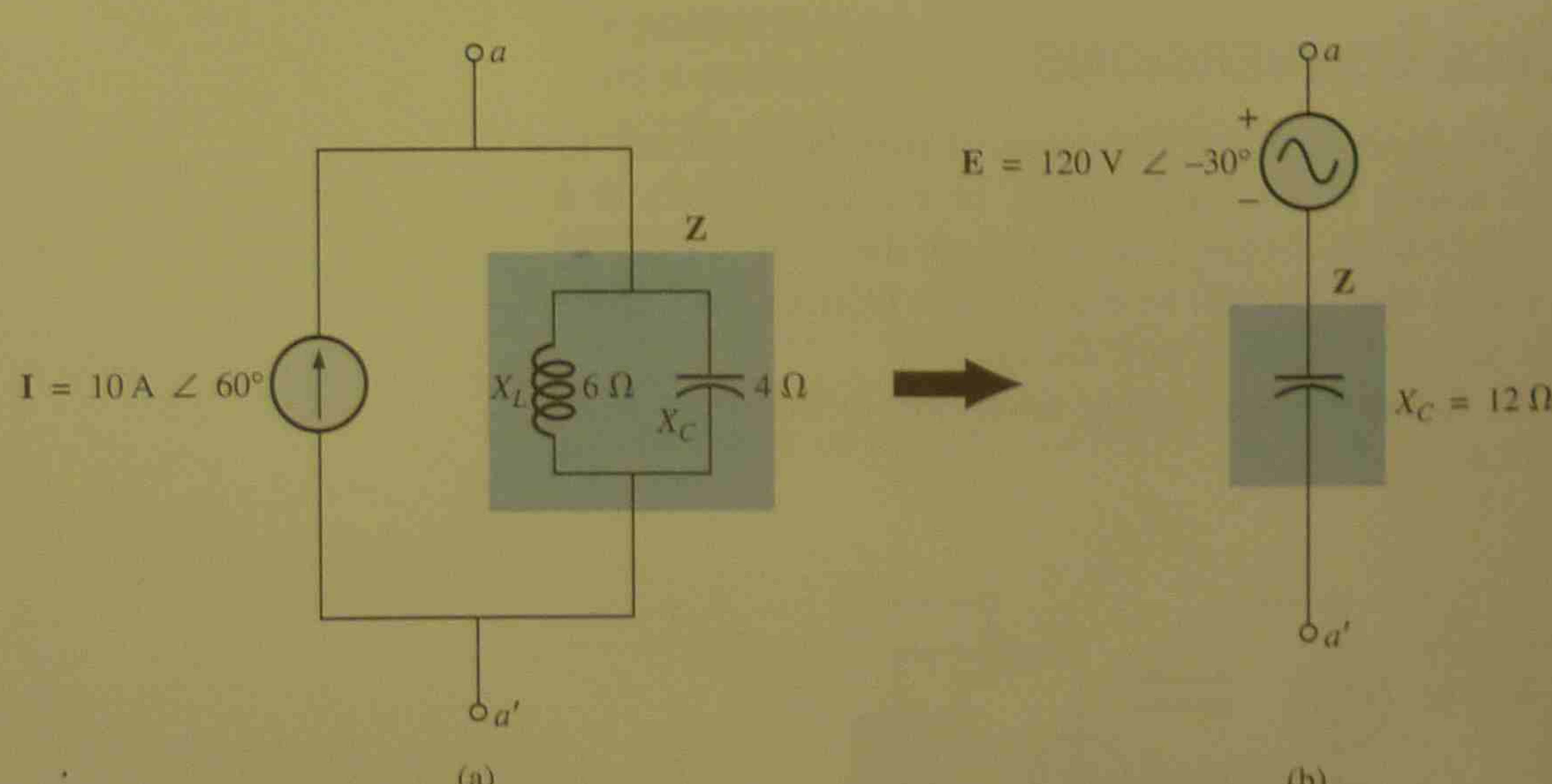


FIG. 17.7

Solution:

$$\mathbf{Z} = \frac{\mathbf{Z}_C\mathbf{Z}_L}{\mathbf{Z}_C + \mathbf{Z}_L} = \frac{X_C\angle -90^\circ)(X_L\angle 90^\circ)}{-jX_C + jX_L}$$

$$= \frac{(4\ \Omega\ \angle -90^\circ)(6\ \Omega\ \angle 90^\circ)}{-j4\ \Omega + j6\ \Omega} = \frac{24\ \Omega\ \angle 0^\circ}{2\ \angle 90^\circ}$$

$$= \mathbf{12\ \Omega\ \angle -90^\circ} \qquad \text{[Fig. 17.7(b)]}$$

$$\mathbf{E} = \mathbf{IZ} = (10\ \text{A}\ \angle 60^\circ)(12\ \Omega\ \angle -90^\circ)$$

$$= \mathbf{120\ V\ \angle -30^\circ} \qquad \text{[Fig. 17.7(b)]}$$

For dependent sources, the direct conversion of Fig. 17.5 can be applied if the controlling variable (**V** or **I** in Fig. 17.4) is not determined by a portion of the network to which the conversion is to be applied. For example, in Figs. 17.8 and 17.9, **V** and **I**, respectively, are controlled by an external portion of the network. Conversions of the other kind, where **V** and **I** are controlled by a portion of the network to be converted, will be considered in Sections 18.3 and 18.4.

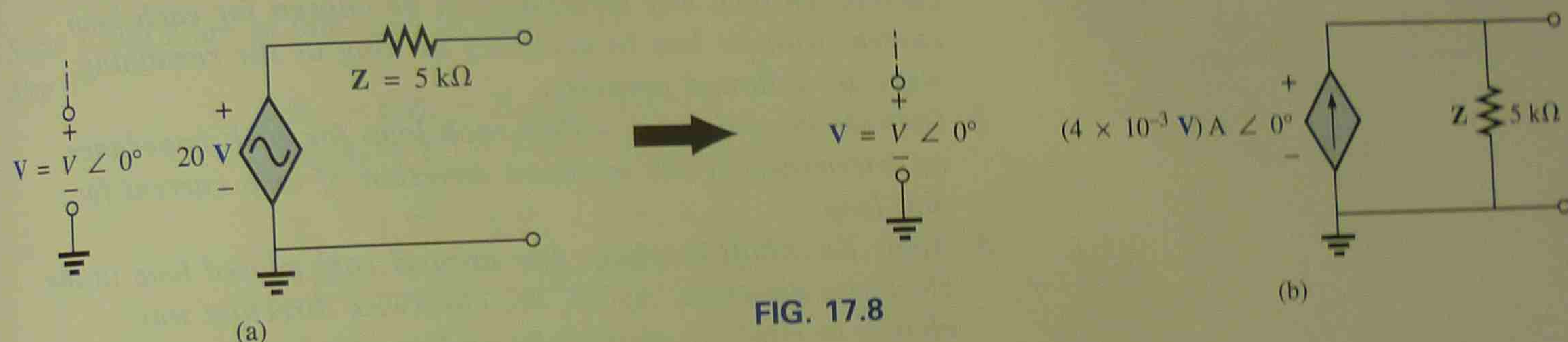


FIG. 17.8

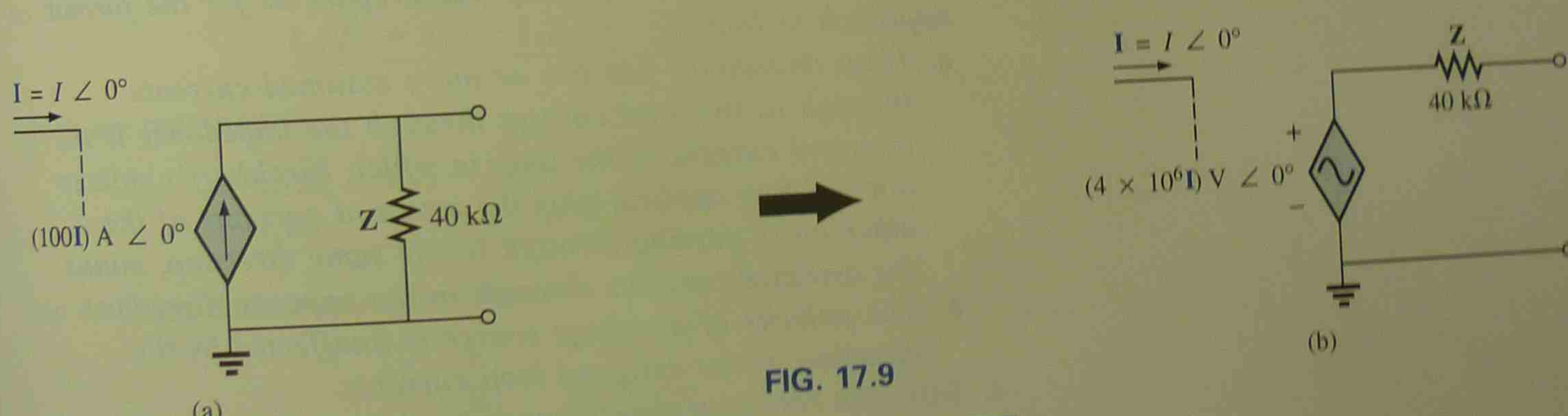


FIG. 17.9

EXAMPLE 17.3 Convert the voltage source of Fig. 17.8(a) to a current source.

Solution:

$$\mathbf{I} = \frac{\mathbf{E}}{\mathbf{Z}} = \frac{(20\mathbf{V})\ \text{V}\ \angle 0^\circ}{5\ \text{k}\Omega\ \angle 0^\circ}$$

$$= \mathbf{(4 \times 10^{-3}\ V)\ A\ \angle 0^\circ} \qquad \text{[Fig. 17.8(b)]}$$

EXAMPLE 17.4 Convert the current source of Fig. 17.9(a) to a voltage source.

Solution:

$$\mathbf{E} = \mathbf{IZ} = [(100\mathbf{I})\ \text{A}\ \angle 0^\circ][40\ \text{k}\Omega\ \angle 0^\circ]$$

$$= \mathbf{(4 \times 10^{6}I)\ V\ \angle 0^\circ} \qquad \text{[Fig. 17.9(b)]}$$

17.4 MESH ANALYSIS

General Approach

Before examining the application of the method to ac networks, the student should first review the appropriate sections on mesh analysis in Chapter 8, since the content of this section will be limited to the general conclusions of Chapter 8.

The general approach to mesh analysis includes the same sequence of steps appearing in Chapter 8. In fact, throughout this section the only change from the dc coverage will be to substitute impedance for resistance and admittance for conductance in the general procedure.

1. ***Assign a distinct current in the clockwise direction to each independent closed loop of the network. It is not absolutely necessary to choose the clockwise direction for each loop current. In fact, any direction can be chosen for each loop current with no loss in accuracy as long as the remaining steps are followed properly.***
2. ***Indicate the polarities within each loop for each impedance as determined by the assumed direction of loop current for that loop.***
3. ***Apply Kirchhoff's voltage law around each closed loop in the clockwise direction. Again, the clockwise direction was chosen to establish uniformity and prepare us for the format approach to follow.***
   a. ***If an impedance has two or more assumed currents through it, the total current through the impedance is the assumed current of the loop in which Kirchhoff's voltage law is being applied, plus the assumed currents of the other loops passing through in the same direction, minus the assumed currents through in the opposite direction.***
   b. ***The polarity of a voltage source is unaffected by the direction of the assigned loop currents.***
4. ***Solve the resulting simultaneous linear equations for the assumed loop currents.***

The technique is applied as above for all networks with independent sources or networks with dependent sources where the controlling variable is not a part of the network under investigation. If the controlling variable is part of the network being examined, additional care must be taken when applying the above steps.

EXAMPLE 17.5 Using the general approach to mesh analysis, find the current $\mathbf{I}_1$ in Fig. 17.10.

Solution: When applying these methods to ac circuits, it is good practice to represent the resistors and reactances (or combinations thereof) by subscripted impedances. When the total solution is found in terms of these subscripted impedances, the numerical values can be substituted to find the unknown quantities.

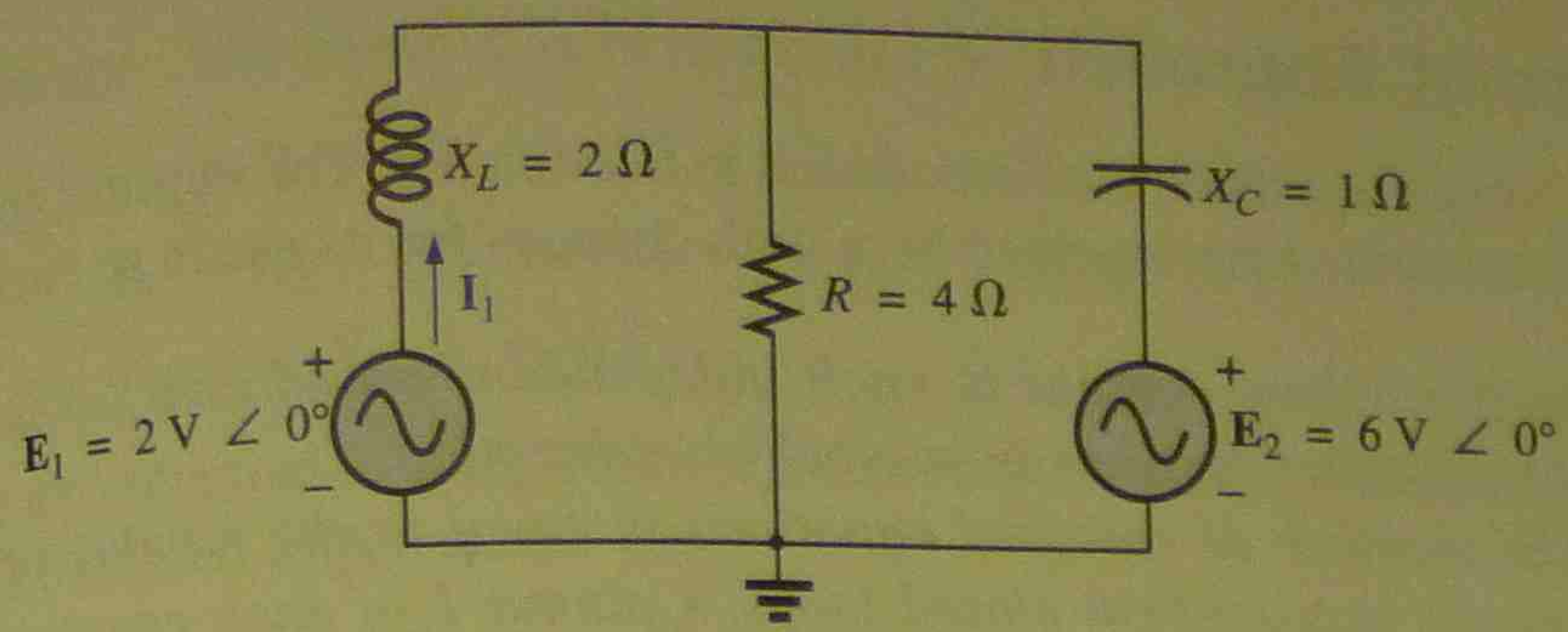


FIG. 17.10

The network is redrawn in Fig. 17.11 with subscripted impedances:

$$\mathbf{Z}_1 = +jX_L = +j2\ \Omega \qquad \mathbf{E}_1 = 2\text{ V}\angle 0^\circ$$
$$\mathbf{Z}_2 = R = 4\ \Omega \qquad \mathbf{E}_2 = 6\text{ V}\angle 0^\circ$$
$$\mathbf{Z}_3 = -jX_C = -j1\ \Omega$$

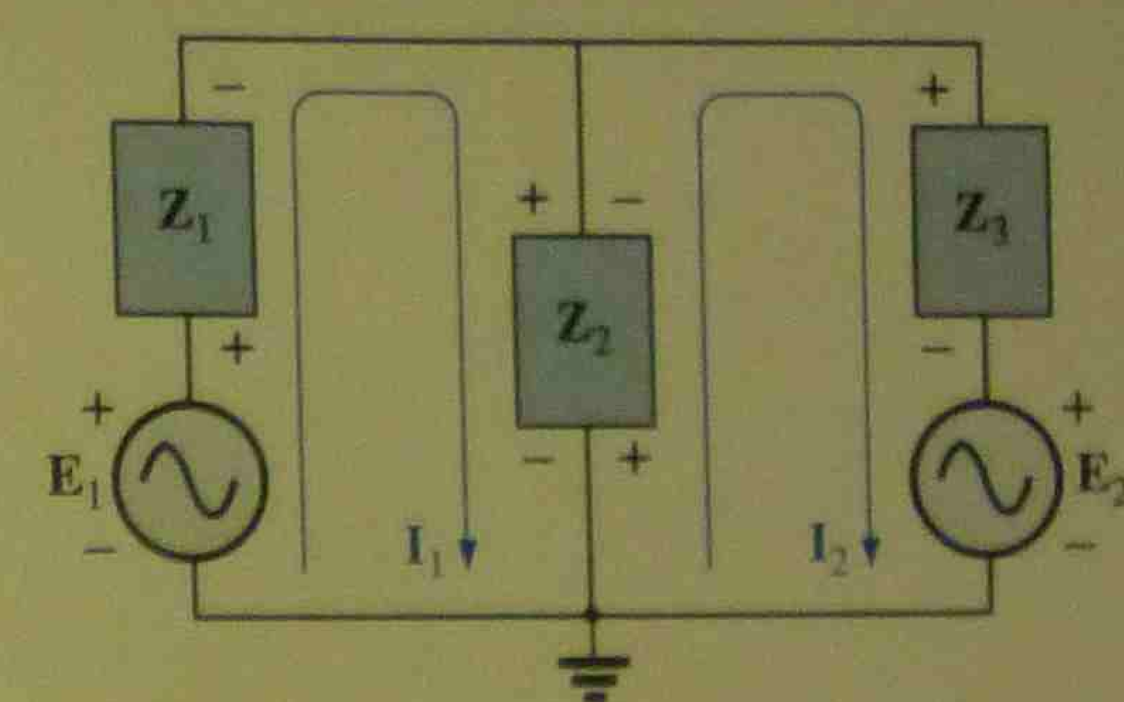


FIG. 17.11

Steps 1 and 2 are as indicated in Fig. 17.11.

Step 3:

$$+\mathbf{E}_1 - \mathbf{I}_1\mathbf{Z}_1 - \mathbf{Z}_2(\mathbf{I}_1 - \mathbf{I}_2) = 0$$
$$-\mathbf{Z}_2(\mathbf{I}_2 - \mathbf{I}_1) - \mathbf{I}_2\mathbf{Z}_3 - \mathbf{E}_2 = 0$$

or

$$\mathbf{E}_1 - \mathbf{I}_1\mathbf{Z}_1 - \mathbf{I}_1\mathbf{Z}_2 + \mathbf{I}_2\mathbf{Z}_2 = 0$$
$$-\mathbf{I}_2\mathbf{Z}_2 + \mathbf{I}_1\mathbf{Z}_2 - \mathbf{I}_2\mathbf{Z}_3 - \mathbf{E}_2 = 0$$

so that

$$\mathbf{I}_1(\mathbf{Z}_1 + \mathbf{Z}_2) - \mathbf{I}_2\mathbf{Z}_2 = \mathbf{E}_1$$
$$\mathbf{I}_2(\mathbf{Z}_2 + \mathbf{Z}_3) - \mathbf{I}_1\mathbf{Z}_2 = -\mathbf{E}_2$$

which are rewritten as

$$\begin{aligned}\mathbf{I}_1(\mathbf{Z}_1 + \mathbf{Z}_2) - \mathbf{I}_2\mathbf{Z}_2 \qquad\qquad &= \mathbf{E}_1\\ -\mathbf{I}_1\mathbf{Z}_2 \qquad + \mathbf{I}_2(\mathbf{Z}_2 + \mathbf{Z}_3) &= -\mathbf{E}_2\end{aligned}$$

Step 4: Using determinants, we obtain

$$\mathbf{I}_1 = \frac{\begin{vmatrix}\mathbf{E}_1 & -\mathbf{Z}_2\\ -\mathbf{E}_2 & \mathbf{Z}_2 + \mathbf{Z}_3\end{vmatrix}}{\begin{vmatrix}\mathbf{Z}_1 + \mathbf{Z}_2 & -\mathbf{Z}_2\\ -\mathbf{Z}_2 & \mathbf{Z}_2 + \mathbf{Z}_3\end{vmatrix}}$$
$$= \frac{\mathbf{E}_1(\mathbf{Z}_2 + \mathbf{Z}_3) - \mathbf{E}_2(\mathbf{Z}_2)}{(\mathbf{Z}_1 + \mathbf{Z}_2)(\mathbf{Z}_2 + \mathbf{Z}_3) - (\mathbf{Z}_2)^2}$$
$$= \frac{(\mathbf{E}_1 - \mathbf{E}_2)\mathbf{Z}_2 + \mathbf{E}_1\mathbf{Z}_3}{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_1\mathbf{Z}_3 + \mathbf{Z}_2\mathbf{Z}_3}$$

Substituting numerical values yields

$$\mathbf{I}_1 = \frac{(2\text{ V} - 6\text{ V})(4\ \Omega) + (2\text{ V})(-j1\ \Omega)}{(+j2\ \Omega)(4\ \Omega) + (+j2\ \Omega)(-j2\ \Omega) + (4\ \Omega)(-j2\ \Omega)}$$
$$= \frac{-16 - j2}{j8 - j^2 2 - j4} = \frac{-16 - j2}{2 + j4} = \frac{16.12\text{ A}\angle -172.87^\circ}{4.47\angle 63.43^\circ}$$
$$= \mathbf{3.61\ A\angle -236.30^\circ} \quad \text{or} \quad \mathbf{3.61\ A\angle 123.70^\circ}$$

Format Approach

The format approach was introduced in Section 8.9. The steps for applying this method are repeated here with changes for its use in ac circuits:

1. ***Assign a loop current to each independent closed loop (as in the previous section) in a clockwise direction.***
2. ***The number of required equations is equal to the number of chosen independent closed loops. Column 1 of each equation is formed by simply summing the impedance values of those impedances through which the loop current of interest passes and multiplying the result by that loop current.***
3. ***We must now consider the mutual terms that are always subtracted from the terms in the first column. It is possible to have more than one mutual term if the loop current of interest has an element in common with more than one other loop current. Each mutual term is the product of the mutual impedance and the other loop current passing through the same element.***
4. ***The column to the right of the equality sign is the algebraic sum of the voltage sources through which the loop current of interest passes. Positive signs are assigned to those sources of voltage having a polarity such that the loop current passes from the negative to the positive terminal. A negative sign is assigned to those potentials for which the reverse is true.***
5. ***Solve resulting simultaneous equations for the desired loop currents.***

The technique is applied as above for all networks with independent sources or networks with dependent sources where the controlling variable is not a part of the network under investigation. If the controlling variable is part of the network being examined, additional care must be taken when applying the above steps.

EXAMPLE 17.6 Using the format approach to mesh analysis, find the current $\mathbf{I}_2$ in Fig. 17.12.

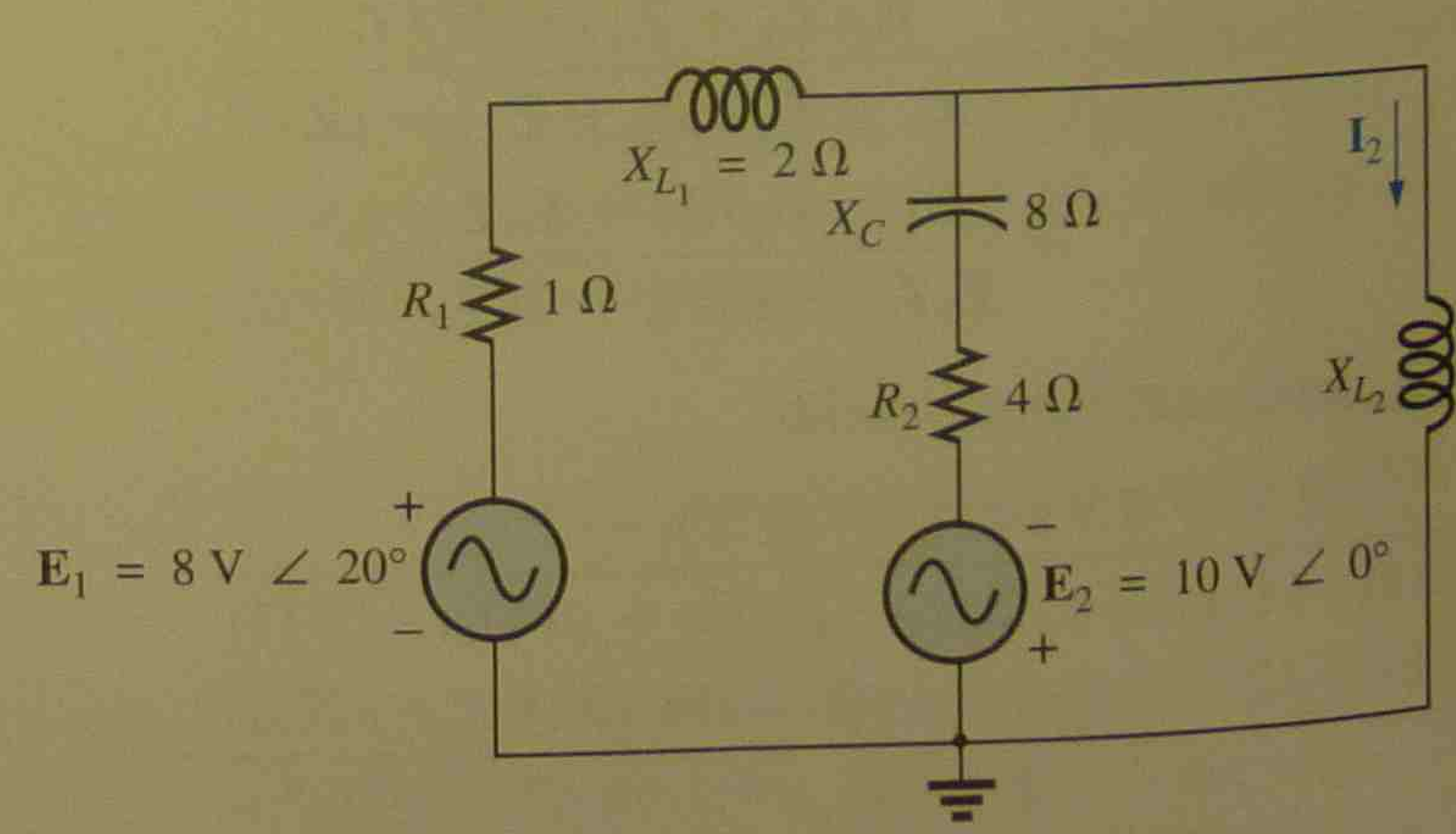


FIG. 17.12

Solution: The network is redrawn in Fig. 17.13:

$$\mathbf{Z}_1 = R_1 + jX_{L_1} = 1\,\Omega + j\,2\,\Omega \qquad \mathbf{E}_1 = 8\text{ V}\angle 20^\circ$$
$$\mathbf{Z}_2 = R_2 - jX_C = 4\,\Omega - j\,8\,\Omega \qquad \mathbf{E}_2 = 10\text{ V}\angle 0^\circ$$
$$\mathbf{Z}_3 = +jX_{L_2} = +j\,6\,\Omega$$

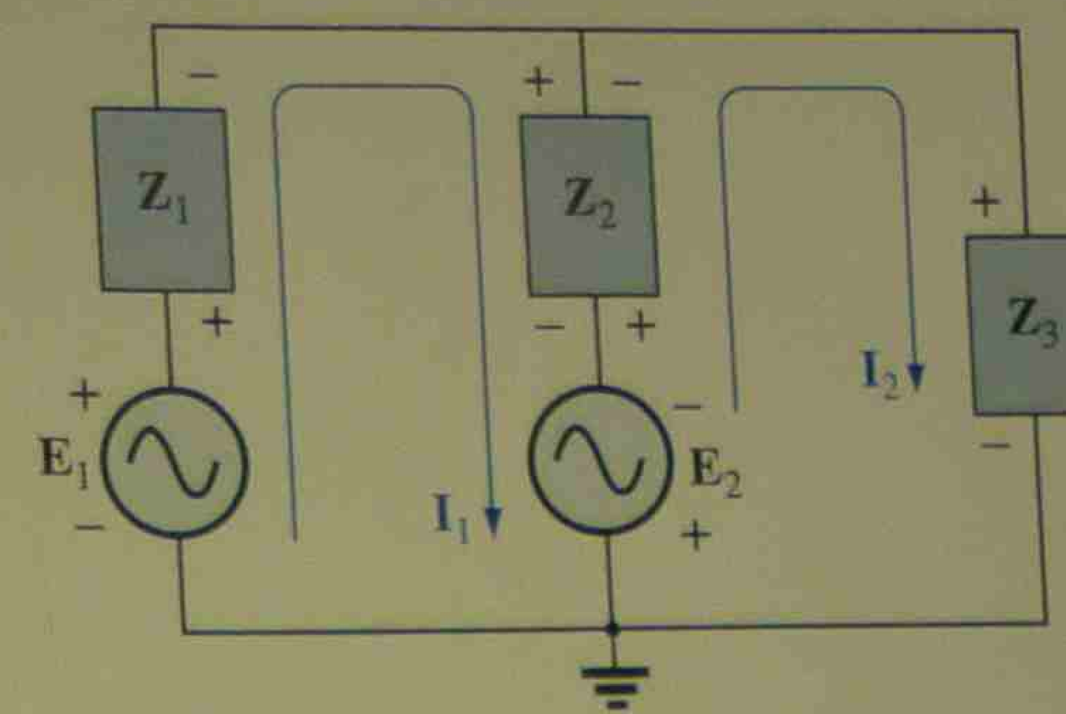


FIG. 17.13

Note the reduction in complexity of the problem with the substitution of the subscripted impedances.

Step 1 is as indicated in Fig. 17.13.

Steps 2 to 4:

$$\mathbf{I}_1(\mathbf{Z}_1 + \mathbf{Z}_2) - \mathbf{I}_2\mathbf{Z}_2 = \mathbf{E}_1 + \mathbf{E}_2$$
$$\mathbf{I}_2(\mathbf{Z}_2 + \mathbf{Z}_3) - \mathbf{I}_1\mathbf{Z}_2 = -\mathbf{E}_2$$

which are rewritten as

$$\begin{aligned} \mathbf{I}_1(\mathbf{Z}_1 + \mathbf{Z}_2) - \mathbf{I}_2\mathbf{Z}_2 \qquad\qquad &= \mathbf{E}_1 + \mathbf{E}_2 \\ -\mathbf{I}_1\mathbf{Z}_2 \qquad + \mathbf{I}_2(\mathbf{Z}_2 + \mathbf{Z}_3) &= -\mathbf{E}_2 \end{aligned}$$

Step 5: Using determinants, we have

$$\mathbf{I}_2 = \frac{\begin{vmatrix} \mathbf{Z}_1 + \mathbf{Z}_2 & \mathbf{E}_1 + \mathbf{E}_2 \\ -\mathbf{Z}_2 & -\mathbf{E}_2 \end{vmatrix}}{\begin{vmatrix} \mathbf{Z}_1 + \mathbf{Z}_2 & -\mathbf{Z}_2 \\ -\mathbf{Z}_2 & \mathbf{Z}_2 + \mathbf{Z}_3 \end{vmatrix}} = \frac{-(\mathbf{Z}_1 + \mathbf{Z}_2)\mathbf{E}_2 + \mathbf{Z}_2(\mathbf{E}_1 + \mathbf{E}_2)}{(\mathbf{Z}_1 + \mathbf{Z}_2)(\mathbf{Z}_2 + \mathbf{Z}_3) - \mathbf{Z}_2^2} = \frac{-\mathbf{Z}_1\mathbf{E}_2 + \mathbf{Z}_2\mathbf{E}_1}{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_1\mathbf{Z}_3 + \mathbf{Z}_2\mathbf{Z}_3}$$

Substituting numerical values yields

$$\mathbf{I}_2 = \frac{-(1\,\Omega + j\,2\,\Omega)(10\text{ V}\angle 0^\circ) + (4\,\Omega - j\,8\,\Omega)(8\text{ V}\angle 20^\circ)}{(1\,\Omega + j\,2\,\Omega)(4\,\Omega - j\,8\,\Omega) + (1\,\Omega + j\,2\,\Omega)(+j\,6\,\Omega) + (4\,\Omega - j\,8\,\Omega)(+j\,6\,\Omega)}$$

$$= \frac{-(10 + j\,20) + (4 - j\,8)(7.52 + j\,2.74)}{20 + (j\,6 - 12) + (j\,24 + 48)}$$

$$= \frac{-(10 + j\,20) + (52.0 - j\,49.20)}{56 + j\,30} = \frac{+42.0 - j\,69.20}{56 + j\,30} = \frac{80.95\text{ A}\angle -58.74^\circ}{63.53\angle 28.18^\circ}$$

$$= \mathbf{1.27\ A\angle -86.92^\circ}$$

EXAMPLE 17.7 Write the mesh equations for the network of Fig. 17.14. Do not solve.

Solution: The network is redrawn in Fig. 17.15. Again note the reduced complexity and increased clarity by use of subscripted impedances:

$$\mathbf{Z}_1 = R_1 + jX_{L_1} \qquad \mathbf{Z}_4 = R_3 - jX_{C_2}$$
$$\mathbf{Z}_2 = R_2 + jX_{L_2} \qquad \mathbf{Z}_5 = R_4$$
$$\mathbf{Z}_3 = jX_{C_1}$$

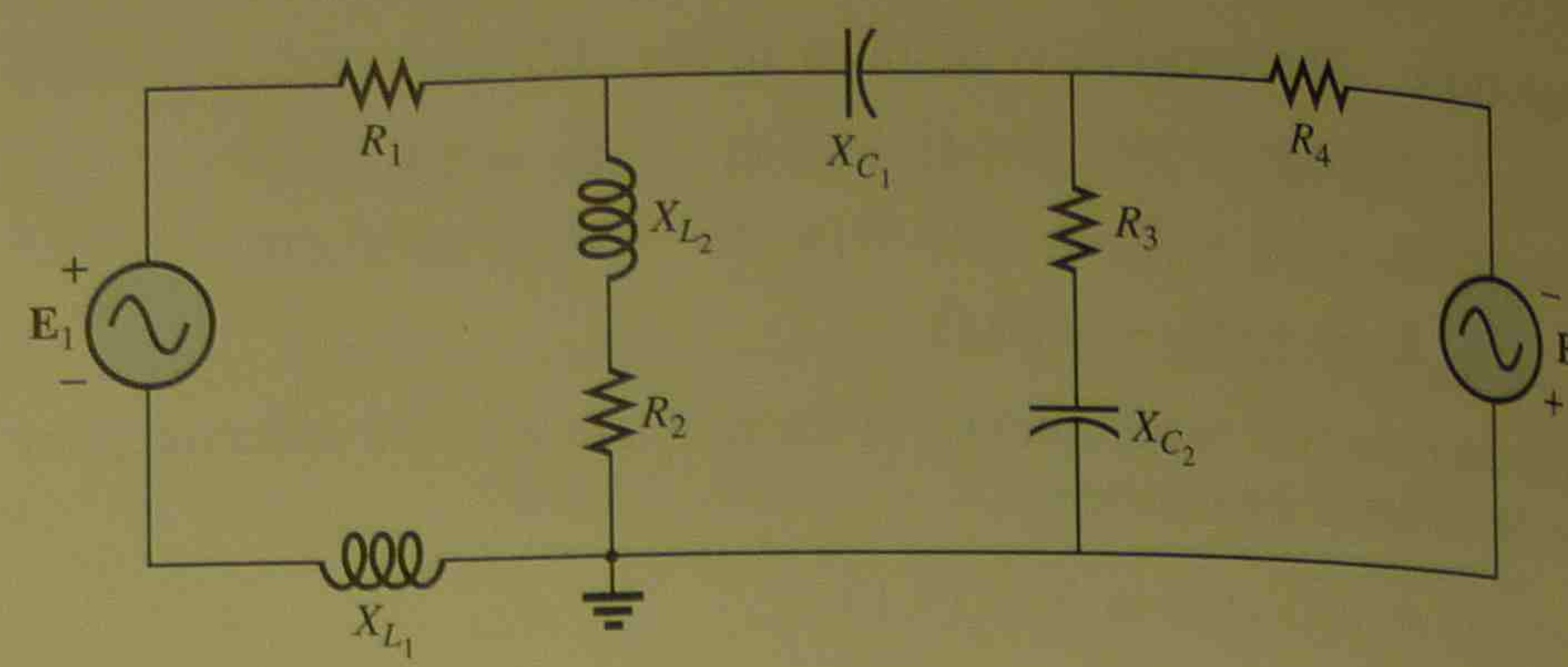


FIG. 17.14

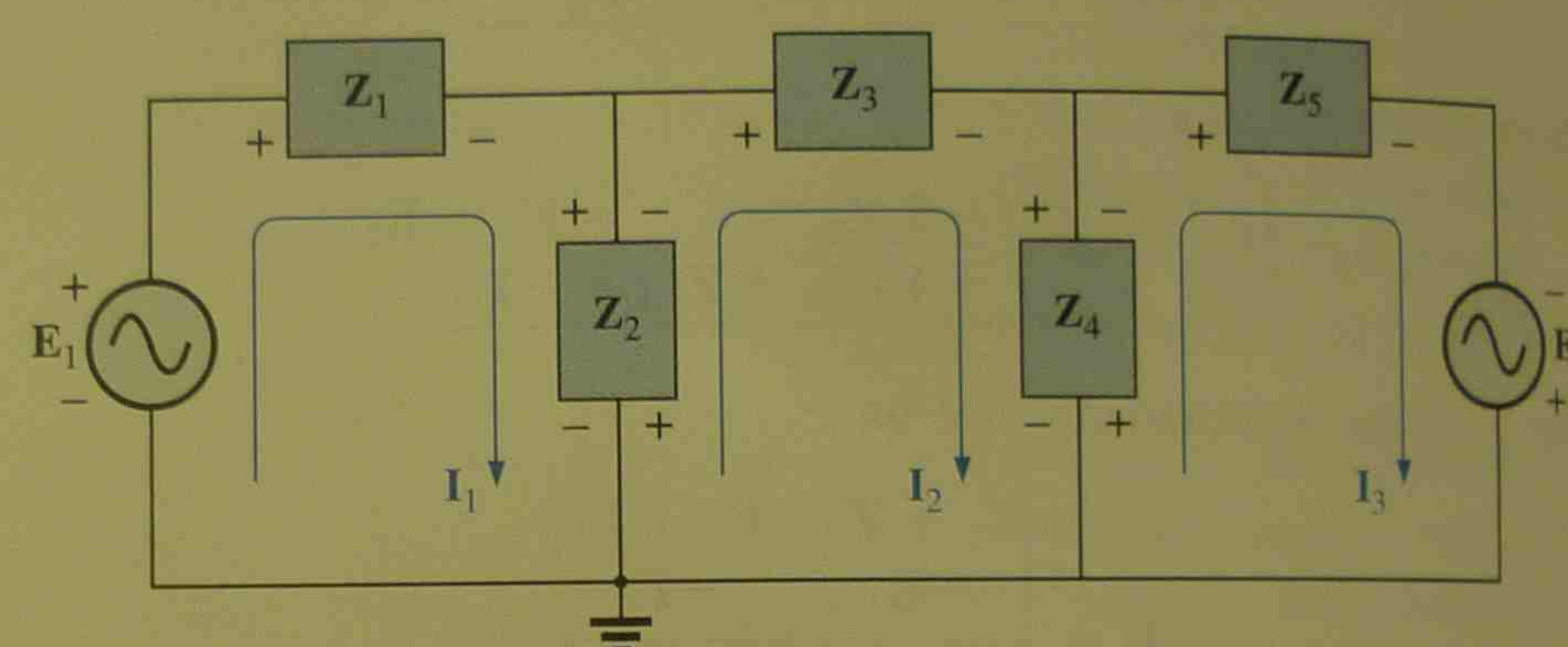


FIG. 17.15

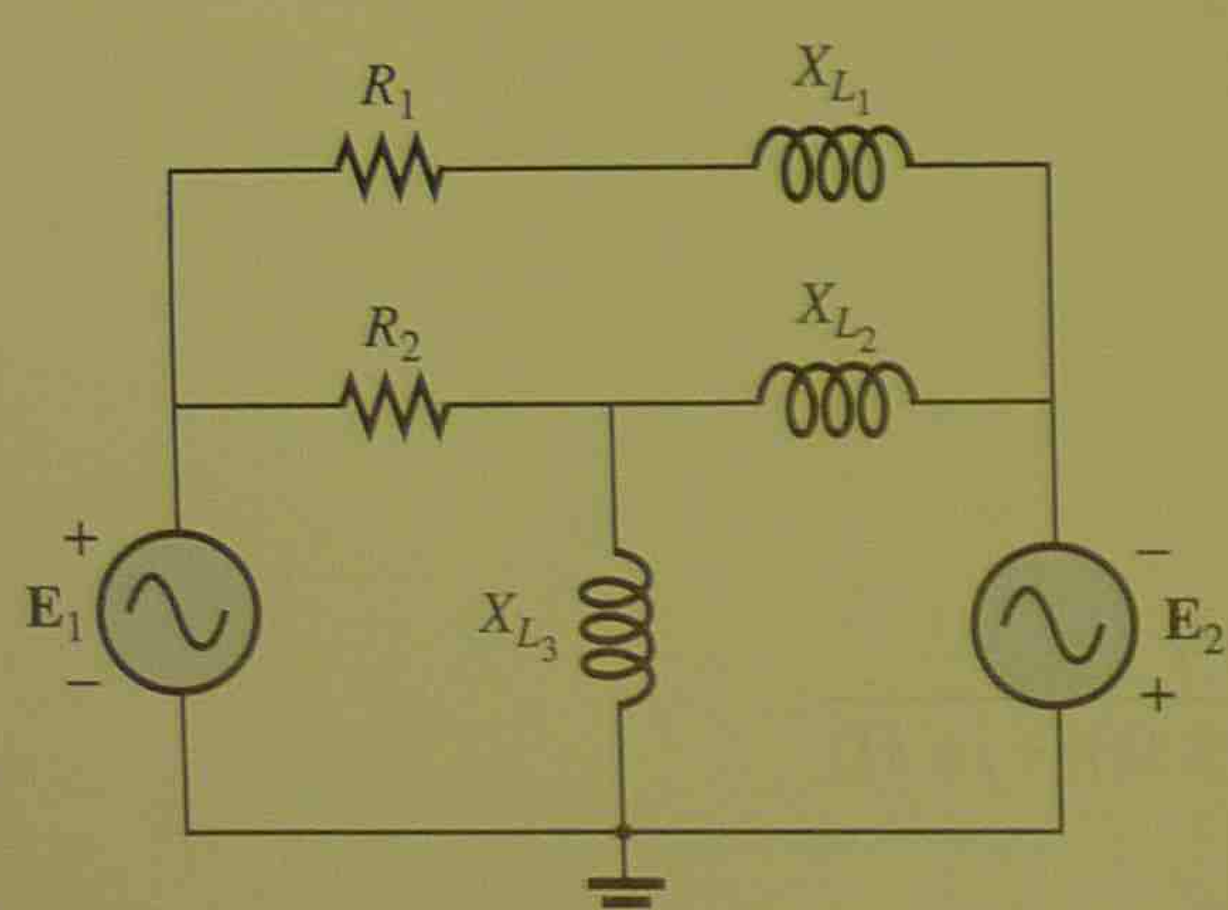


FIG. 17.16

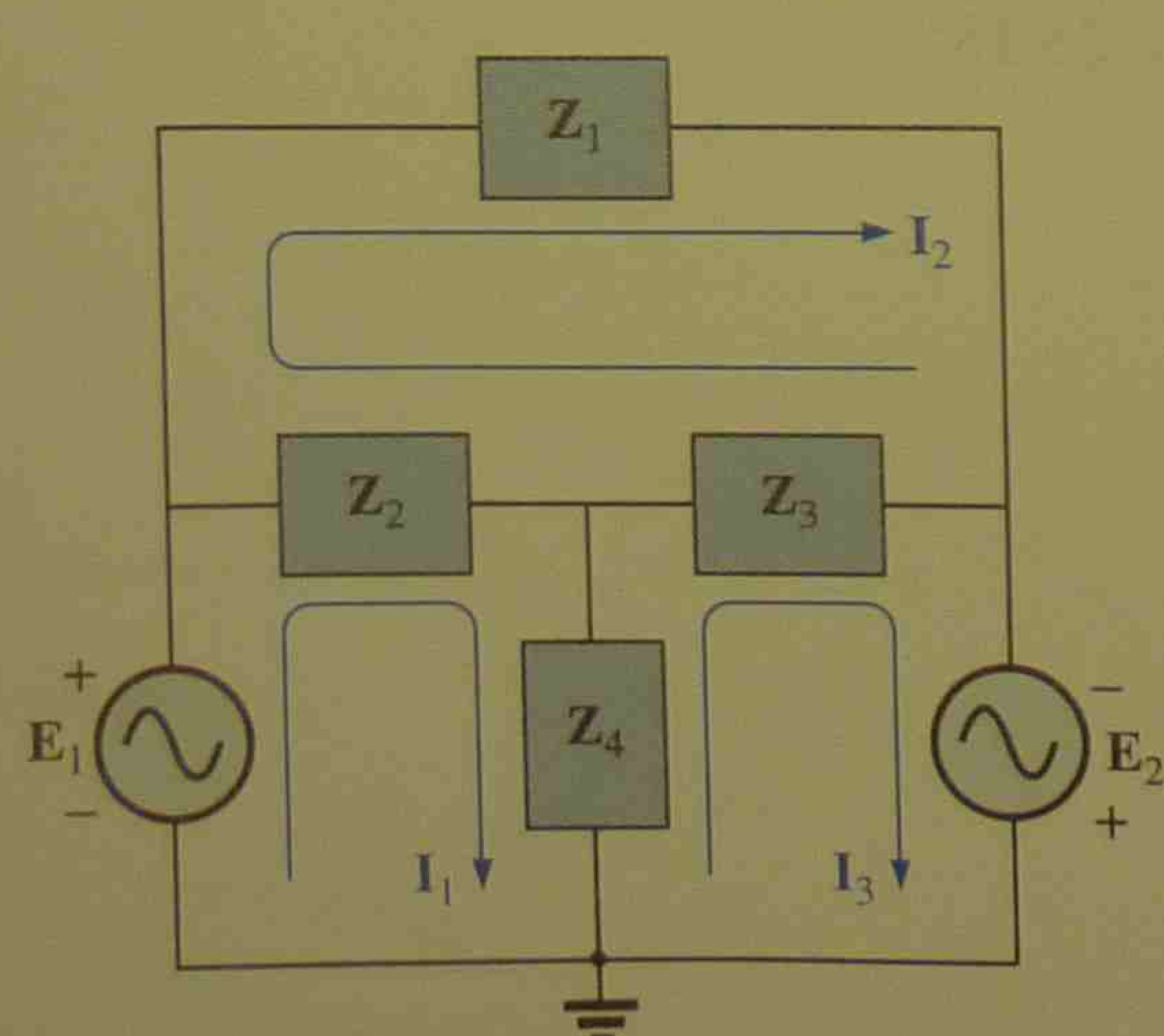


FIG. 17.17

and

$$\begin{aligned} \mathbf{I}_1(\mathbf{Z}_1 + \mathbf{Z}_2) - \mathbf{I}_2\mathbf{Z}_2 &= \mathbf{E}_1 \\ \mathbf{I}_2(\mathbf{Z}_2 + \mathbf{Z}_3 + \mathbf{Z}_4) - \mathbf{I}_1\mathbf{Z}_2 - \mathbf{I}_3\mathbf{Z}_4 &= 0 \\ \mathbf{I}_3(\mathbf{Z}_4 + \mathbf{Z}_5) - \mathbf{I}_2\mathbf{Z}_4 &= \mathbf{E}_2 \end{aligned}$$

or

$$\begin{array}{llll} \mathbf{I}_1(\mathbf{Z}_1 + \mathbf{Z}_2) - \mathbf{I}_2(\mathbf{Z}_2) & & + 0 & = \mathbf{E}_1 \\ \mathbf{I}_1(\mathbf{Z}_2) & - \mathbf{I}_2(\mathbf{Z}_2 + \mathbf{Z}_3 + \mathbf{Z}_4) & + \mathbf{I}_3(\mathbf{Z}_4) & = 0 \\ 0 & - \mathbf{I}_2(\mathbf{Z}_4) & + \mathbf{I}_3(\mathbf{Z}_4 + \mathbf{Z}_5) & = \mathbf{E}_2 \end{array}$$

EXAMPLE 17.8 Using the format approach, write the mesh equations for the network of Fig. 17.16.

Solution: The network is redrawn as shown in Fig. 17.17, where

$$\mathbf{Z}_1 = R_1 + jX_{L_1} \qquad \mathbf{Z}_3 = jX_{L_2}$$
$$\mathbf{Z}_2 = R_2 \qquad \mathbf{Z}_4 = jX_{L_3}$$

and

$$\begin{aligned} \mathbf{I}_1(\mathbf{Z}_2 + \mathbf{Z}_4) - \mathbf{I}_2\mathbf{Z}_2 - \mathbf{I}_3\mathbf{Z}_4 &= \mathbf{E}_1 \\ \mathbf{I}_2(\mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3) - \mathbf{I}_1\mathbf{Z}_2 - \mathbf{I}_3\mathbf{Z}_3 &= 0 \\ \mathbf{I}_3(\mathbf{Z}_3 + \mathbf{Z}_4) - \mathbf{I}_2\mathbf{Z}_3 - \mathbf{I}_1\mathbf{Z}_4 &= \mathbf{E}_2 \end{aligned}$$

or

$$\begin{array}{llll} \mathbf{I}_1(\mathbf{Z}_2 + \mathbf{Z}_4) - \mathbf{I}_2\mathbf{Z}_2 & & - \mathbf{I}_3\mathbf{Z}_4 & = \mathbf{E}_1 \\ -\mathbf{I}_1\mathbf{Z}_2 & + \mathbf{I}_2(\mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3) & - \mathbf{I}_3\mathbf{Z}_3 & = 0 \\ -\mathbf{I}_1\mathbf{Z}_4 & - \mathbf{I}_2\mathbf{Z}_3 & + \mathbf{I}_3(\mathbf{Z}_3 + \mathbf{Z}_4) & = \mathbf{E}_2 \end{array}$$

Note the symmetry *about* the diagonal axis. That is, note the location of $-\mathbf{Z}_2$, $-\mathbf{Z}_4$, and $-\mathbf{Z}_3$ off the diagonal.

17.5 NODAL ANALYSIS

General Approach

Before examining the application of the method to ac networks, the student should first review the appropriate sections on nodal analysis in Chapter 8, since the content of this section will be limited to the general conclusions of Chapter 8.

The fundamental steps are the following:

1. ***Determine the number of nodes within the network.***
2. ***Pick a reference node and label each remaining node with a subscripted value of voltage: $\mathbf{V}_1$, $\mathbf{V}_2$, and so on.***
3. ***Assume a direction of current for each branch.***
4. ***Apply Kirchhoff's current law at each node except the reference.***
5. ***Solve the resulting equations for the nodal voltages.***

A few examples will refresh your memory about the content of Chapter 8 and the general approach to a nodal analysis solution.

EXAMPLE 17.9 Determine the nodal voltages for the network of Fig. 17.18.

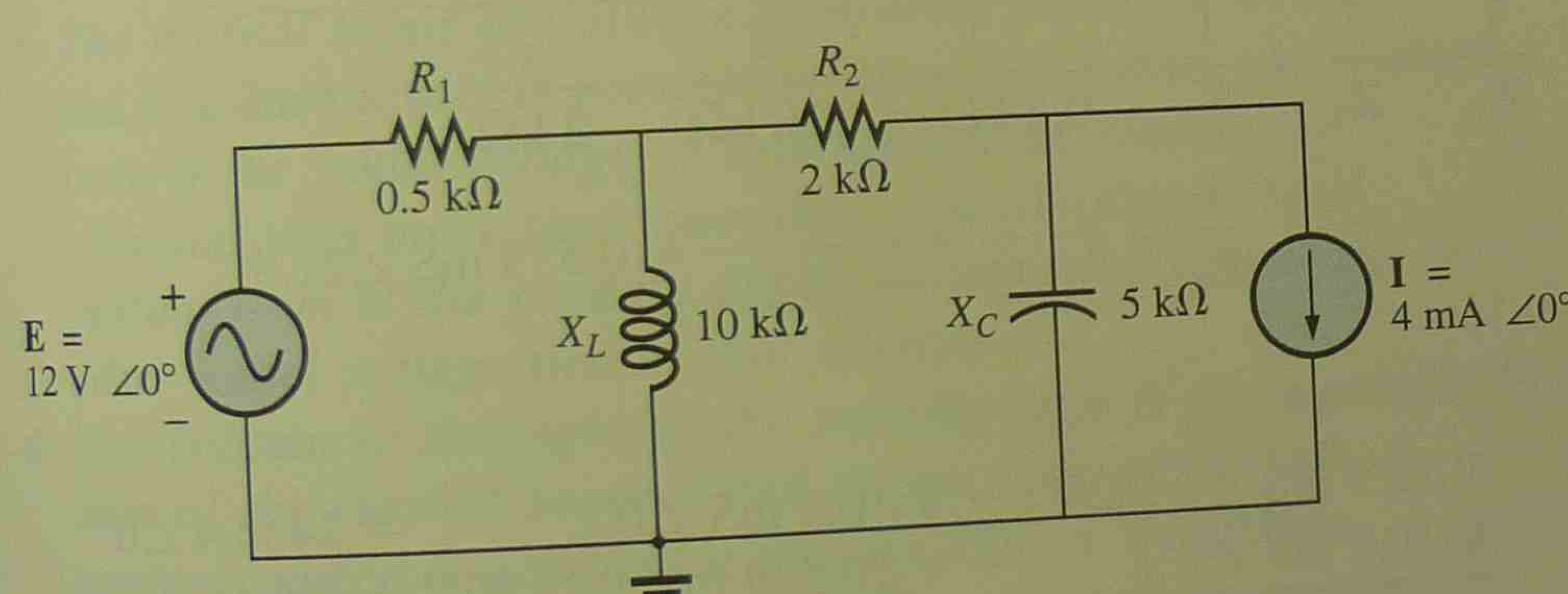


FIG. 17.18

Solution: Choosing nodes, defining the impedances, and choosing (arbitrary) the current directions will result in the network of Fig. 17.19.

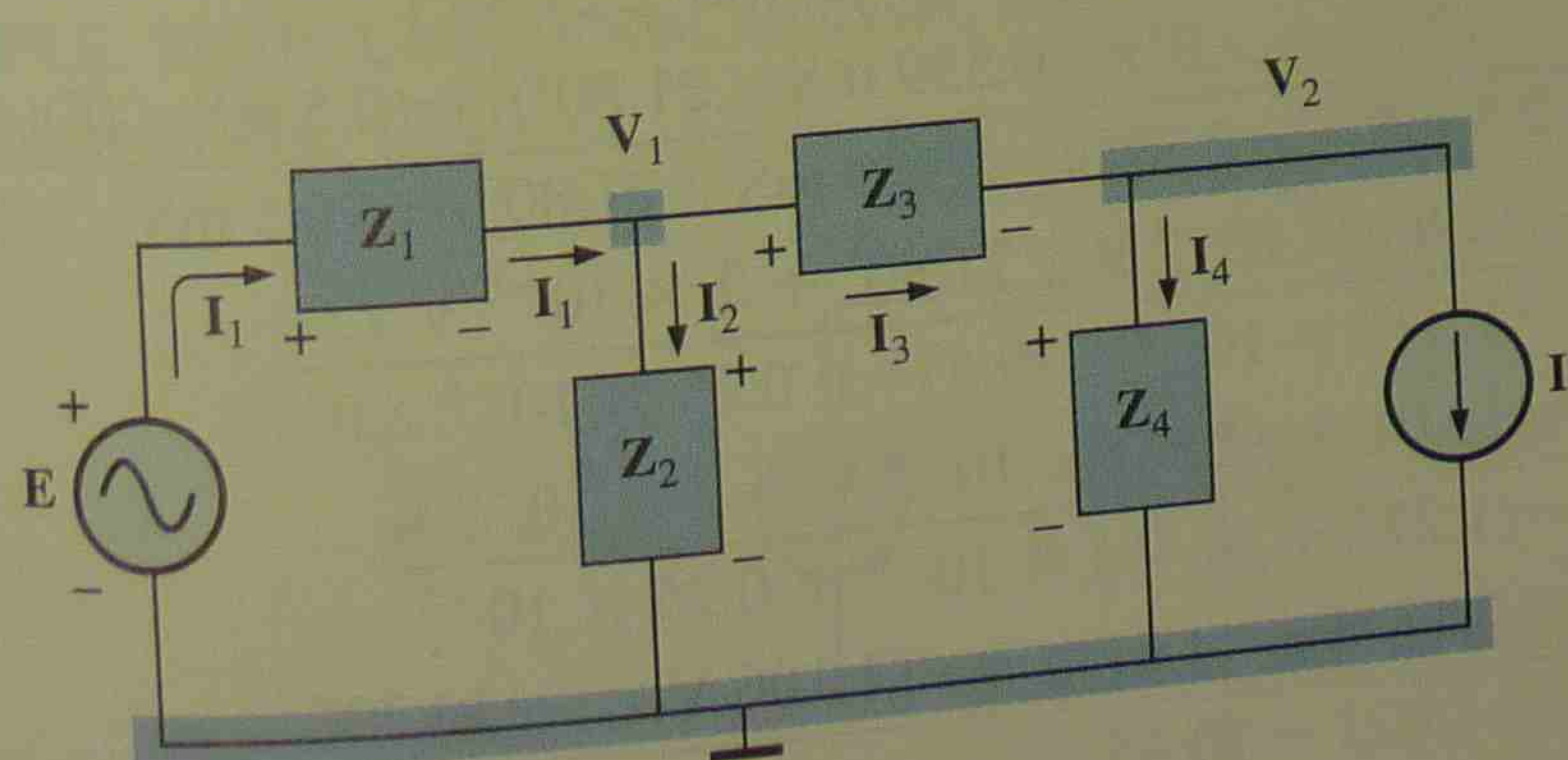


FIG. 17.19

At node 1: $\mathbf{I}_1 = \mathbf{I}_2 + \mathbf{I}_3$

Substituting:
$$\frac{\mathbf{E} - \mathbf{V}_1}{\mathbf{Z}_1} = \frac{\mathbf{V}_1}{\mathbf{Z}_2} + \frac{(\mathbf{V}_1 - \mathbf{V}_2)}{\mathbf{Z}_3}$$

Rearranging:
$$\mathbf{V}_1\left[\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \frac{1}{\mathbf{Z}_3}\right] - \mathbf{V}_2\left[\frac{1}{\mathbf{Z}_3}\right] = \frac{\mathbf{E}}{\mathbf{Z}_1} \qquad (17.1)$$

At node 2: $\mathbf{I}_3 = \mathbf{I}_4 + \mathbf{I}$

Substituting:
$$\frac{\mathbf{V}_1 - \mathbf{V}_2}{\mathbf{Z}_3} = \frac{\mathbf{V}_2}{\mathbf{Z}_4} + \mathbf{I}$$

Rearranging:
$$\mathbf{V}_2\left[\frac{1}{\mathbf{Z}_3} + \frac{1}{\mathbf{Z}_4}\right] - \mathbf{V}_1\left[\frac{1}{\mathbf{Z}_3}\right] = -\mathbf{I} \qquad (17.2)$$

Grouping equations:
$$\begin{aligned}
\mathbf{V}_1\left[\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \frac{1}{\mathbf{Z}_3}\right] - \mathbf{V}_2\left[\frac{1}{\mathbf{Z}_3}\right] &= \frac{\mathbf{E}}{\mathbf{Z}_1}\\
\mathbf{V}_1\left[\frac{1}{\mathbf{Z}_3}\right] - \mathbf{V}_2\left[\frac{1}{\mathbf{Z}_3} + \frac{1}{\mathbf{Z}_4}\right] &= \mathbf{I}
\end{aligned}$$

$$\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \frac{1}{\mathbf{Z}_3} = \frac{1}{0.5\text{ k}\Omega} + \frac{1}{j10\text{ k}\Omega} + \frac{1}{2\text{ k}\Omega} = 2.5\text{ mS}\ \angle -2.29^\circ$$

$$\frac{1}{\mathbf{Z}_3} + \frac{1}{\mathbf{Z}_4} = \frac{1}{2\text{ k}\Omega} + \frac{1}{-j5\text{ k}\Omega} = 0.539\text{ mS}\ \angle 21.80^\circ$$

and

$$\begin{aligned}
\mathbf{V}_1[2.5\text{ mS}\ \angle -2.29^\circ] - \mathbf{V}_2[0.5\text{ mS}\ \angle 0^\circ] &= 24\text{ mA}\ \angle 0^\circ\\
\mathbf{V}_1[0.5\text{ mS}\ \angle 0^\circ] - \mathbf{V}_2[0.539\text{ mS}\ \angle 21.80^\circ] &= 4\text{ mA}\ \angle 0^\circ
\end{aligned}$$

with

$$\mathbf{V}_1 = \frac{\begin{vmatrix} 24\text{ mA}\ \angle 0^\circ & -0.5\text{ mS}\ \angle 0^\circ \\ 4\text{ mA}\ \angle 0^\circ & -0.539\text{ mS}\ \angle 21.80^\circ \end{vmatrix}}{\begin{vmatrix} 2.5\text{ mS}\ \angle -2.29^\circ & -0.5\text{ mS}\ \angle 0^\circ \\ 0.5\text{ mS}\ \angle 0^\circ & -0.539\text{ mS}\ \angle 21.80^\circ \end{vmatrix}}$$

$$= \frac{(24\text{ mA}\ \angle 0^\circ)(-0.539\text{ mS}\ \angle 21.80^\circ) + (0.5\text{ mS}\ \angle 0^\circ)(4\text{ mA}\ \angle 0^\circ)}{(2.5\text{ mS}\ \angle -2.29^\circ)(-0.539\text{ mS}\ \angle 21.80^\circ) + (0.5\text{ mS}\ \angle 0^\circ)(0.5\text{ mS}\ \angle 0^\circ)}$$

$$= \frac{-12.94 \times 10^{-6}\text{ V}\ \angle 21.80^\circ + 2 \times 10^{-6}\text{ V}\ \angle 0^\circ}{-1.348 \times 10^{-6}\ \angle 19.51^\circ + 0.25 \times 10^{-6}\ \angle 0^\circ}$$

$$= \frac{-(12.01 + j4.81) \times 10^{-6}\text{ V} + 2 \times 10^{-6}\text{ V}}{-(1.271 + j0.45) \times 10^{-6} + 0.25 \times 10^{-6}}$$

$$= \frac{-10.01\text{ V} - j4.81\text{ V}}{-1.021 - j0.45} = \frac{11.106\text{ V}\ \angle -154.33^\circ}{1.116\ \angle -156.21^\circ}$$

$$\mathbf{V}_1 = \mathbf{9.95\ V\ \angle 1.88^\circ}$$

Similarly,

$$\mathbf{V}_2 = 1.828 \text{ V} \angle -12.49°$$

Format Approach

A close examination of Eqs. (17.1) and (17.2) in Example 17.9 will reveal that they are the same equations that would have been obtained using the format approach introduced in Chapter 8. Recall that the approach required that the voltage source first be converted to a current source, but the writing of the equations was quite direct and minimized and chances of an error due to a lost sign or missing term.

The sequence of steps required to apply the format approach is the following:

1. ***Choose a reference node and assign a subscripted voltage label to the (N − 1) remaining independent nodes of the network.***
2. ***The number of equations required for a complete solution is equal to the number of subscripted voltages (N − 1). Column 1 of each equation is formed by summing the admittances tied to the node of interest and multiplying the result by that subscripted nodal voltage.***
3. ***The mutual terms are always subtracted from the terms of the first column. It is possible to have more than one mutual term if the nodal voltage of interest has an element in common with more than one other nodal voltage. Each mutual term is the product of the mutual admittance and the other nodal voltage tied to that admittance.***
4. ***The column to the right of the equality sign is the algebraic sum of the current sources tied to the node of interest. A current source is assigned a positive sign if it supplies current to a node, and a negative sign if it draws current from the node.***
5. ***Solve resulting simultaneous equations for the desired nodal voltages. The comments offered for mesh analysis regarding independent and dependent sources apply here also.***

EXAMPLE 17.10 Using nodal analysis, find the voltage across the 4-Ω resistor in Fig. 17.20.

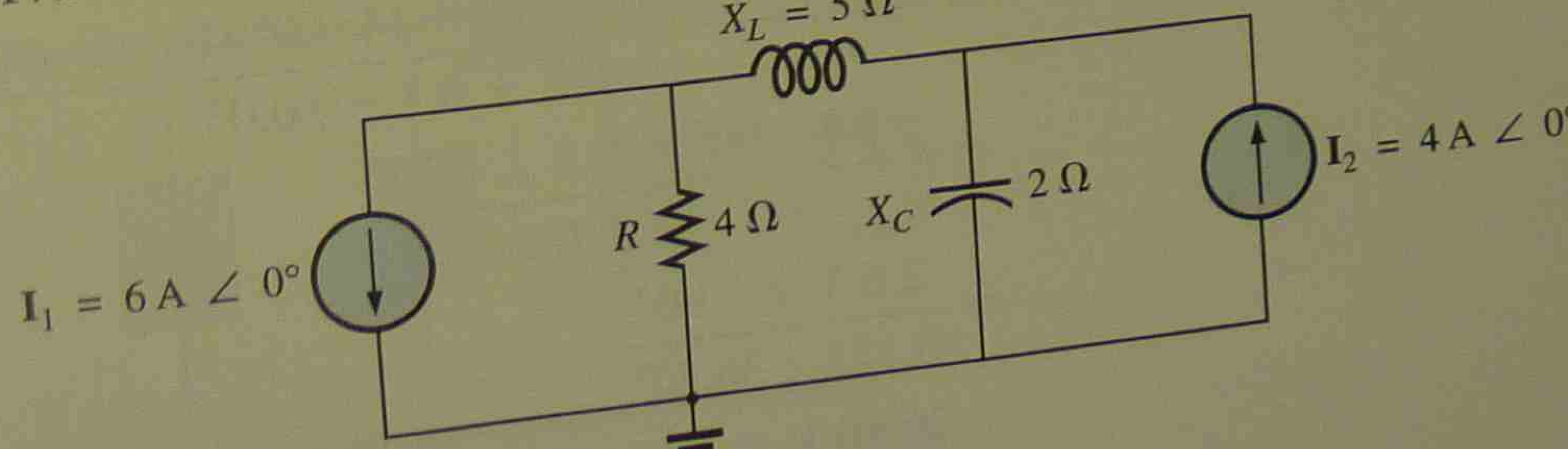


FIG. 17.20

Solution: Choosing nodes (Fig. 17.21) and writing the nodal equations, we have

$$\mathbf{Z}_1 = R = 4\,\Omega \qquad \mathbf{Z}_2 = jX_L = j5\,\Omega \qquad \mathbf{Z}_3 = -jX_C = -j2\,\Omega$$

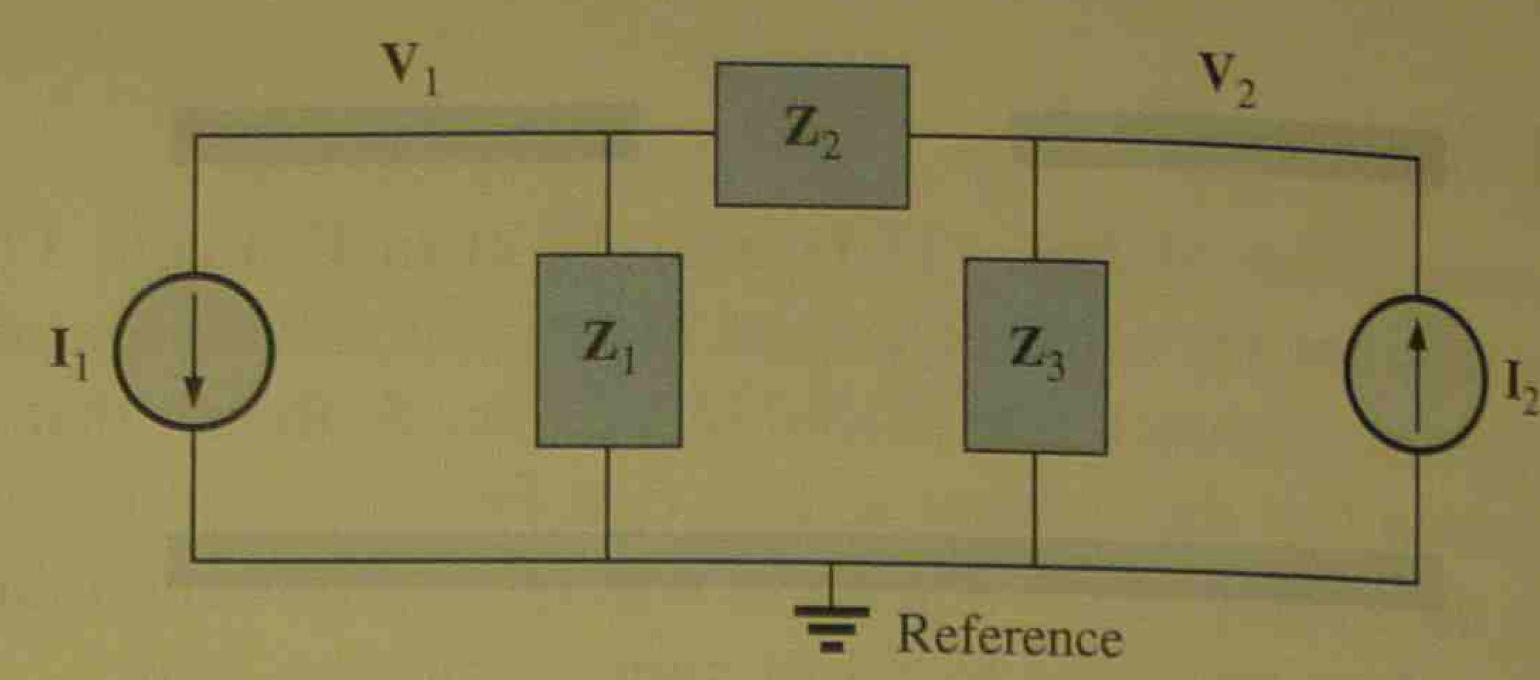


FIG. 17.21

$$\begin{aligned}\mathbf{V}_1(\mathbf{Y}_1 + \mathbf{Y}_2) - \mathbf{V}_2(\mathbf{Y}_2) &= -\mathbf{I}_1\\ \mathbf{V}_2(\mathbf{Y}_3 + \mathbf{Y}_2) - \mathbf{V}_1(\mathbf{Y}_2) &= +\mathbf{I}_2\end{aligned}$$

or

$$\begin{aligned}\mathbf{V}_1(\mathbf{Y}_1 + \mathbf{Y}_2) - \mathbf{V}_2(\mathbf{Y}_2) \qquad\qquad &= -\mathbf{I}_1\\ -\mathbf{V}_1(\mathbf{Y}_2) \qquad + \mathbf{V}_2(\mathbf{Y}_3 + \mathbf{Y}_2) &= +\mathbf{I}_2\end{aligned}$$

$$\mathbf{Y}_1 = \frac{1}{\mathbf{Z}_1} \qquad \mathbf{Y}_2 = \frac{1}{\mathbf{Z}_2} \qquad \mathbf{Y}_3 = \frac{1}{\mathbf{Z}_3}$$

Using determinants yields

$$\begin{aligned}\mathbf{V}_1 &= \frac{\begin{vmatrix} -\mathbf{I}_1 & -\mathbf{Y}_2 \\ +\mathbf{I}_2 & \mathbf{Y}_3 + \mathbf{Y}_2 \end{vmatrix}}{\begin{vmatrix} \mathbf{Y}_1 + \mathbf{Y}_2 & -\mathbf{Y}_2 \\ -\mathbf{Y}_2 & \mathbf{Y}_3 + \mathbf{Y}_2 \end{vmatrix}}\\ &= \frac{-(\mathbf{Y}_3 + \mathbf{Y}_2)\mathbf{I}_1 + \mathbf{I}_2\mathbf{Y}_2}{(\mathbf{Y}_1 + \mathbf{Y}_2)(\mathbf{Y}_3 + \mathbf{Y}_2) - \mathbf{Y}_2^2}\\ &= \frac{-(\mathbf{Y}_3 + \mathbf{Y}_2)\mathbf{I}_1 + \mathbf{I}_2\mathbf{Y}_2}{\mathbf{Y}_1\mathbf{Y}_3 + \mathbf{Y}_2\mathbf{Y}_3 + \mathbf{Y}_1\mathbf{Y}_2}\end{aligned}$$

Substituting numerical values, we have

$$\begin{aligned}\mathbf{V}_1 &= \frac{-[(1/-j2\,\Omega) + (1/j5\,\Omega)]6\text{ A }\angle 0^\circ + 4\text{ A }\angle 0^\circ(1/j5\,\Omega)}{(1/4\,\Omega)(1/-j2\,\Omega) + (1/j5\,\Omega)(1/-j2\,\Omega) + (1/4\,\Omega)(1/j5\,\Omega)}\\ &= \frac{-(+j0.5 - j0.2)6\angle 0^\circ + 4\angle 0^\circ(-j0.2)}{(1/-j8) + (1/10) + (1/j20)}\\ &= \frac{(-0.3\angle 90^\circ)(6\angle 0^\circ) + (4\angle 0^\circ)(0.2\angle -90^\circ)}{j0.125 + 0.1 - j0.05}\\ &= \frac{-1.8\angle 90^\circ + 0.8\angle -90^\circ}{0.1 + j0.075}\\ &= \frac{2.6\text{ V}\angle -90^\circ}{0.125\angle 36.87^\circ}\\ &= \mathbf{20.80\text{ V}\angle -126.87^\circ}\end{aligned}$$

EXAMPLE 17.11 Write the nodal equations for the network of Fig. 17.22. In this case, a voltage source appears in the network. Do not solve.

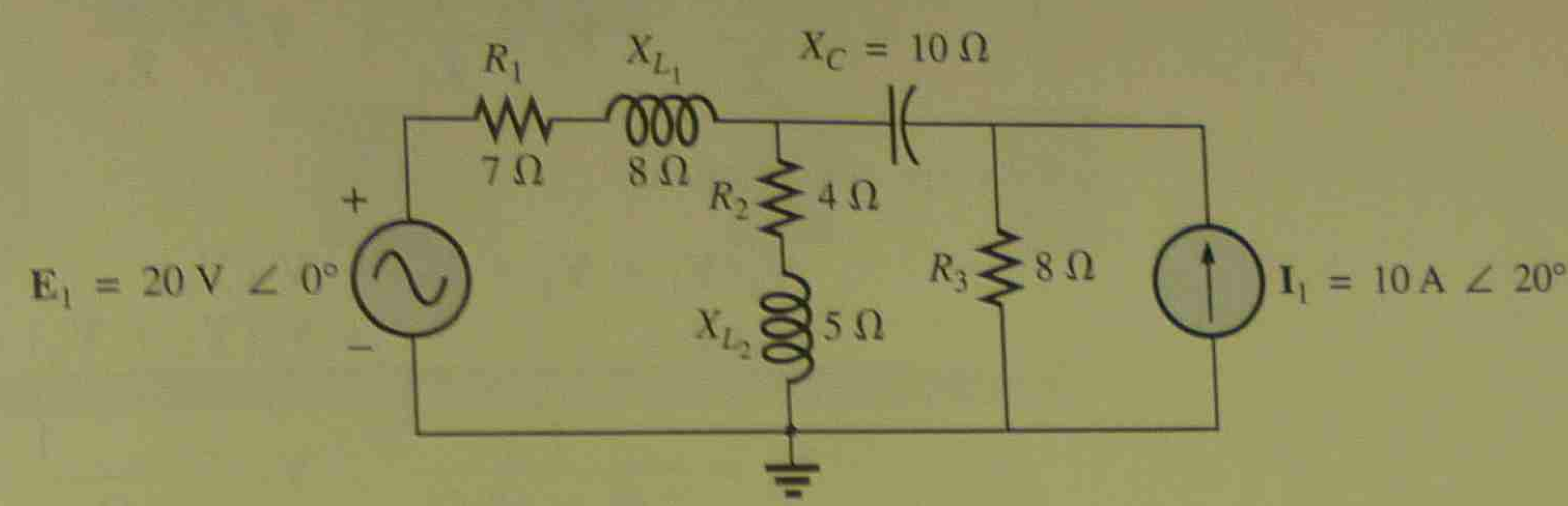


FIG. 17.22

Solution: The circuit is redrawn in Fig. 17.23, where

$$\mathbf{Z}_1 = R_1 + jX_{L_1} = 7\ \Omega + j\,8\ \Omega \qquad \mathbf{E}_1 = 20\ \text{V} \angle 0°$$
$$\mathbf{Z}_2 = R_2 + jX_{L_2} = 4\ \Omega + j\,5\ \Omega \qquad \mathbf{I}_1 = 10\ \text{A} \angle 20°$$
$$\mathbf{Z}_3 = -jX_C = -j\,10\ \Omega$$
$$\mathbf{Z}_4 = R_3 = 8\ \Omega$$

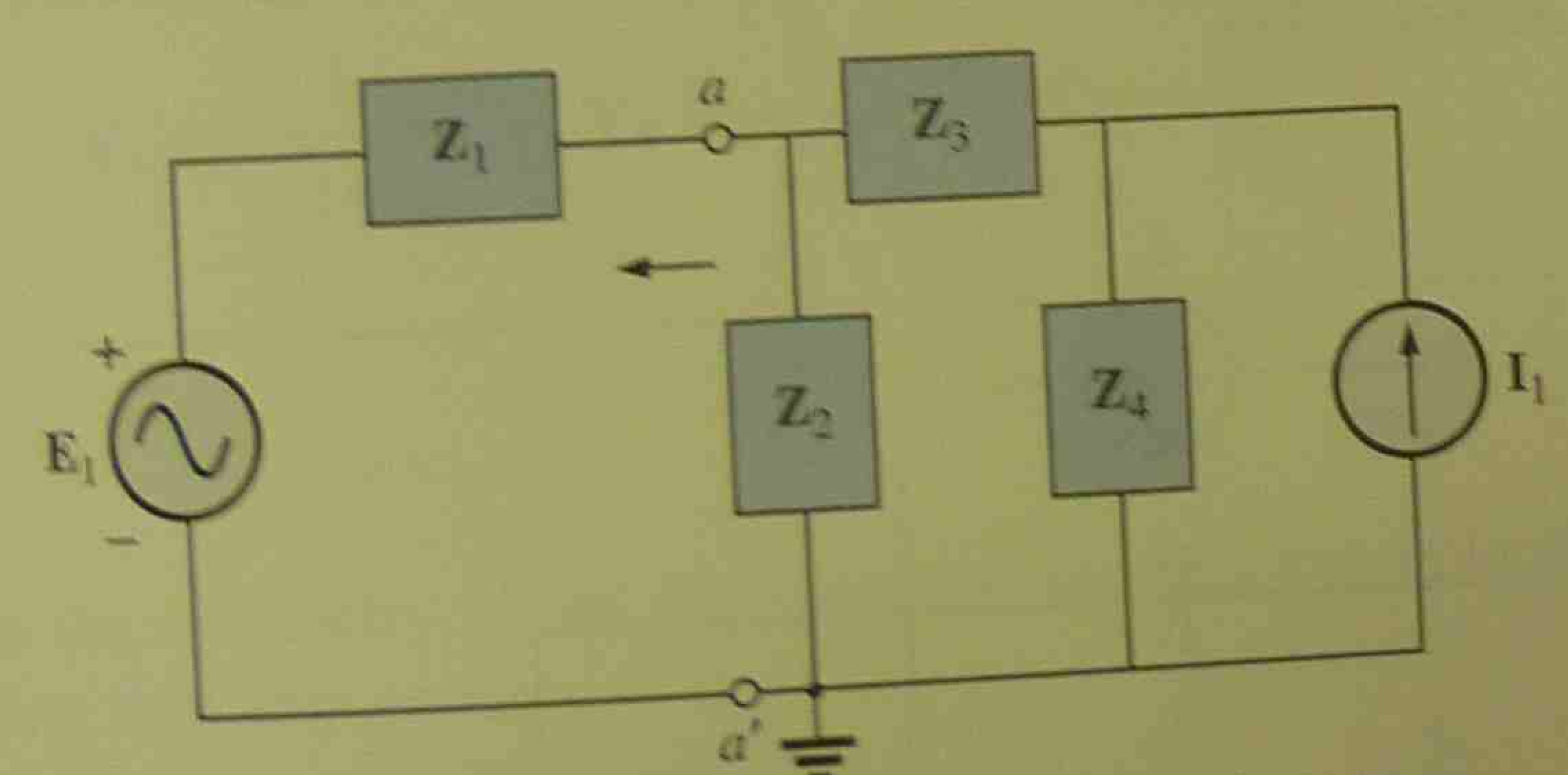


FIG. 17.23

Converting the voltage source to a current source and choosing nodes, we obtain Fig. 17.24. Note the "neat" appearance of the network using the subscripted impedances. Working directly with Fig. 17.22 would be difficult and may produce errors.

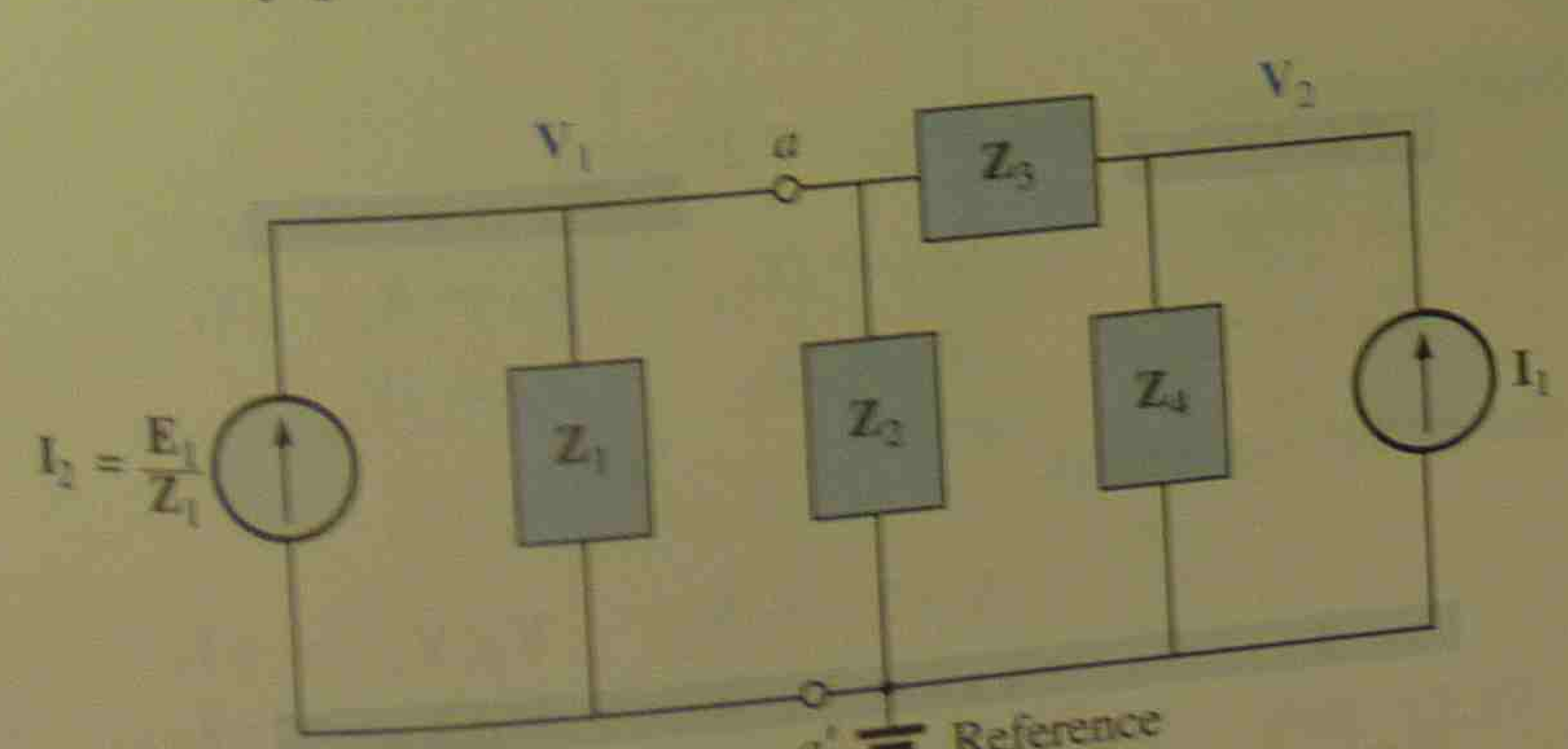


FIG. 17.24

Write the nodal equations:

$$\mathbf{V}_1(\mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3) - \mathbf{V}_2(\mathbf{Y}_3) = +\mathbf{I}_2$$
$$\mathbf{V}_2(\mathbf{Y}_3 + \mathbf{Y}_4) - \mathbf{V}_1(\mathbf{Y}_3) = +\mathbf{I}_1$$

$$\mathbf{Y}_1 = \frac{1}{\mathbf{Z}_1} \qquad \mathbf{Y}_2 = \frac{1}{\mathbf{Z}_2} \qquad \mathbf{Y}_3 = \frac{1}{\mathbf{Z}_3} \qquad \mathbf{Y}_4 = \frac{1}{\mathbf{Z}_4}$$

which are rewritten as

$$\begin{aligned} \mathbf{V}_1(\mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3) - \mathbf{V}_2(\mathbf{Y}_3) \qquad &= +\mathbf{I}_2 \\ -\mathbf{V}_1(\mathbf{Y}_3) \qquad + \mathbf{V}_2(\mathbf{Y}_3 + \mathbf{Y}_4) &= +\mathbf{I}_1 \end{aligned}$$

$$\mathbf{Y}_1 = \frac{1}{7\ \Omega + j8\ \Omega} \qquad \mathbf{Y}_2 = \frac{1}{4\ \Omega + j5\ \Omega}$$

$$\mathbf{Y}_3 = \frac{1}{-j10\ \Omega} \qquad \mathbf{Y}_4 = \frac{1}{8\ \Omega}$$

$$\mathbf{I}_2 = \frac{20\ \text{V} \angle 0^\circ}{7\ \Omega + j8\ \Omega} \qquad \mathbf{I}_1 = 10\ \text{A} \angle 20^\circ$$

EXAMPLE 17.12 Write the nodal equations for the network of Fig. 17.25. Do not solve.

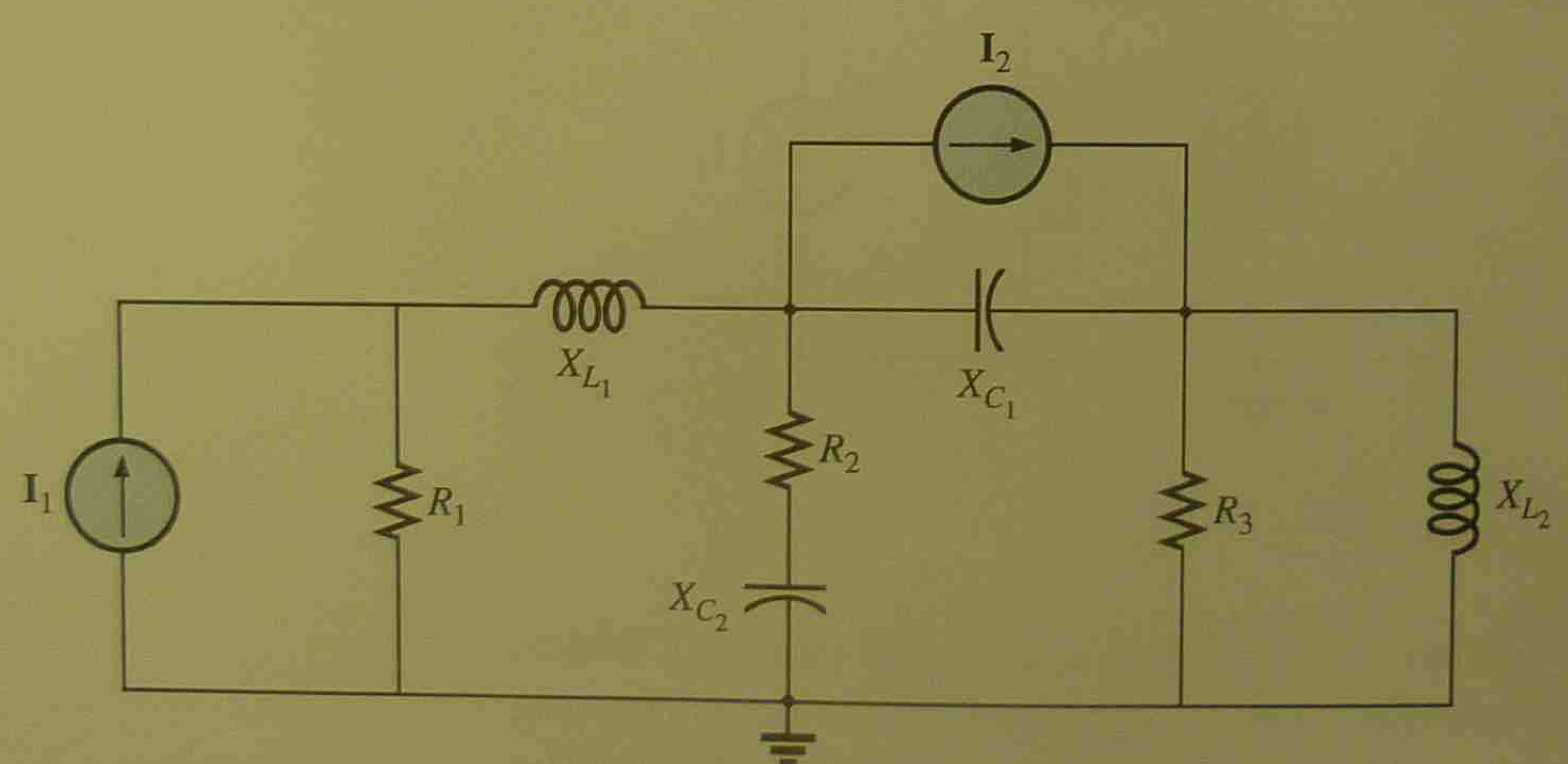


FIG. 17.25

Solution: Choose nodes (Fig. 17.26):

$$\mathbf{Z}_1 = R_1 \qquad \mathbf{Z}_2 = jX_{L_1} \qquad \mathbf{Z}_3 = R_2 - jX_{C_2}$$
$$\mathbf{Z}_4 = -jX_{C_1} \qquad \mathbf{Z}_5 = R_3 \qquad \mathbf{Z}_6 = jX_{L_2}$$

and write nodal equations:

$$\mathbf{V}_1(\mathbf{Y}_1 + \mathbf{Y}_2) - \mathbf{V}_2(\mathbf{Y}_2) = +\mathbf{I}_1$$
$$\mathbf{V}_2(\mathbf{Y}_2 + \mathbf{Y}_3 + \mathbf{Y}_4) - \mathbf{V}_1(\mathbf{Y}_2) - \mathbf{V}_3(\mathbf{Y}_4) = -\mathbf{I}_2$$
$$\mathbf{V}_3(\mathbf{Y}_4 + \mathbf{Y}_5 + \mathbf{Y}_6) - \mathbf{V}_2(\mathbf{Y}_4) = +\mathbf{I}_2$$

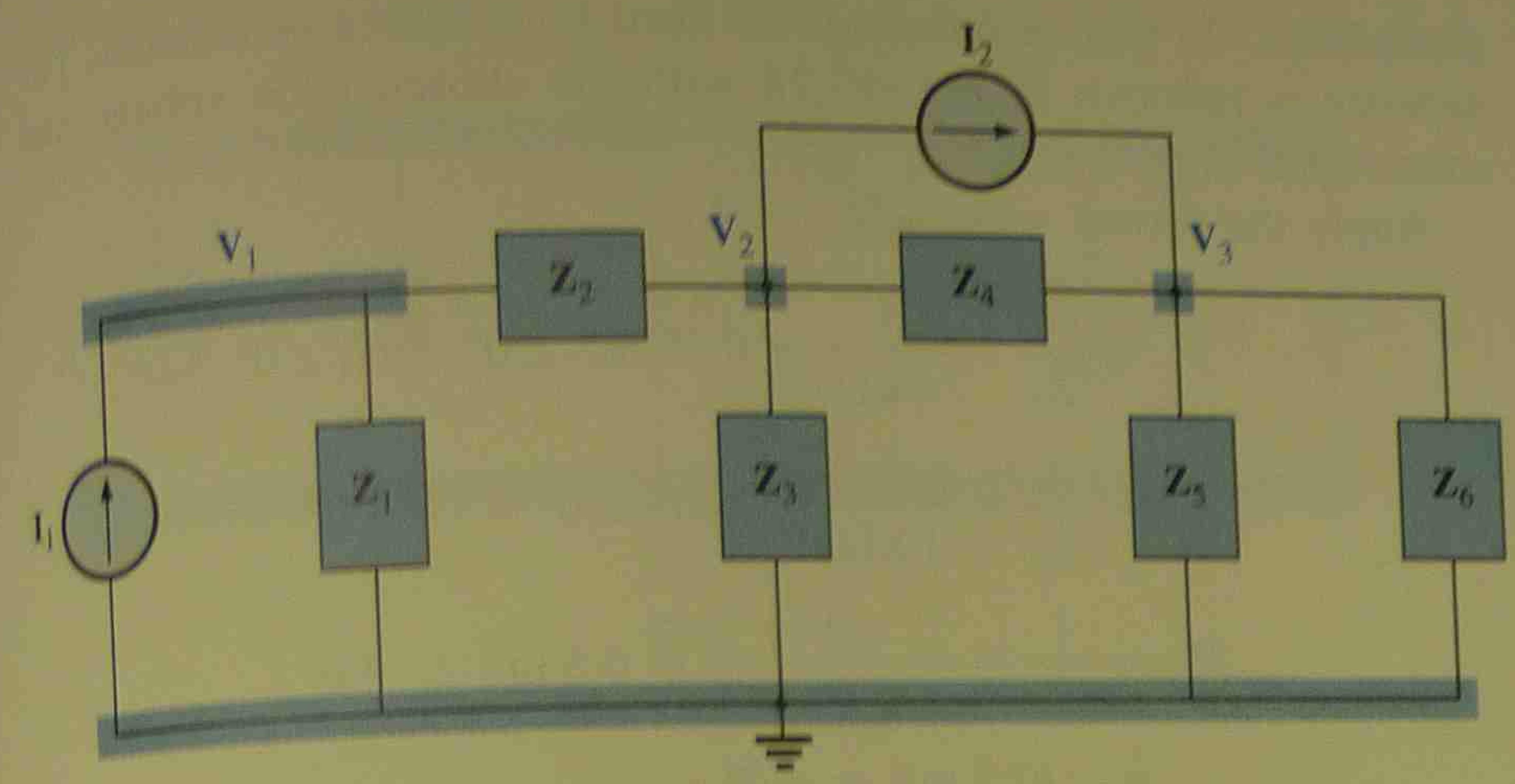


FIG. 17.26

which are rewritten as

$$\begin{array}{lll} \mathbf{V}_1(\mathbf{Y}_1 + \mathbf{Y}_2) - \mathbf{V}_2(\mathbf{Y}_2) & + 0 & = +\mathbf{I}_1 \\ -\mathbf{V}_1(\mathbf{Y}_2) + \mathbf{V}_2(\mathbf{Y}_2 + \mathbf{Y}_3 + \mathbf{Y}_4) & - \mathbf{V}_3(\mathbf{Y}_4) & = -\mathbf{I}_2 \\ 0 - \mathbf{V}_2(\mathbf{Y}_4) & + \mathbf{V}_3(\mathbf{Y}_4 + \mathbf{Y}_5 + \mathbf{Y}_6) & = +\mathbf{I}_2 \end{array}$$

$$\mathbf{Y}_1 = \frac{1}{R_1} \qquad \mathbf{Y}_2 = \frac{1}{jX_{L_1}} \qquad \mathbf{Y}_3 = \frac{1}{R_2 - jX_{C_2}}$$

$$\mathbf{Y}_4 = \frac{1}{-jX_{C_1}} \qquad \mathbf{Y}_5 = \frac{1}{R_3} \qquad \mathbf{Y}_6 = \frac{1}{jX_{L_2}}$$

Note the symmetry about the diagonal for this example and those preceding it in this section.

EXAMPLE 17.13 Apply nodal analysis to the network of Fig. 17.27. Determine the voltage $\mathbf{V}_L$.

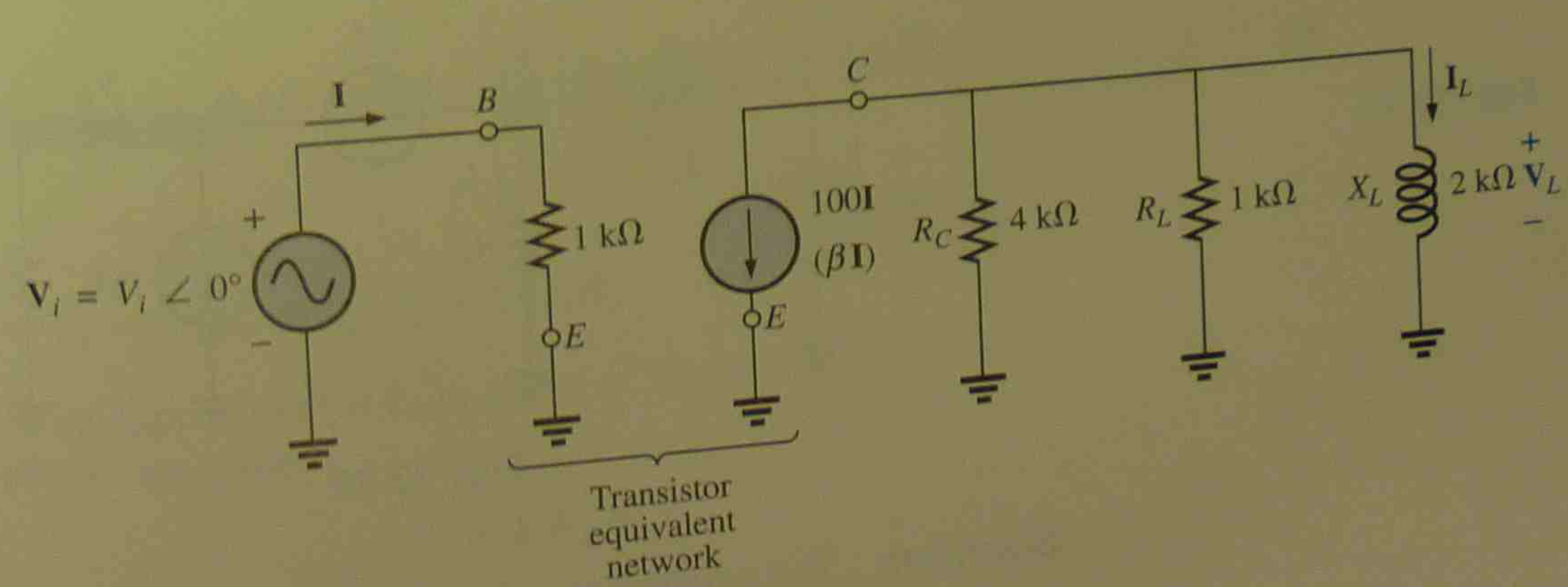


FIG. 17.27

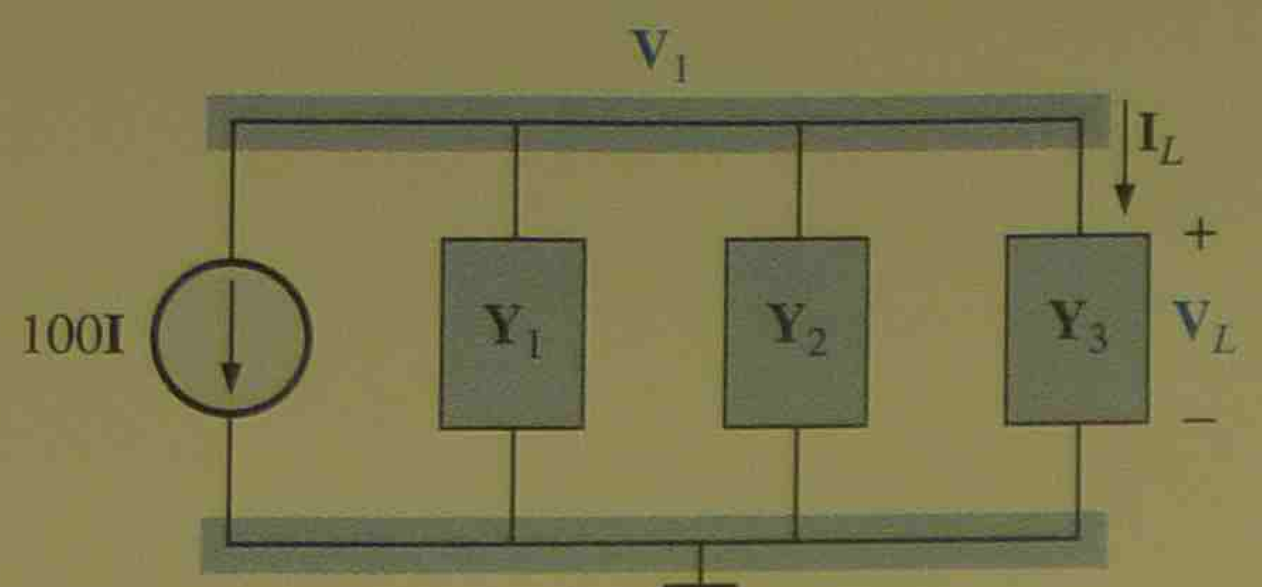


FIG. 17.28

Solution: In this case there is no need for a source conversion. The network is redrawn in Fig. 17.28 with the chosen node voltage and subscripted impedances.

Apply the format approach:

$$\mathbf{Y}_1 = \frac{1}{\mathbf{Z}_1} = \frac{1}{4\text{ k}\Omega} = 0.25\text{ mS }\angle 0^\circ = G_1 \angle 0^\circ$$

$$\mathbf{Y}_2 = \frac{1}{\mathbf{Z}_2} = \frac{1}{1\text{ k}\Omega} = 1\text{ mS }\angle 0^\circ = G_2 \angle 0^\circ$$

$$\mathbf{Y}_3 = \frac{1}{\mathbf{Z}_3} = \frac{1}{2\text{ k}\Omega \angle 90^\circ} = 0.5\text{ mS }\angle -90^\circ$$

$$= -j0.5\text{ mS} = -jB_L$$

$$\mathbf{V}_1\!: (\mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3)\mathbf{V}_1 = -100\mathbf{I}$$

and

$$\mathbf{V}_1 = \frac{-100\mathbf{I}}{\mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3}$$

$$= \frac{-100\mathbf{I}}{0.25\text{ mS} + 1\text{ mS} - j0.5\text{ mS}}$$

$$= \frac{-100 \times 10^3\mathbf{I}}{1.25 - j0.5} = \frac{-100 \times 10^3\mathbf{I}}{1.3463 \angle -21.80^\circ}$$

$$= -74.28 \times 10^3\mathbf{I} \angle 21.80^\circ$$

$$= -74.28 \times 10^3\left(\frac{\mathbf{V}_i}{1\text{ k}\Omega}\right) \angle 21.80^\circ$$

$$\mathbf{V}_1 = \mathbf{V}_L = \mathbf{-(74.28V}_i\mathbf{)\ V} \angle \mathbf{21.80^\circ}$$

EXAMPLE 17.14 The transistor configuration of Fig. 17.29 will result in a network very similar in appearance to Fig. 17.30 when the transistor equivalent circuits are substituted. The quantities β_1 and β_2 are the amplification factors of the transistors. Determine $\mathbf{V}_L$ for the network of Fig. 17.30. In this case, both of the controlling variables $\mathbf{I}_1$ and $\mathbf{I}_2$ are part of the network to be analyzed. Care must be exercised when applying the method.

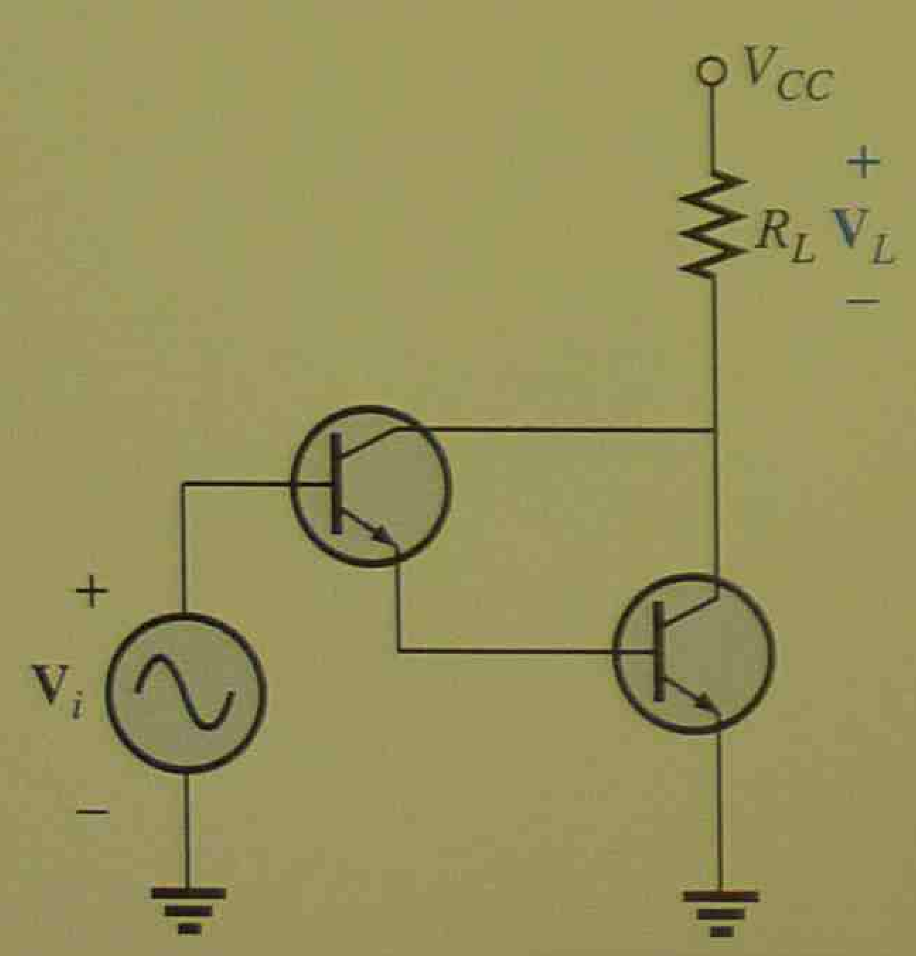


FIG. 17.29

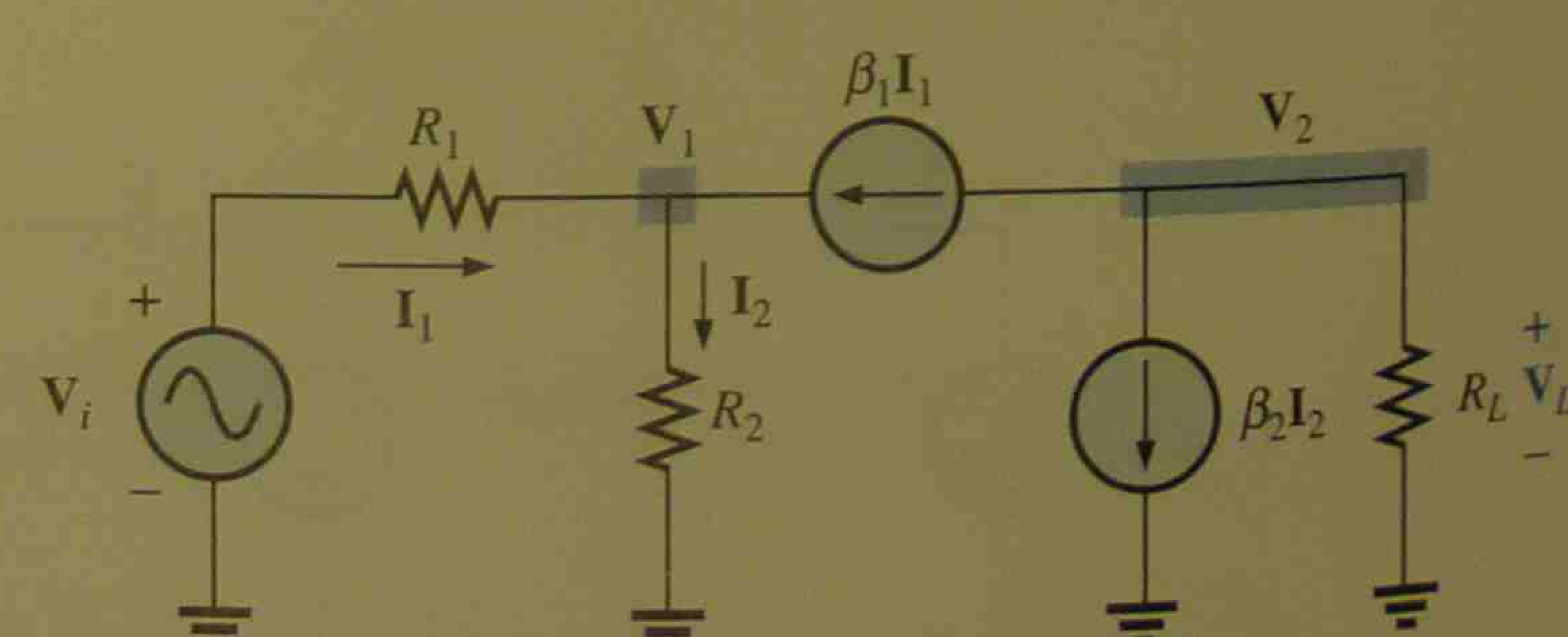


FIG. 17.30

Solution: The network is redrawn in Fig. 17.31 after converting the input voltage source to a current source. Before applying nodal analysis,

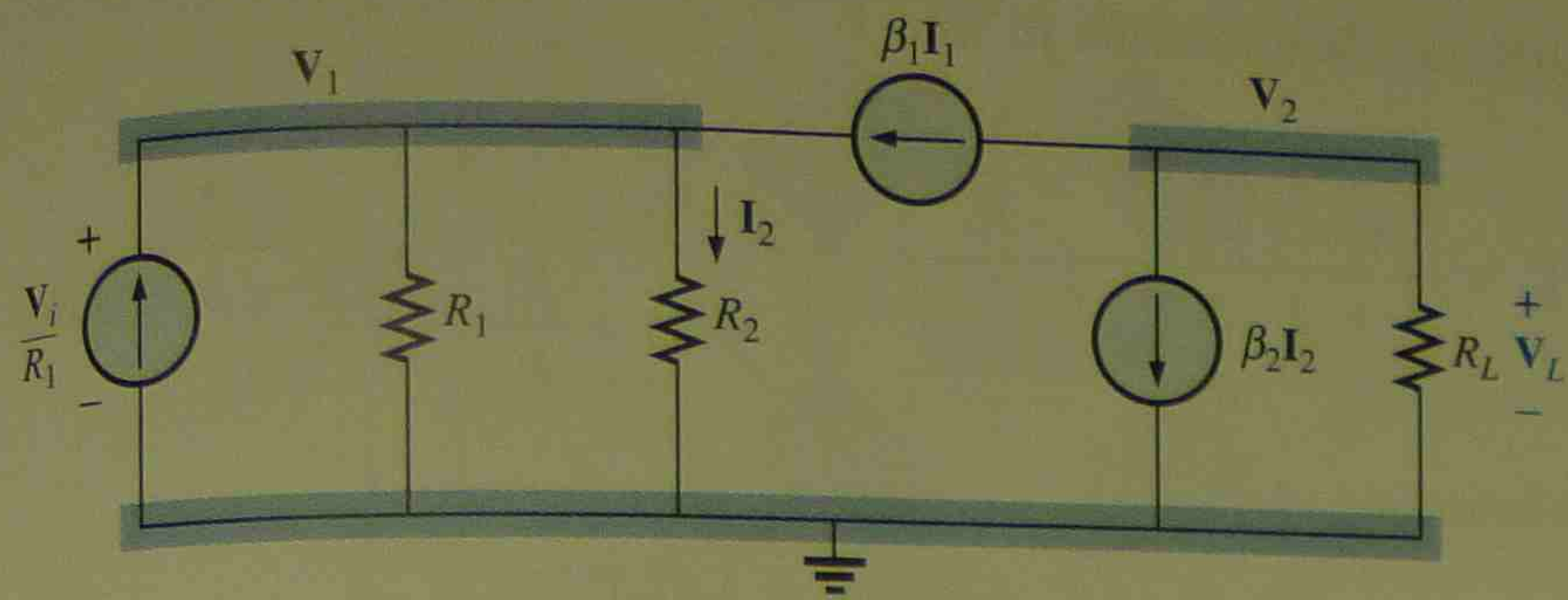


FIG. 17.31

note in Fig. 17.30 that

$$\mathbf{I}_1 + \beta_1\mathbf{I}_1 = \mathbf{I}_2 \qquad \text{(Kirchhoff's current law)}$$

and

$$\mathbf{I}_2 = (\beta_1 + 1)\mathbf{I}_1$$

Also recognize from Fig. 17.30 that

$$\mathbf{I}_1 = \frac{\mathbf{V}_i - \mathbf{V}_1}{R_1}$$

or

$$\mathbf{V}_1 + \mathbf{I}_1 R_1 = \mathbf{V}_i$$

Applying nodal analysis to node 1:

$$\mathbf{V}_1\left[\frac{1}{R_1} + \frac{1}{R_2}\right] = \frac{\mathbf{V}_i}{R_1} + \beta_1\mathbf{I}_1$$

or

$$\mathbf{V}_1\left[\frac{1}{R_1} + \frac{1}{R_2}\right] - \beta_1\mathbf{I}_1 = \frac{\mathbf{V}_i}{R_1}$$

Applying nodal analysis to node 2:

$$\mathbf{V}_2\left[\frac{1}{R_L}\right] = -\beta_1\mathbf{I}_1 - \beta_2\mathbf{I}_2$$

Substituting $\mathbf{I}_2 = (\beta_1 + 1)\mathbf{I}_1$ from above,

$$\frac{\mathbf{V}_2}{R_L} = -\beta_1\mathbf{I}_1 - \beta_2(\beta_1 + 1)\mathbf{I}_1 = -[\beta_1 + \beta_2(\beta_1 + 1)]\mathbf{I}_1$$

and

$$\mathbf{V}_2 = \mathbf{V}_L = -[\beta_1 + \beta_2(\beta_1 + 1)]\mathbf{I}_1 R_L$$

We now need an expression for $\mathbf{I}_1$ in terms of V_i. Rewriting two equations from above, we have

$$\mathbf{V}_1 + \mathbf{I}_1 R_1 = \mathbf{V}_i$$

$$\mathbf{V}_1\left[\frac{1}{R_1} + \frac{1}{R_2}\right] - \beta_1\mathbf{I}_1 = \frac{\mathbf{V}_i}{R_1}$$

Applying determinants to find $\mathbf{I}_1$:

$$\mathbf{I}_1 = \frac{\left(\dfrac{1}{R_1} + \dfrac{1}{R_2}\right)\mathbf{V}_i - \dfrac{\mathbf{V}_i}{R_1}}{\left(\dfrac{1}{R_1} + \dfrac{1}{R_2}\right)R_1 + \beta_1} = \frac{\dfrac{\mathbf{V}_i}{R_2}}{(\beta_1 + 1) + \dfrac{R_1}{R_2}} = \frac{\mathbf{V}_i}{R_1 + (\beta_1 + 1)R_2}$$

Substituting above:

$$\mathbf{V}_L = -[\beta_1 + \beta_2(\beta_1 + 1)]\mathbf{I}_1 R_L$$

$$= -[\beta_1 + \beta_2(\beta_1 + 1)]\frac{\mathbf{V}_i R_L}{R_1 + (\beta_1 + 1)R_2}$$

and
$$\mathbf{V}_L = -\frac{[\beta_1 + \beta_2(\beta_1 + 1)]R_L}{R_1 + (\beta_1 + 1)R_2}\mathbf{V}_i$$

Since β_1 is usually much greater than 1, $\beta_1 + 1 \cong \beta_1$

and
$$\mathbf{V}_L = -\frac{(\beta_1 + \beta_2\beta_1)R_L}{R_1 + \beta_1 R_2}\mathbf{V}_i = -\frac{\beta_1(\beta_2 + 1)R_L}{R_1 + \beta_1 R_2}\mathbf{V}_i$$

But β_2 is usually much greater than 1, so $\beta_2 + 1 \cong \beta_2$. Therefore,

$$\mathbf{V}_L \cong -\frac{\beta_1\beta_2 R_L}{R_1 + \beta_1 R_2}\mathbf{V}_i$$

For most applications $\beta_1 R_2 \gg R_1$ and

$$\mathbf{V}_L \cong -\frac{\beta_1\beta_2 R_2}{\beta_1 R_2}\mathbf{V}_i = -\frac{\beta_2 R_L}{R_2}\mathbf{V}_i$$

which is certainly less complex than the original solution for $\mathbf{V}_L$.

For the typical values of $\beta_2 = 100$, $R_L = 2\text{ k}\Omega$ and $R_2 = 1\text{ k}\Omega$

$$\mathbf{V}_L = \frac{-(100)(2\text{ k}\Omega)\mathbf{V}_i}{1\text{ k}\Omega} = -200\mathbf{V}_i$$

17.6 BRIDGE NETWORKS (ac)

The basic bridge configuration was discussed in some detail in Section 8.11 for dc networks. We now continue to examine bridge networks by considering those that have reactive components and a sinusoidal ac voltage or current applied.

We will first analyze various familiar forms of the bridge network using mesh analysis and nodal analysis (the format approach). The balance conditions will be investigated throughout the section.

Apply *mesh analysis* to the network of Fig. 17.32. The network is redrawn in Fig. 17.33, where

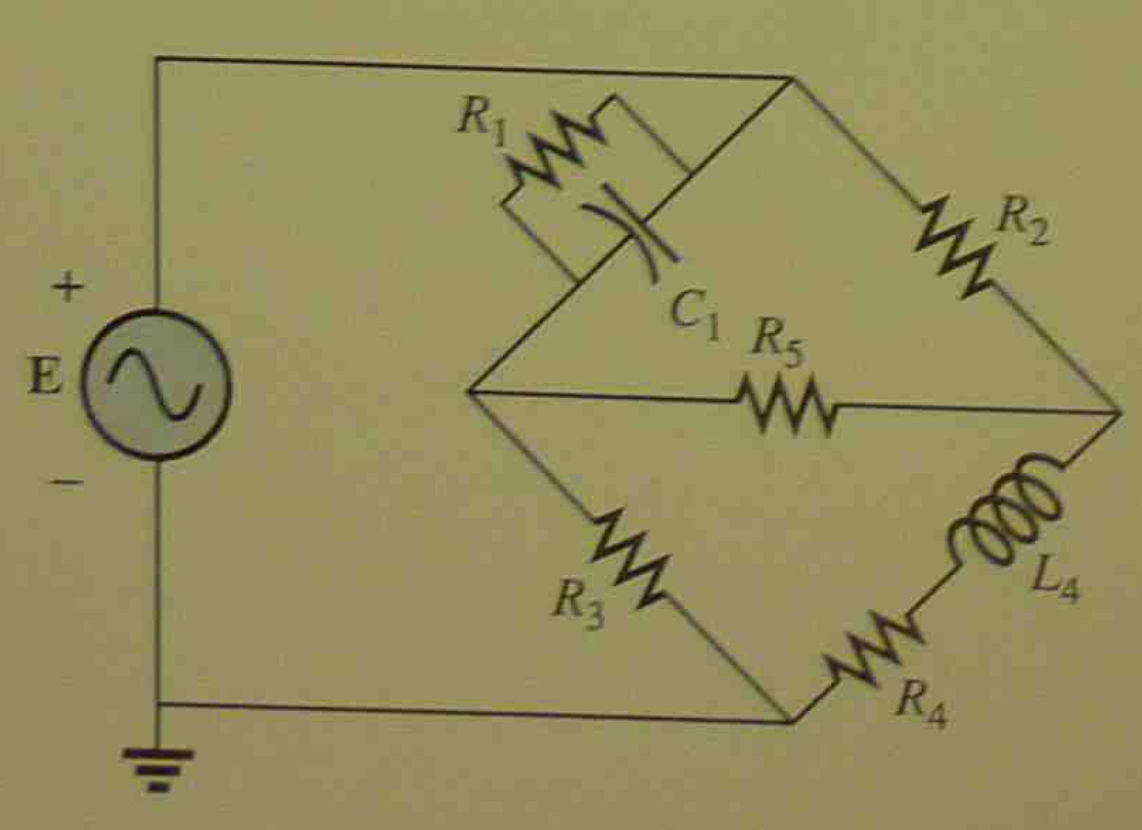


FIG. 17.32
Maxwell bridge.

$$\mathbf{Z}_1 = \frac{1}{\mathbf{Y}_1} = \frac{1}{G_1 + jB_C} = \frac{G_1}{G_1^2 + B_C^2} - j\frac{B_C}{G_1^2 + B_C^2}$$

$$\mathbf{Z}_2 = R_2 \qquad \mathbf{Z}_3 = R_3 \qquad \mathbf{Z}_4 = R_4 + jX_L \qquad \mathbf{Z}_5 = R_5$$

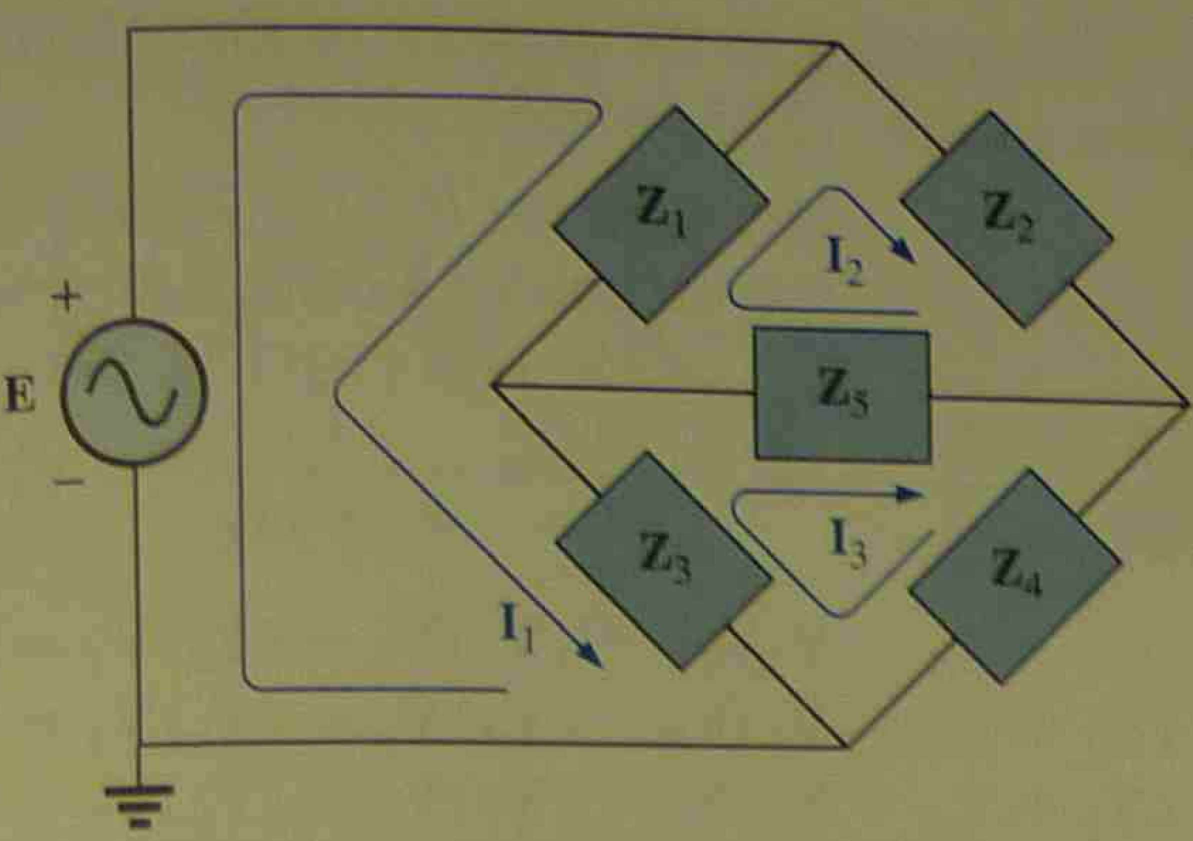


FIG. 17.33

Apply the format approach:

$$\begin{aligned}
(\mathbf{Z}_1 + \mathbf{Z}_3)\mathbf{I}_1 - (\mathbf{Z}_1)\mathbf{I}_2 - (\mathbf{Z}_3)\mathbf{I}_3 &= \mathbf{E}\\
(\mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_5)\mathbf{I}_2 - (\mathbf{Z}_1)\mathbf{I}_1 - (\mathbf{Z}_5)\mathbf{I}_3 &= 0\\
(\mathbf{Z}_3 + \mathbf{Z}_4 + \mathbf{Z}_5)\mathbf{I}_3 - (\mathbf{Z}_3)\mathbf{I}_1 - (\mathbf{Z}_5)\mathbf{I}_2 &= 0
\end{aligned}$$

which are rewritten as

$$\begin{array}{lllr}
\mathbf{I}_1(\mathbf{Z}_1 + \mathbf{Z}_3) & - \mathbf{I}_2\mathbf{Z}_1 & - \mathbf{I}_3\mathbf{Z}_3 & = \mathbf{E}\\
-\mathbf{I}_1\mathbf{Z}_1 & + \mathbf{I}_2(\mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_5) & - \mathbf{I}_3\mathbf{Z}_5 & = 0\\
-\mathbf{I}_1\mathbf{Z}_3 & - \mathbf{I}_2\mathbf{Z}_5 & + \mathbf{I}_3(\mathbf{Z}_3 + \mathbf{Z}_4 + \mathbf{Z}_5) & = 0
\end{array}$$

Note the symmetry about the diagonal of the above equations. For balance, $\mathbf{I}_{\mathbf{Z}_5} = 0$ A, and

$$\mathbf{I}_{\mathbf{Z}_5} = \mathbf{I}_2 - \mathbf{I}_3 = 0$$

From the above equations,

$$\mathbf{I}_2 = \frac{\begin{vmatrix} \mathbf{Z}_1 + \mathbf{Z}_3 & \mathbf{E} & -\mathbf{Z}_3 \\ -\mathbf{Z}_1 & 0 & -\mathbf{Z}_5 \\ -\mathbf{Z}_3 & 0 & (\mathbf{Z}_3 + \mathbf{Z}_4 + \mathbf{Z}_5) \end{vmatrix}}{\begin{vmatrix} \mathbf{Z}_1 + \mathbf{Z}_3 & -\mathbf{Z}_1 & -\mathbf{Z}_3 \\ -\mathbf{Z}_1 & (\mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_5) & -\mathbf{Z}_5 \\ -\mathbf{Z}_3 & -\mathbf{Z}_5 & (\mathbf{Z}_3 + \mathbf{Z}_4 + \mathbf{Z}_5) \end{vmatrix}}$$

$$= \frac{\mathbf{E}(\mathbf{Z}_1\mathbf{Z}_3 + \mathbf{Z}_1\mathbf{Z}_4 + \mathbf{Z}_1\mathbf{Z}_5 + \mathbf{Z}_3\mathbf{Z}_5)}{\Delta}$$

where Δ signifies the determinant of the denominator (or coefficients). Similarly,

$$\mathbf{I}_3 = \frac{\mathbf{E}(\mathbf{Z}_1\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_2 + \mathbf{Z}_1\mathbf{Z}_5 + \mathbf{Z}_3\mathbf{Z}_5)}{\Delta}$$

and

$$\mathbf{I}_{\mathbf{Z}_5} = \mathbf{I}_2 - \mathbf{I}_3 = \frac{\mathbf{E}(\mathbf{Z}_1\mathbf{Z}_4 - \mathbf{Z}_3\mathbf{Z}_2)}{\Delta}$$

For $\mathbf{I}_{\mathbf{Z}_5} = 0$, the following must be satisfied (for a finite Δ not equal to zero):

$$\boxed{\mathbf{Z}_1\mathbf{Z}_4 = \mathbf{Z}_3\mathbf{Z}_2} \qquad \mathbf{I}_{\mathbf{Z}_5} = 0 \qquad (17.3)$$

This condition will be analyzed in greater depth later in this section.

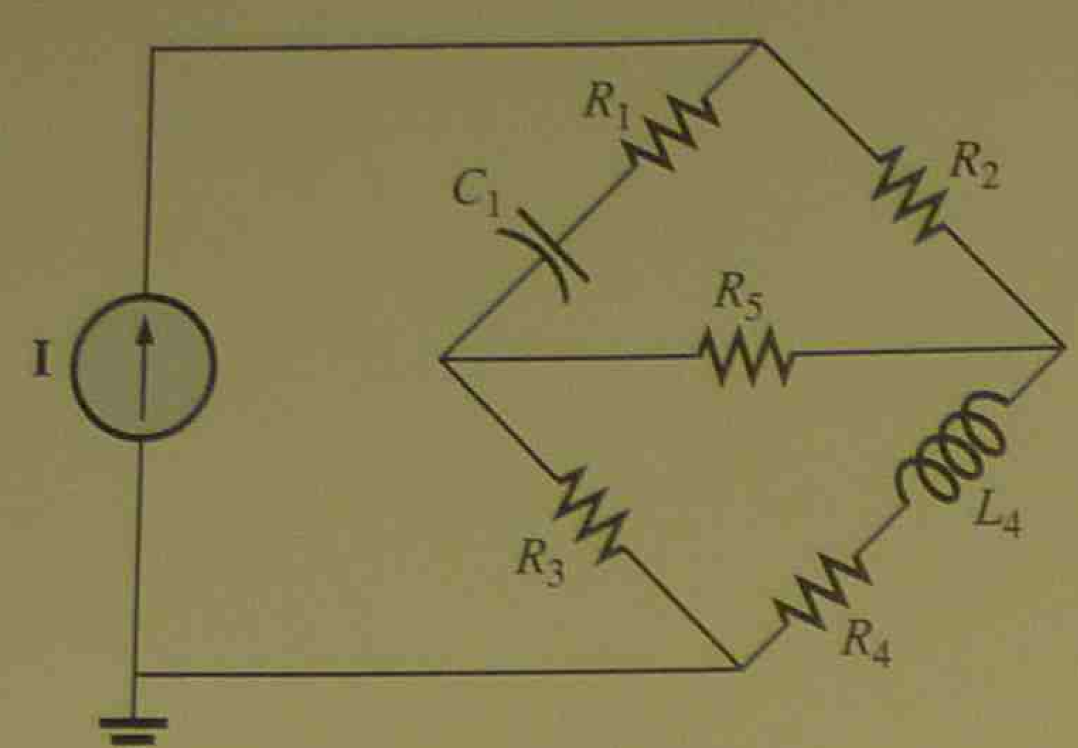


FIG. 17.34
Hay bridge.

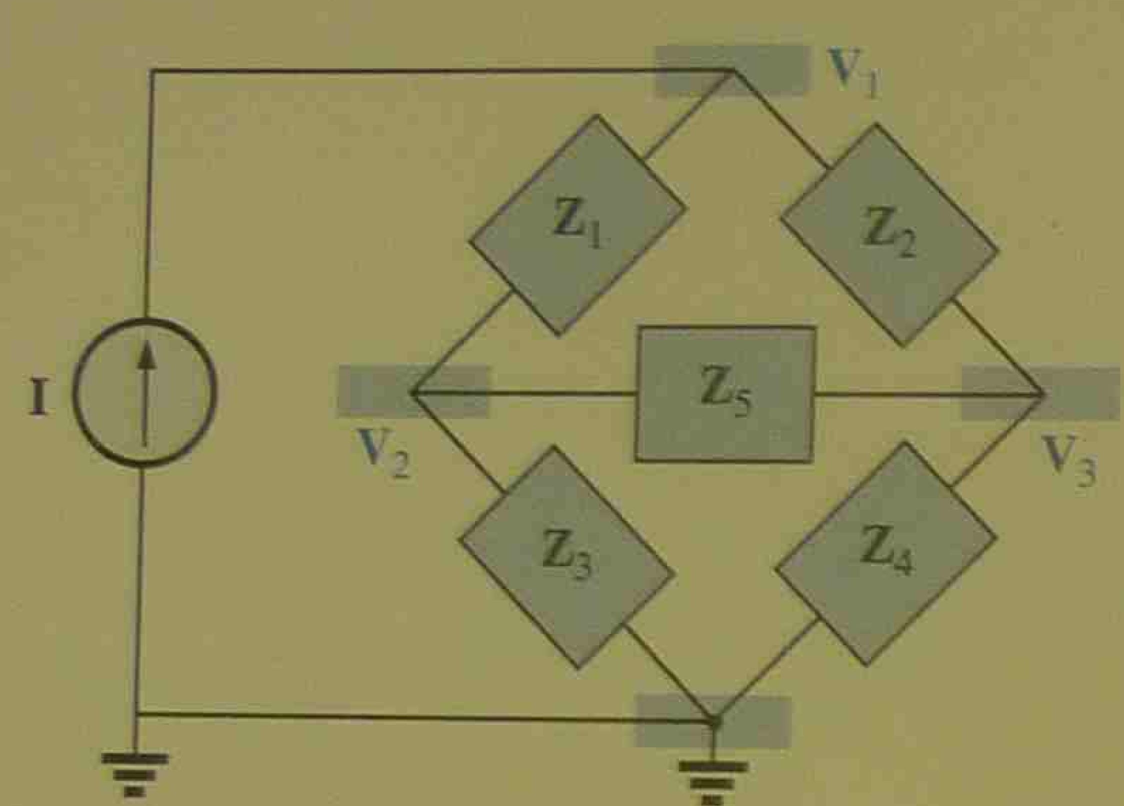


FIG. 17.35

Applying *nodal analysis* to the network of Fig. 17.34 will result in the configuration of Fig. 17.35, where

$$\mathbf{Y}_1 = \frac{1}{\mathbf{Z}_1} = \frac{1}{R_1 - jX_C} \qquad \mathbf{Y}_2 = \frac{1}{\mathbf{Z}_2} = \frac{1}{R_2}$$

$$\mathbf{Y}_3 = \frac{1}{\mathbf{Z}_3} = \frac{1}{R_3} \qquad \mathbf{Y}_4 = \frac{1}{\mathbf{Z}_4} = \frac{1}{R_4 + jX_L} \qquad \mathbf{Y}_5 = \frac{1}{R_5}$$

and

$$\begin{aligned}(\mathbf{Y}_1 + \mathbf{Y}_2)\mathbf{V}_1 - (\mathbf{Y}_1)\mathbf{V}_2 - (\mathbf{Y}_2)\mathbf{V}_3 &= \mathbf{I}\\ (\mathbf{Y}_1 + \mathbf{Y}_3 + \mathbf{Y}_5)\mathbf{V}_2 - (\mathbf{Y}_1)\mathbf{V}_1 - (\mathbf{Y}_5)\mathbf{V}_3 &= 0\\ (\mathbf{Y}_2 + \mathbf{Y}_4 + \mathbf{Y}_5)\mathbf{V}_3 - (\mathbf{Y}_2)\mathbf{V}_1 - (\mathbf{Y}_5)\mathbf{V}_2 &= 0\end{aligned}$$

which are rewritten as

$$\begin{array}{llll}\mathbf{V}_1(\mathbf{Y}_1 + \mathbf{Y}_2) - \mathbf{V}_2\mathbf{Y}_1 & & - \mathbf{V}_3\mathbf{Y}_2 & = \mathbf{I}\\ -\mathbf{V}_1\mathbf{Y}_1 & + \mathbf{V}_2(\mathbf{Y}_1 + \mathbf{Y}_3 + \mathbf{Y}_5) & - \mathbf{V}_3\mathbf{Y}_5 & = 0\\ -\mathbf{V}_1\mathbf{Y}_2 & - \mathbf{V}_2\mathbf{Y}_5 & + \mathbf{V}_3(\mathbf{Y}_2 + \mathbf{Y}_4 + \mathbf{Y}_5) & = 0\end{array}$$

Again, note the symmetry about the diagonal axis. For balance, $\mathbf{V}_{\mathbf{Z}_5} = 0$ V, and

$$\mathbf{V}_{\mathbf{Z}_5} = \mathbf{V}_2 - \mathbf{V}_3 = 0$$

From the above equations,

$$\mathbf{V}_2 = \frac{\begin{vmatrix}\mathbf{Y}_1 + \mathbf{Y}_2 & \mathbf{I} & -\mathbf{Y}_2\\ -\mathbf{Y}_1 & 0 & -\mathbf{Y}_5\\ -\mathbf{Y}_2 & 0 & (\mathbf{Y}_2 + \mathbf{Y}_4 + \mathbf{Y}_5)\end{vmatrix}}{\begin{vmatrix}\mathbf{Y}_1 + \mathbf{Y}_2 & -\mathbf{Y}_1 & -\mathbf{Y}_2\\ -\mathbf{Y}_1 & (\mathbf{Y}_1 + \mathbf{Y}_3 + \mathbf{Y}_5) & -\mathbf{Y}_5\\ -\mathbf{Y}_2 & -\mathbf{Y}_5 & (\mathbf{Y}_2 + \mathbf{Y}_4 + \mathbf{Y}_5)\end{vmatrix}}$$

$$= \frac{\mathbf{I}(\mathbf{Y}_1\mathbf{Y}_3 + \mathbf{Y}_1\mathbf{Y}_4 + \mathbf{Y}_1\mathbf{Y}_5 + \mathbf{Y}_3\mathbf{Y}_5)}{\Delta}$$

Similarly,

$$\mathbf{V}_3 = \frac{\mathbf{I}(\mathbf{Y}_1\mathbf{Y}_3 + \mathbf{Y}_3\mathbf{Y}_2 + \mathbf{Y}_1\mathbf{Y}_5 + \mathbf{Y}_3\mathbf{Y}_5)}{\Delta}$$

Note the similarities between the above equations and those obtained for mesh analysis. Then

$$\mathbf{V}_{\mathbf{Z}_5} = \mathbf{V}_2 - \mathbf{V}_3 = \frac{\mathbf{I}(\mathbf{Y}_1\mathbf{Y}_4 - \mathbf{Y}_3\mathbf{Y}_2)}{\Delta}$$

For $\mathbf{V}_{\mathbf{Z}_5} = 0$, the following must be satisfied for a finite Δ not equal to zero:

$$\boxed{\mathbf{Y}_1\mathbf{Y}_4 = \mathbf{Y}_3\mathbf{Y}_2} \qquad \mathbf{V}_{\mathbf{Z}_5} = 0 \tag{17.4}$$

However, substituting $\mathbf{Y}_1 = 1/\mathbf{Z}_1$, $\mathbf{Y}_2 = 1/\mathbf{Z}_2$, $\mathbf{Y}_3 = 1/\mathbf{Z}_3$, and $\mathbf{Y}_4 = 1/\mathbf{Z}_4$, we have

$$\frac{1}{\mathbf{Z}_1\mathbf{Z}_4} = \frac{1}{\mathbf{Z}_3\mathbf{Z}_2}$$

or

$$\mathbf{Z}_1\mathbf{Z}_4 = \mathbf{Z}_3\mathbf{Z}_2 \qquad \mathbf{V}_{\mathbf{Z}_5} = 0$$

corresponding with Eq. (17.3) obtained earlier.

Let us now investigate the balance criteria in more detail by considering the network of Fig. 17.36, where it is specified that **I**, **V** = 0.

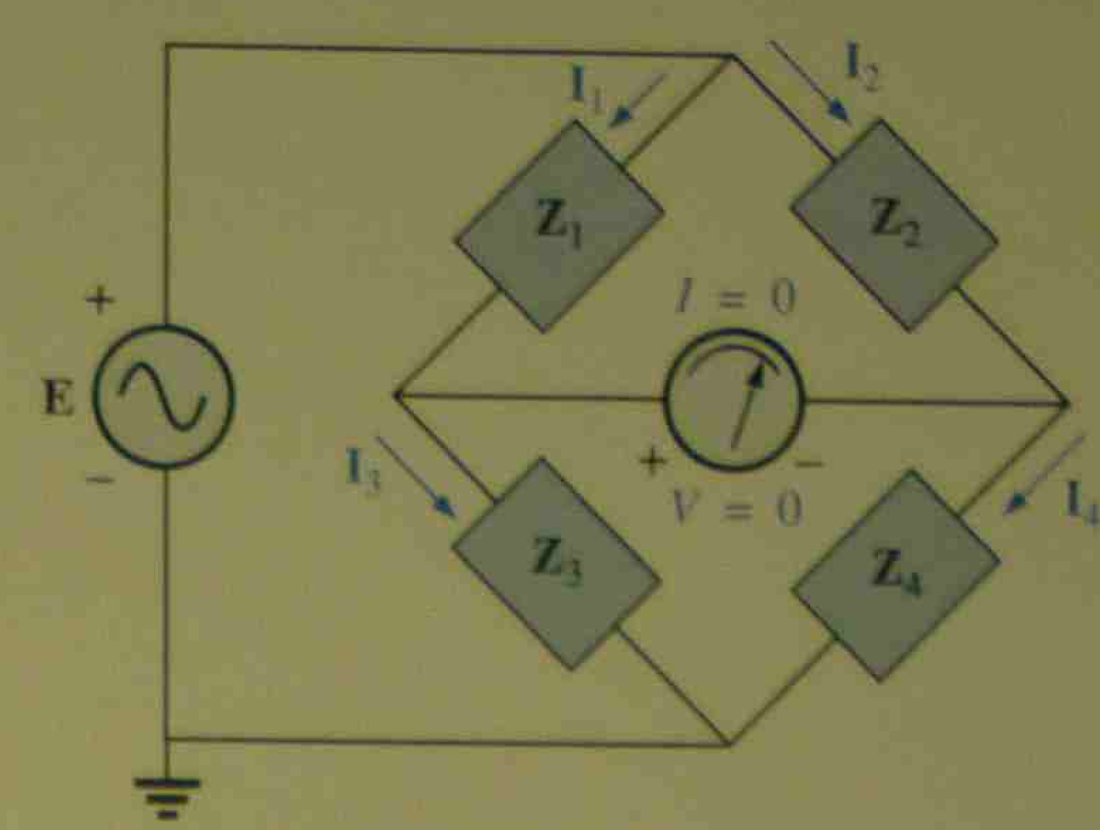


FIG. 17.36

Since **I** = 0,

$$\mathbf{I}_1 = \mathbf{I}_3 \tag{17.5a}$$

and

$$\mathbf{I}_2 = \mathbf{I}_4 \tag{17.5b}$$

In addition, for **V** = 0,

$$\mathbf{I}_1\mathbf{Z}_1 = \mathbf{I}_2\mathbf{Z}_2 \tag{17.5c}$$

and

$$\mathbf{I}_3\mathbf{Z}_3 = \mathbf{I}_4\mathbf{Z}_4 \tag{17.5d}$$

Substituting the preceding current relations into Eq. (17.5d), we have

$$\mathbf{I}_1\mathbf{Z}_3 = \mathbf{I}_2\mathbf{Z}_4$$

and

$$\mathbf{I}_2 = \frac{\mathbf{Z}_3}{\mathbf{Z}_4}\mathbf{I}_1$$

Substituting this relationship for $\mathbf{I}_2$ into Eq. (17.5c) yields

$$\mathbf{I}_1\mathbf{Z}_1 = \left(\frac{\mathbf{Z}_3}{\mathbf{Z}_4}\mathbf{I}_1\right)\mathbf{Z}_2$$

and

$$\mathbf{Z}_1\mathbf{Z}_4 = \mathbf{Z}_2\mathbf{Z}_3$$

as obtained earlier. Rearranging, we have

$$\frac{\mathbf{Z}_1}{\mathbf{Z}_3} = \frac{\mathbf{Z}_2}{\mathbf{Z}_4} \tag{17.6}$$

corresponding with Eq. (8.4) for dc resistive networks.

For the network of Fig. 17.34, which is referred to as a *Hay bridge* when $\mathbf{Z}_5$ is replaced by a sensitive galvanometer,

$$\mathbf{Z}_1 = R_1 - jX_C$$
$$\mathbf{Z}_2 = R_2$$
$$\mathbf{Z}_3 = R_3$$
$$\mathbf{Z}_4 = R_4 + jX_L$$

This particular network is used for measuring the resistance and inductance of coils in which the resistance is a small fraction of the reactance X_L.

Substitute into Eq. (17.6) in the following form:

$$\mathbf{Z}_2\mathbf{Z}_3 = \mathbf{Z}_4\mathbf{Z}_1$$
$$R_2R_3 = (R_4 + jX_L)(R_1 - jX_C)$$

or
$$R_2R_3 = R_1R_4 + j(R_1X_L - R_4X_C) + X_CX_L$$

so that
$$R_2R_3 + j0 = (R_1R_4 + X_CX_L) + j(R_1X_L - R_4X_C)$$

In order for the equations to be equal, *the real and imaginary parts must be equal*. Therefore, for a balanced Hay bridge,

$$\boxed{R_2R_3 = R_1R_4 + X_CX_L} \tag{17.7a}$$

and
$$\boxed{0 = R_1X_L - R_4X_C} \tag{17.7b}$$

or substituting
$$X_L = \omega L \text{ and } X_C = \frac{1}{\omega C}$$

we have
$$X_CX_L = \left(\frac{1}{\omega C}\right)(\omega L) = \frac{L}{C}$$

and
$$R_2R_3 = R_1R_4 + \frac{L}{C}$$

with
$$R_1\omega L = \frac{R_4}{\omega C}$$

Solving for R_4 in the last equation yields

$$R_4 = \omega^2 LCR_1$$

and substituting into the previous equation, we have

$$R_2R_3 = R_1(\omega^2 LCR_1) + \frac{L}{C}$$

Multiply through by C and factor:

$$CR_2R_3 = L(\omega^2C^2R_1^2 + 1)$$

and
$$\boxed{L = \frac{CR_2R_3}{1 + \omega^2C^2R_1^2}} \tag{17.8a}$$

with further algebra yielding

$$\boxed{R_4 = \frac{\omega^2C^2R_1R_2R_3}{1 + \omega^2C^2R_1^2}} \tag{17.8b}$$

Equations (17.7) and (17.8) are the balance conditions for the Hay bridge. Note that each is frequency dependent. For different frequencies, the resistive and capacitive elements must vary for a particular coil to achieve balance. For a coil placed in the Hay bridge as shown in Fig. 17.35, the resistance and inductance of the coil can be determined by Eqs. (17.8a) and (17.8b) when balance is achieved.

The bridge of Fig. 17.32 is referred to as a *Maxwell bridge* when $\mathbf{Z}_5$ is replaced by a sensitive galvanometer. This setup is used for inductance measurements when the resistance of the coil is large enough not to require a Hay bridge.

Application of Eq. (17.6) will yield the following results for the inductance and resistance of the inserted coil:

$$L = CR_2R_3 \tag{17.9}$$

$$R_4 = \frac{R_2R_3}{R_1} \tag{17.10}$$

The derivation of these equations is quite similar to that employed for the Hay bridge. Keep in mind that the real and imaginary parts must be equal.

One remaining popular bridge is the *capacitance comparison bridge* of Fig. 17.37. An unknown capacitance and its associated resistance can be determined using this bridge. Application of Eq. (17.6) will yield the following results:

$$C_4 = C_3\frac{R_1}{R_2} \tag{17.11}$$

$$R_4 = \frac{R_2R_3}{R_1} \tag{17.12}$$

The derivation of these equations will appear as a problem at the end of the chapter.

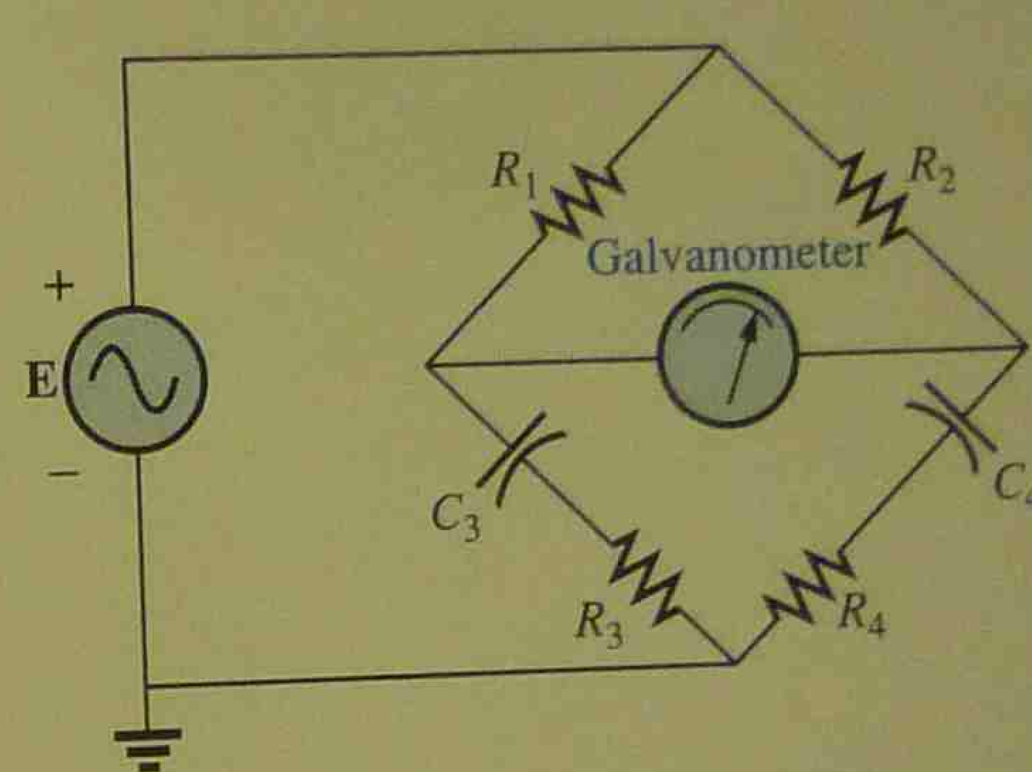


FIG. 17.37
Capacitance comparison bridge.

17.7 Δ-Y, Y-Δ CONVERSIONS

The Δ-Y, Y-Δ (or π-T, T-π as defined in Section 8.12) conversions for ac circuits will not be derived here since the development corresponds exactly with that for dc circuits. Taking the Δ-Y configuration shown in Fig. 17.38, we find the general equations for the impedances of the Y in terms of those for the Δ:

$$\mathbf{Z}_1 = \frac{\mathbf{Z}_B\mathbf{Z}_C}{\mathbf{Z}_A + \mathbf{Z}_B + \mathbf{Z}_C} \tag{17.13}$$

$$\mathbf{Z}_2 = \frac{\mathbf{Z}_A\mathbf{Z}_C}{\mathbf{Z}_A + \mathbf{Z}_B + \mathbf{Z}_C} \tag{17.14}$$

$$\mathbf{Z}_3 = \frac{\mathbf{Z}_A\mathbf{Z}_B}{\mathbf{Z}_A + \mathbf{Z}_B + \mathbf{Z}_C} \tag{17.15}$$

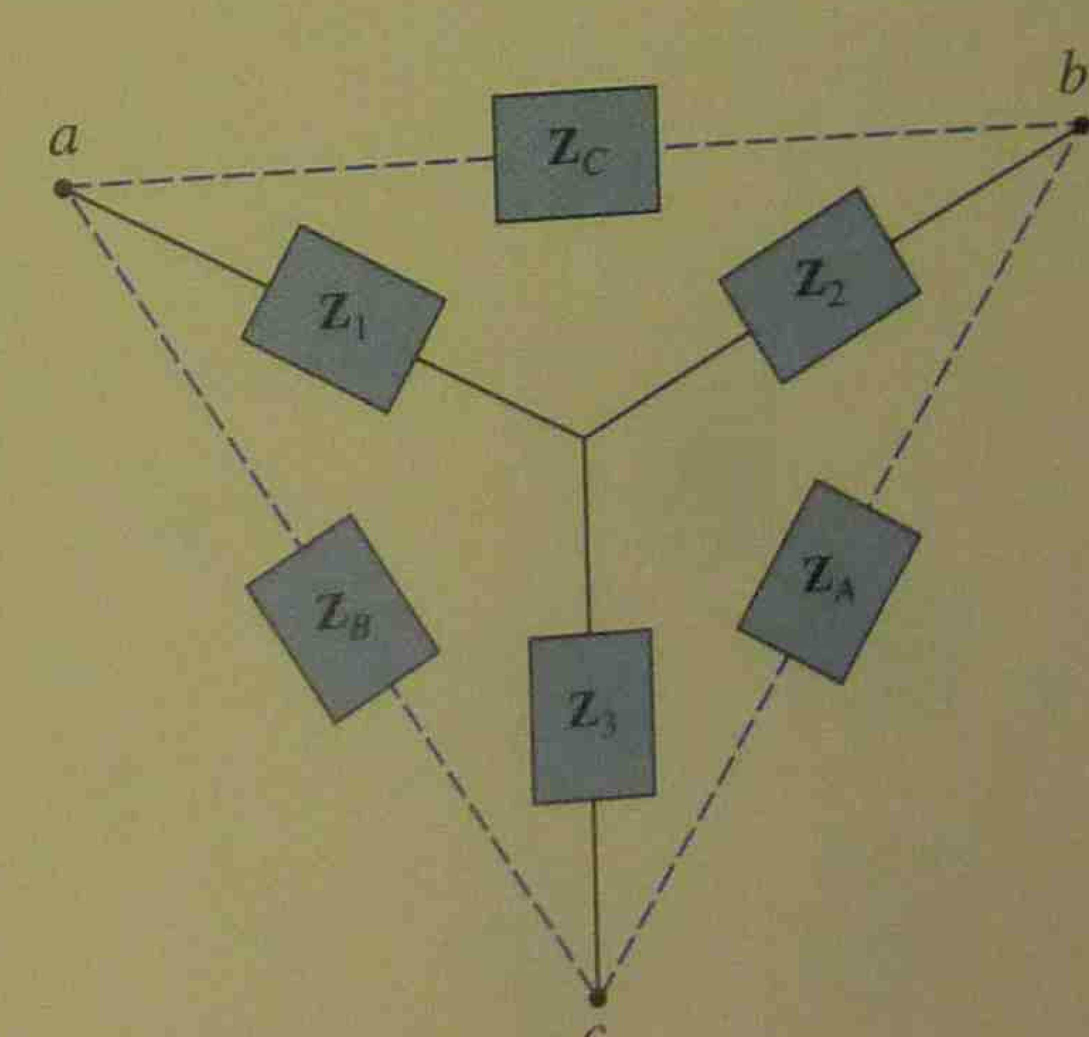


FIG. 17.38
Δ-Y configuration.

For the impedances of the Δ in terms of those for the Y, the equations are

$$\mathbf{Z}_B = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_1\mathbf{Z}_3 + \mathbf{Z}_2\mathbf{Z}_3}{\mathbf{Z}_2} \qquad (17.16)$$

$$\mathbf{Z}_A = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_1\mathbf{Z}_3 + \mathbf{Z}_2\mathbf{Z}_3}{\mathbf{Z}_1} \qquad (17.17)$$

$$\mathbf{Z}_C = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_1\mathbf{Z}_3 + \mathbf{Z}_2\mathbf{Z}_3}{\mathbf{Z}_3} \qquad (17.18)$$

Note that each impedance of the Y is equal to the product of the impedances in the two closest branches of the Δ, divided by the sum of the impedances in the Δ.

Further, the value of each impedance of the Δ is equal to the sum of the possible product combinations of the impedances of the Y, divided by the impedances of the Y farthest from the impedance to be determined.

Drawn in different forms (Fig. 17.39), they are also referred to as the T and π configurations.

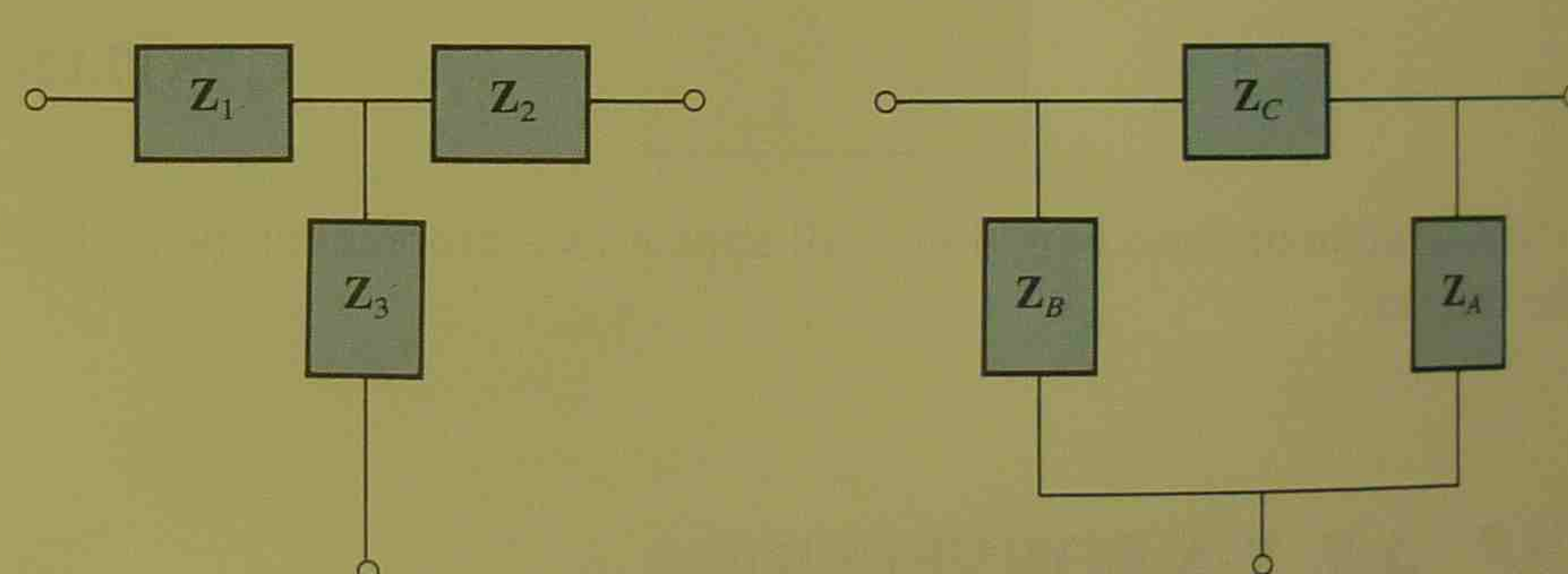


FIG. 17.39

In the study of dc networks, we found that if all of the resistors of the Δ or Y were the same, the conversion from one to the other could be accomplished using the equation

$$R_\Delta = 3R_Y \quad \text{or} \quad R_Y = \frac{R_\Delta}{3}$$

For ac networks,

$$\mathbf{Z}_\Delta = 3\mathbf{Z}_Y \quad \text{or} \quad \mathbf{Z}_Y = \frac{\mathbf{Z}_\Delta}{3} \qquad (17.19)$$

Be careful when using this simplified form. It is not sufficient for all the impedances of the Δ or Y to be of the same magnitude: *The angle associated with each must also be the same.*

EXAMPLE 17.15 Find the total impedance $\mathbf{Z}_T$ of the network of Fig. 17.40.

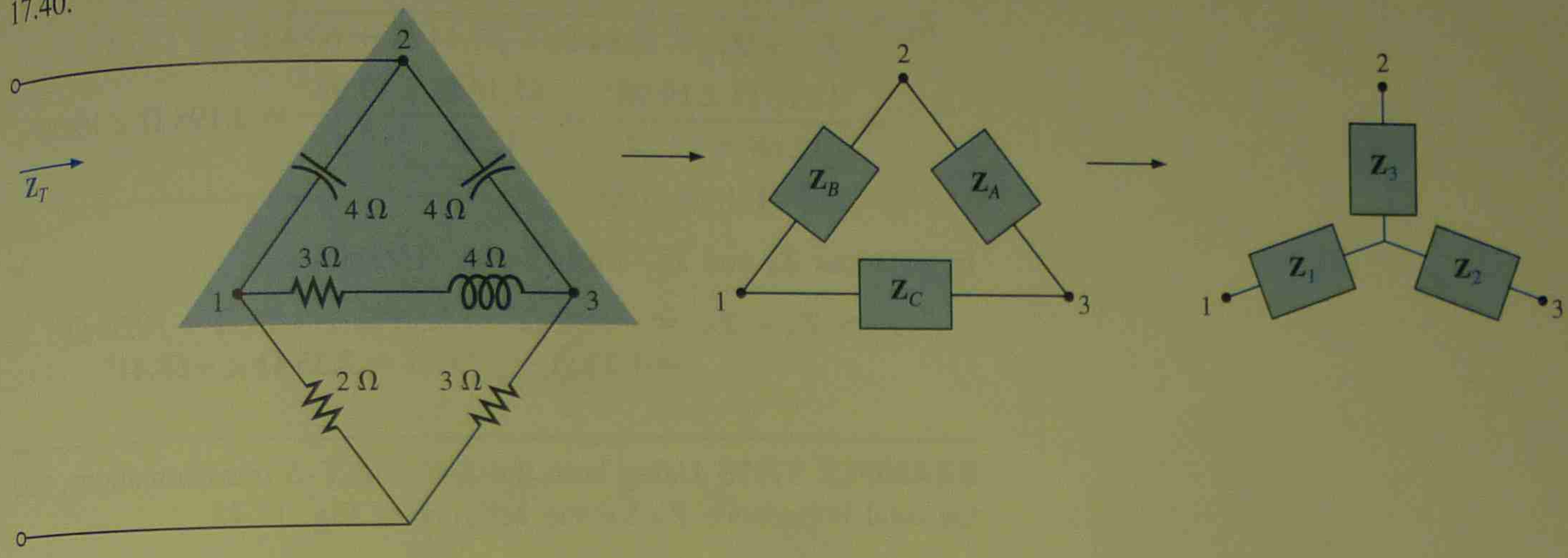


FIG. 17.40

Solution:

$$\mathbf{Z}_B = -j4 \qquad \mathbf{Z}_A = -j4 \qquad \mathbf{Z}_C = 3 + j4$$

$$\mathbf{Z}_1 = \frac{\mathbf{Z}_B\mathbf{Z}_C}{\mathbf{Z}_A + \mathbf{Z}_B + \mathbf{Z}_C} = \frac{(-j4\ \Omega)(3\ \Omega + j4\ \Omega)}{(-j4\ \Omega) + (-j4\ \Omega) + (3\ \Omega + j4\ \Omega)}$$

$$= \frac{(4\ \angle -90^\circ)(5\ \angle 53.13^\circ)}{3 - j4} = \frac{20\ \angle -36.87^\circ}{5\ \angle -53.13^\circ}$$

$$= 4\ \Omega\ \angle 16.13^\circ = 3.84\ \Omega + j1.11\ \Omega$$

$$\mathbf{Z}_2 = \frac{\mathbf{Z}_A\mathbf{Z}_C}{\mathbf{Z}_A + \mathbf{Z}_B + \mathbf{Z}_C} = \frac{(-j4\ \Omega)(3\ \Omega + j4\ \Omega)}{5\ \Omega\ \angle -53.13^\circ}$$

$$= 4\ \Omega\ \angle 16.13^\circ = 3.84\ \Omega + j1.11\ \Omega$$

Recall from the study of dc circuits that if two branches of the Y or Δ were the same, the corresponding Δ or Y, respectively, would also have two similar branches. In this example, $\mathbf{Z}_A = \mathbf{Z}_B$. Therefore, $\mathbf{Z}_1 = \mathbf{Z}_2$, and

$$\mathbf{Z}_3 = \frac{\mathbf{Z}_A\mathbf{Z}_B}{\mathbf{Z}_A + \mathbf{Z}_B + \mathbf{Z}_C} = \frac{(-j4\ \Omega)(-j4\ \Omega)}{5\ \Omega\ \angle -53.13^\circ}$$

$$= \frac{16\ \angle -180^\circ}{5\ \angle -53.13^\circ} = 3.2\ \Omega\ \angle -126.87^\circ = -1.92\ \Omega - j2.56\ \Omega$$

Replace the Δ by the Y (Fig. 17.41):

$$\mathbf{Z}_1 = 3.84\ \Omega + j1.11\ \Omega \qquad \mathbf{Z}_2 = 3.84\ \Omega + j1.11\ \Omega$$
$$\mathbf{Z}_3 = -1.92\ \Omega - j2.56\ \Omega \qquad \mathbf{Z}_4 = 2\ \Omega \qquad \mathbf{Z}_5 = 3\ \Omega$$

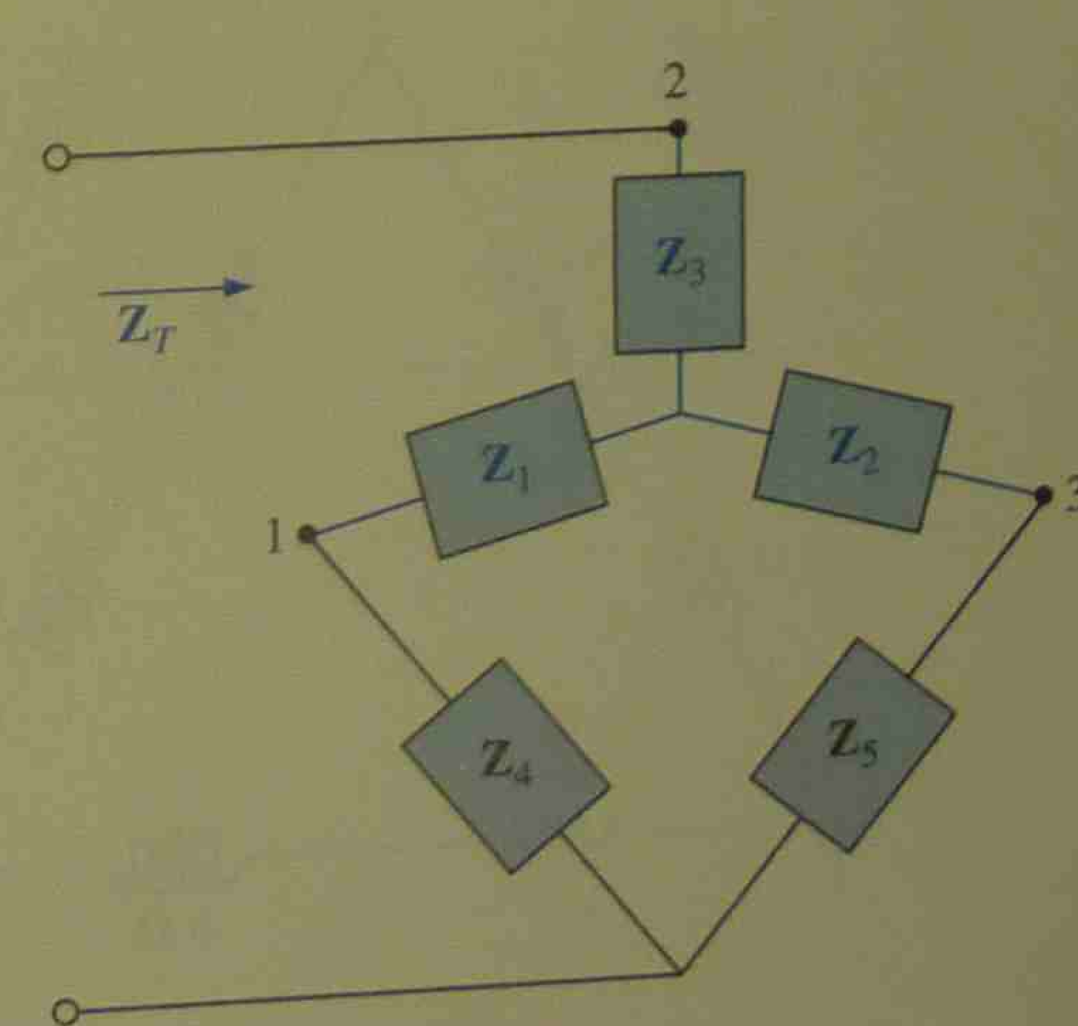


FIG. 17.41

Impedances $\mathbf{Z}_1$ and $\mathbf{Z}_4$ are in series:

$$\mathbf{Z}_{T_1} = \mathbf{Z}_1 + \mathbf{Z}_4 = 3.84\ \Omega + j1.11\ \Omega + 2\ \Omega = 5.84\ \Omega + j1.11\ \Omega$$
$$= 5.94\ \Omega\ \angle 10.76^\circ$$

Impedances $\mathbf{Z}_2$ and $\mathbf{Z}_5$ are in series:

$$\mathbf{Z}_{T_2} = \mathbf{Z}_2 + \mathbf{Z}_5 = 3.84\ \Omega + j1.11\ \Omega + 3\ \Omega = 6.84\ \Omega + j1.11\ \Omega$$
$$= 6.93\ \Omega\ \angle 9.22^\circ$$

Impedances $\mathbf{Z}_{T_1}$ and $\mathbf{Z}_{T_2}$ are in parallel:

$$\begin{aligned}\mathbf{Z}_{T_3} &= \frac{\mathbf{Z}_{T_1}\mathbf{Z}_{T_2}}{\mathbf{Z}_{T_1}+\mathbf{Z}_{T_2}} = \frac{(5.94\ \Omega\ \angle 10.76^\circ)(6.93\ \Omega\ \angle 9.22^\circ)}{5.84\ \Omega + j1.11\ \Omega + 6.84\ \Omega + j1.11\ \Omega} \\ &= \frac{41.16\ \Omega\ \angle 19.98^\circ}{12.68 + j2.22} = \frac{41.16\ \Omega\ \angle 19.98^\circ}{12.87\ \angle 9.93^\circ} = 3.198\ \Omega\ \angle 10.05^\circ \\ &= 3.15\ \Omega + j0.56\ \Omega\end{aligned}$$

Impedances $\mathbf{Z}_3$ and $\mathbf{Z}_{T_3}$ are in series. Therefore,

$$\begin{aligned}\mathbf{Z}_T = \mathbf{Z}_3 + \mathbf{Z}_{T_3} &= -1.92\ \Omega - j2.56\ \Omega + 3.15\ \Omega + j0.56\ \Omega \\ &= 1.23\ \Omega - j2.0\ \Omega = \mathbf{2.35\ \Omega\ \angle -58.41^\circ}\end{aligned}$$

EXAMPLE 17.16 Using both the Δ-Y and Y-Δ transformations, find the total impedance $\mathbf{Z}_T$ for the network of Fig. 17.42.

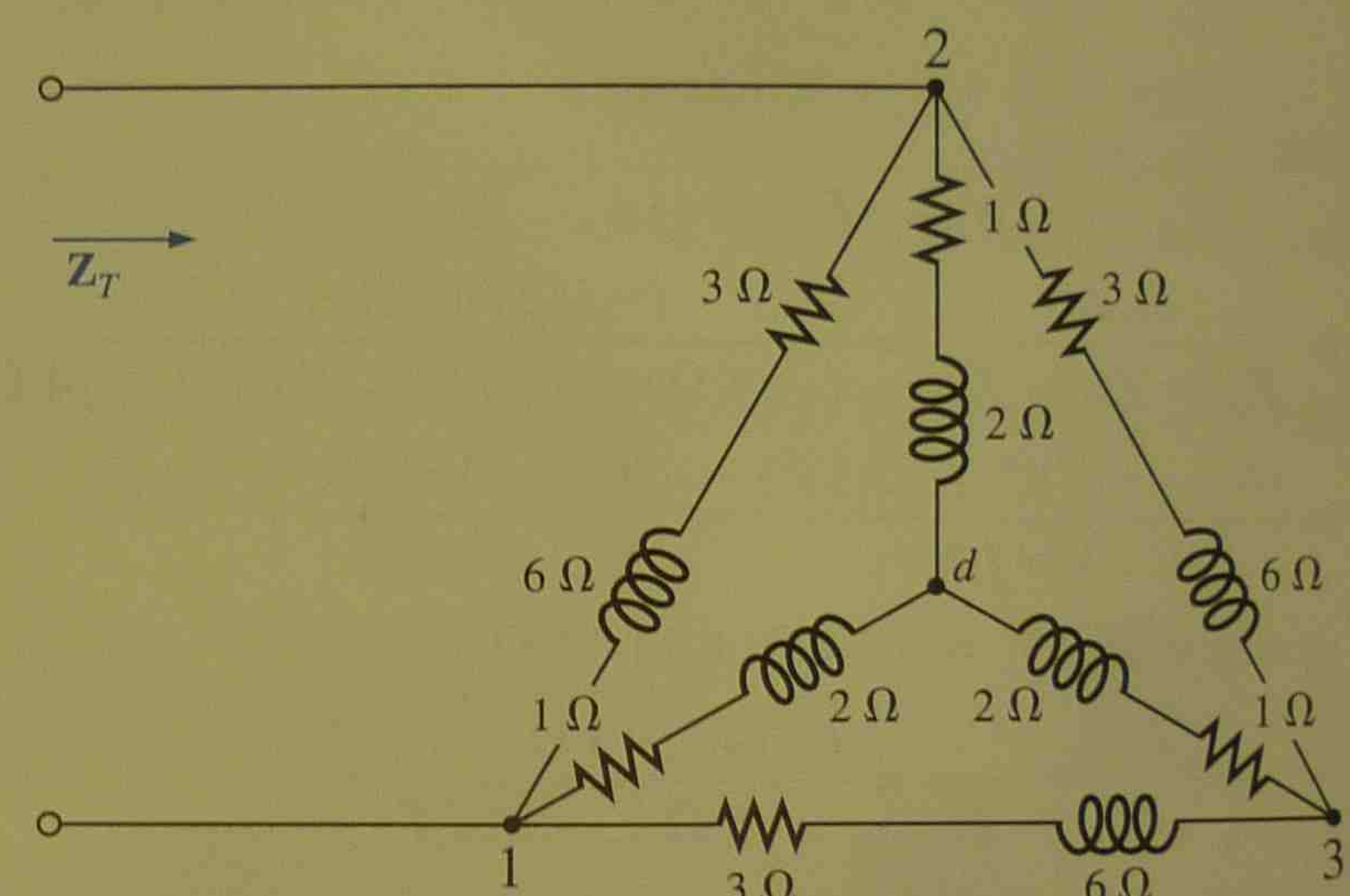


FIG. 17.42

Solution: *Using the Δ-Y transformation,* we obtain Fig. 17.43. In this case, since both systems are balanced (same impedance in each branch),

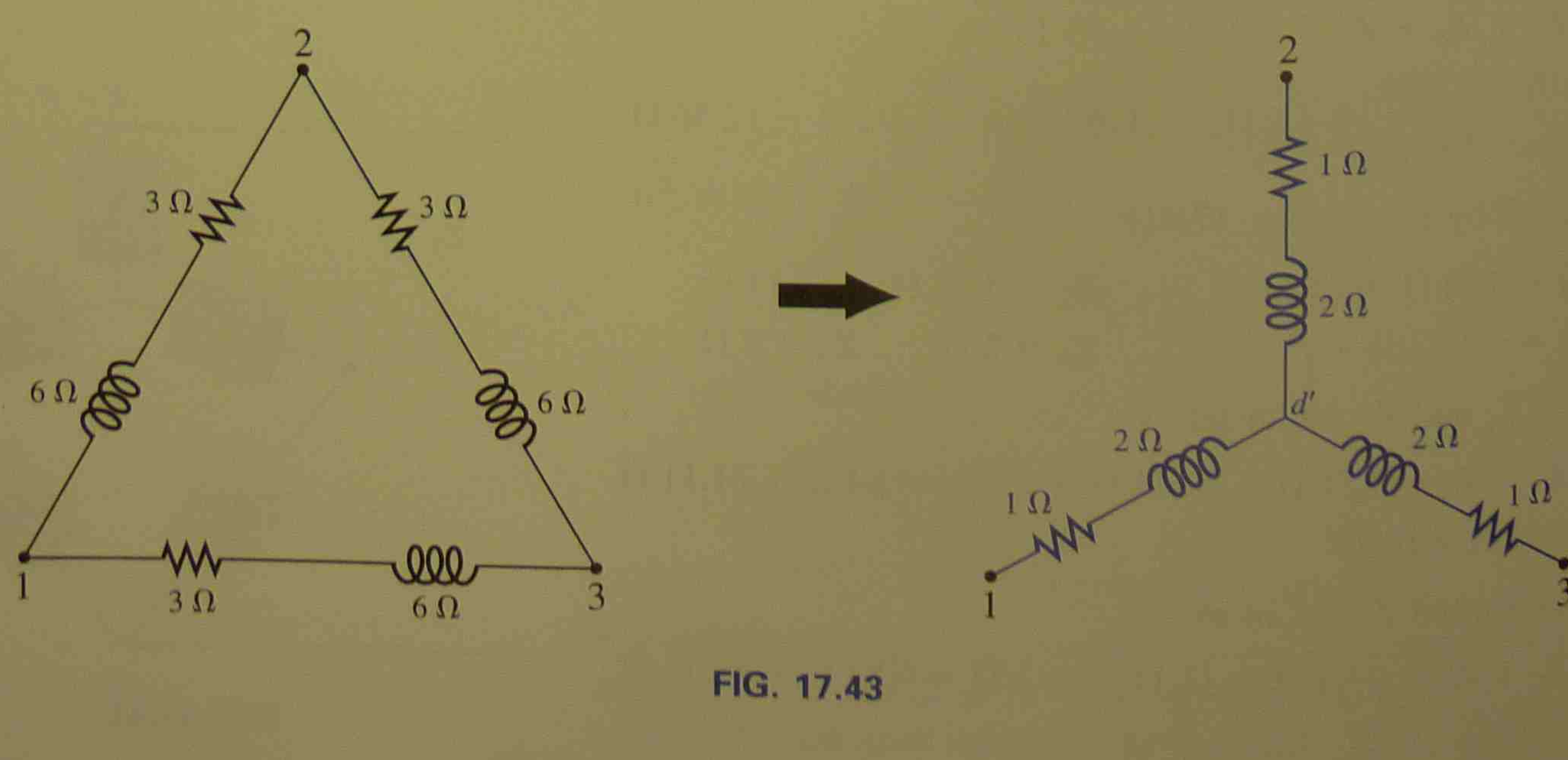


FIG. 17.43

the center point d' of the transformed Δ will be the same as the point d of the original Y:

$$\mathbf{Z}_Y = \frac{\mathbf{Z}_\Delta}{3} = \frac{3\ \Omega + j6\ \Omega}{3} = 1\ \Omega + j2\ \Omega$$

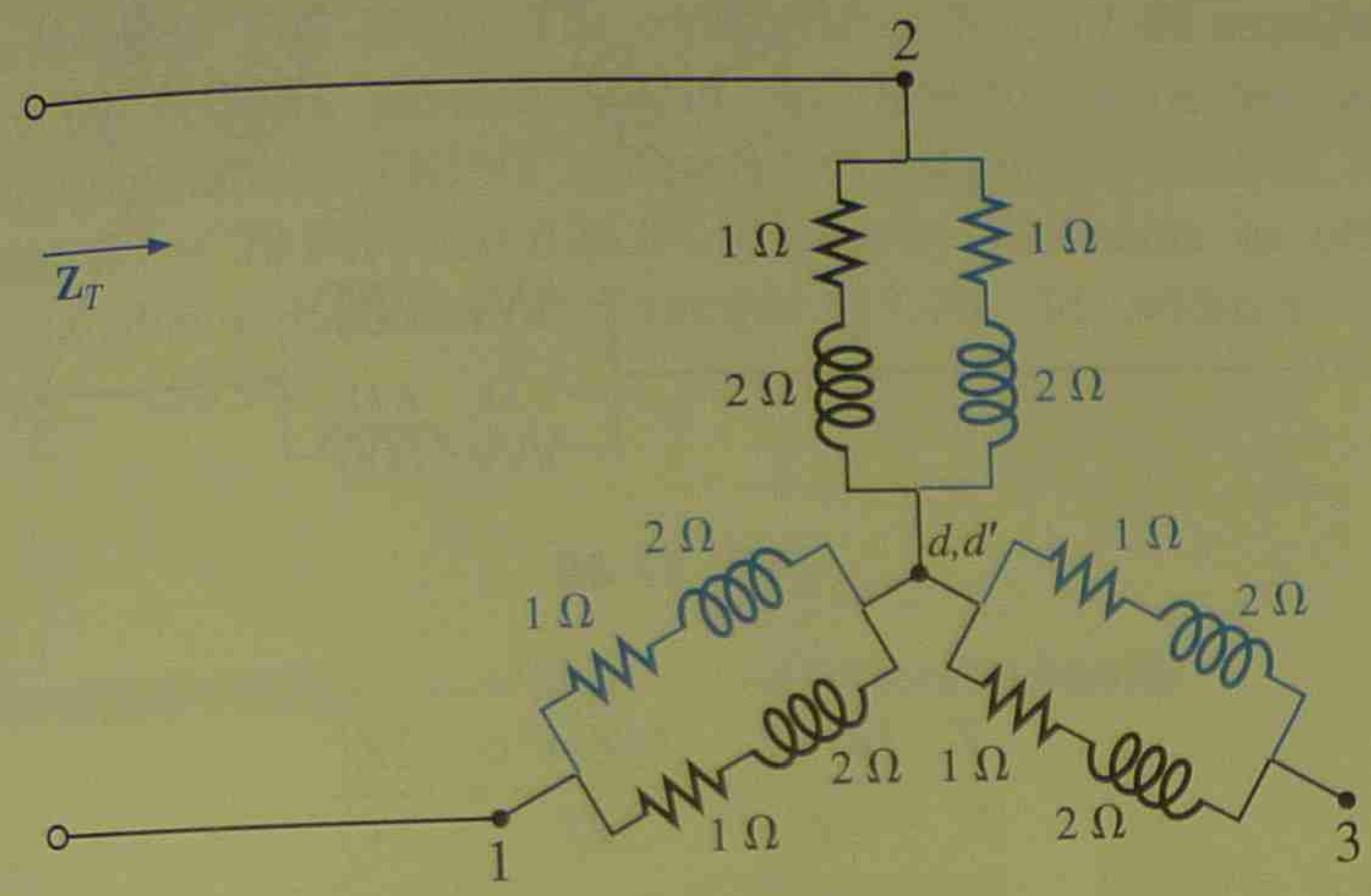


FIG. 17.44

and (Fig. 17.44)

$$\mathbf{Z}_T = 2\left(\frac{1\ \Omega + j2\ \Omega}{2}\right) = \mathbf{1\ \Omega + \mathit{j}2\ \Omega}$$

Using the Y-Δ transformation (Fig. 17.45), we obtain

$$\mathbf{Z}_\Delta = 3\mathbf{Z}_Y = 3(1\ \Omega + j2\ \Omega) = 3\ \Omega + j6\ \Omega$$

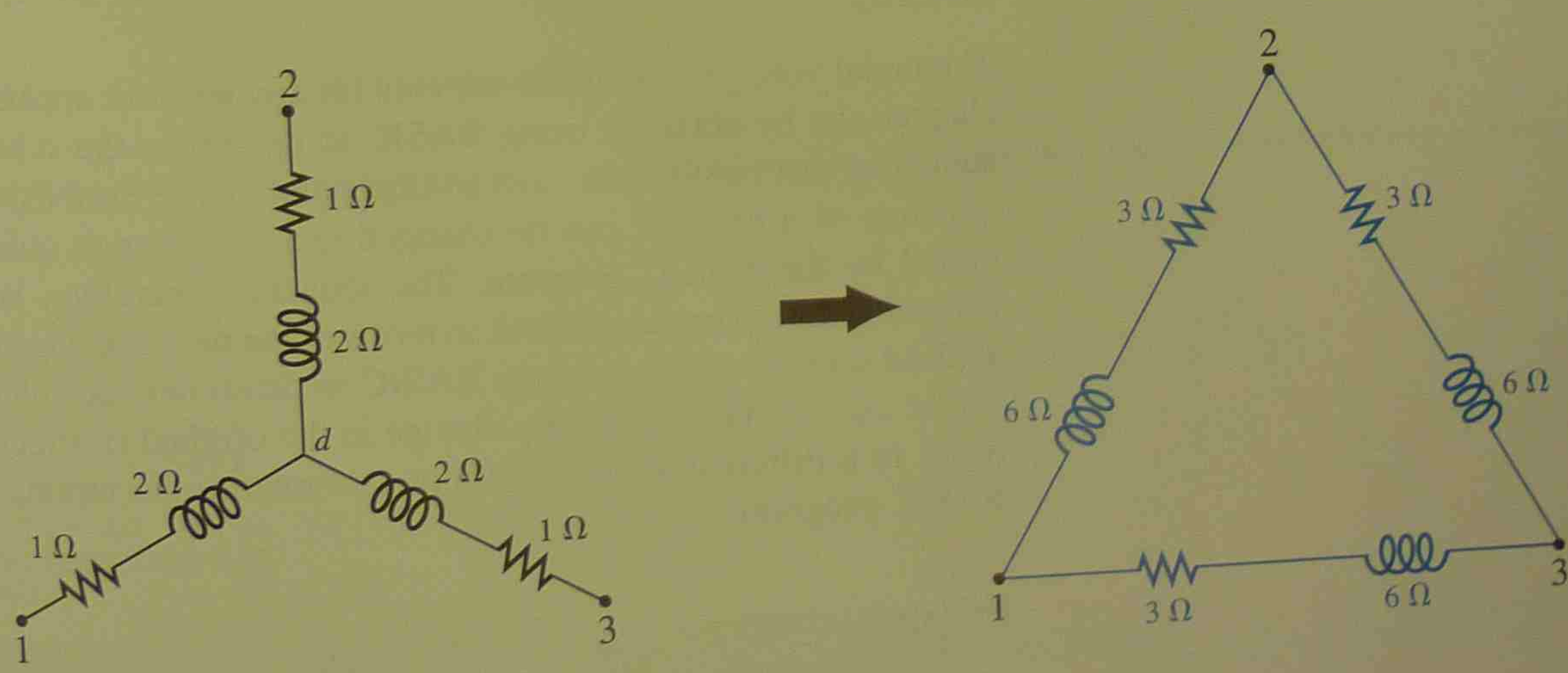


FIG. 17.45

Each resulting parallel combination in Fig. 17.46 will have the following impedance:

$$\mathbf{Z}' = \frac{3\ \Omega + j6\ \Omega}{2} = 1.5\ \Omega + j3\ \Omega$$

18

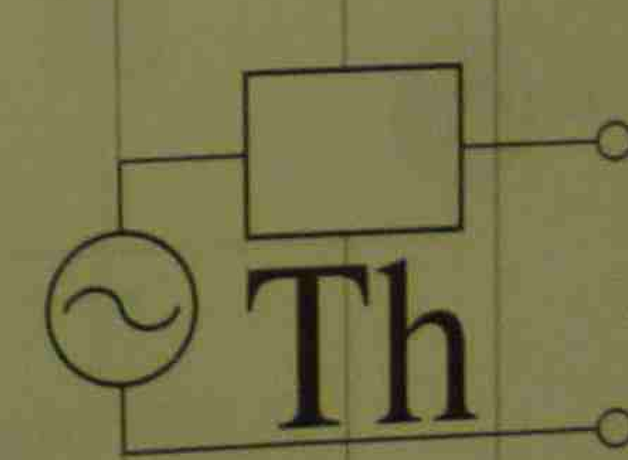
Network Theorems (ac)

18.1 INTRODUCTION

This chapter will parallel Chapter 9, which dealt with network theorems as applied to dc networks. It would be time well spent to review each theorem in Chapter 9 before beginning this chapter, as many of the comments offered there will not be repeated.

Due to the need for developing confidence in the application of the various theorems to networks with controlled (dependent) sources, some sections have been divided into two parts: independent sources and dependent sources.

Theorems to be considered in detail include the superposition theorem, Thevenin and Norton theorems, and the maximum power theorem. The substitution and reciprocity theorems and Millman's theorem are not discussed in detail here, since a review of Chapter 9 will enable you to apply them to sinusoidal ac networks with little difficulty.



18.2 SUPERPOSITION THEOREM

You will recall from Chapter 9 that the superposition theorem eliminated the need for solving simultaneous linear equations by considering the effects of each source independently. To consider the effects of each source, we had to remove the remaining sources. This was accomplished by setting voltage sources to zero (short-circuit representation) and current sources to zero (open-circuit representation). The current through, or voltage across, a portion of the network produced by each source was then added algebraically to find the total solution for the current or voltage.

The only variation in applying this method to ac networks with independent sources is that we will now be working with impedances and phasors instead of just resistors and real numbers.

The superposition theorem is not applicable to power effects in ac networks, since we are still dealing with a nonlinear relationship. It can be applied to networks with sources of different frequencies only if the total response for *each* frequency is found independently and the results are expanded in a nonsinusoidal expression as appearing in Chapter 24.

One of the most frequent applications of the superposition theorem is to electronic systems in which the dc and ac analysis are treated separately and the total solution is the sum of the two. It is an important application of the theorem because the impact of the reactive elements changes dramatically in response to the two types of independent sources. In addition, the dc analysis of an electronic system can often define important parameters for the ac analysis. The fourth example will demonstrate the impact of the applied source on the general configuration of the network.

We will first consider networks with only independent sources to provide a close association with the analysis of Chapter 9.

Independent Sources

EXAMPLE 18.1 Using the superposition theorem, find the current **I** through the 4-Ω reactance (X_{L_2}) of Fig. 18.1.

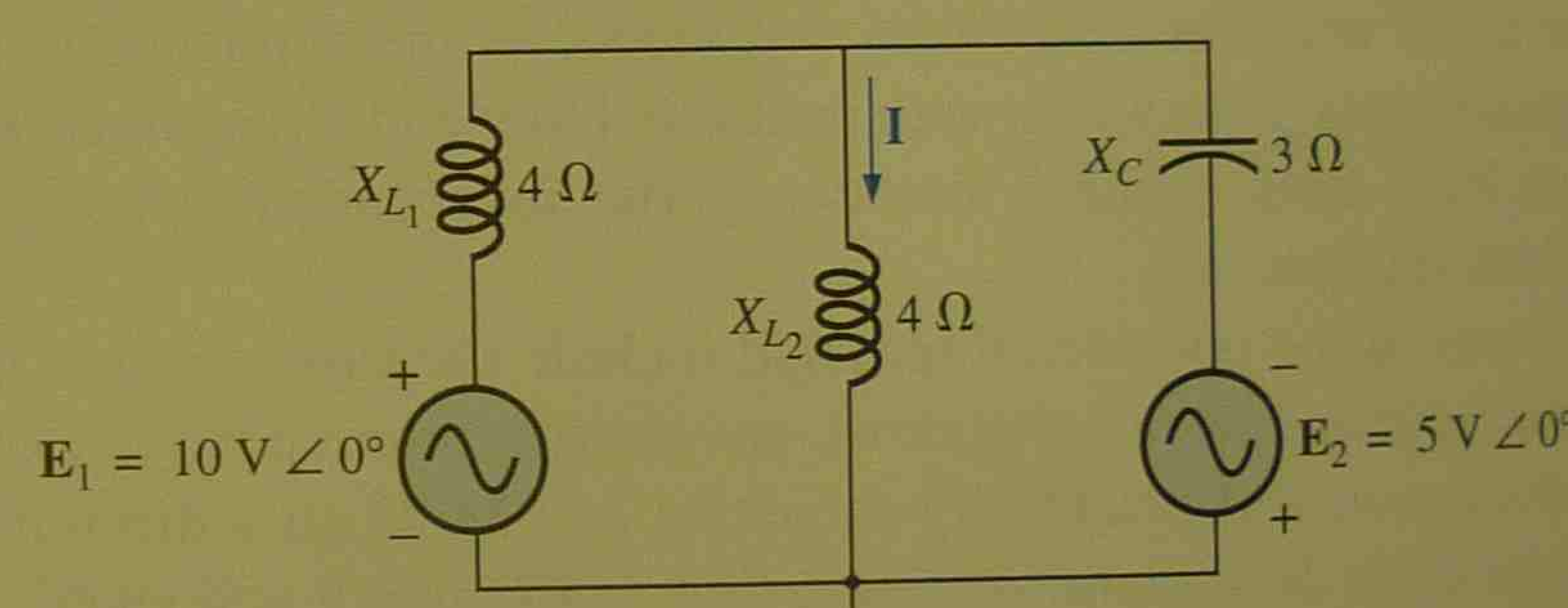


FIG. 18.1

Solution: For the redrawn circuit (Fig. 18.2),

$$\mathbf{Z}_1 = +jX_{L_1} = j4\ \Omega$$
$$\mathbf{Z}_2 = +jX_{L_2} = j4\ \Omega$$
$$\mathbf{Z}_3 = -jX_C = -j3\ \Omega$$

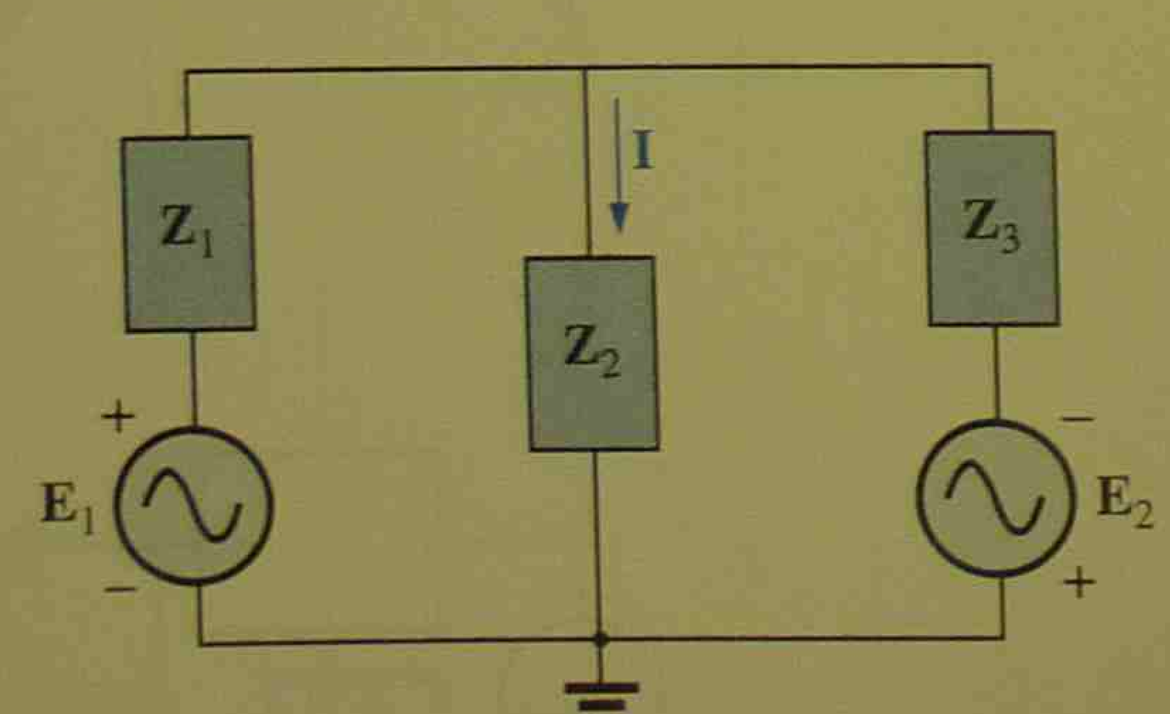


FIG. 18.2

Considering the effects of the voltage source $\mathbf{E}_1$ (Fig. 18.3), we have

$$\mathbf{Z}_{2\|3} = \frac{\mathbf{Z}_2\mathbf{Z}_3}{\mathbf{Z}_2 + \mathbf{Z}_3} = \frac{(j4\ \Omega)(-j3\ \Omega)}{j4\ \Omega - j3\ \Omega} = \frac{12\ \Omega}{j} = -j12\ \Omega$$
$$= 12\ \Omega\ \angle -90°$$

$$I_{s_1} = \frac{\mathbf{E}_1}{\mathbf{Z}_{2\|3} + \mathbf{Z}_1} = \frac{10\ \text{V}\ \angle 0°}{-j12\ \Omega + j4\ \Omega} = \frac{10\ \text{V}\ \angle 0°}{8\ \Omega\ \angle -90°}$$
$$= 1.25\ \text{A}\ \angle 90°$$

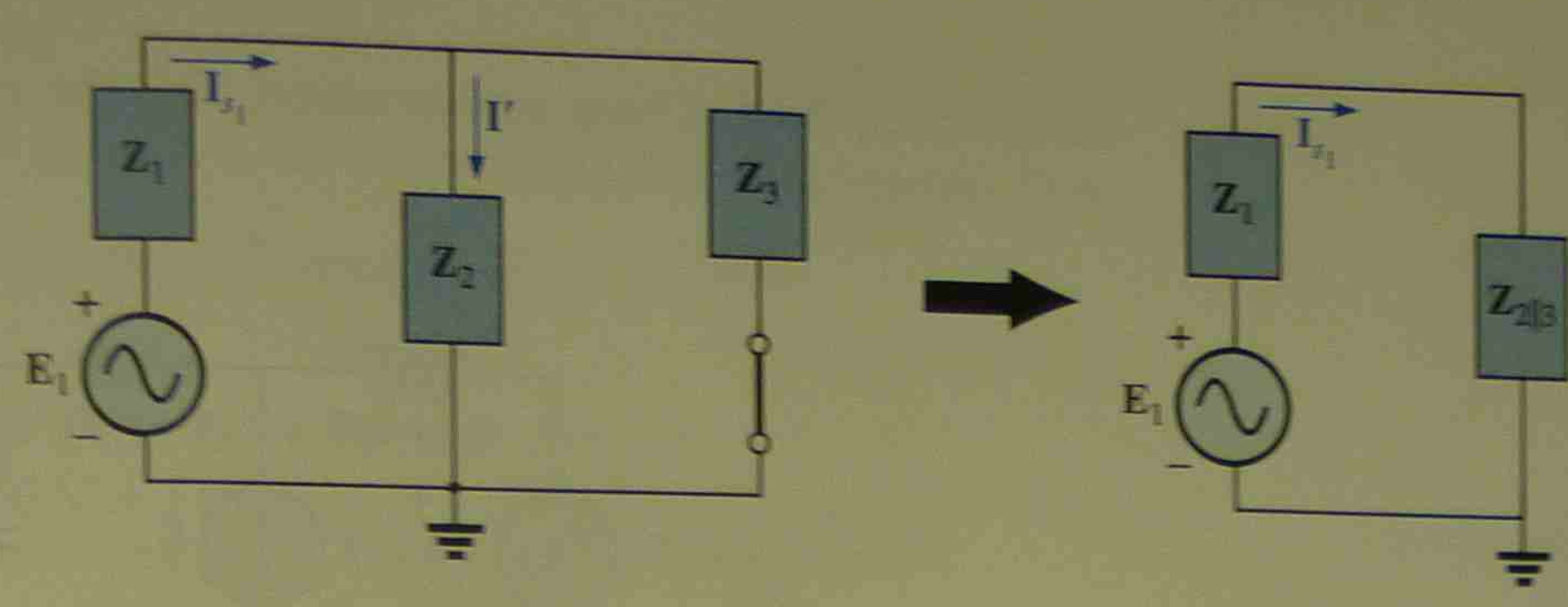


FIG. 18.3

and

$$I' = \frac{Z_3 I_{s_1}}{Z_2 + Z_3} \qquad \text{(current divider rule)}$$

$$= \frac{(-j3\,\Omega)(j1.25\text{ A})}{j4\,\Omega - j3\,\Omega} = \frac{3.75\text{ A}}{j1} = 3.75\text{ A}\angle -90^\circ$$

Considering the effects of the voltage source E_2 (Fig. 18.4), we have

$$Z_{1\|2} = \frac{Z_1}{N} = \frac{j4\,\Omega}{2} = j2\,\Omega$$

$$I_{s_2} = \frac{E_2}{Z_{1\|2} + Z_3} = \frac{5\text{ V}\angle 0^\circ}{j2\,\Omega - j3\,\Omega} = \frac{5\text{ V}\angle 0^\circ}{1\,\Omega\angle -90^\circ} = 5\text{ A}\angle 90^\circ$$

and

$$I'' = \frac{I_{s_2}}{2} = 2.5\text{ A}\angle 90^\circ$$

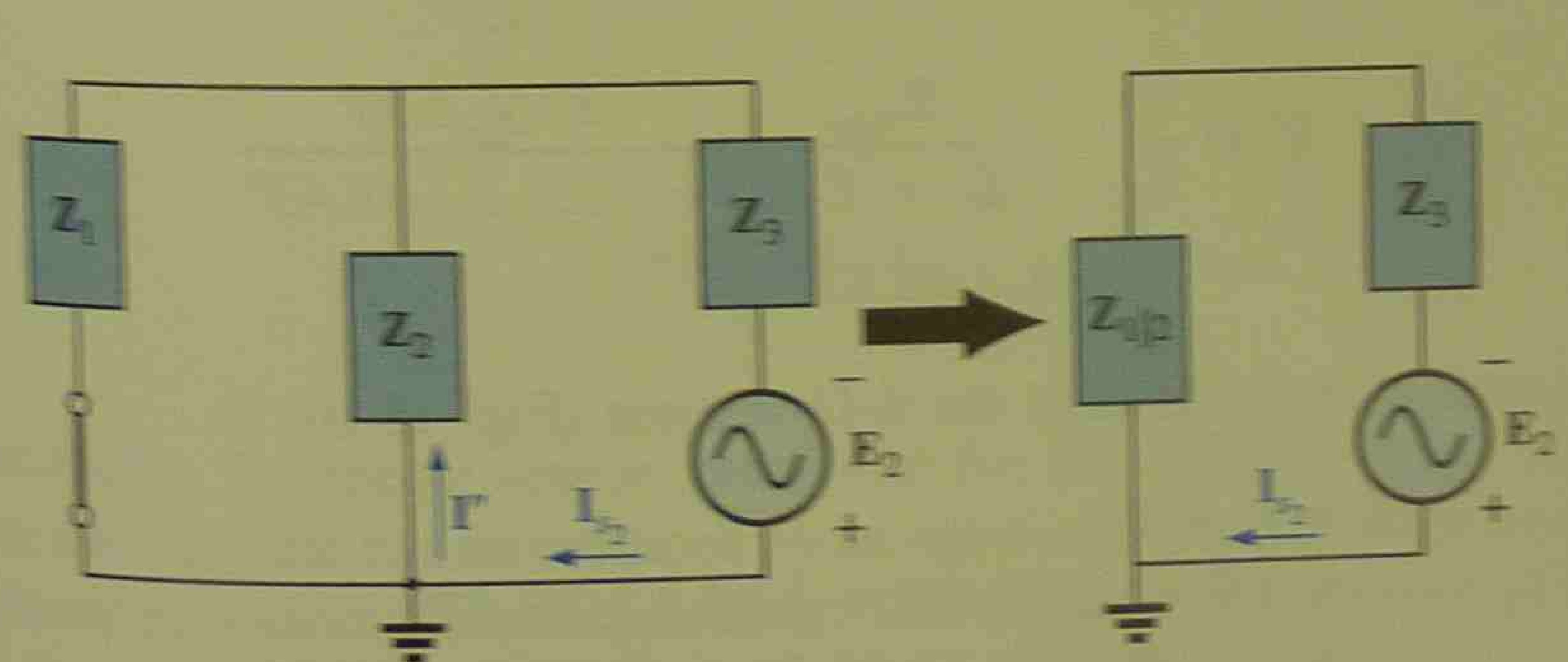


FIG. 18.4

The total current through the 4-Ω reactance X_{L_2} (Fig. 18.5) is

$$I = I' - I''$$
$$= 3.75\text{ A}\angle -90^\circ - 2.50\text{ A}\angle 90^\circ = -j3.75\text{ A} - j2.50\text{ A}$$
$$= -j6.25\text{ A}$$
$$I = 6.25\text{ A}\angle -90^\circ$$

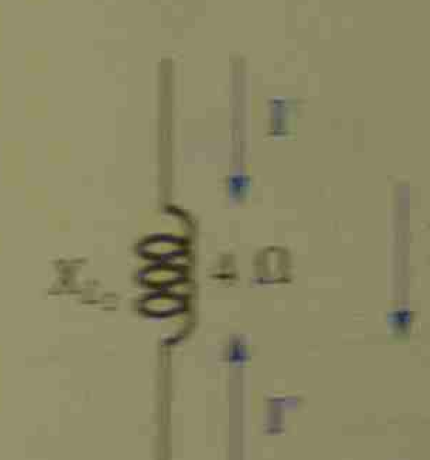


FIG. 18.5

EXAMPLE 18.2 Using superposition, find the current **I** through the 6-Ω resistor of Fig. 18.6.

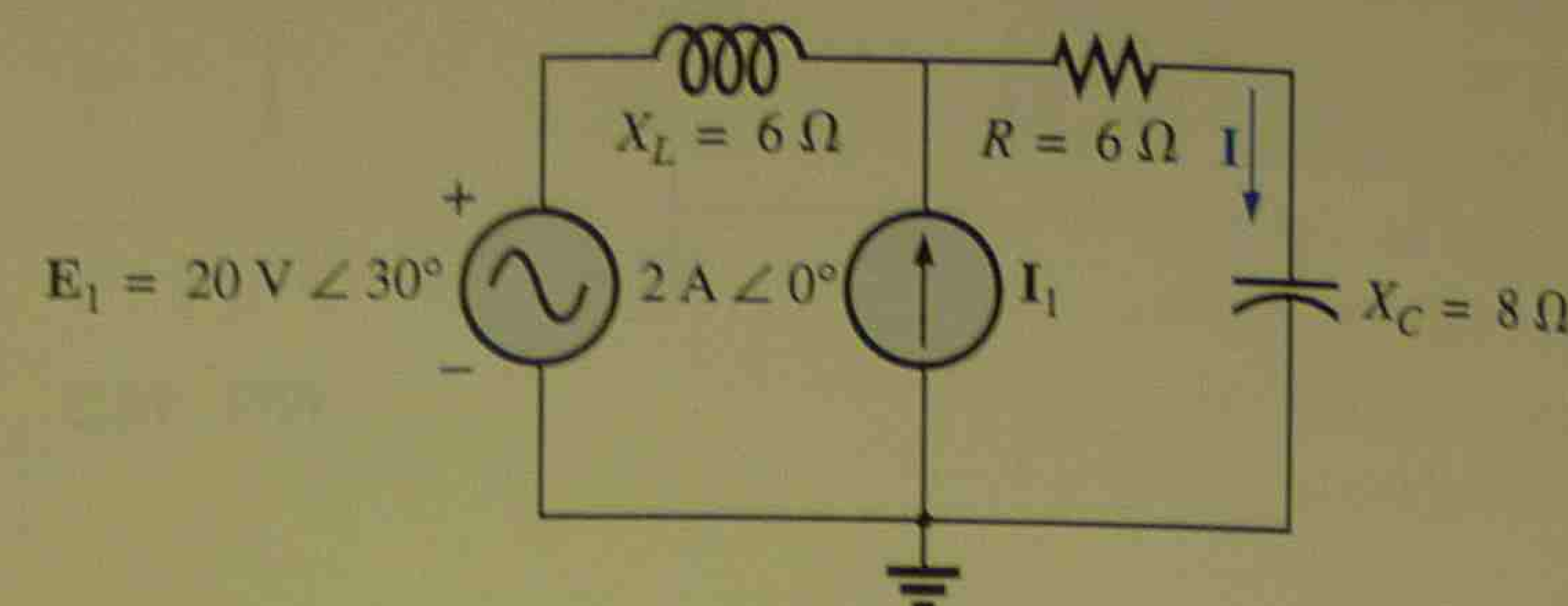


FIG. 18.6

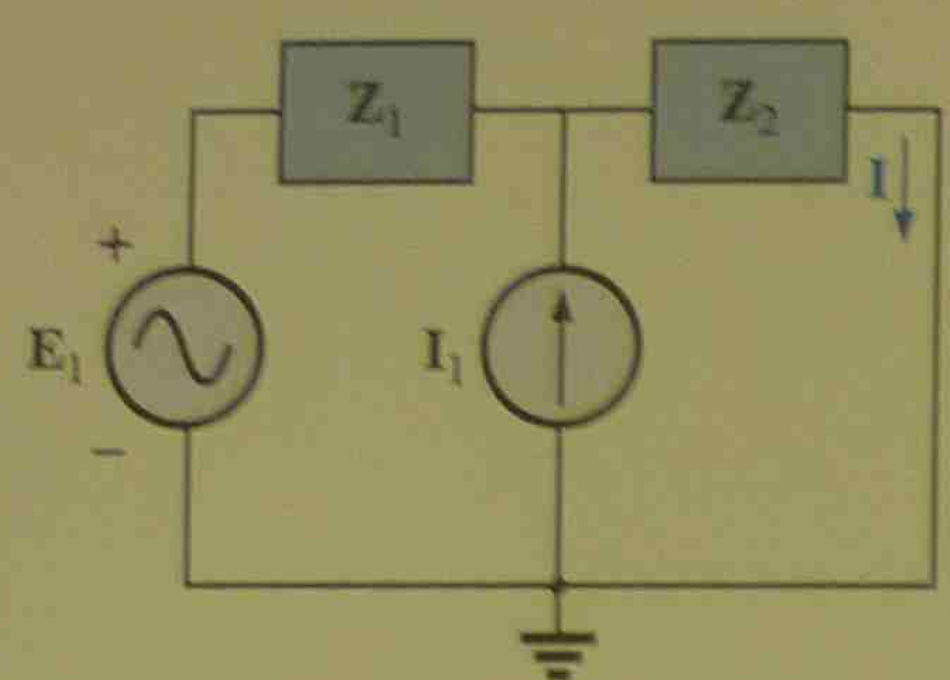


FIG. 18.7

Solution: For the redrawn circuit (Fig. 18.7),

$$\mathbf{Z}_1 = j6\ \Omega \qquad \mathbf{Z}_2 = 6 - j8\ \Omega$$

Consider the effects of the current source (Fig. 18.8). Applying the current divider rule, we have

$$\mathbf{I}' = \frac{\mathbf{Z}_1\mathbf{I}_1}{\mathbf{Z}_1 + \mathbf{Z}_2} = \frac{(j6\ \Omega)(2\ \text{A})}{j6\ \Omega + 6\ \Omega - j8\ \Omega} = \frac{j12}{6 - j2}$$
$$= \frac{12 \angle 90°}{6.32 \angle -18.43°}$$
$$\mathbf{I}' = 1.9\ \text{A} \angle 108.43°$$

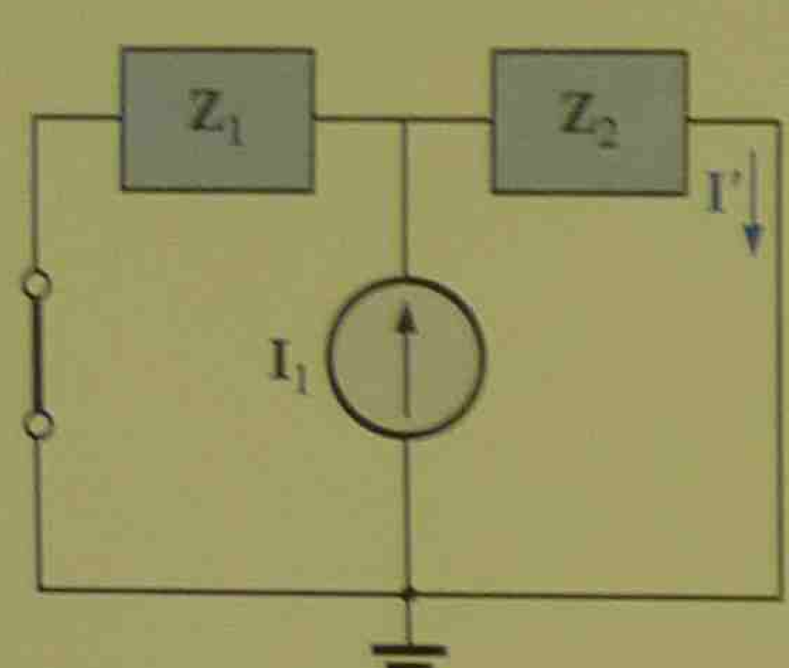


FIG. 18.8

Consider the effects of the voltage source (Fig. 18.9). Applying Ohm's law gives us

$$\mathbf{I}'' = \frac{\mathbf{E}_1}{\mathbf{Z}_T} = \frac{\mathbf{E}_1}{\mathbf{Z}_1 + \mathbf{Z}_2} = \frac{20\ \text{V} \angle 30°}{6.32\ \Omega \angle -18.43°}$$
$$= 3.16\ \text{A} \angle 48.43°$$

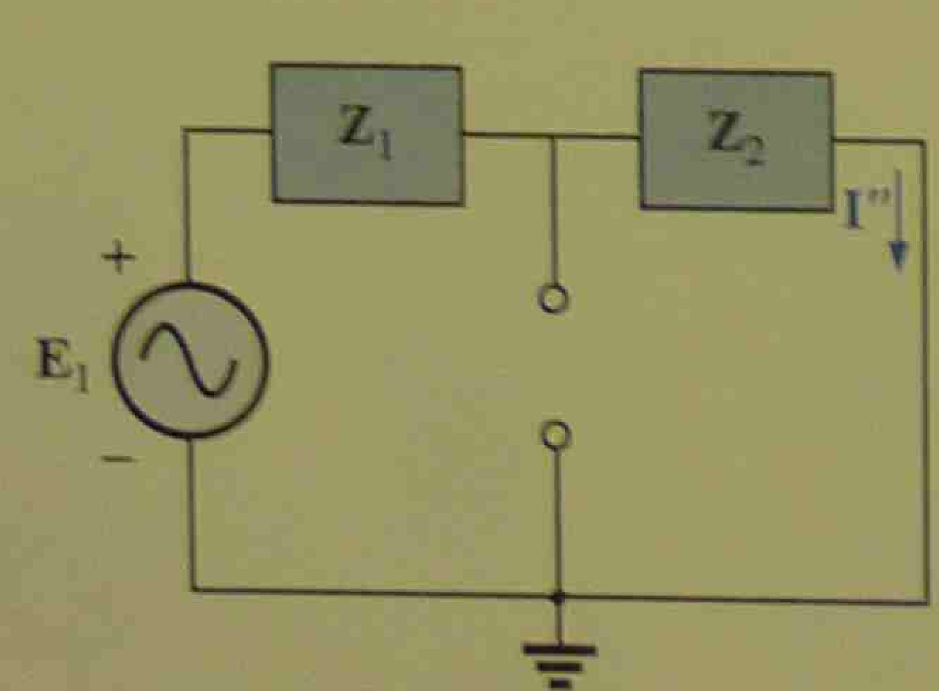


FIG. 18.9

The total current through the 6-Ω resistor (Fig. 18.10) is

$$\mathbf{I} = \mathbf{I}' + \mathbf{I}''$$
$$= 1.9\ \text{A} \angle 108.43° + 3.16\ \text{A} \angle 48.43°$$
$$= (-0.60\ \text{A} + j1.80\ \text{A}) + (2.10\ \text{A} + j2.36\ \text{A})$$
$$= 1.50\ \text{A} + j4.16\ \text{A}$$
$$\mathbf{I} = \mathbf{4.42\ A \angle 70.2°}$$

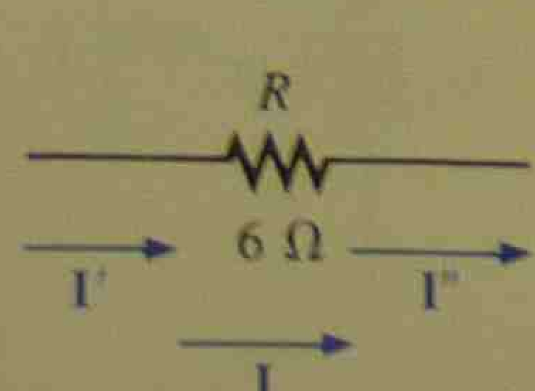


FIG. 18.10

EXAMPLE 18.3 Using superposition, find the voltage across the 6-Ω resistor in Fig. 18.6. Check the results against $\mathbf{V}_{6\Omega} = \mathbf{I}(6\ \Omega)$, where **I** is the current found through the 6-Ω resistor in the previous example.

Solution: For the current source,

$$\mathbf{V}'_{6\Omega} = \mathbf{I}'(6\ \Omega) = (1.9\ \text{A} \angle 108.43°)(6\ \Omega) = 11.4\ \text{V} \angle 108.43°$$

For the voltage source,

$$\mathbf{V}''_{6\Omega} = \mathbf{I}''(6) = (3.16\ \text{A}\ \angle 48.43^\circ)(6\ \Omega) = 18.96\ \text{V}\ \angle 48.43^\circ$$

The total voltage across the 6-Ω resistor (Fig. 18.11) is

$$\begin{aligned}\mathbf{V}_{6\Omega} &= \mathbf{V}'_{6\Omega} + \mathbf{V}''_{6\Omega}\\ &= 11.4\ \text{V}\ \angle 108.43^\circ + 18.96\ \text{V}\ \angle 48.43^\circ\\ &= (-3.60\ \text{V} + j\,10.82\ \text{V}) + (12.58\ \text{V} + j\,14.18\ \text{V})\\ &= 8.98\ \text{V} + j\,25.0\ \text{V}\\ \mathbf{V}_{6\Omega} &= \mathbf{26.5\ V\ \angle 70.2^\circ}\end{aligned}$$

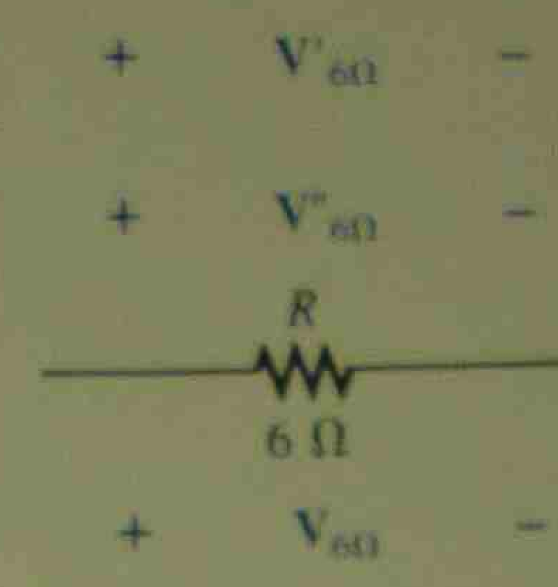


FIG. 18.11

Checking the result, we have

$$\begin{aligned}\mathbf{V}_{6\Omega} &= \mathbf{I}(6\ \Omega) = (4.42\ \text{A}\ \angle 70.2^\circ)(6\ \Omega)\\ &= \mathbf{26.5\ V\ \angle 70.2^\circ} \qquad \text{(checks)}\end{aligned}$$

EXAMPLE 18.4 For the network of Fig. 18.12, determine the sinusoidal expression for the voltage v_3 using superposition.

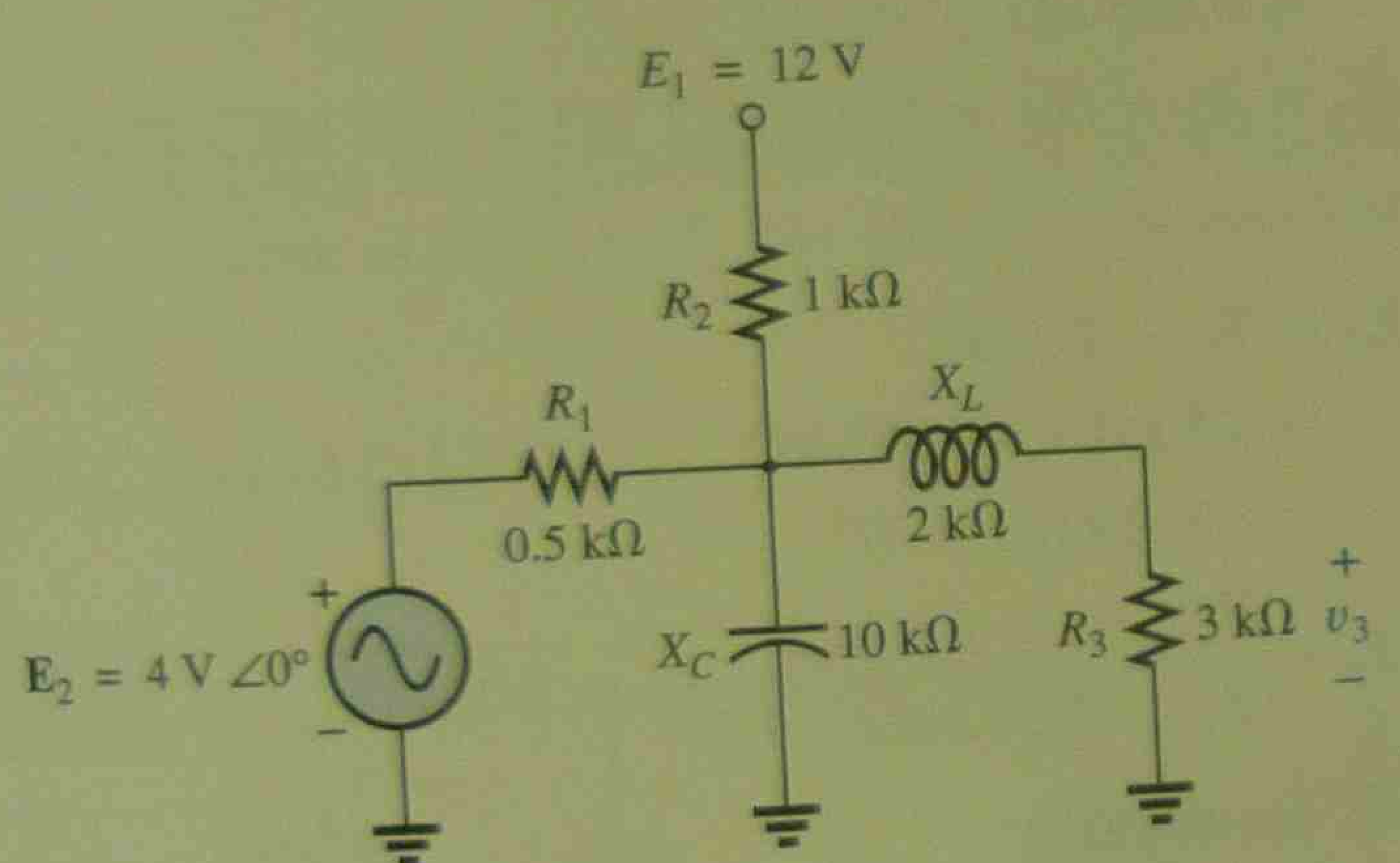


FIG. 18.12

Solution: For the dc source, recall that for dc analysis, in the steady state the capacitor can be replaced by an open-circuit equivalent and the inductor by a short-circuit equivalent. The result is the network of Fig. 18.13.

The resistors R_1 and R_3 are then in parallel and the voltage V_3 can be determined using the voltage divider rule:

$$R' = R_1 \| R_3 = 0.5\ \text{k}\Omega \| 3\ \text{k}\Omega = 0.429\ \text{k}\Omega$$

and

$$\begin{aligned}V_3 &= \frac{R'E_1}{R' + R_2}\\ &= \frac{(0.429\ \text{k}\Omega)(12\ \text{V})}{0.429\ \text{k}\Omega + 1\ \text{k}\Omega} = \frac{5.148\ \text{V}}{1.429}\\ V_3 &\cong \mathbf{3.6\ V}\end{aligned}$$

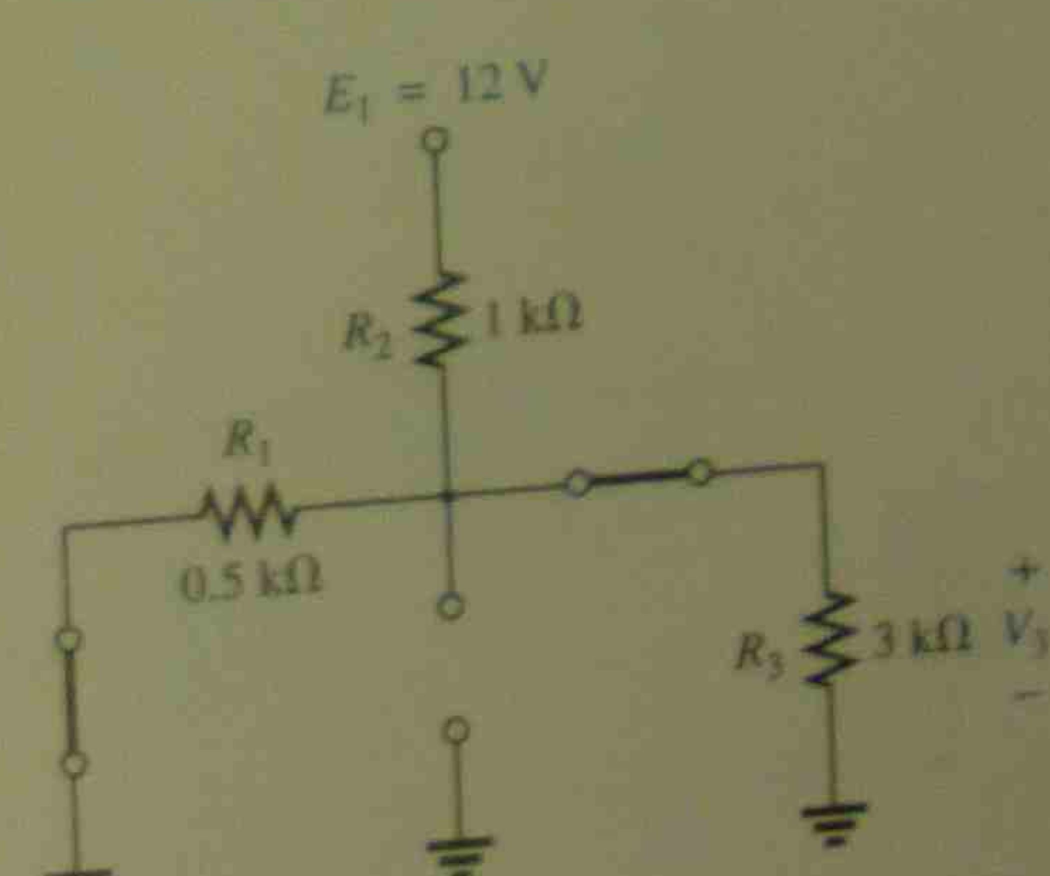


FIG. 18.13

For ac analysis, the dc source is set to zero and the network is redrawn, as shown in Fig. 18.14.

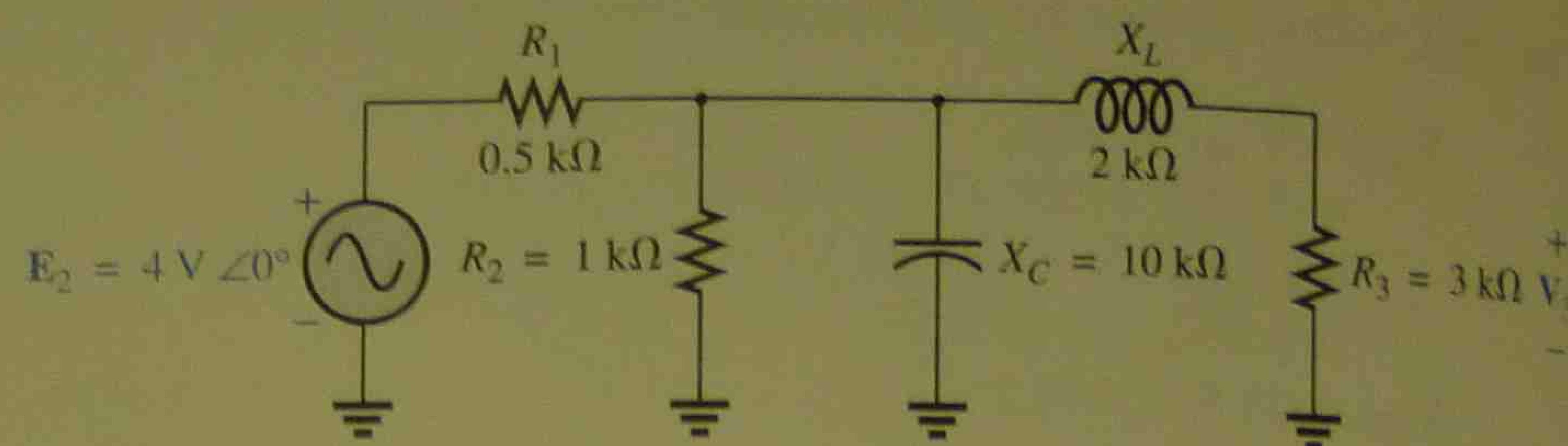


FIG. 18.14

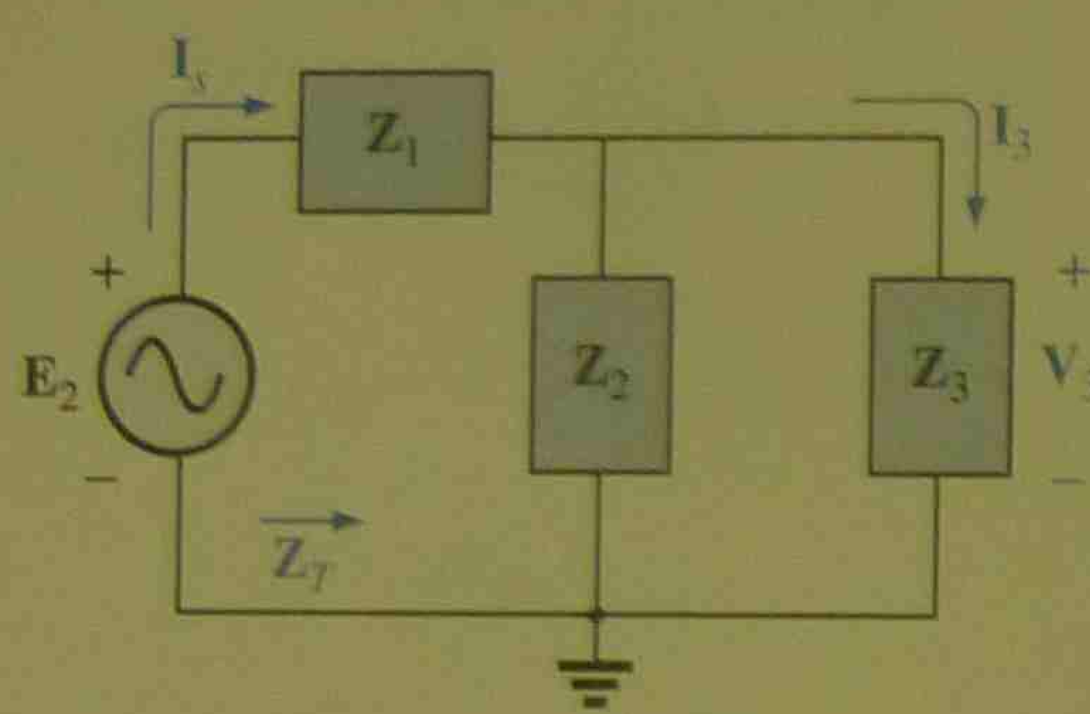


FIG. 18.15

The block impedances are then defined as in Fig. 18.15 and series-parallel techniques are applied as follows:

$$\mathbf{Z}_1 = 0.5\text{ k}\Omega\angle 0^\circ$$

$$\mathbf{Z}_2 = (R_2\angle 0^\circ) \parallel (X_C\angle -90^\circ)$$

$$= \frac{(1\text{ k}\Omega\angle 0^\circ)(10\text{ k}\Omega\angle -90^\circ)}{1\text{ k}\Omega - j10\text{ k}\Omega} = \frac{10\text{ k}\Omega\angle -90^\circ}{10.05\angle -84.29^\circ}$$

$$= 0.995\text{ k}\Omega\angle -5.71^\circ$$

$$\mathbf{Z}_3 = R_3 + jX_L = 3\text{ k}\Omega + j2\text{ k}\Omega = 3.61\text{ k}\Omega\angle 33.69^\circ$$

and

$$\mathbf{Z}_T = \mathbf{Z}_1 + \mathbf{Z}_2 \parallel \mathbf{Z}_3$$

$$= 0.5\text{ k}\Omega + (0.995\text{ k}\Omega\angle -5.71^\circ) \parallel (3.61\text{ k}\Omega\angle 33.69^\circ)$$

$$= 1.312\text{ k}\Omega\angle 1.57^\circ$$

$$\mathbf{I}_s = \frac{\mathbf{E}_2}{\mathbf{Z}_T} = \frac{4\text{ V}\angle 0^\circ}{1.312\text{ k}\Omega\angle 1.57^\circ} = 3.05\text{ mA}\angle -1.57^\circ$$

Current divider rule:

$$\mathbf{I}_3 = \frac{\mathbf{Z}_2\mathbf{I}_s}{\mathbf{Z}_2 + \mathbf{Z}_3} = \frac{(0.995\text{ k}\Omega\angle -5.71^\circ)(3.05\text{ mA}\angle -1.57^\circ)}{0.995\text{ k}\Omega\angle -5.71^\circ + 3.61\text{ k}\Omega\angle 33.69^\circ}$$

$$= 0.686\text{ mA}\angle -32.74^\circ$$

with

$$\mathbf{V}_3 = (I_3\angle\theta)(R_3\angle 0^\circ)$$

$$= (0.686\text{ mA}\angle -32.74^\circ)(3\text{ k}\Omega\angle 0^\circ)$$

$$= \mathbf{2.06\text{ V}\angle -32.74^\circ}$$

The total solution:

$$\upsilon_3 = \upsilon_3\text{ (dc)} + \upsilon_3\text{ (ac)}$$

$$= 3.6\text{ V} + 2.06\text{ V}\angle -32.74^\circ$$

$$\upsilon_3 = \mathbf{3.6 + 2.91\sin(\omega t - 32.74^\circ)}$$

The result is a sinusoidal voltage having a peak value of 2.91 V riding on an average value of 3.6 V, as shown in Fig. 18.16.

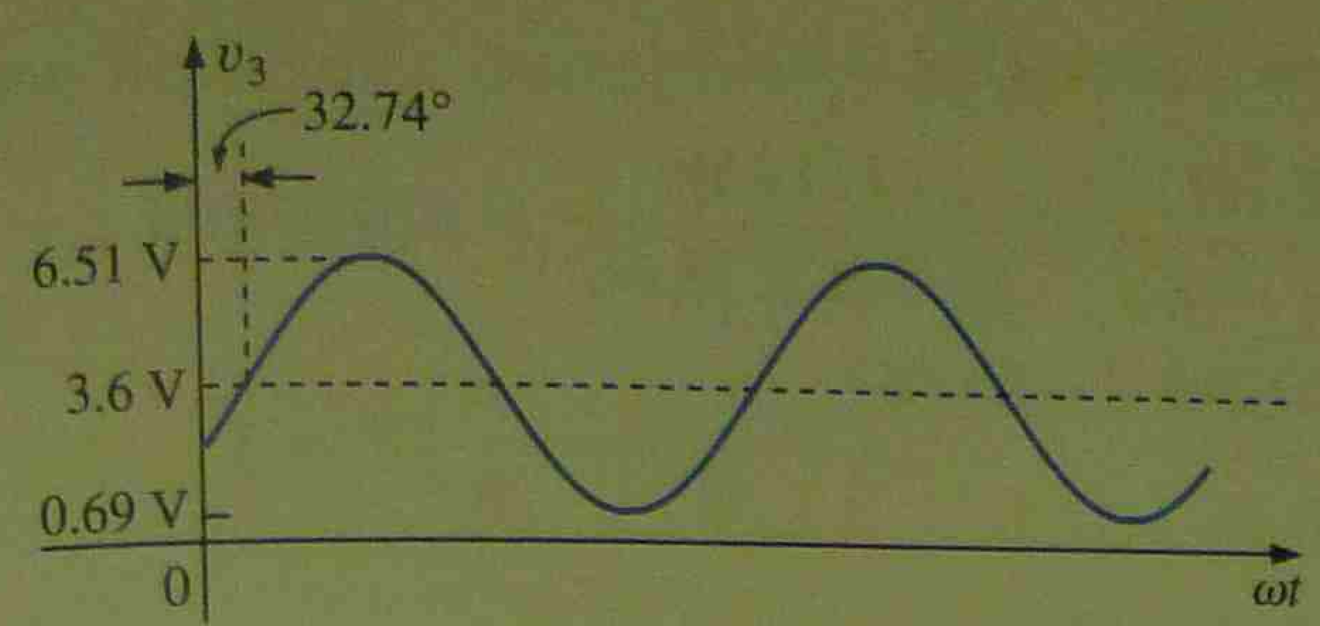


FIG. 18.16

Dependent Sources

For dependent sources in which the controlling variable is not determined by the network to which the superposition theorem is to be applied, the application of the theorem is basically the same as for independent sources. The solution obtained will simply be in terms of the controlling variables.

EXAMPLE 18.5 Using the superposition theorem, determine the current $\mathbf{I}_2$ for the network of Fig. 18.17. The quantities μ and h are constants.

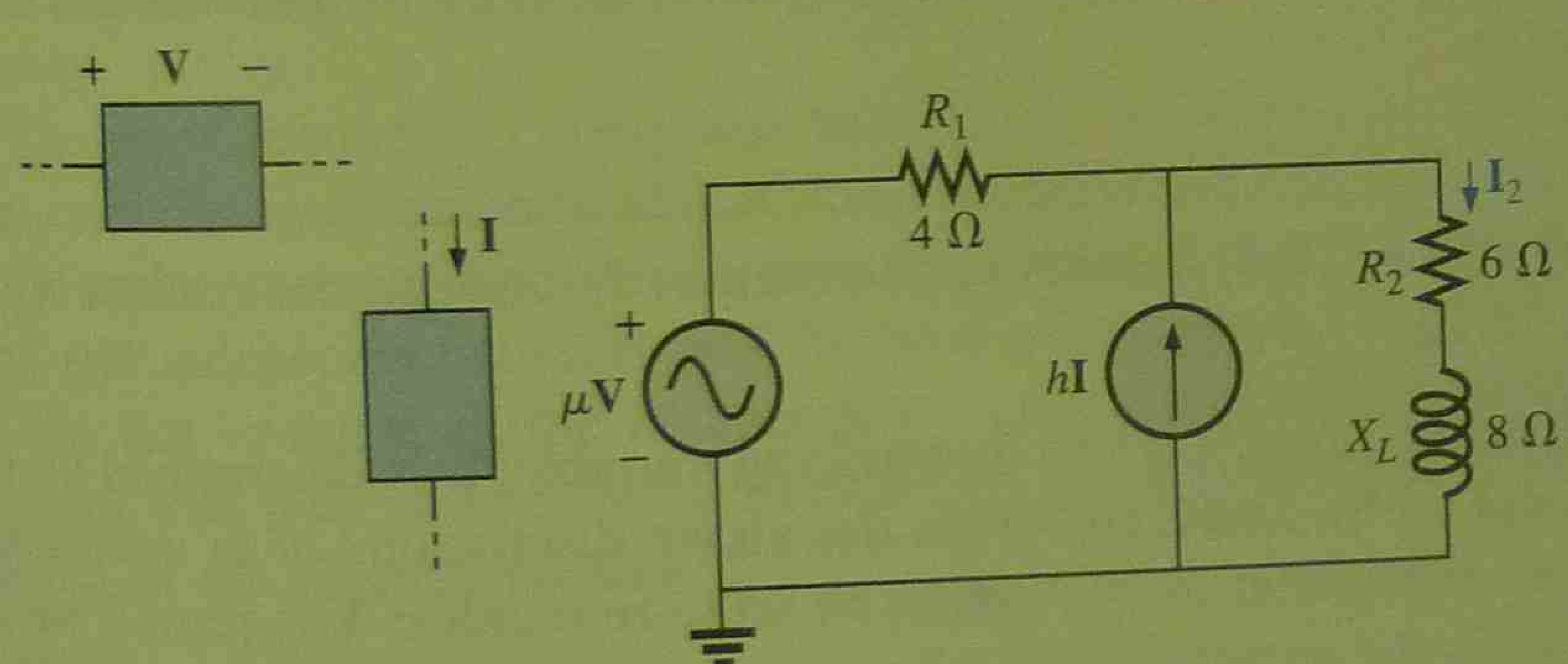


FIG. 18.17

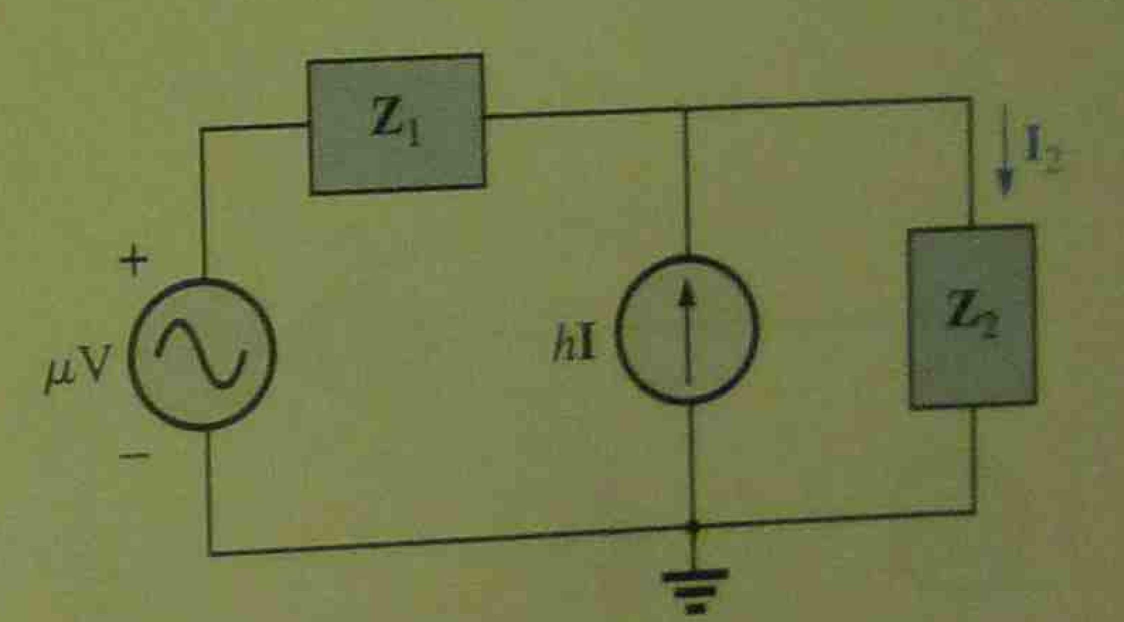


FIG. 18.18

Solution: With a portion of the system redrawn (Fig. 18.18),

$$\mathbf{Z}_1 = R_1 = 4\ \Omega \qquad \mathbf{Z}_2 = R_2 + jX_L = 6\ \Omega + j8\ \Omega$$

For the voltage source (Fig. 18.19),

$$\mathbf{I}' = \frac{\mu\mathbf{V}}{\mathbf{Z}_1 + \mathbf{Z}_2} = \frac{\mu\mathbf{V}}{4\ \Omega + 6\ \Omega + j8\ \Omega} = \frac{\mu\mathbf{V}}{10\ \Omega + j8\ \Omega}$$

$$= \frac{\mu\mathbf{V}}{12.8\ \Omega\ \angle 38.66^\circ} = 0.078\ \mu\mathbf{V}/\Omega\angle -38.66^\circ$$

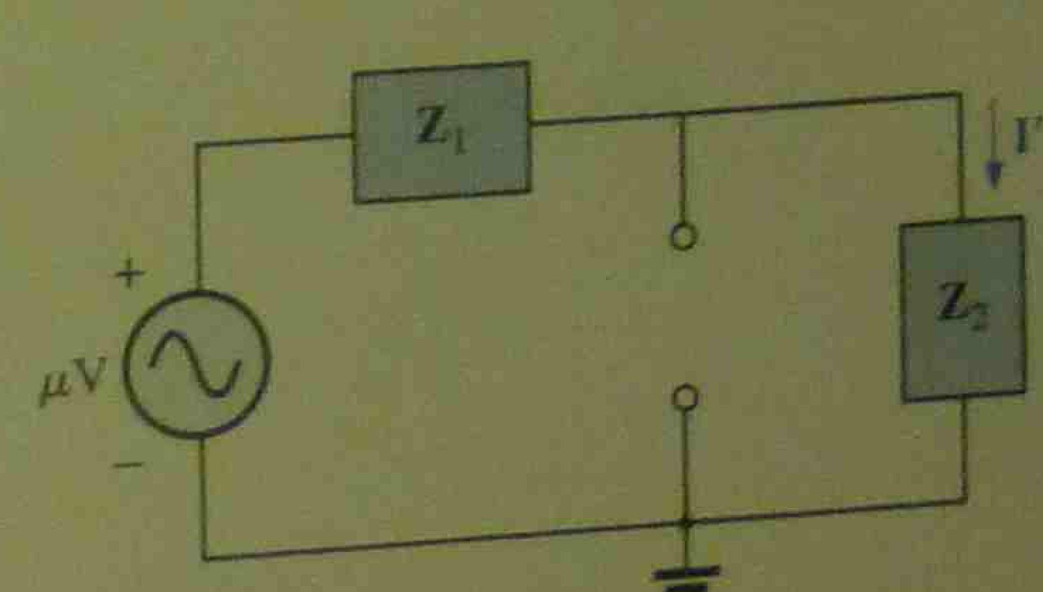


FIG. 18.19

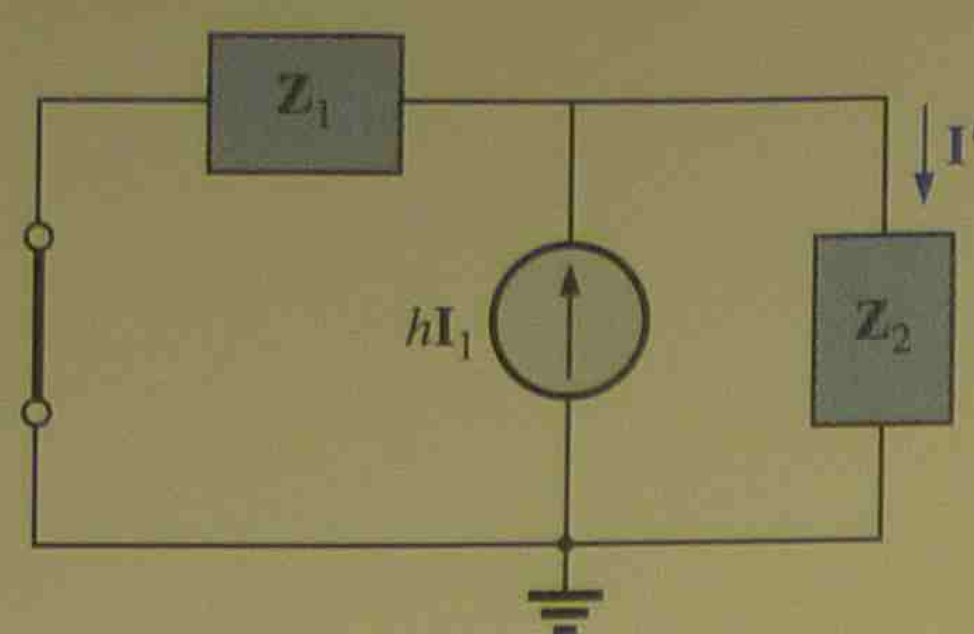


FIG. 18.20

For the current source (Fig. 18.20),

$$\mathbf{I}'' = \frac{\mathbf{Z}_1(h\mathbf{I})}{\mathbf{Z}_1 + \mathbf{Z}_2} = \frac{(4\ \Omega)(h\mathbf{I})}{12.8\ \Omega\ \angle 38.66^\circ} = 4(0.078)h\mathbf{I}\ \angle -38.66^\circ$$
$$= 0.312h\mathbf{I}\ \angle -38.66^\circ$$

The current $\mathbf{I}_2$ is

$$\mathbf{I}_2 = \mathbf{I}' + \mathbf{I}''$$
$$= 0.078\ \mu\mathbf{V}/\Omega\angle -38.66^\circ + 0.312h\mathbf{I}\angle \mathbf{-38.66^\circ}$$

For $\mathbf{V} = 10\ \text{V}\ \angle 0^\circ$, $\mathbf{I} = 20\ \text{mA}\ \angle 0^\circ$, $\mu = 20$, $h = 100$,

$$\mathbf{I}_2 = 0.078(20)(10\ \text{V}\angle 0^\circ)/\Omega\angle -38.66^\circ + 0.312(100)(20\ \text{mA}\angle 0^\circ)\angle -38.66^\circ$$
$$= 15.60\ \text{A}\ \angle -38.66^\circ + 0.62\ \text{A}\ \angle -38.66^\circ$$
$$\mathbf{I}_2 = \mathbf{16.22\ A\ \angle -38.66^\circ}$$

For dependent sources in which the controlling variable is determined by the network to which the theorem is to be applied, the dependent source cannot be set to zero unless the controlling variable is also zero. For networks containing dependent sources such as indicated in Example 18.5 and dependent sources of the type just introduced above, the superposition theorem is applied for each independent source and each dependent source not having a controlling variable in the portions of the network under investigation. It must be reemphasized that dependent sources are not sources of energy in the sense that if all independent sources are removed from a system, all currents and voltages must be zero.

EXAMPLE 18.6 Determine the current $\mathbf{I}_L$ through the resistor R_L of Fig. 18.21.

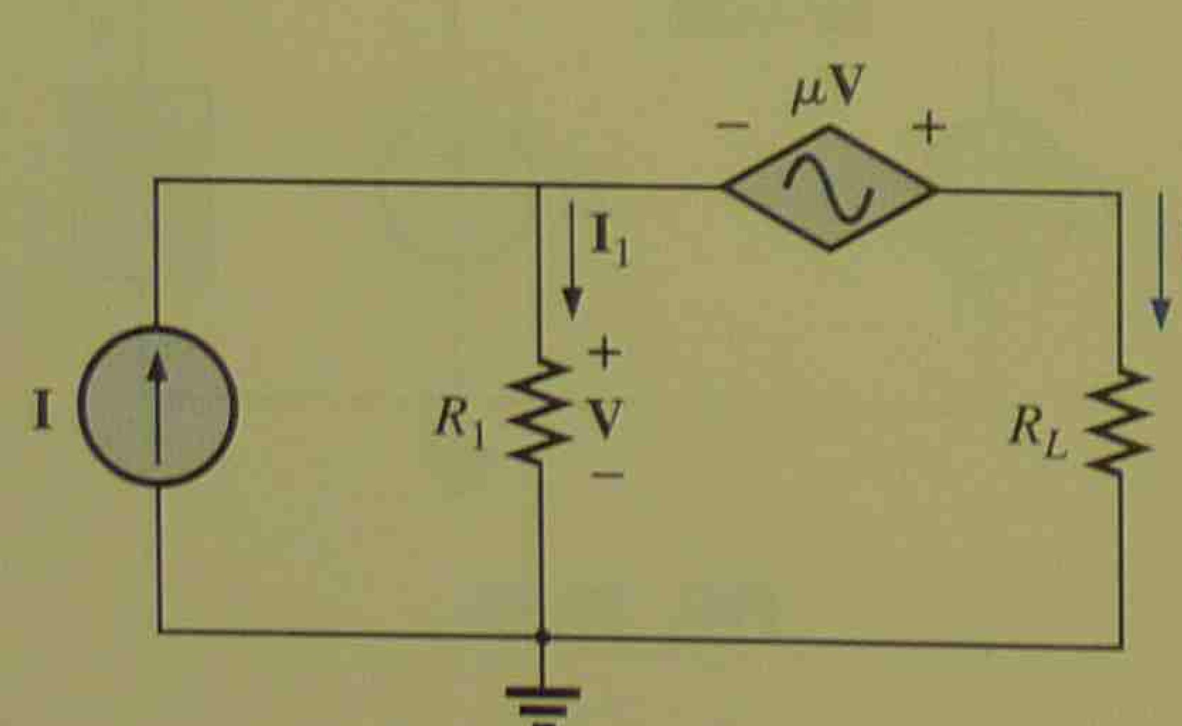


FIG. 18.21

Solution: Note that the controlling variable $\mathbf{V}$ is determined by the network to be analyzed. From the above discussions, it is understood that the dependent source cannot be set to zero unless $\mathbf{V}$ is zero. If we set $\mathbf{I}$ to zero, the network lacks a source of voltage, and $\mathbf{V} = 0$ with $\mu\mathbf{V} = 0$. The resulting $\mathbf{I}_L$ under this condition is zero. Obviously, therefore, the network must be analyzed as it appears in Fig. 18.21, with the result that neither source can be eliminated, as is normally done using the superposition theorem.

Applying Kirchhoff's voltage law, we have

$$\mathbf{V}_L = \mathbf{V} + \mu\mathbf{V} = (1 + \mu)\mathbf{V}$$

and

$$\mathbf{I}_L = \frac{\mathbf{V}_L}{R_L} = \frac{(1 + \mu)\mathbf{V}}{R_L}$$

The result, however, must be found in terms of $\mathbf{I}$ since $\mathbf{V}$ and $\mu\mathbf{V}$ are only dependent variables.

Applying Kirchhoff's current law gives us

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_L = \frac{\mathbf{V}}{R_1} + \frac{(1+\mu)\mathbf{V}}{R_L}$$

and

$$\mathbf{I} = \mathbf{V}\left(\frac{1}{R_1} + \frac{1+\mu}{R_L}\right)$$

or

$$\mathbf{V} = \frac{\mathbf{I}}{(1/R_1) + [(1+\mu)/R_L]}$$

Substituting into the above yields

$$\mathbf{I}_L = \frac{(1+\mu)\mathbf{V}}{R_L} = \frac{(1+\mu)}{R_L}\left(\frac{\mathbf{I}}{(1/R_1) + [(1+\mu)/R_L]}\right)$$

Therefore,

$$\mathbf{I}_L = \frac{(1+\mu)R_1\mathbf{I}}{R_L + (1+\mu)R_1}$$

18.3 THEVENIN'S THEOREM

Thevenin's theorem, as stated for sinusoidal ac circuits, is changed only to include the term *impedance* instead of *resistance;* that is,

any two-terminal linear ac network can be replaced by an equivalent circuit consisting of a voltage source and an impedance in series as shown in Fig. 18.22.

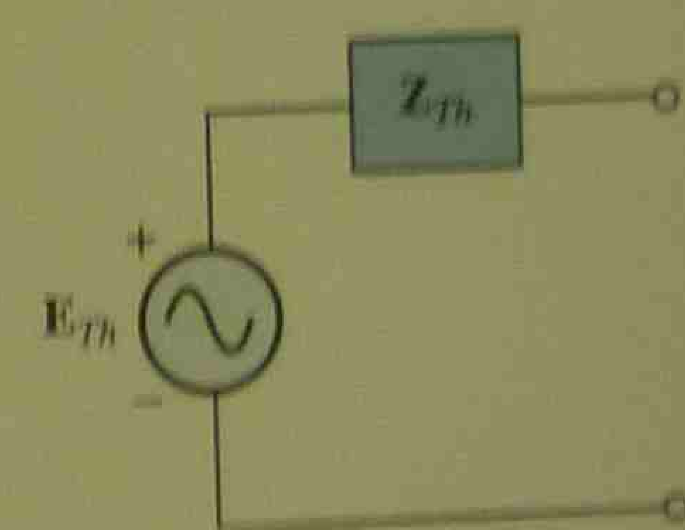


FIG. 18.22

Since the reactances of a circuit are frequency dependent, the Thevenin circuit found for a particular network is applicable only at *one* frequency.

The steps required to apply this method to dc circuits are repeated here with changes for sinusoidal ac circuits. As before, the only change is the replacement of the term *resistance* by *impedance*. Again, dependent and independent sources will be treated separately.

The last example of the independent source section will include a network with a dc and ac source to establish the groundwork for possible use in the electronics area.

Independent Sources

1. ***Remove that portion of the network across which the Thevenin equivalent circuit is to be found.***
2. ***Mark (○, •, and so on) the terminals of the remaining two-terminal network.***
3. ***Calculate $\mathbf{Z}_{Th}$ by first setting all voltage and current sources to zero (short circuit and open circuit, respectively) and then finding the resulting impedance between the two marked terminals.***
4. ***Calculate $\mathbf{E}_{Th}$ by first replacing the voltage and current sources and then finding the open-circuit voltage between the marked terminals.***

5. ***Draw the Thevenin equivalent circuit with the portion of the circuit previously removed replaced between the terminals of the Thevenin equivalent circuit.***

EXAMPLE 18.7 Find the Thevenin equivalent circuit for the network external to resistor R in Fig. 18.23.

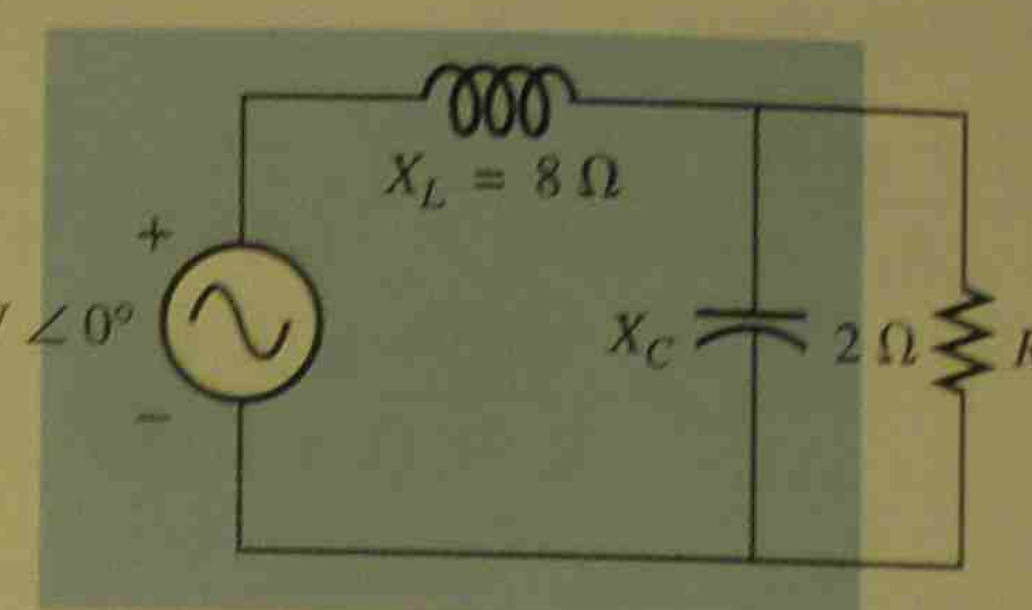


FIG. 18.23

Solution:

Steps 1 and 2 (Fig. 18.24):

$$\mathbf{Z}_1 = jX_L = j8\ \Omega \qquad \mathbf{Z}_2 = -jX_C = -j2\ \Omega$$

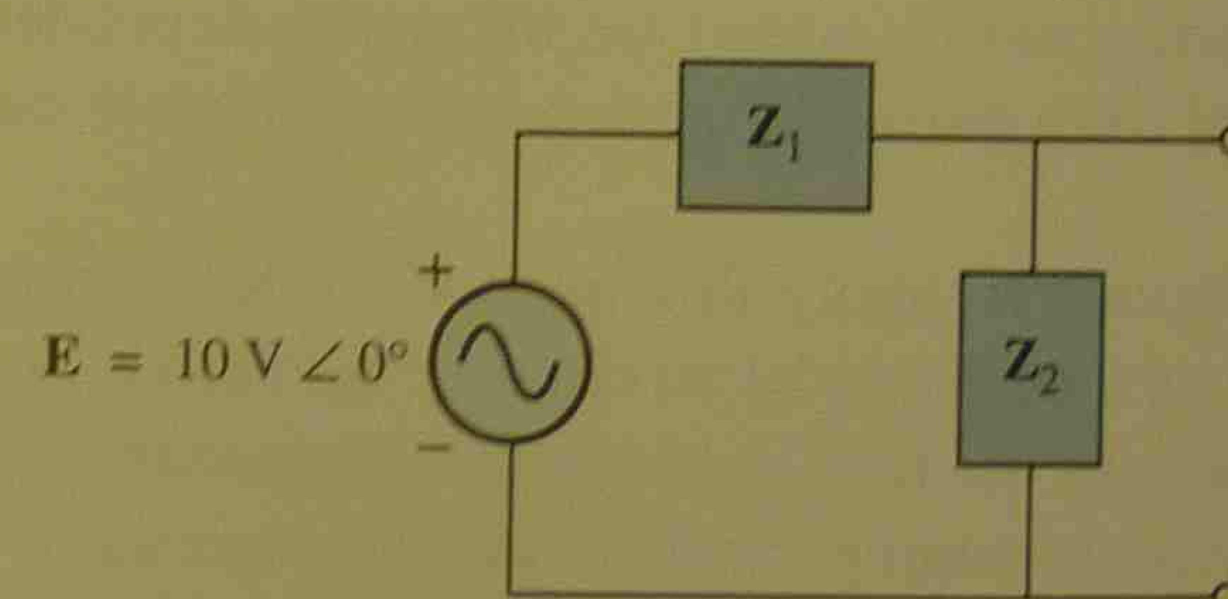


FIG. 18.24

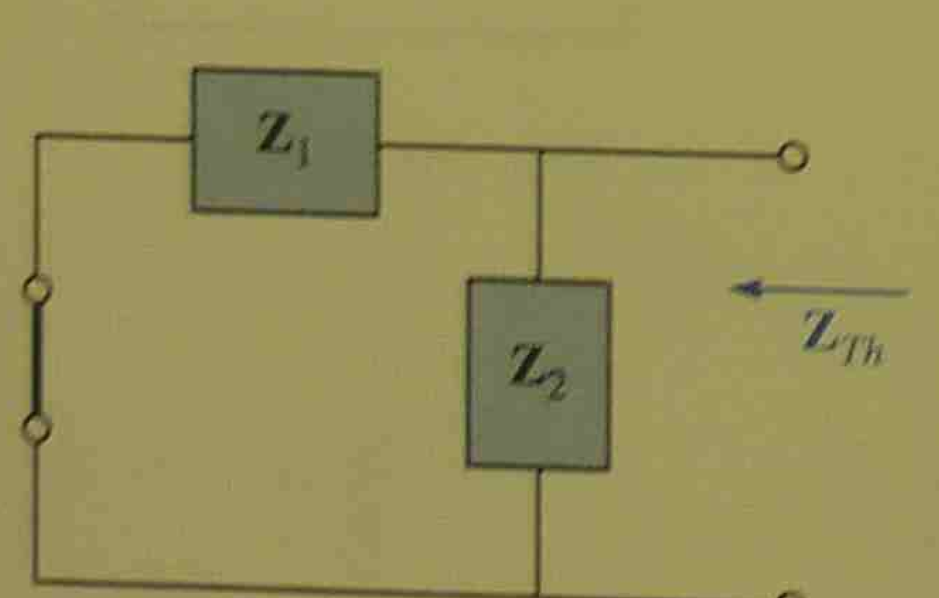


FIG. 18.25

Step 3 (Fig. 18.25):

$$\mathbf{Z}_{Th} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2} = \frac{(j8\ \Omega)(-j2\ \Omega)}{j8\ \Omega - j2\ \Omega} = \frac{-j^2 16\ \Omega}{j6} = \frac{16\ \Omega}{6\angle 90^\circ}$$
$$= \mathbf{2.67\ \Omega\ \angle -90^\circ}$$

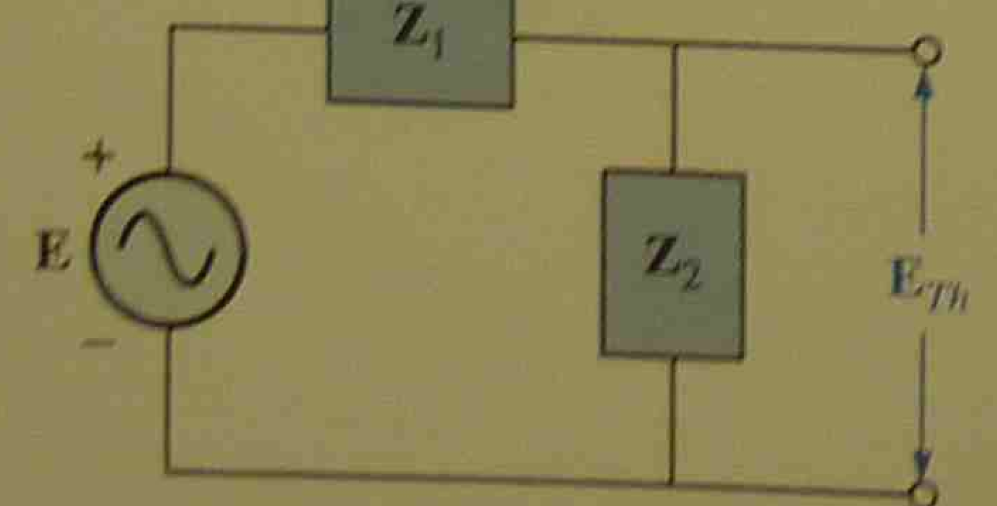


FIG. 18.26

Step 4 (Fig. 18.26):

$$\mathbf{E}_{Th} = \frac{\mathbf{Z}_2\mathbf{E}}{\mathbf{Z}_1 + \mathbf{Z}_2} \qquad \text{(voltage divider rule)}$$

$$= \frac{(-j2\ \Omega)(10\ \text{V})}{j8\ \Omega - j2\ \Omega} = \frac{-j20\ \text{V}}{j6} = \mathbf{3.33\ V\ \angle -180^\circ}$$

Step 3: The Thevenin equivalent circuit is shown in Fig. 18.27.

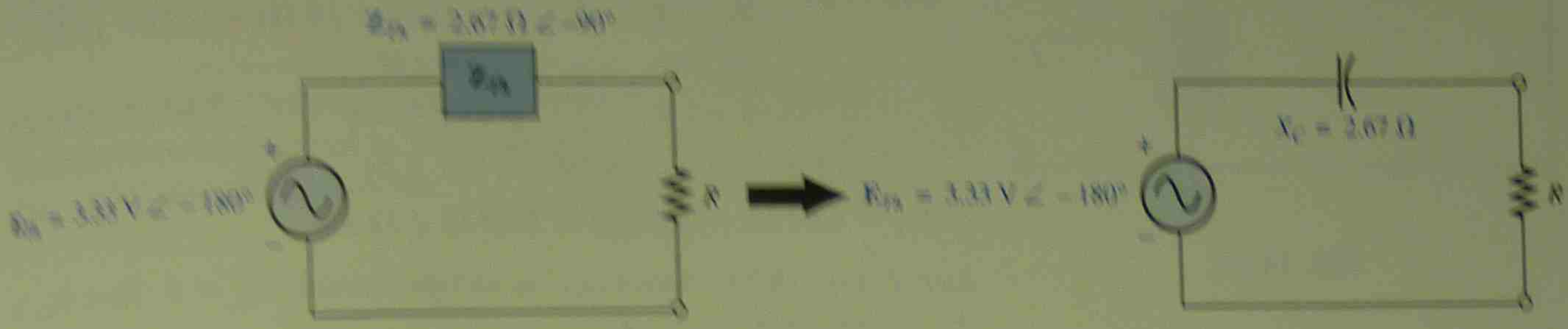


FIG. 18.27

EXAMPLE 18.8 Find the Thevenin equivalent circuit for the network external to branch *a-a'* in Fig. 18.28.

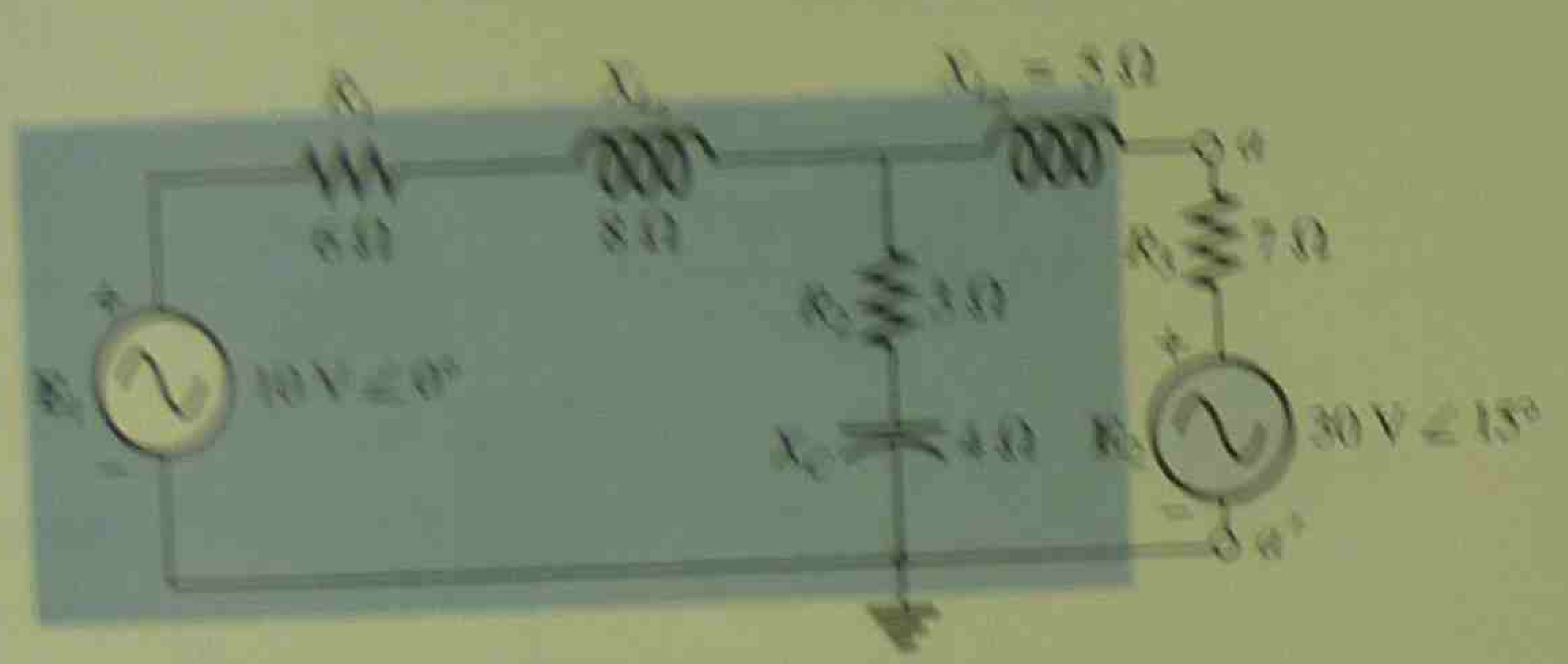


FIG. 18.28

Solution:

Steps 1 and 2 (Fig. 18.29): Note the reduced complexity with subscripted impedances:

$$\mathbf{Z}_1 = R_1 + jX_{L_1} = 6\ \Omega + j8\ \Omega$$
$$\mathbf{Z}_2 = R_2 - jX_C = 3\ \Omega - j4\ \Omega$$
$$\mathbf{Z}_3 = +jX_{L_2} = j5\ \Omega$$

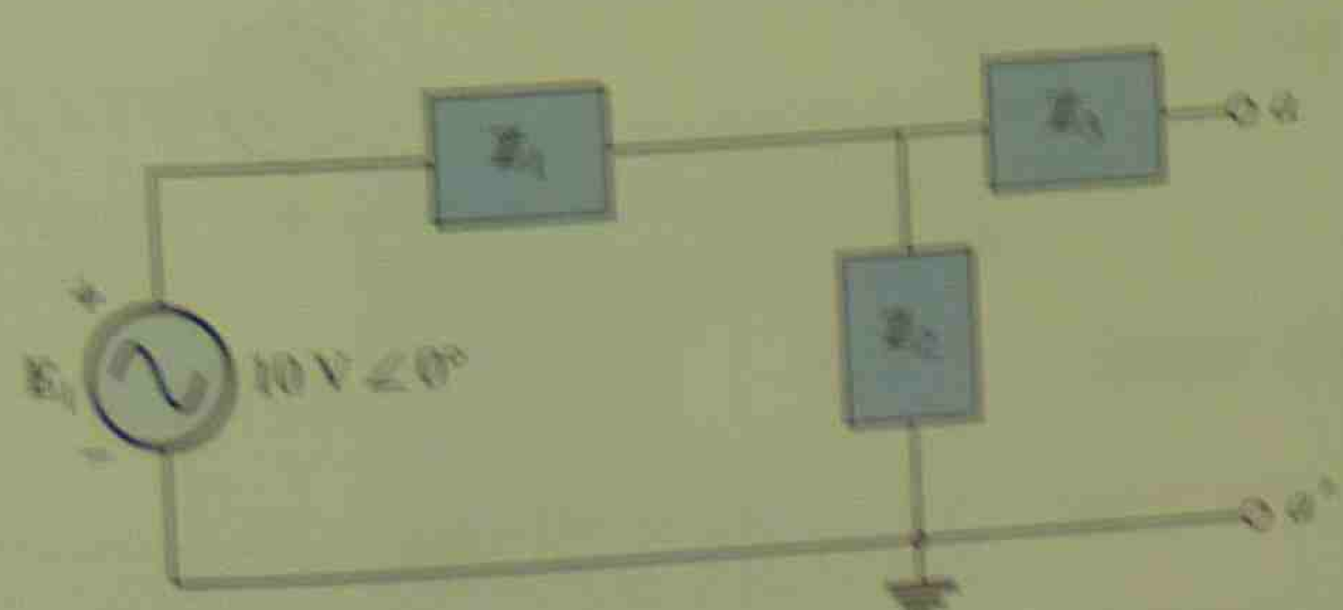


FIG. 18.29

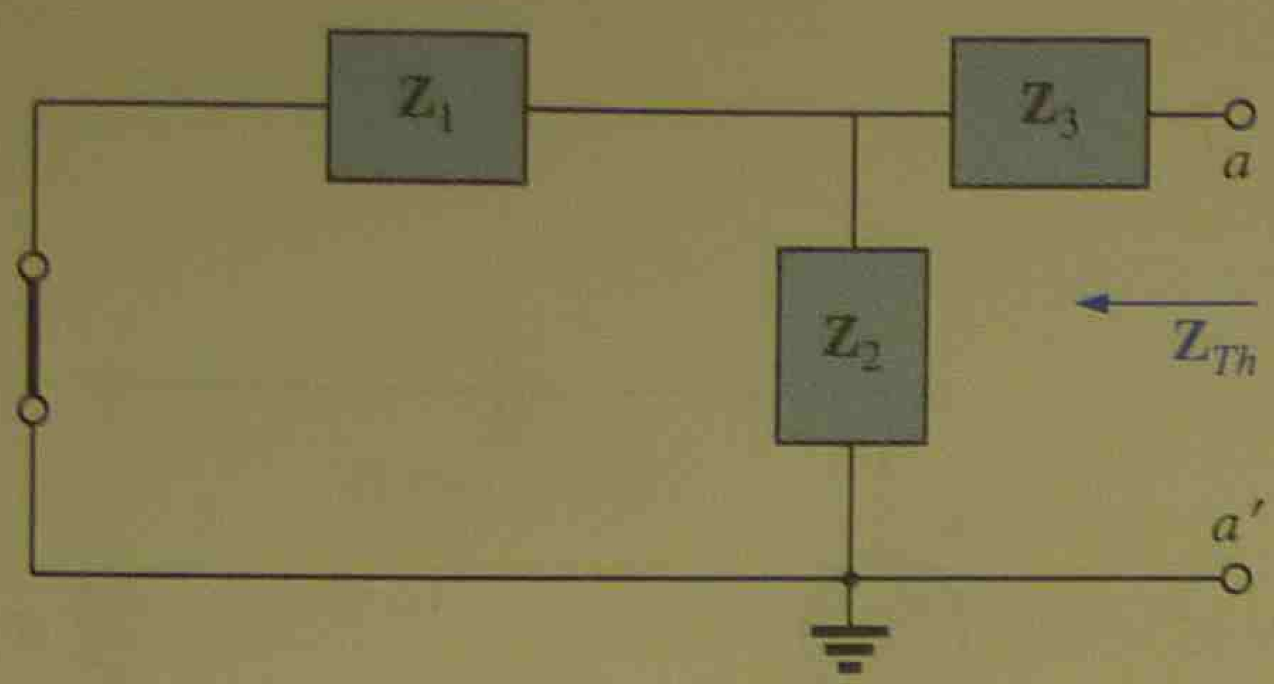


FIG. 18.30

Step 3 (Fig. 18.30):

$$\mathbf{Z}_{Th} = \mathbf{Z}_3 + \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2} = j5\ \Omega + \frac{(10\ \Omega\ \angle 53.13^\circ)(5\ \Omega\ \angle -53.13^\circ)}{(6\ \Omega + j8\ \Omega) + (3\ \Omega - j4\ \Omega)}$$

$$= j5 + \frac{50\ \angle 0^\circ}{9 + j4} = j5 + \frac{50\ \angle 0^\circ}{9.85\ \angle 23.96^\circ}$$

$$= j5 + 5.08\ \angle -23.96^\circ = j5 + 4.64 - j2.06$$

$$\mathbf{Z}_{Th} = \mathbf{4.64\ \Omega + \mathit{j}2.94\ \Omega = 5.49\ \Omega\ \angle 32.36^\circ}$$

Step 4 (Fig. 18.31): Since a-a' is an open circuit, $\mathbf{I}_{\mathbf{Z}_3} = 0$. Then $\mathbf{E}_{Th}$ is the voltage drop across $\mathbf{Z}_2$:

$$\mathbf{E}_{Th} = \frac{\mathbf{Z}_2\mathbf{E}}{\mathbf{Z}_2 + \mathbf{Z}_1} \qquad \text{(voltage divider rule)}$$

$$= \frac{(5\ \Omega\ \angle -53.13^\circ)(10\ \text{V}\ \angle 0^\circ)}{9.85\ \Omega\ \angle 23.96^\circ}$$

$$\mathbf{E}_{Th} = \frac{50\ \text{V}\ \angle -53.13^\circ}{9.85\ \angle 23.96^\circ} = \mathbf{5.08\ V\ \angle -77.09^\circ}$$

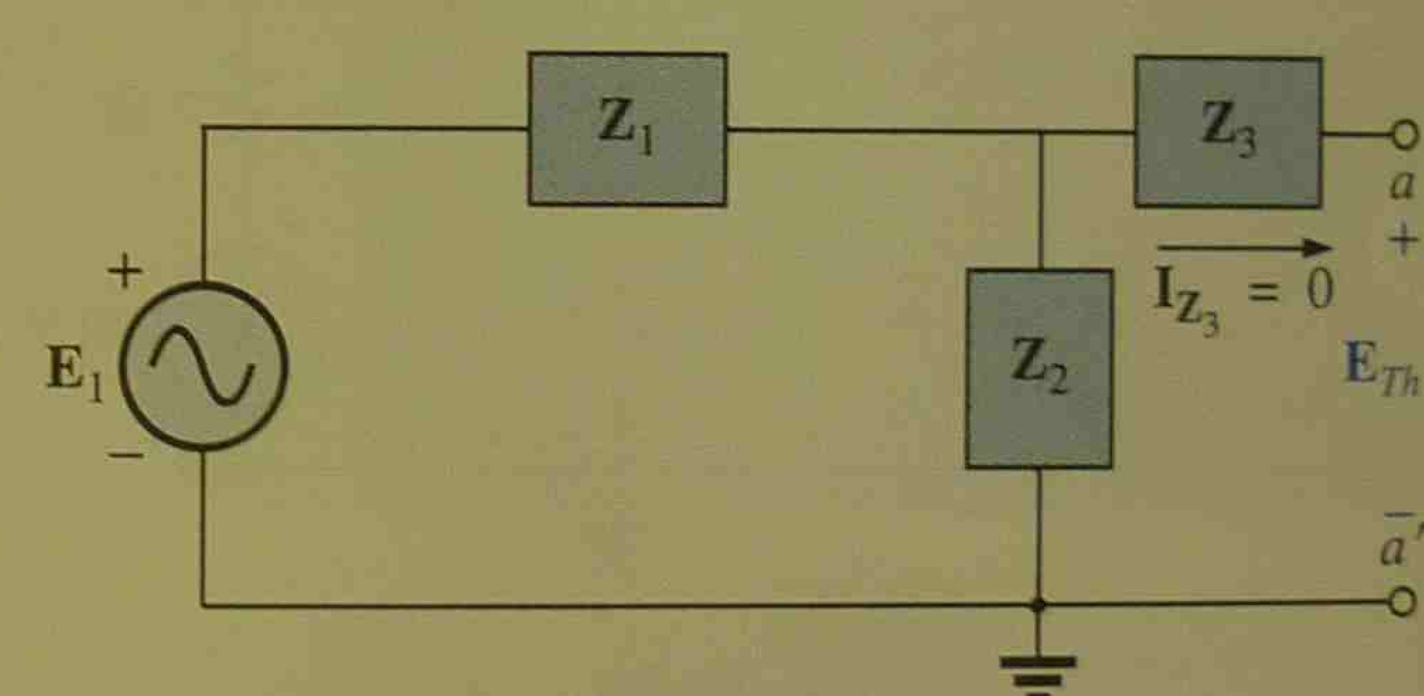


FIG. 18.31

Step 5: The Thevenin equivalent circuit is shown in Fig. 18.32.

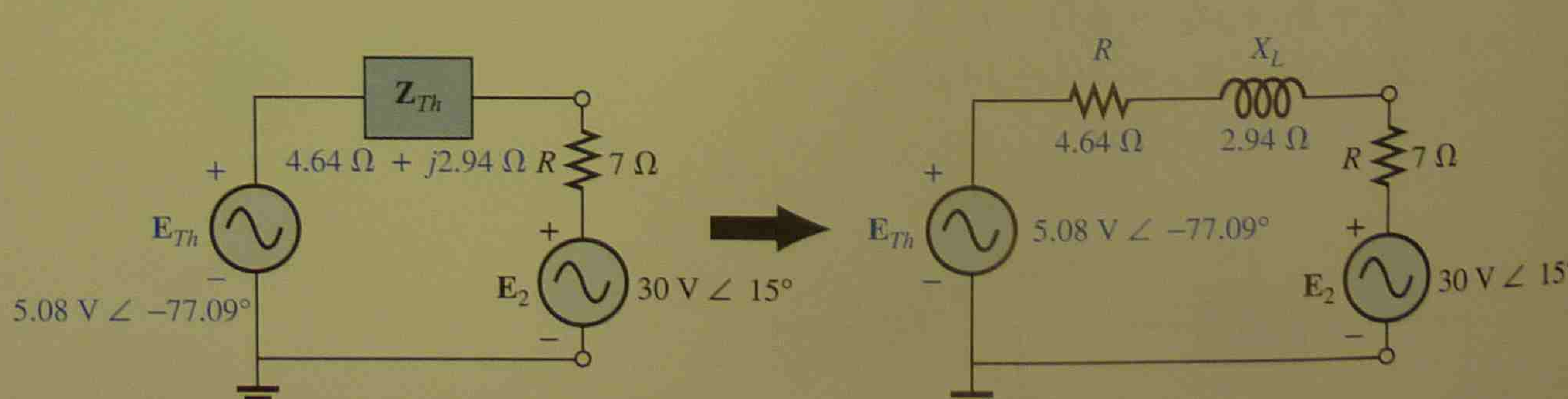


FIG. 18.32

The next example demonstrates how superposition is applied to electronic circuits to permit a separation of the dc and ac analysis. The fact that the controlling variable in this analysis is not in the portion of the network connected directly to the terminals of interest permits an analysis of the network in the same manner as applied above for independent sources.

EXAMPLE 18.9 Determine the Thevenin equivalent circuit for the transistor network external to the resistor R_L in the following network (Fig. 18.33) and then determine $\mathbf{V}_L$.

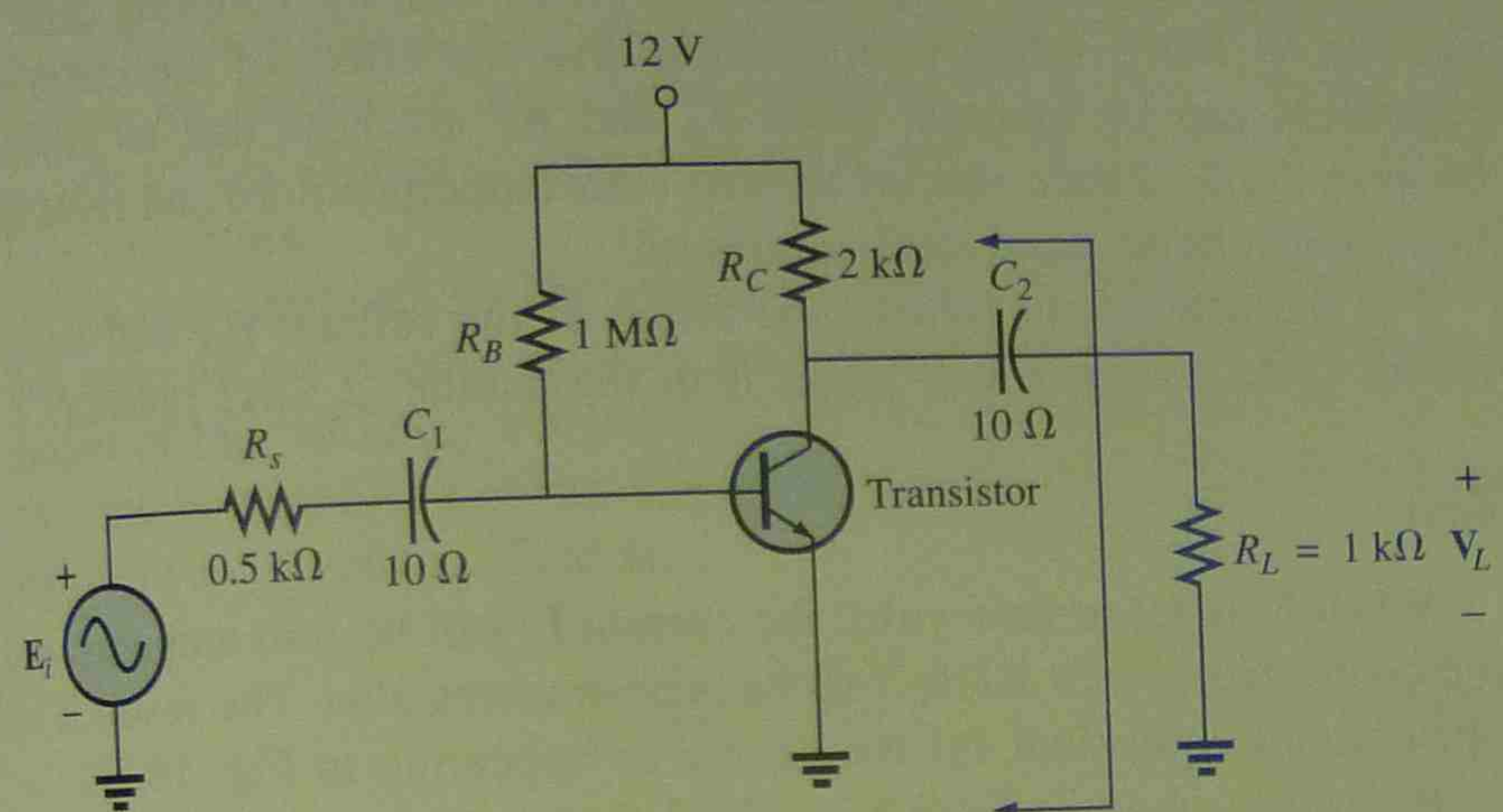


FIG. 18.33

Solution: Applying superposition.

dc conditions:

Substituting the open-circuit equivalent for the coupling capacitor C_2 will isolate the dc source and the resulting currents from the load resistor. The result is for dc conditions that $V_L = 0$ V. Although the output dc voltage is zero, the application of the dc voltage is important to the basic operation of the transistor in a number of important ways, one of which is to determine the parameters of the "equivalent circuit" to appear in the ac analysis to follow.

ac conditions:

For the ac analysis an equivalent circuit is substituted for the transistor as established by the dc conditions above that will behave like the actual transistor. A great deal more will be said about equivalent circuits and the operations performed to obtain the network of Fig. 18.34, but for

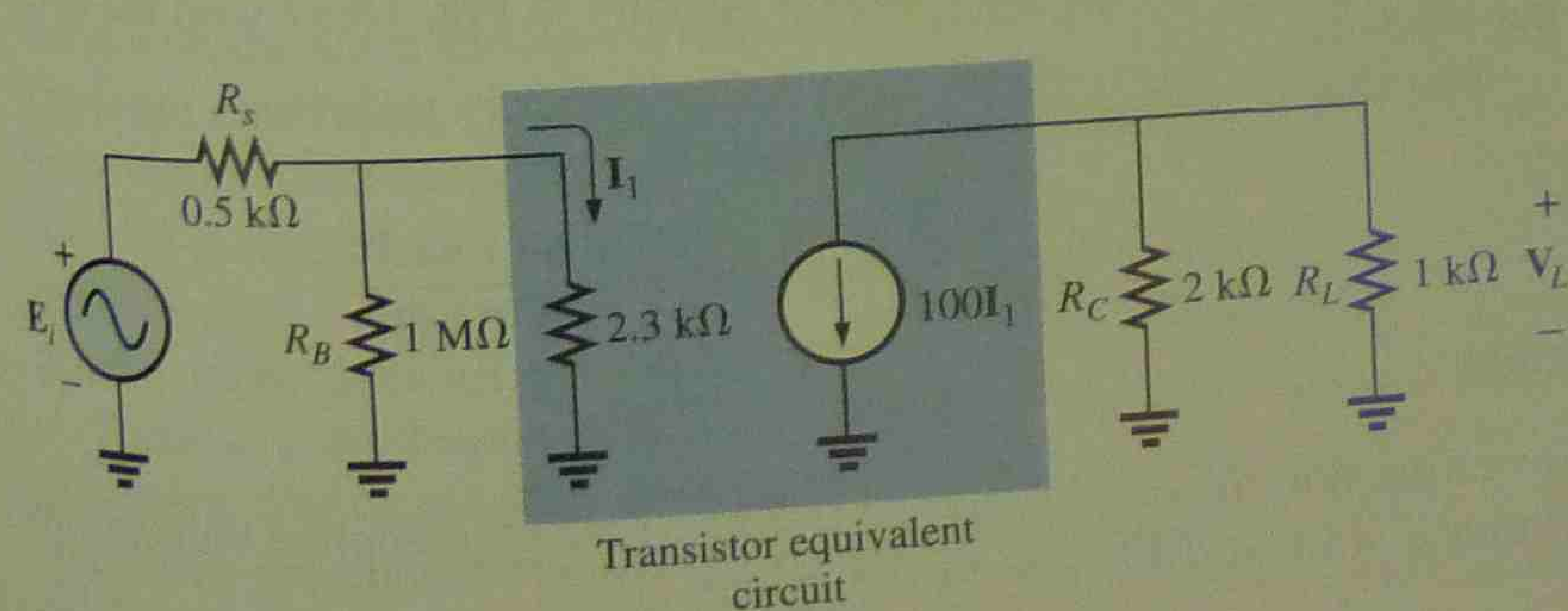


FIG. 18.34

now let us limit our attention to the manner in which the Thevenin equivalent circuit is obtained. Note in Fig. 18.34 that the equivalent circuit includes a resistor of 2.3 kΩ and a controlled current source whose magnitude is determined by the product of a factor of 100 and the current I_1 in another part of the network.

Note in Fig. 18.34 the absence of the coupling capacitors for the ac analysis. In general, coupling capacitors are designed to be open circuits for dc and short circuits for ac analysis. The short-circuit equivalent is valid because the other impedances in series with the coupling capacitors are so much larger in magnitude that the effect of the coupling capacitors can be ignored. Both R_B and R_C are now tied to ground because the dc source was set to zero volts (superposition) and replaced by a short-circuit equivalent to ground.

For the analysis to follow, the effect of the resistor R_B will be ignored, since it is so much larger than the parallel 2.3-kΩ resistor.

Z_{Th}:

When $\mathbf{E}_i$ is set to zero volts, the current $\mathbf{I}_1$ will be zero amperes, and the controlled source $100\mathbf{I}_1$ will be zero amperes also. The result is an open-circuit equivalent for the source as appearing in Fig. 18.35.

It is fairly obvious from Fig. 18.35 that

$$\mathbf{Z}_{Th} = 2\ \mathrm{k\Omega}$$

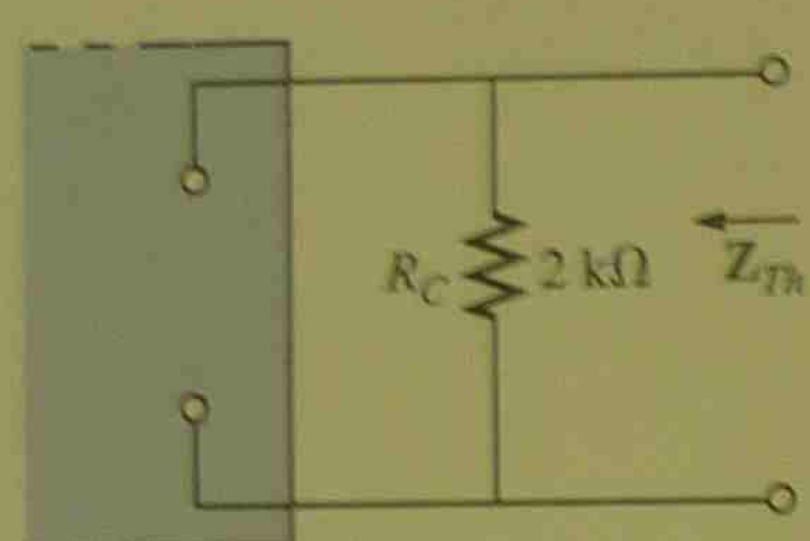


FIG. 18.35

$\mathbf{E}_{Th}$:

For $\mathbf{E}_{Th}$, the current $\mathbf{I}_1$ of Fig. 18.34 will be

$$\mathbf{I}_1 = \frac{\mathbf{E}_i}{R_s + 2.3\ \mathrm{k\Omega}} = \frac{\mathbf{E}_i}{0.5\ \mathrm{k\Omega} + 2.3\ \mathrm{k\Omega}} = \frac{\mathbf{E}_i}{2.8\ \mathrm{k\Omega}}$$

and

$$100\mathbf{I}_1 = (100)\left(\frac{\mathbf{E}_i}{2.8\ \mathrm{k\Omega}}\right) = 35.71 \times 10^{-3}/\Omega\ \mathbf{E}_i$$

Referring to Fig. 18.36, we find that

$$\mathbf{E}_{Th} = -(100\mathbf{I}_1)R_C$$
$$= -(35.71 \times 10^{-3}/\Omega\ \mathbf{E}_i)(2 \times 10^3\ \Omega)$$
$$\mathbf{E}_{Th} = -71.42\mathbf{E}_i$$

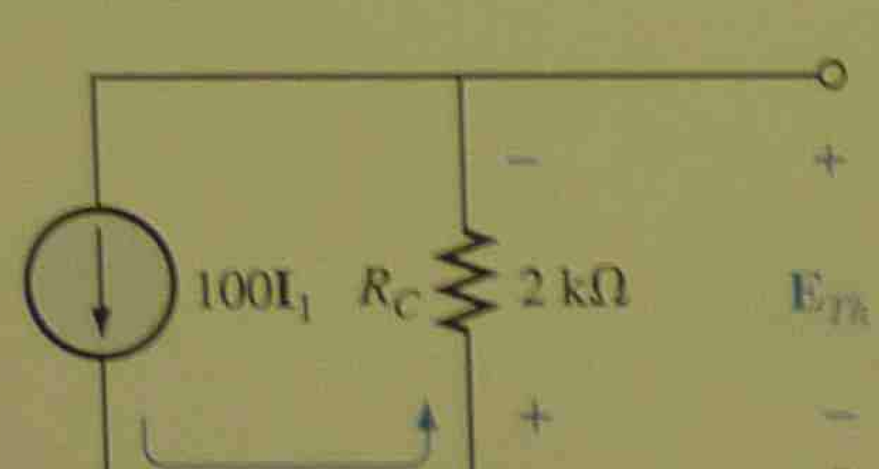


FIG. 18.36

The Thevenin equivalent circuit appears in Fig. 18.37 with the original load R_L.

The output voltage V_L:

$$\mathbf{V}_L = \frac{-R_L\mathbf{E}_{Th}}{R_L + \mathbf{Z}_{Th}} = \frac{-(1\ \mathrm{k\Omega})(71.42\ \mathbf{E}_i)}{1\ \mathrm{k\Omega} + 2\ \mathrm{k\Omega}}$$

and

$$\mathbf{V}_L = -23.81\mathbf{E}_i$$

revealing that the output voltage is 23.81 times the applied voltage with a phase shift of 180° due to the minus sign.

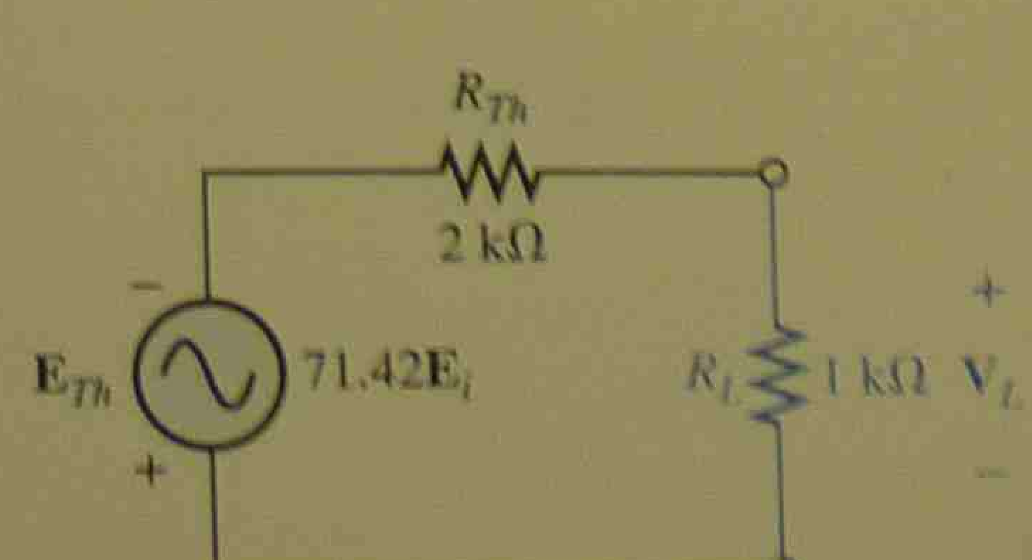


FIG. 18.37

Dependent Sources

For dependent sources with a controlling variable not in the network under investigation, the procedure indicated above can be applied. However, for dependent sources of the other type, where the controlling variable is part of the network to which the theorem is to be applied, another approach must be employed. The necessity for a different approach will be demonstrated in an example to follow. The method is not limited to dependent sources of the latter type. It can also be applied to any dc or sinusoidal ac network. However, for networks of independent sources, the method of application employed in Chapter 9 and the first portion of this section is generally more direct, with usual savings in time and errors.

The new approach to Thevenin's theorem can best be introduced at this stage in the development by considering the Thevenin equivalent circuit of Fig. 18.38(a). As indicated in Fig. 18.38(b), the open-circuit terminal voltage ($\mathbf{E}_{oc}$) of the Thevenin equivalent circuit is the Thevenin equivalent voltage. That is,

$$\mathbf{E}_{oc} = \mathbf{E}_{Th} \quad (18.1)$$

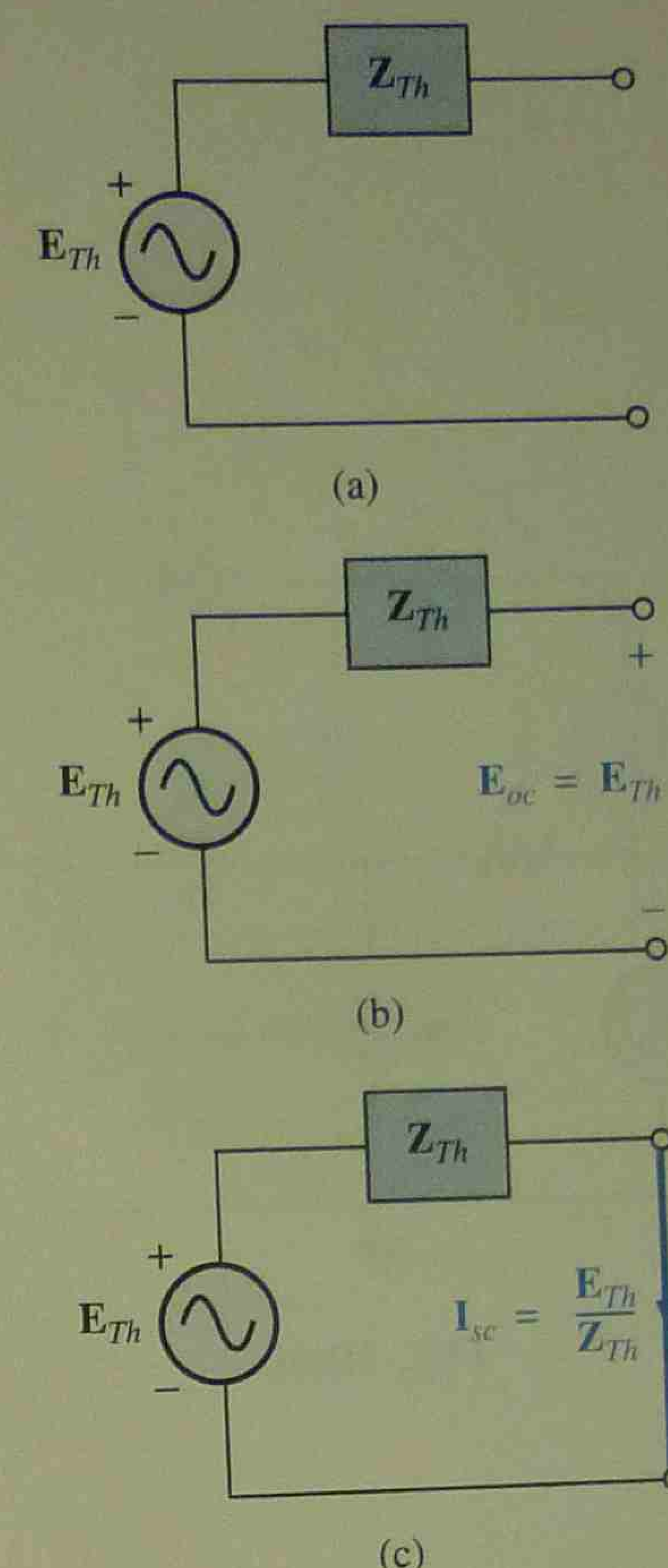


FIG. 18.38

If the external terminals are short circuited as in Fig. 18.38(c), the resulting short-circuit current is determined by

$$\mathbf{I}_{sc} = \frac{\mathbf{E}_{Th}}{\mathbf{Z}_{Th}} \quad (18.2)$$

or, rearranged,

$$\mathbf{Z}_{Th} = \frac{\mathbf{E}_{Th}}{\mathbf{I}_{sc}}$$

and

$$\mathbf{Z}_{Th} = \frac{\mathbf{E}_{oc}}{\mathbf{I}_{sc}} \quad (18.3)$$

Equations (18.1) and (18.3) indicate that for any linear bilateral dc or ac network with or without dependent sources of any type, if the open-circuit terminal voltage of a portion of a network can be determined along with the short-circuit current between the same two terminals, the Thevenin equivalent circuit is effectively known. A few examples will make the method quite clear. The advantage of the method, which was stressed earlier in this section for independent sources, should now be more obvious. The current $\mathbf{I}_{sc}$, which is necessary to find $\mathbf{Z}_{Th}$, is in general more difficult to obtain since all of the sources are present.

There is a third approach to the Thevenin equivalent circuit that is also useful from a practical viewpoint. The Thevenin voltage is found as in the two previous methods. However, the Thevenin impedance is obtained by applying a source of voltage to the terminals of interest and determining the source current as indicated in Fig. 18.39. For this

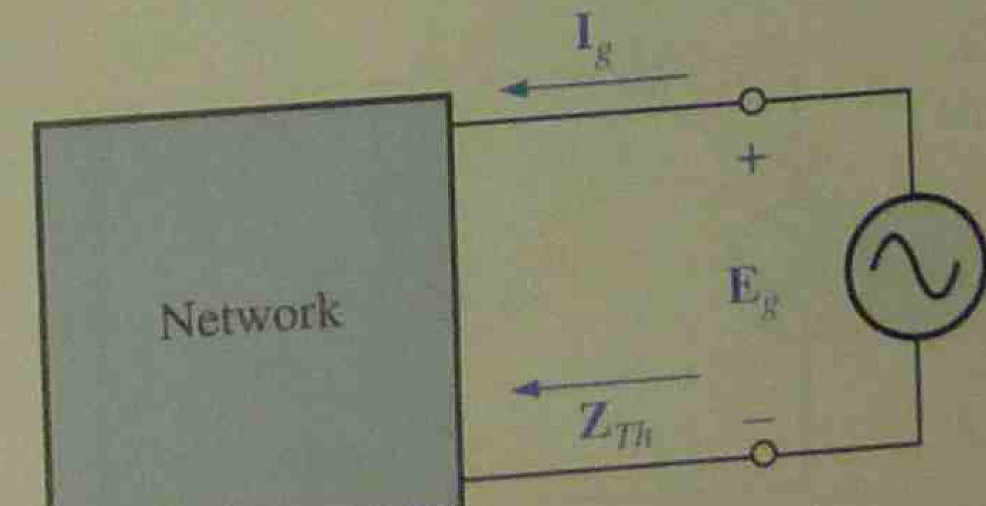


FIG. 18.39
Determining $\mathbf{Z}_{Th}$.

method, the source voltage of the original network is set to zero. The Thevenin impedance is then determined by the following equation:

$$\mathbf{Z}_{Th} = \frac{\mathbf{E}_g}{\mathbf{I}_g} \qquad (18.4)$$

Note that for each technique, $\mathbf{E}_{Th} = \mathbf{E}_{oc}$, but the Thevenin impedance is found in different ways.

EXAMPLE 18.10 Using each of the three techniques described in this section, determine the Thevenin equivalent circuit for the network of Fig. 18.40.

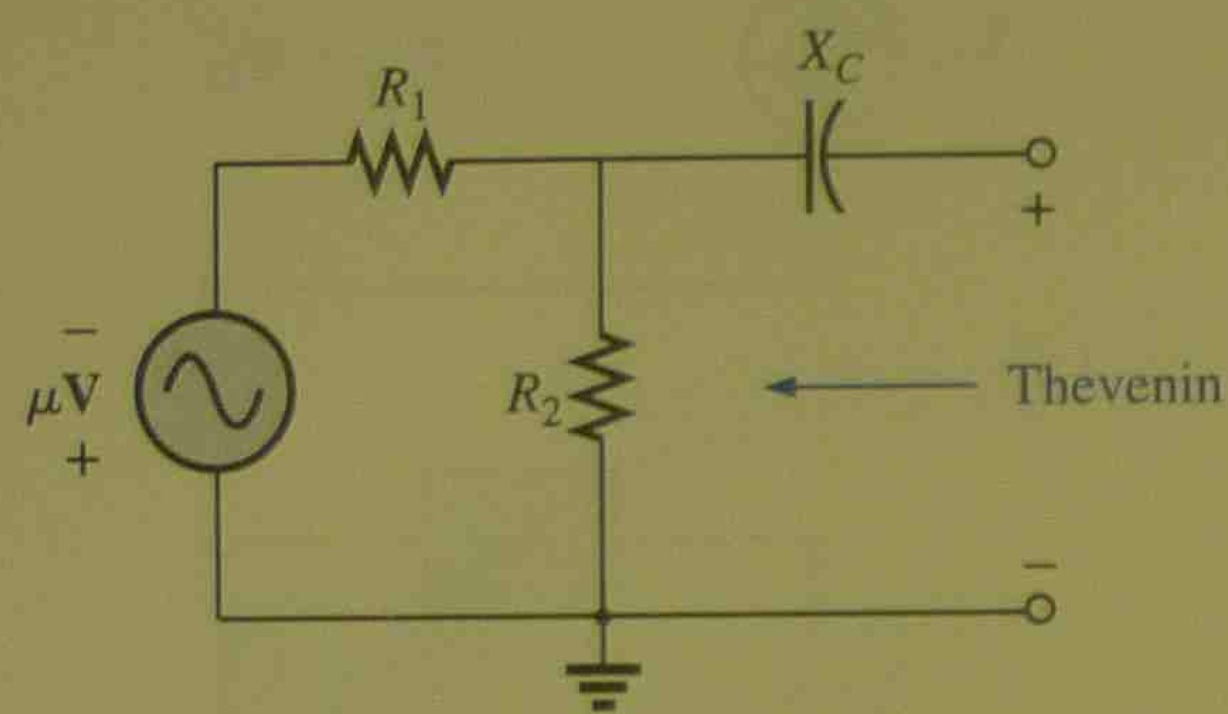


FIG. 18.40

Solution: Since for each approach the Thevenin voltage is found in exactly the same manner, it will be determined first. From Fig. 18.40, where $\mathbf{I}_{X_C} = 0$,

Due to the polarity for V and defined terminal polarities

$$\mathbf{V}_{R_1} = \mathbf{E}_{Th} = \mathbf{E}_{oc} = -\frac{R_2(\mu\mathbf{V})}{R_1 + R_2} = -\frac{\mu R_2 \mathbf{V}}{R_1 + R_2}$$

The following three methods for determining the Thevenin impedance appear in the order in which they were introduced in this section.

Method 1 (Fig. 18.41):

$$\mathbf{Z}_{Th} = R_1 \| R_2 - jX_C$$

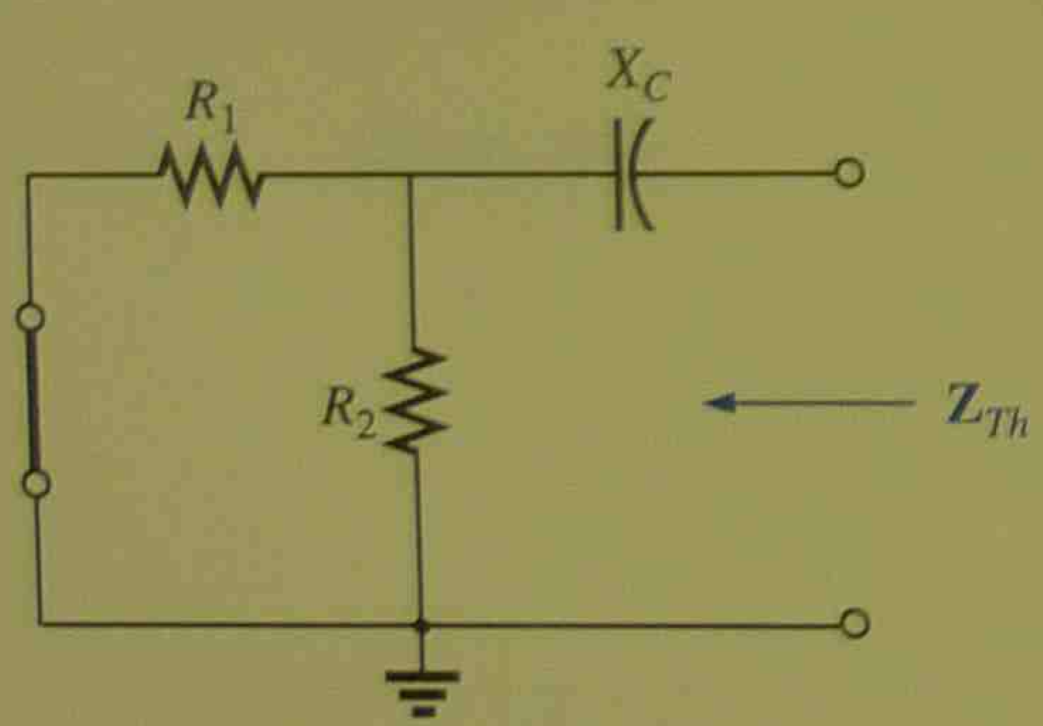


FIG. 18.41

Method 2 (Fig. 18.42): Converting the voltage source to a current source (Fig. 18.43), we have (current divider rule)

$$\mathbf{I}_{sc} = \frac{-(R_1 \| R_2)\dfrac{\mu\mathbf{V}}{R_1}}{(R_1 \| R_2) - jX_C} = \frac{-\dfrac{R_1R_2}{R_1 + R_2}\left(\dfrac{\mu\mathbf{V}}{R_1}\right)}{(R_1 \| R_2) - jX_C}$$

$$= \frac{\dfrac{-\mu R_2\mathbf{V}}{R_1 + R_2}}{(R_1 \| R_2) - jX_C}$$

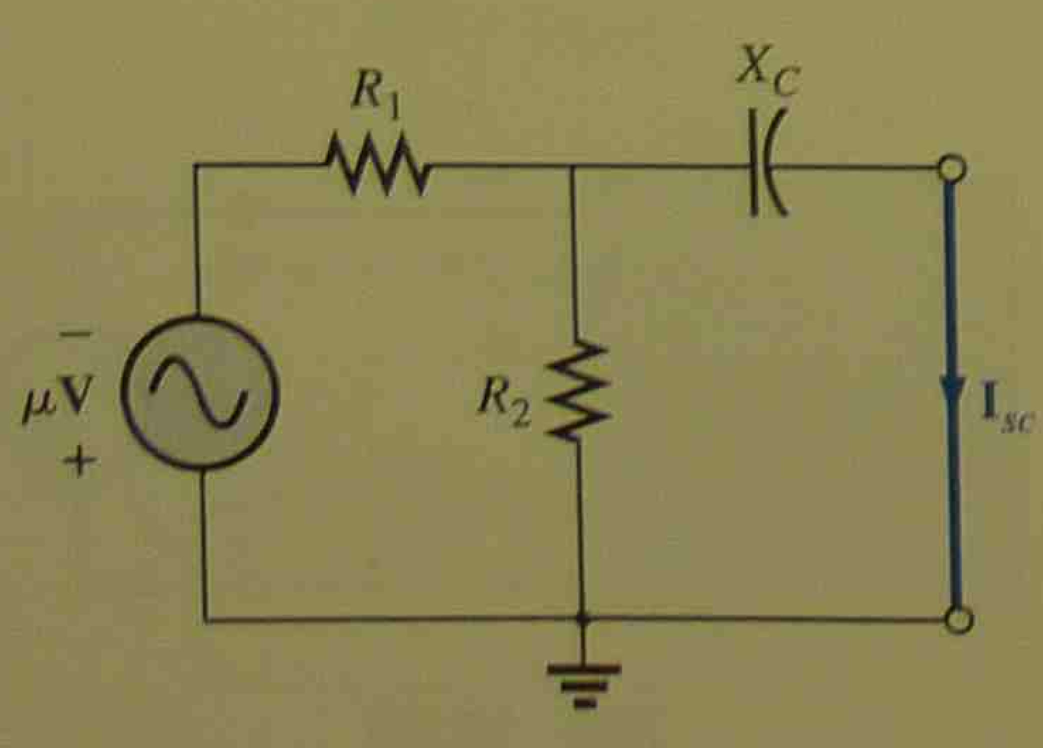


FIG. 18.42

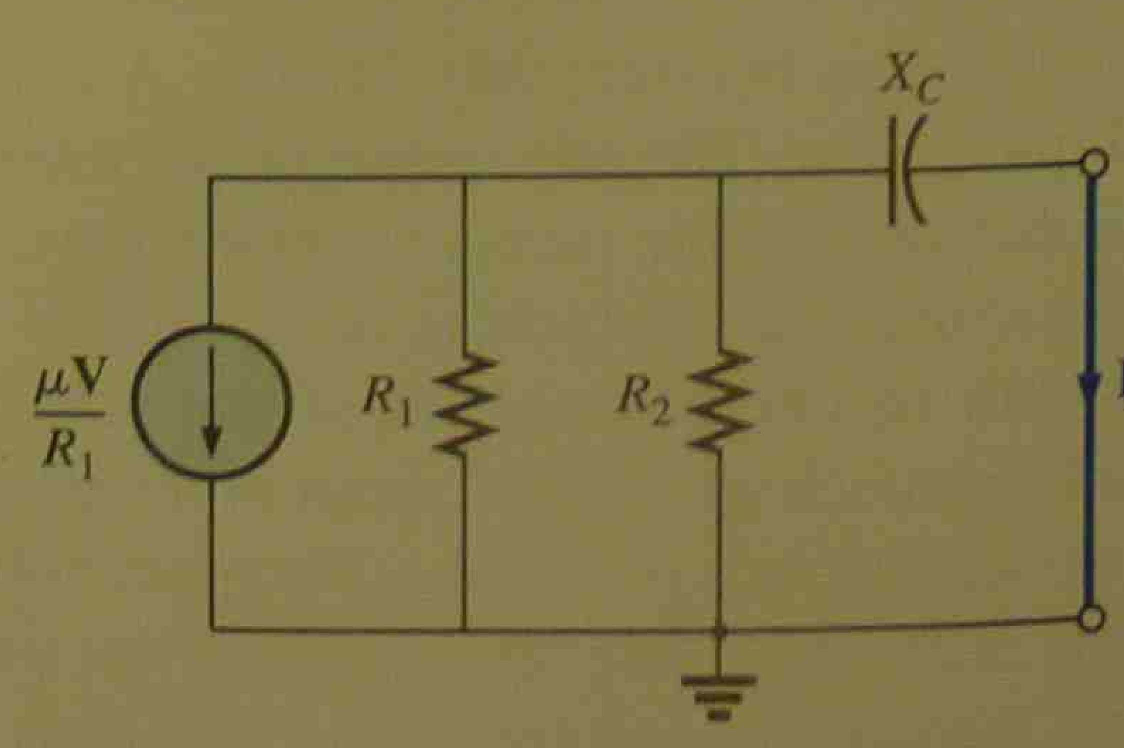


FIG. 18.43

and

$$Z_{Th} = \frac{E_{oc}}{I_{sc}} = \frac{\dfrac{-\mu R_2 \mathbf{V}}{R_1 + R_2}}{\dfrac{\dfrac{-\mu R_2 \mathbf{V}}{R_1 + R_2}}{(R_1 \parallel R_2) - jX_C}} = \frac{1}{\dfrac{1}{(R_1 \parallel R_2) - jX_C}}$$

$$= R_1 \parallel R_2 - jX_C$$

Method 3 (Fig. 18.44):

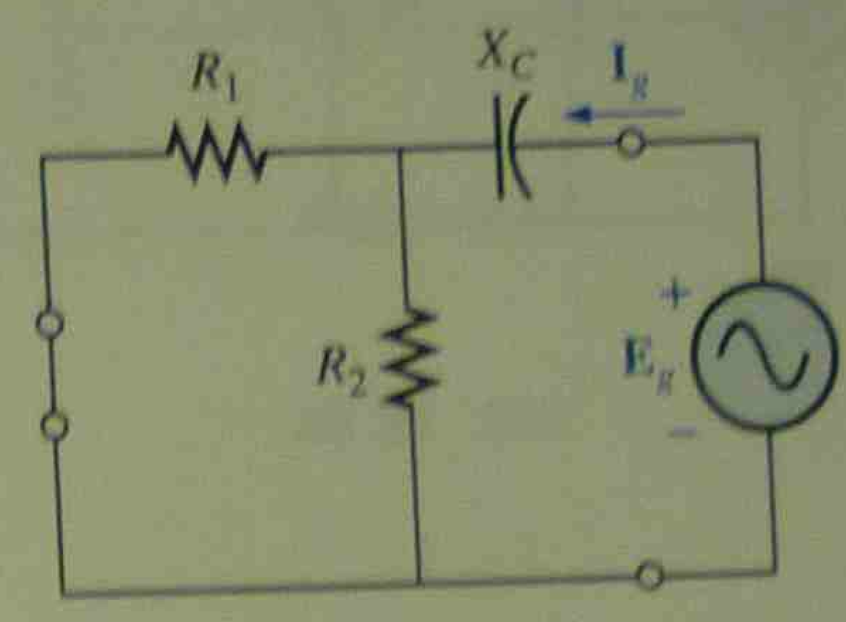


FIG. 18.44

$$\mathbf{I}_g = \frac{\mathbf{E}_g}{(R_1 \parallel R_2) - jX_C}$$

and

$$\mathbf{Z}_{Th} = \frac{\mathbf{E}_g}{\mathbf{I}_g} = R_1 \parallel R_2 - jX_C$$

In each case, the Thevenin impedance is the same. The resulting Thevenin equivalent circuit is shown in Fig. 18.45.

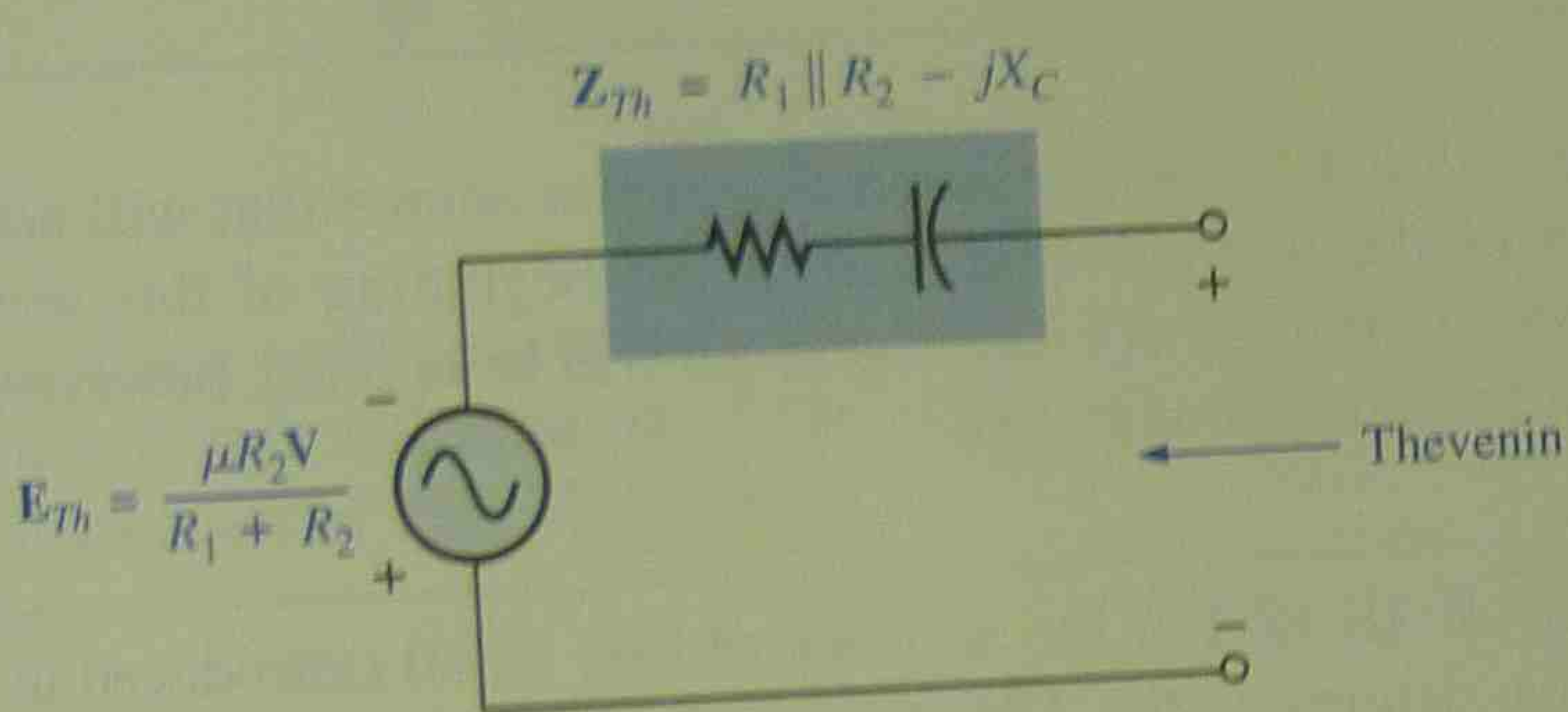


FIG. 18.45

EXAMPLE 18.11 Repeat Example 18.10 for the network of Fig. 18.46.

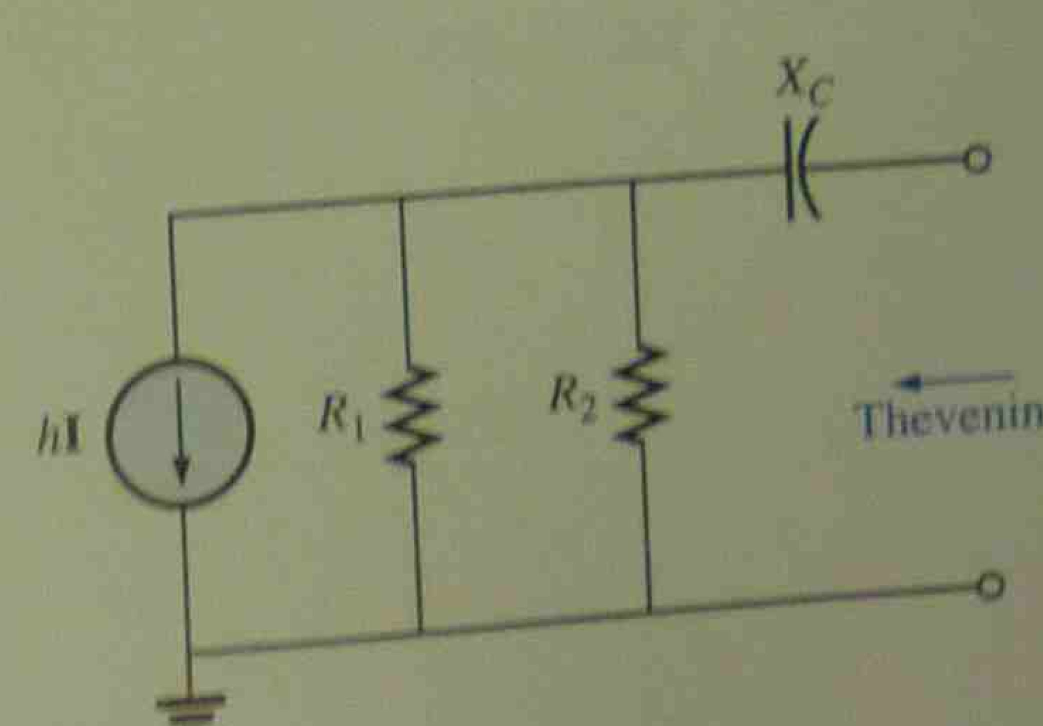


FIG. 18.46

Solution: From Fig. 18.46, $\mathbf{E}_{Th}$ is

$$\mathbf{E}_{Th} = \mathbf{E}_{oc} = -h\mathbf{I}(R_1 \parallel R_2) = -\frac{hR_1R_2\mathbf{I}}{R_1 + R_2}$$

Method 1 (Fig. 18.47):

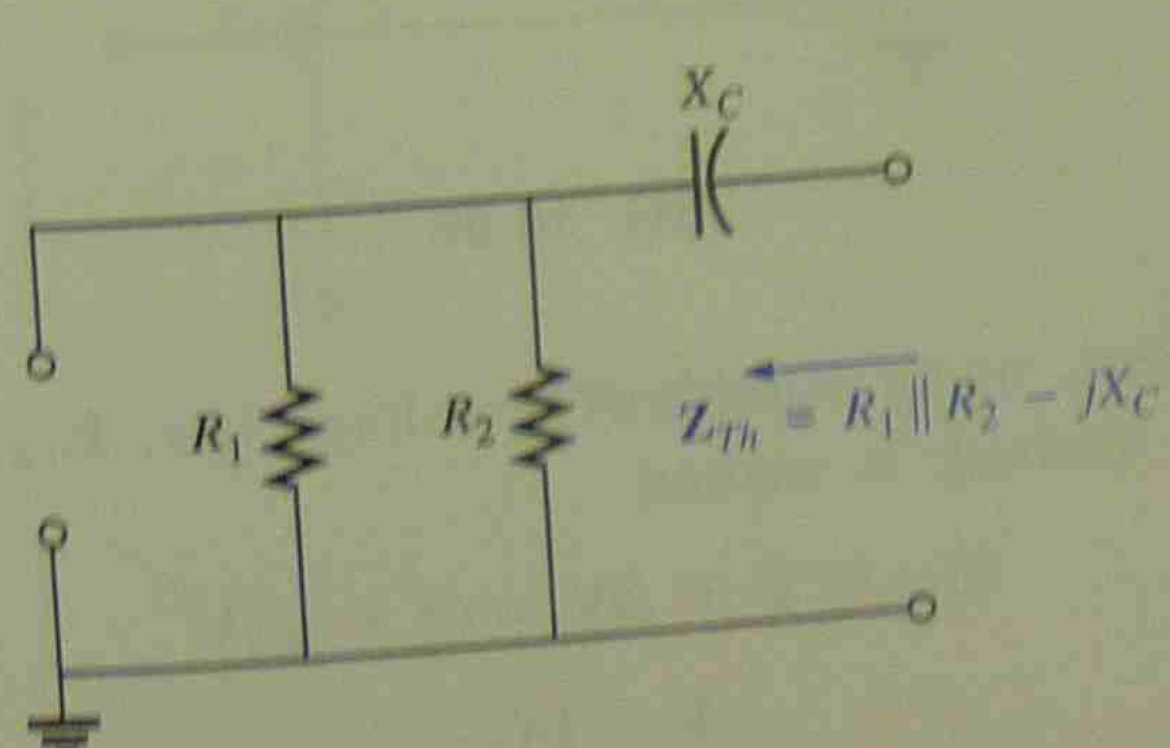


FIG. 18.47

$$\mathbf{Z}_{Th} = R_1 \| R_2 - jX_C$$

Note the similarity between this solution and that obtained for the previous example.

Method 2 (Fig. 18.48):

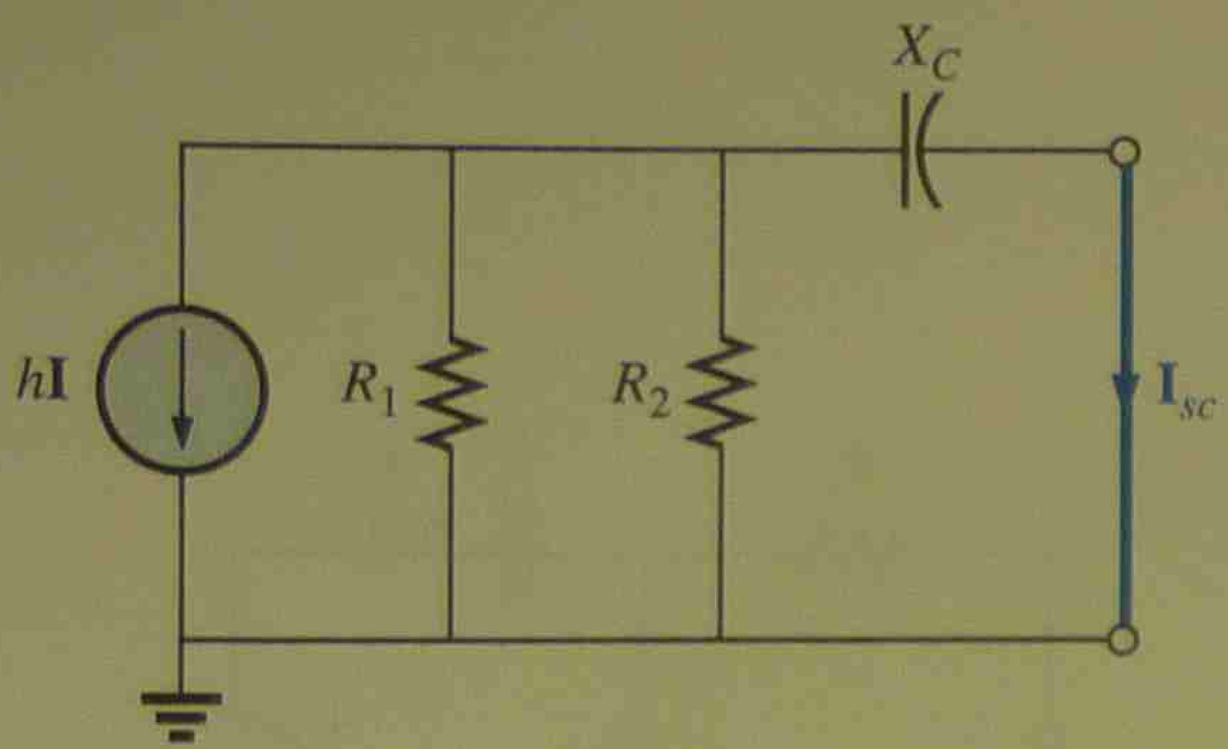


FIG. 18.48

$$\mathbf{I}_{sc} = \frac{-(R_1 \| R_2)h\mathbf{I}}{(R_1 \| R_2) - jX_C}$$

and

$$\mathbf{Z}_{Th} = \frac{\mathbf{E}_{oc}}{\mathbf{I}_{sc}} = \frac{-h\mathbf{I}(R_1 \| R_2)}{\dfrac{-(R_1 \| R_2)h\mathbf{I}}{(R_1 \| R_2) - jX_C}} = R_1 \| R_2 - jX_C$$

Method 3 (Fig. 18.49):

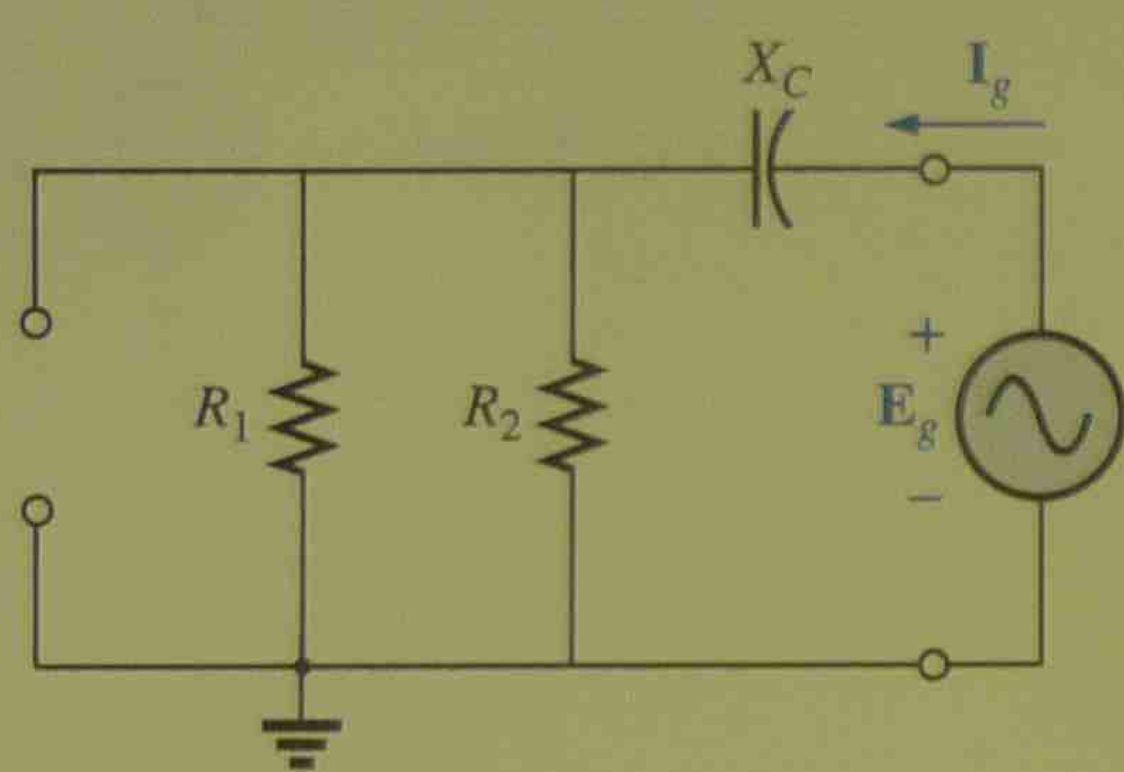


FIG. 18.49

$$\mathbf{I}_g = \frac{\mathbf{E}_g}{(R_1 \| R_2) - jX_C}$$

and

$$\mathbf{Z}_{Th} = \frac{\mathbf{E}_g}{\mathbf{I}_g} = R_1 \| R_2 - jX_C$$

The following example has a dependent source that will not permit the use of the method described in the beginning of this section for independent sources. All three methods will be applied, however, so that the results can be compared.

EXAMPLE 18.12 For the network of Fig. 18.50 (introduced in Example 18.6), determine the Thevenin equivalent circuit between the indicated terminals using each method described in this section. Compare your results.

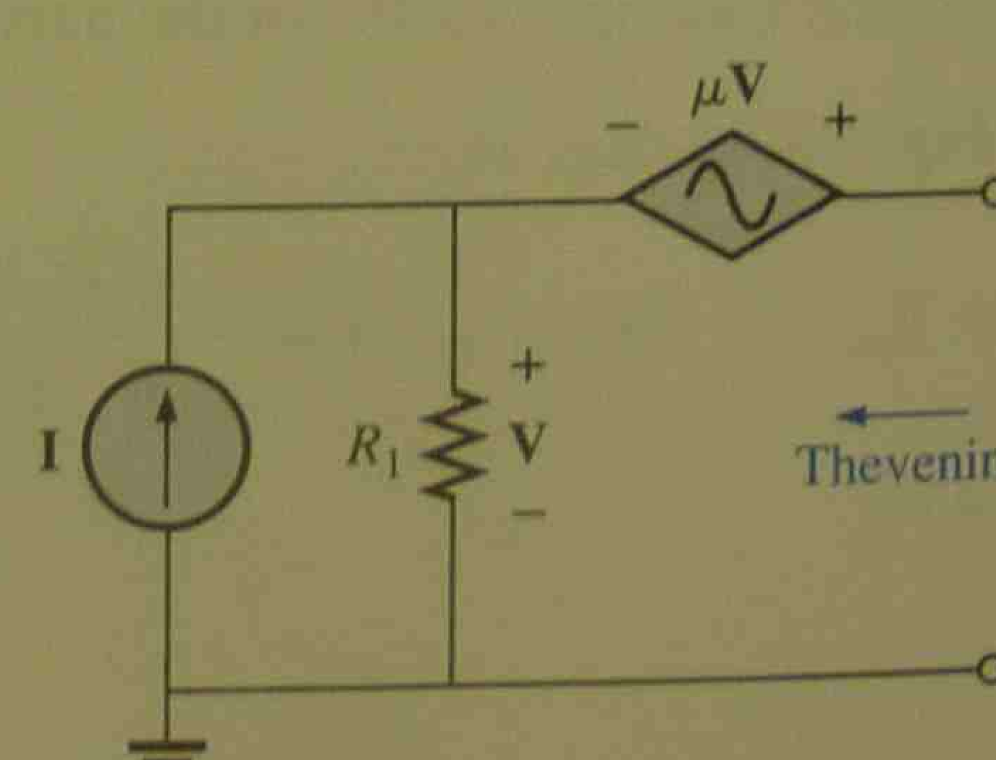


FIG. 18.50

Solution: First, using Kirchhoff's voltage law, $\mathbf{E}_{Th}$ (which is the same for each method) is written

$$\mathbf{E}_{Th} = \mathbf{V} + \mu\mathbf{V} = (1 + \mu)\mathbf{V}$$

However,

$$\mathbf{V} = \mathbf{I}R_1$$

so

$$\mathbf{E}_{Th} = (1 + \mu)\mathbf{I}R_1$$

$\mathbf{Z}_{Th}$:

Method 1 (Fig. 18.51): Since $\mathbf{I} = 0$, $\mathbf{V}$ and $\mu\mathbf{V} = 0$, and

~~$\mathbf{Z}_{Th} = R_1$~~ (incorrect)

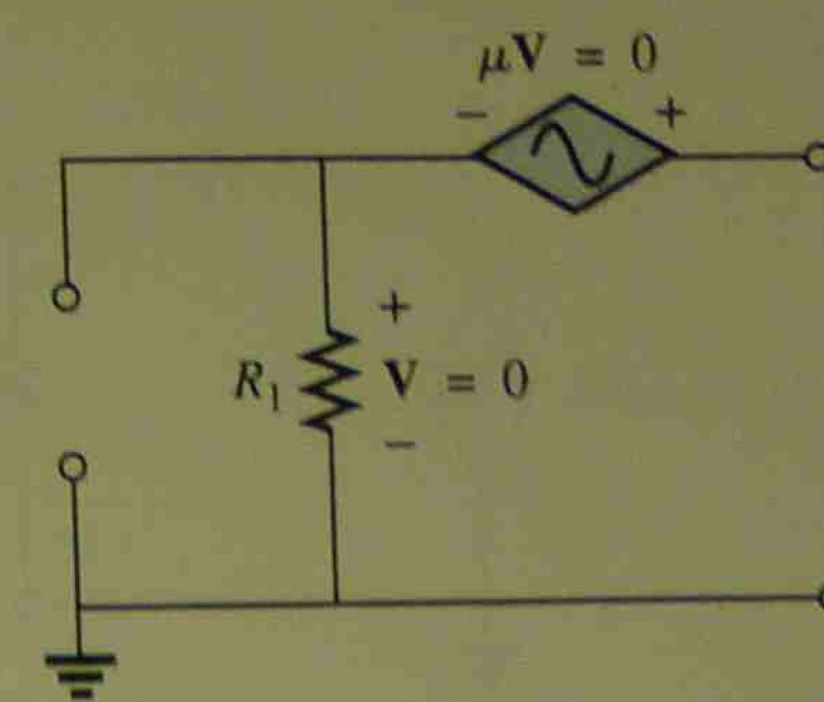


FIG. 18.51

Method 2 (Fig. 18.52): Kirchhoff's voltage law around the indicated loop gives us

$$\mathbf{V} + \mu\mathbf{V} = 0$$

and

$$\mathbf{V}(1 + \mu) = 0$$

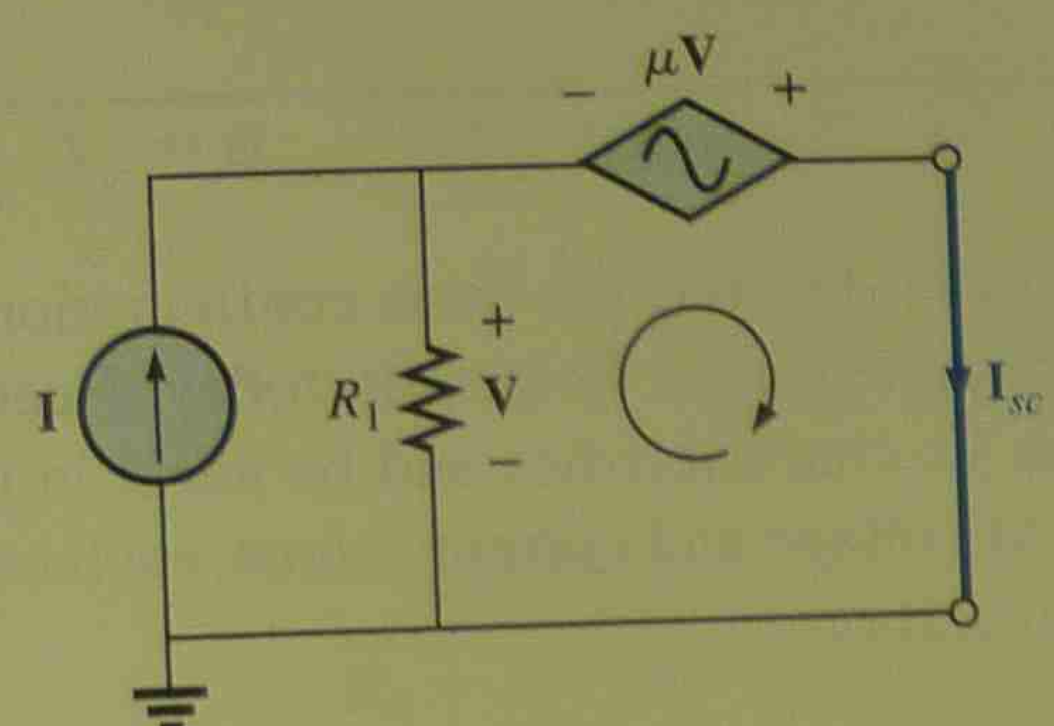


FIG. 18.52

Since μ is a positive constant, the above equation can be satisfied only when $\mathbf{V} = 0$. Substitution of this result into Fig. 18.52 will yield the configuration of Fig. 18.53, and

$$\mathbf{I}_{sc} = \mathbf{I}$$

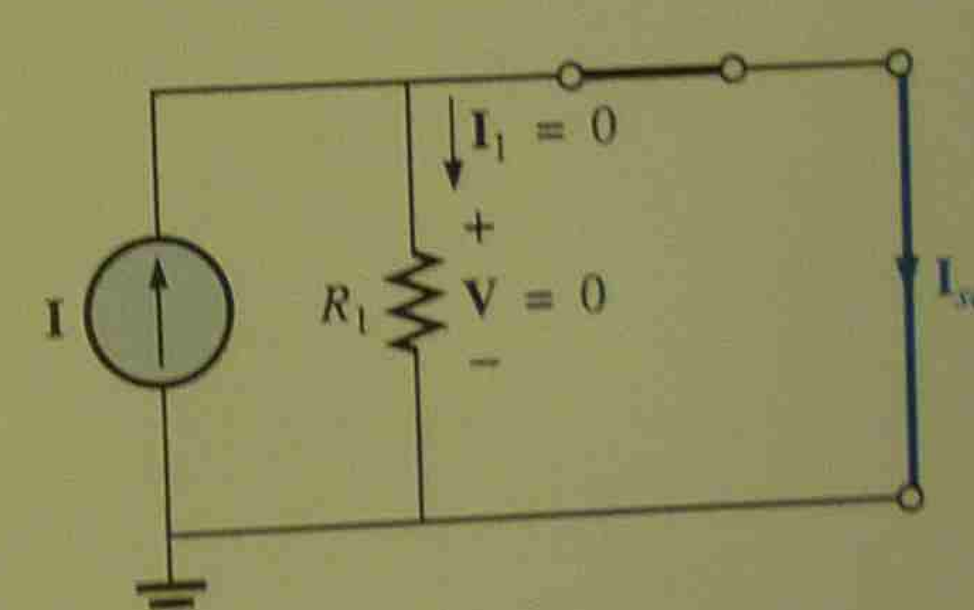


FIG. 18.53

with

$$\mathbf{Z}_{Th} = \frac{\mathbf{E}_{oc}}{\mathbf{I}_{sc}} = \frac{(1 + \mu)\mathbf{I}R_1}{\mathbf{I}} = (1 + \mu)R_1 \qquad \text{(correct)}$$

Method 3 (Fig. 18.54):

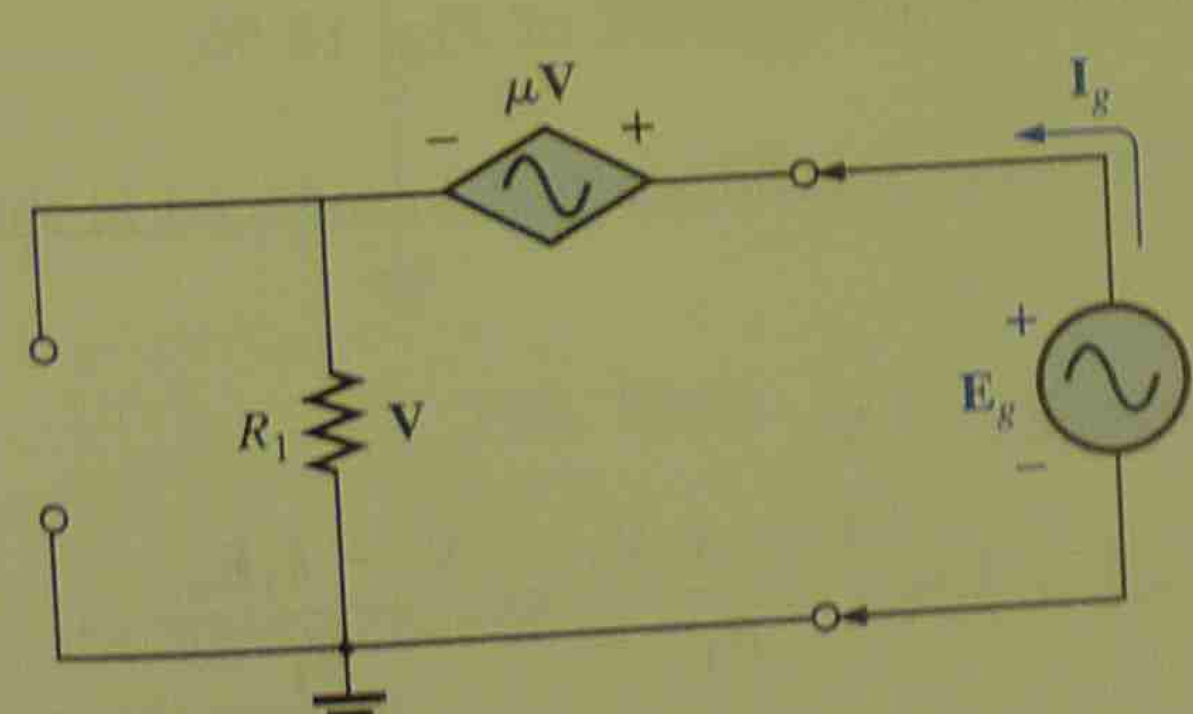


FIG. 18.54

$$\mathbf{E}_g = \mathbf{V} + \mu\mathbf{V} = (1 + \mu)\mathbf{V}$$

or

$$\mathbf{V} = \frac{\mathbf{E}_g}{1 + \mu}$$

and
$$\mathbf{I}_g = \frac{\mathbf{V}}{R_1} = \frac{\mathbf{E}_g}{(1 + \mu)R_1}$$

and
$$\mathbf{Z}_{Th} = \frac{\mathbf{E}_g}{\mathbf{I}_g} = (1 + \mu)R_1 \qquad \text{(correct)}$$

The Thevenin equivalent circuit appears in Fig. 18.55, and

$$\mathbf{I}_L = \frac{(1 + \mu)R_1\mathbf{I}}{R_L + (1 + \mu)R_1}$$

which compares with the result of Example 18.6.

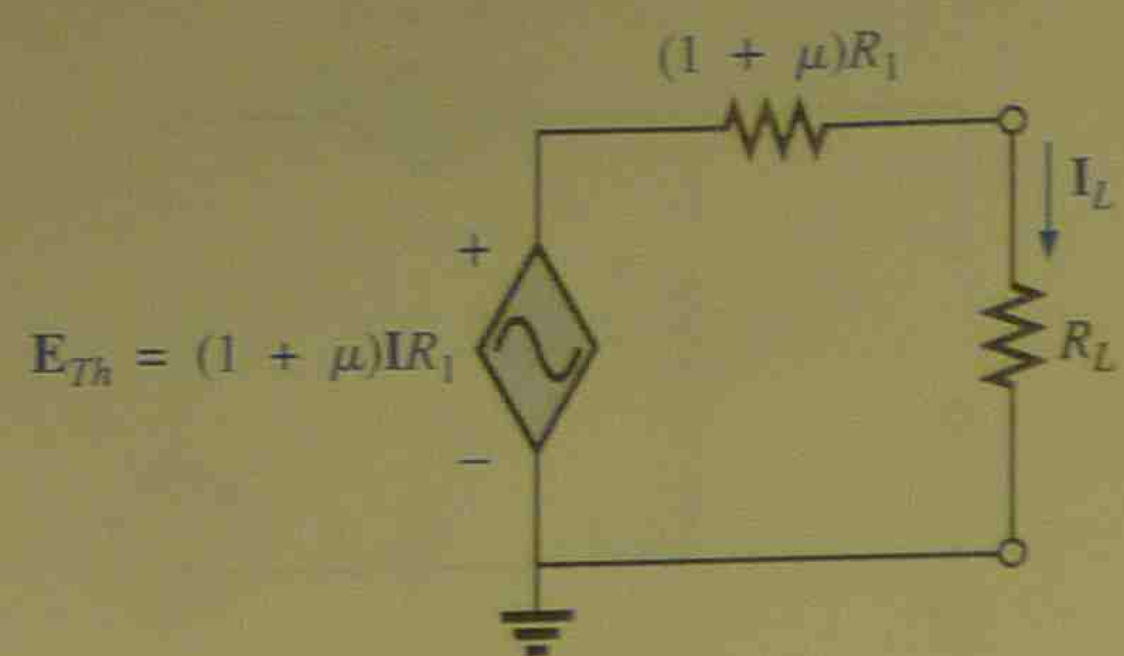


FIG. 18.55

The network of Fig. 18.56 is the basic configuration of the transistor equivalent circuit applied most frequently today. Needless to say, it is necessary to know its characteristics and be adept in its use. Note that there is a controlled voltage and current source, each controlled by variables in the configuration.

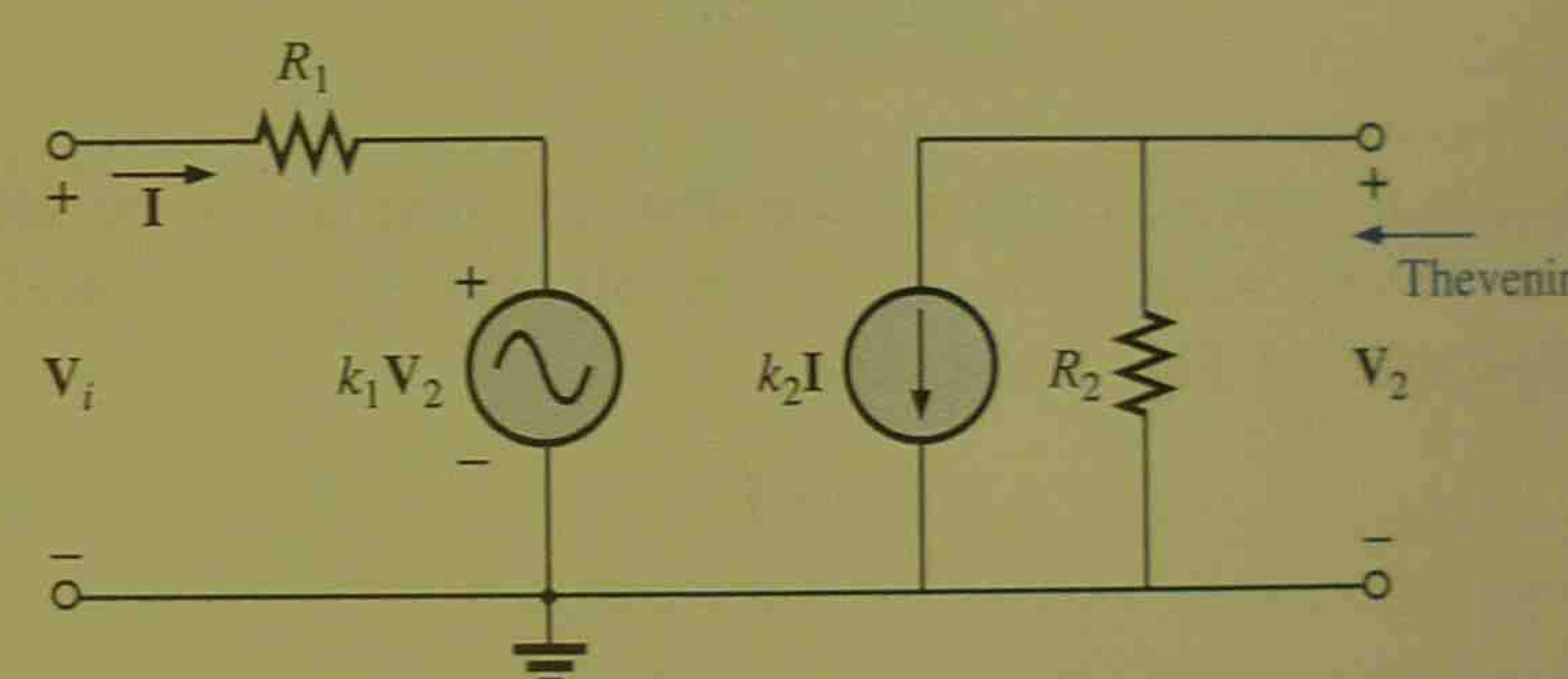


FIG. 18.56

EXAMPLE 18.13 Determine the Thevenin equivalent circuit for the indicated terminals of the network of Fig. 18.56.

Solution: Apply the second method introduced in this section.

$\mathbf{E}_{Th}$:

$$\mathbf{E}_{oc} = \mathbf{V}_2$$

$$\mathbf{I} = \frac{\mathbf{V}_i - k_1\mathbf{V}_2}{R_1} = \frac{\mathbf{V}_i - k_1\mathbf{E}_{oc}}{R_1}$$

and
$$\mathbf{E}_{oc} = -k_2\mathbf{I}R_2 = -k_2R_2\left(\frac{\mathbf{V}_i - k_1\mathbf{E}_{oc}}{R_1}\right)$$

$$= \frac{-k_2R_2\mathbf{V}_i}{R_1} + \frac{k_1k_2R_2\mathbf{E}_{oc}}{R_1}$$

or
$$\mathbf{E}_{oc}\left(1 - \frac{k_1k_2R_2}{R_1}\right) = \frac{-k_2R_2\mathbf{V}_i}{R_1}$$

and
$$\mathbf{E}_{oc}\left(\frac{R_1 - k_1k_2R_2}{\not{R}_1}\right) = \frac{-k_2R_2\mathbf{V}_i}{\not{R}_1}$$

so
$$\boxed{\mathbf{E}_{oc} = \frac{-k_2R_2\mathbf{V}_i}{R_1 - k_1k_2R_2} = \mathbf{E}_{Th}} \tag{18.5}$$

$\mathbf{I}_{sc}$:

For the network of Fig. 18.57, where

$$\mathbf{V}_2 = 0 \qquad k_1\mathbf{V}_2 = 0 \qquad \mathbf{I} = \frac{\mathbf{V}_i}{R_1}$$

and
$$\mathbf{I}_{sc} = -k_2\mathbf{I} = \frac{-k_2\mathbf{V}_i}{R_1}$$

so
$$\mathbf{Z}_{Th} = \frac{\mathbf{E}_{oc}}{\mathbf{I}_{sc}} = \frac{\dfrac{-k_2R_2\mathbf{V}_i}{R_1 - k_1k_2R_2}}{\dfrac{-k_2\mathbf{V}_i}{R_1}} = \frac{R_1R_2}{R_1 - k_1k_2R_2}$$

and
$$\boxed{\mathbf{Z}_{Th} = \frac{R_2}{1 - \dfrac{k_1k_2R_2}{R_1}}} \tag{18.6}$$

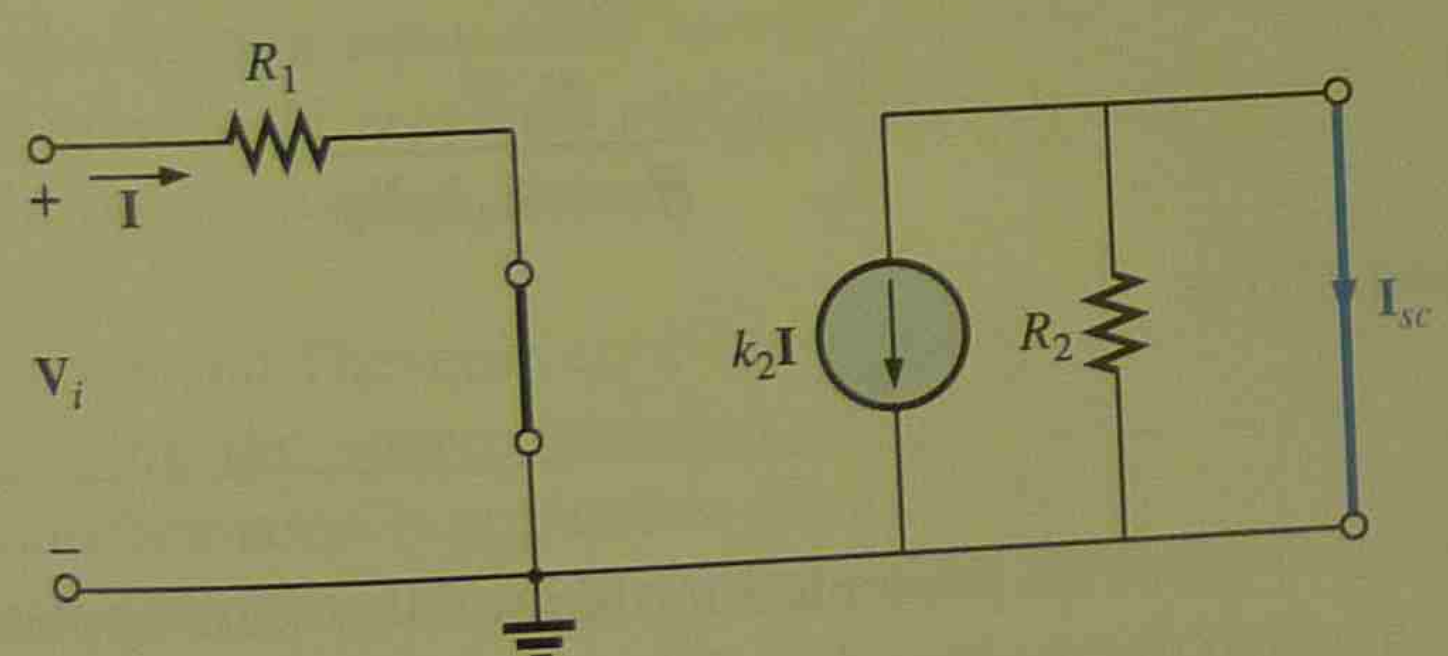


FIG. 18.57

Frequently, the approximation $k_1 \cong 0$ is applied. Then, the Thevenin voltage and impedance are

$$\boxed{\mathbf{E}_{Th} = \frac{-k_2R_2\mathbf{V}_i}{R_1}} \qquad k_1 = 0 \tag{18.7}$$

$$\boxed{\mathbf{Z}_{Th} = R_2} \qquad k_1 = 0 \tag{18.8}$$

Apply $\mathbf{Z}_{Th} = \mathbf{E}_g/\mathbf{I}_g$ to the network of Fig. 18.58, where

$$\mathbf{I} = \frac{-k_1\mathbf{V}_2}{R_1}$$

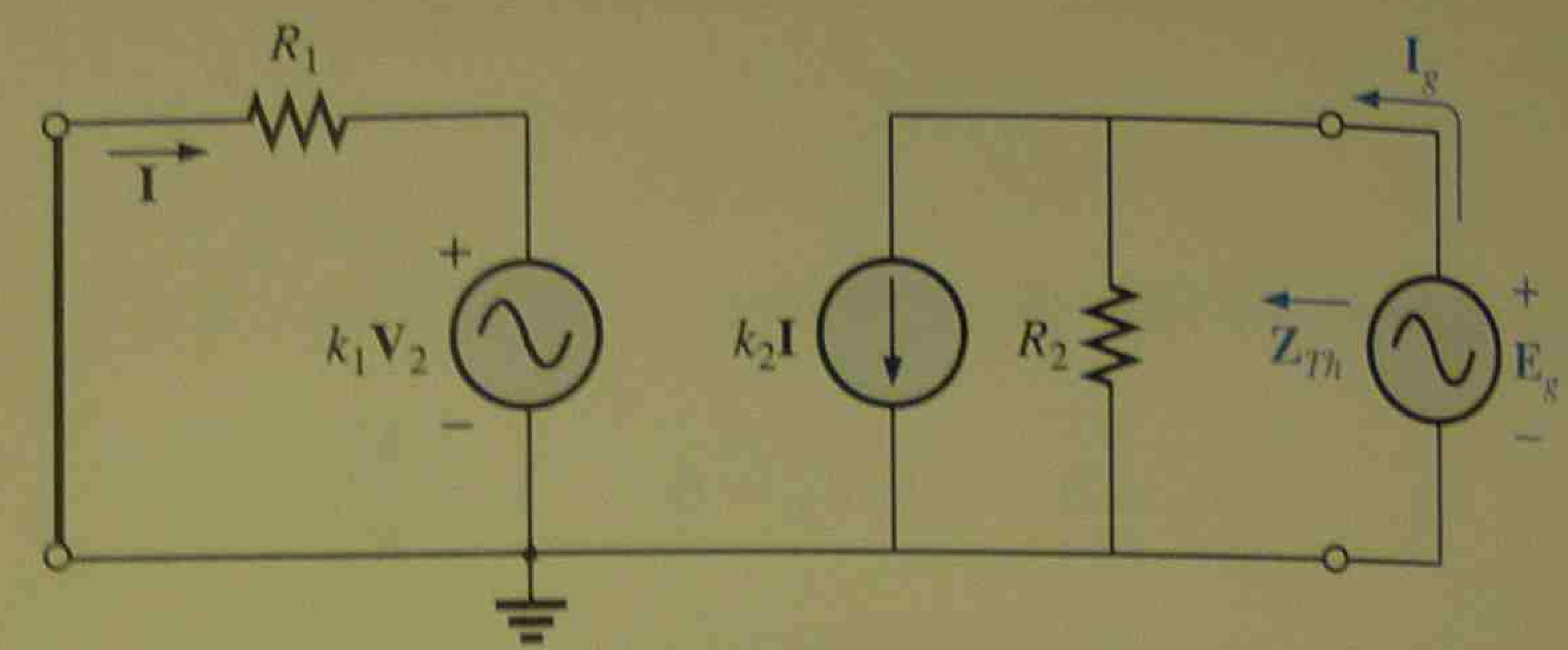


FIG. 18.58

But
$$\mathbf{V}_2 = \mathbf{E}_g$$

so
$$\mathbf{I} = \frac{-k_1\mathbf{E}_g}{R_1}$$

Applying Kirchhoff's current law, we have

$$\mathbf{I}_g = k_2\mathbf{I} + \frac{\mathbf{E}_g}{R_2} = k_2\left(-\frac{k_1\mathbf{E}_g}{R_1}\right) + \frac{\mathbf{E}_g}{R_2}$$

$$= \mathbf{E}_g\left(\frac{1}{R_2} - \frac{k_1k_2}{R_1}\right)$$

and
$$\frac{\mathbf{I}_g}{\mathbf{E}_g} = \frac{R_1 - k_1k_2R_2}{R_1R_2}$$

or
$$\mathbf{Z}_{Th} = \frac{\mathbf{E}_g}{\mathbf{I}_g} = \frac{R_1R_2}{R_1 - k_1k_2R_2}$$

as obtained above.

The last two methods presented in this section were applied only to networks in which the magnitudes of the controlled sources were dependent on a variable within the network for which the Thevenin equivalent circuit was to be obtained. Understand that both of those methods can also be applied to any dc or sinusoidal ac network containing only independent sources or dependent sources of the other kind.

18.4 NORTON'S THEOREM

The three methods described for Thevenin's theorem will each be altered to permit their use with Norton's theorem. Since the Thevenin and Norton impedances are the same for a particular network, certain portions of the discussion will be quite similar to those encountered in the previous section. We will first consider independent sources and the approach developed in Chapter 9, followed by dependent sources and the new techniques developed for Thevenin's theorem.

You will recall from Chapter 9 that Norton's theorem allows us to replace any two-terminal linear bilateral ac network by an equivalent circuit consisting of a current source and impedance, as in Fig. 18.59.

The Norton equivalent circuit, like the Thevenin equivalent circuit, is applicable at only one frequency since the reactances are frequency dependent.

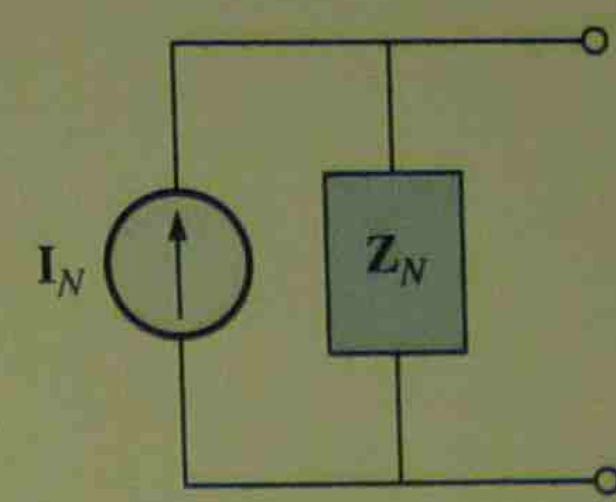


FIG. 18.59

Independent Sources

The procedure outlined below to find the Norton equivalent of a sinusoidal ac network is changed (from that in Chapter 9) in only one respect: to replace the term *impedance* with the term *resistance*.

1. *Remove that portion of the network across which the Norton equivalent circuit is to be found.*
2. *Mark (○, •, and so on) the terminals of the remaining two-terminal network.*
3. *Calculate $\mathbf{Z}_N$ by first setting all voltage and current sources to zero (short circuit and open circuit, respectively) and then finding the resulting impedance between the two marked terminals.*
4. *Calculate $\mathbf{I}_N$ by first replacing the voltage and current sources and then finding the short-circuit current between the marked terminals.*
5. *Draw the Norton equivalent circuit with the portion of the circuit previously removed replaced between the terminals of the Norton equivalent circuit.*

The Norton and Thevenin equivalent circuits can be found from each other by using the source transformation shown in Fig. 18.60. The source transformation is applicable for any Thevenin or Norton equivalent circuit determined from a network with any combination of independent or dependent sources.

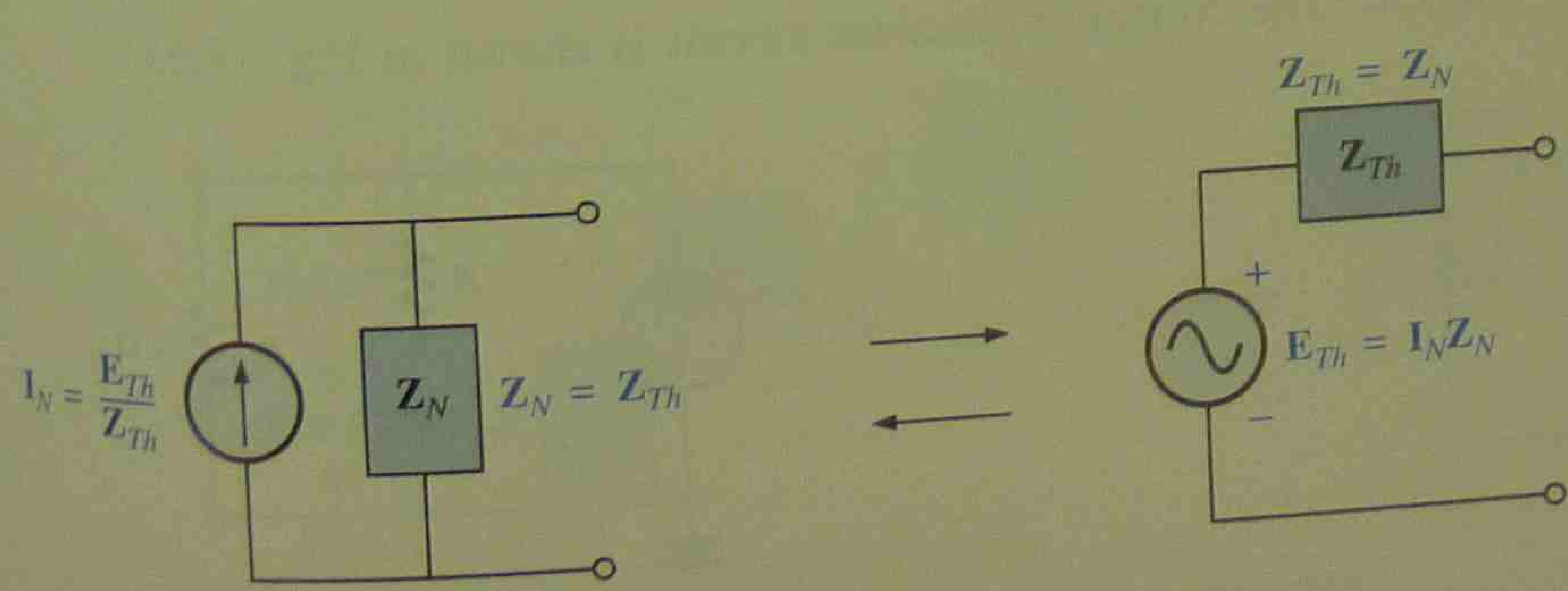


FIG. 18.60

EXAMPLE 18.14 Determine the Norton equivalent circuit for the network external to the 6-Ω resistor of Fig. 18.61.

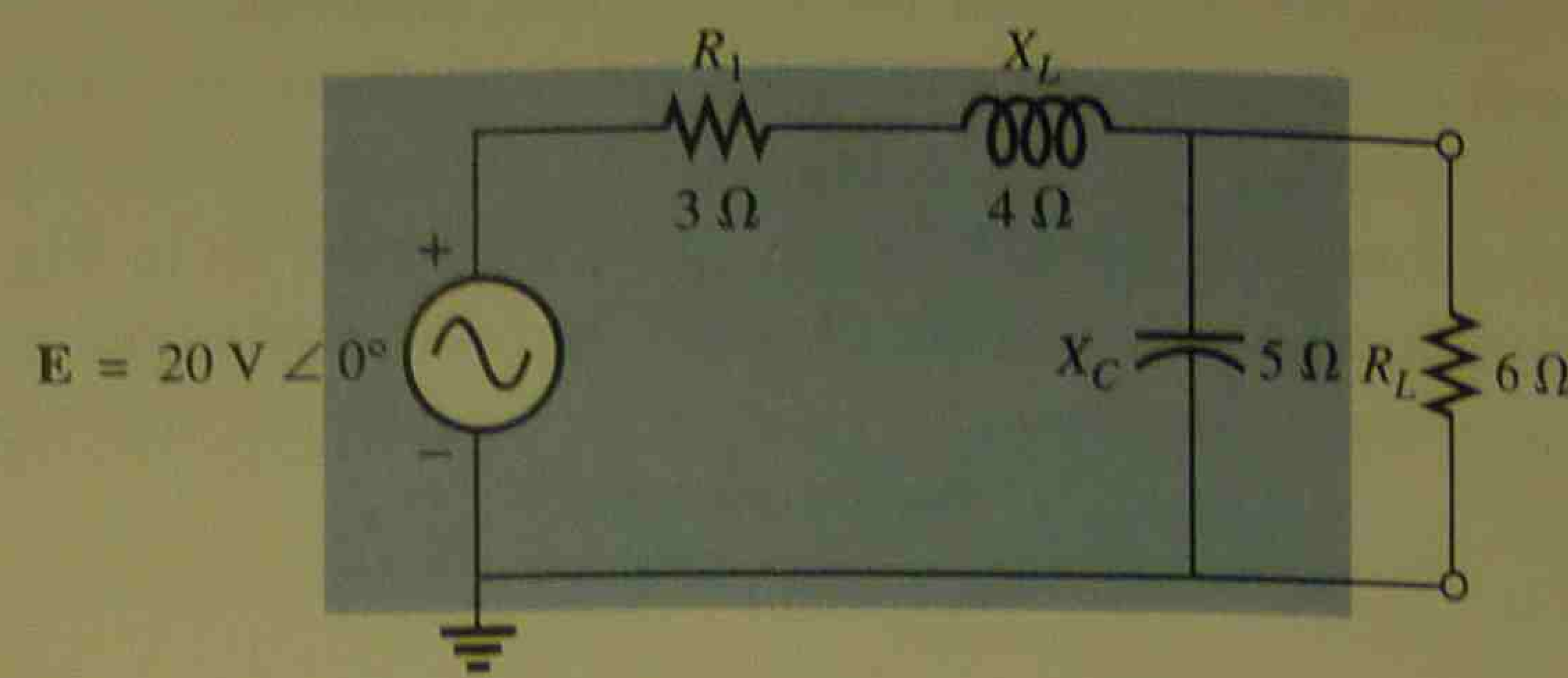


FIG. 18.61

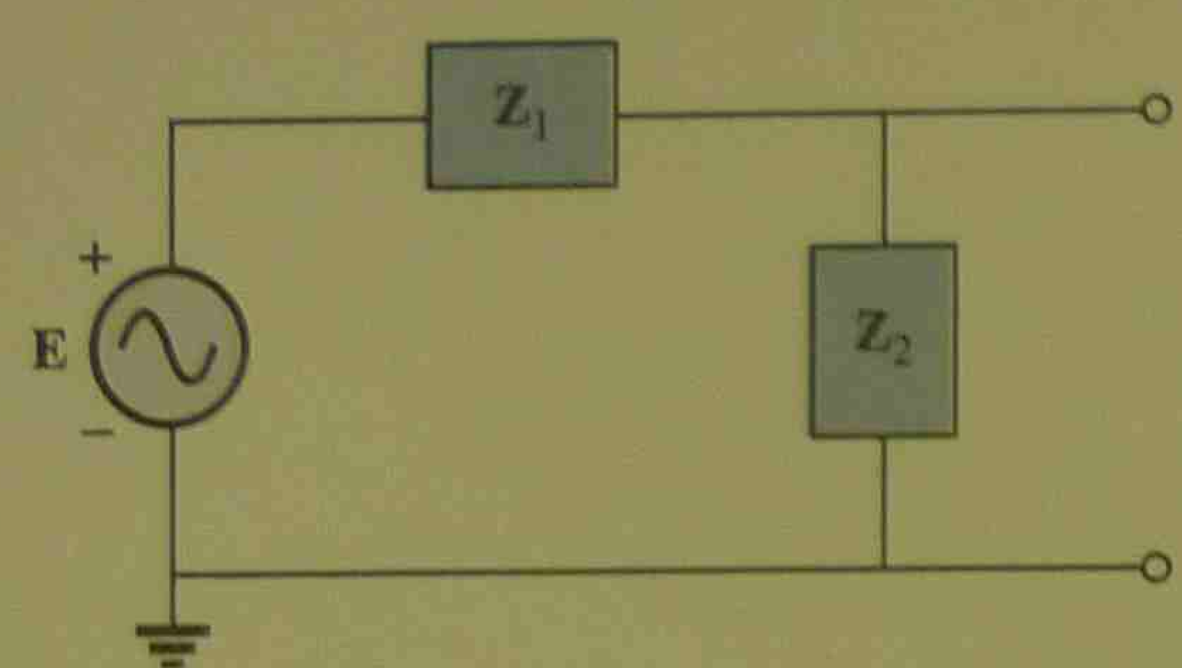


FIG. 18.62

Solution:

Steps 1 and 2 (Fig. 18.62):

$$\mathbf{Z}_1 = R_1 + jX_L = 3\ \Omega + j4\ \Omega = 5\ \Omega\ \angle 53.13^\circ$$
$$\mathbf{Z}_2 = -jX_C = -j5\ \Omega$$

Step 3 (Fig. 18.63):

$$\mathbf{Z}_N = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2} = \frac{(5\ \Omega\ \angle 53.13^\circ)(5\ \Omega\ \angle -90^\circ)}{3\ \Omega + j4\ \Omega - j5\ \Omega} = \frac{25\ \Omega\ \angle -36.87^\circ}{3 - j1}$$
$$= \frac{25\ \Omega\ \angle -36.87^\circ}{3.16\ \angle -18.43^\circ} = 7.91\ \Omega\ \angle -18.44^\circ = \mathbf{7.50\ \Omega - j2.50\ \Omega}$$

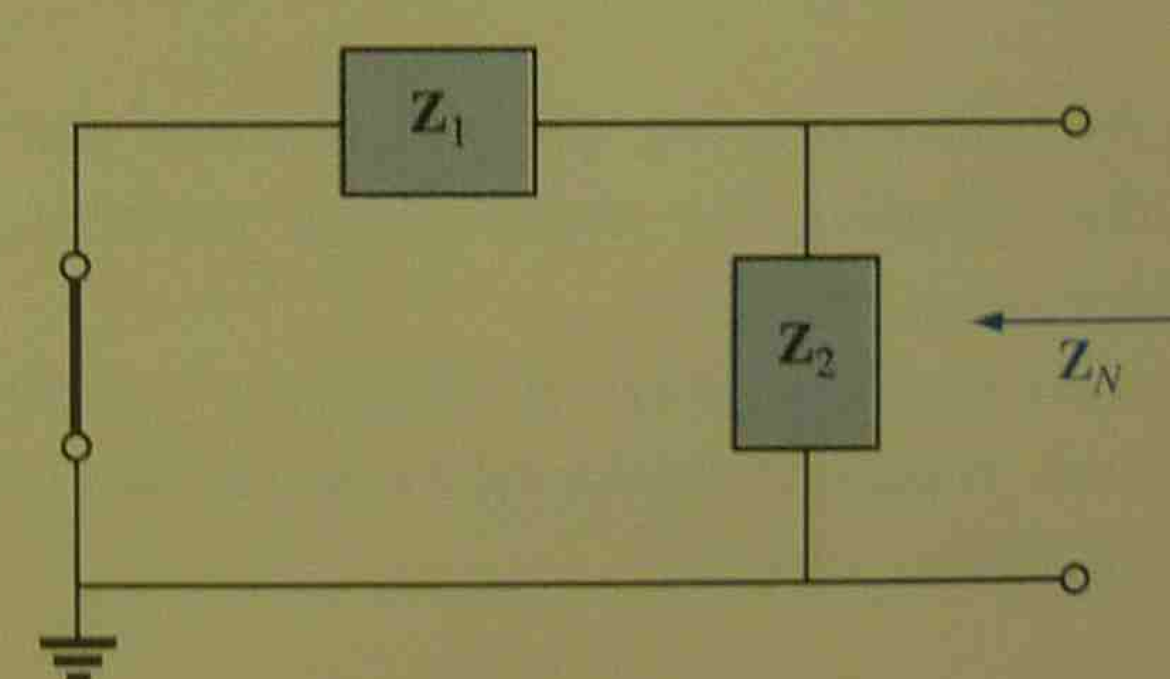


FIG. 18.63

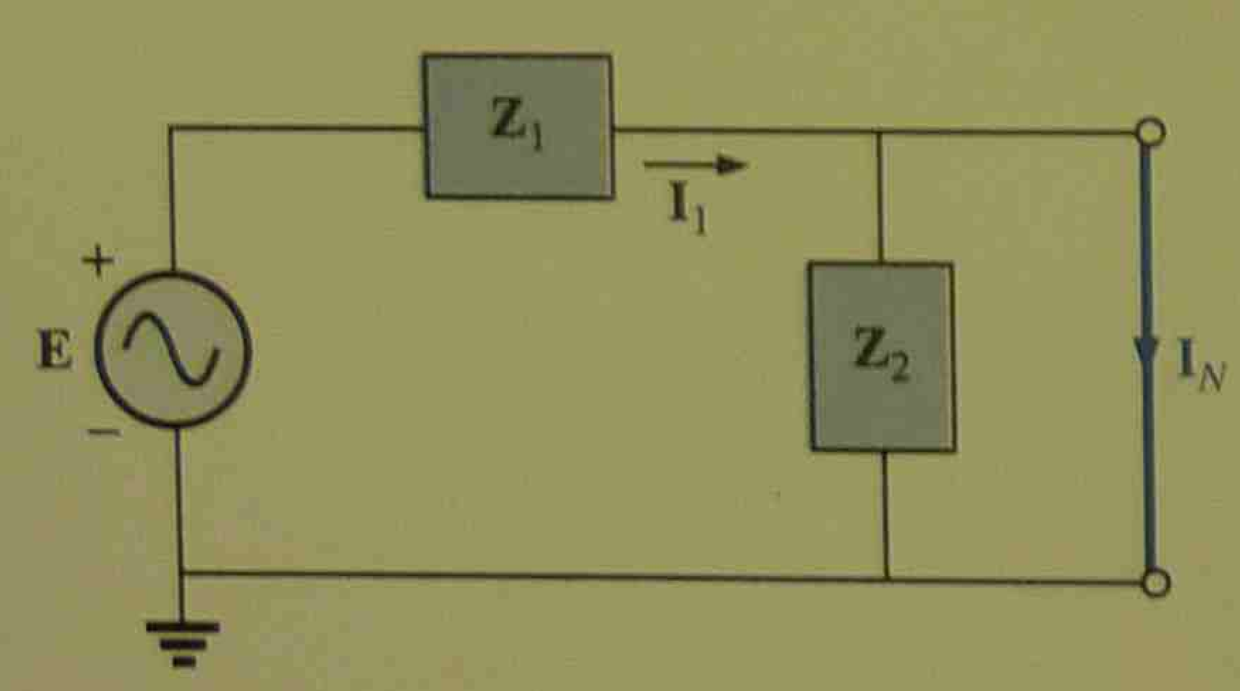


FIG. 18.64

Step 4 (Fig. 18.64):

$$\mathbf{I}_N = \mathbf{I}_1 = \frac{\mathbf{E}}{\mathbf{Z}_1} = \frac{20\ \text{V}\ \angle 0^\circ}{5\ \Omega\ \angle 53.13^\circ} = \mathbf{4\ A\ \angle -53.13^\circ}$$

Step 5: The Norton equivalent circuit is shown in Fig. 18.65.

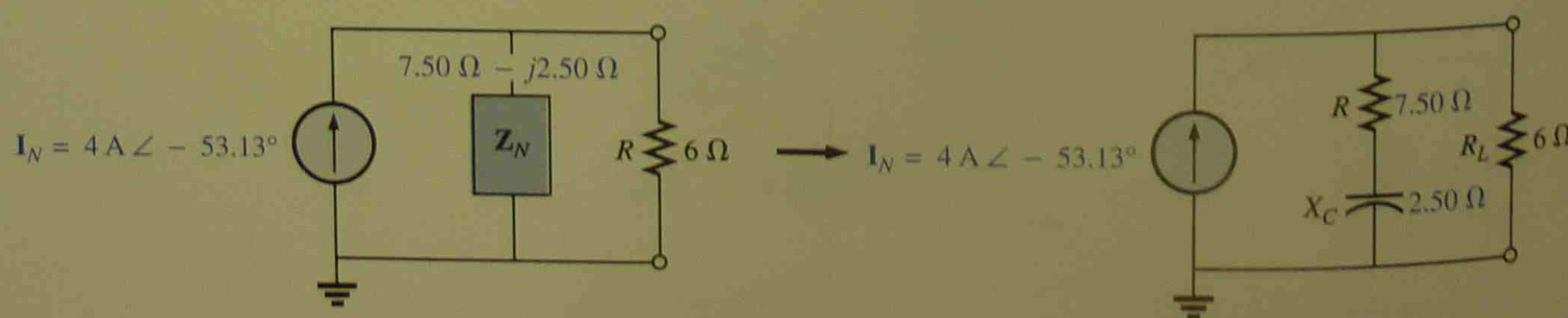


FIG. 18.65

EXAMPLE 18.15 Find the Norton equivalent circuit for the network external to the 7-Ω capacitive reactance in Fig. 18.66.

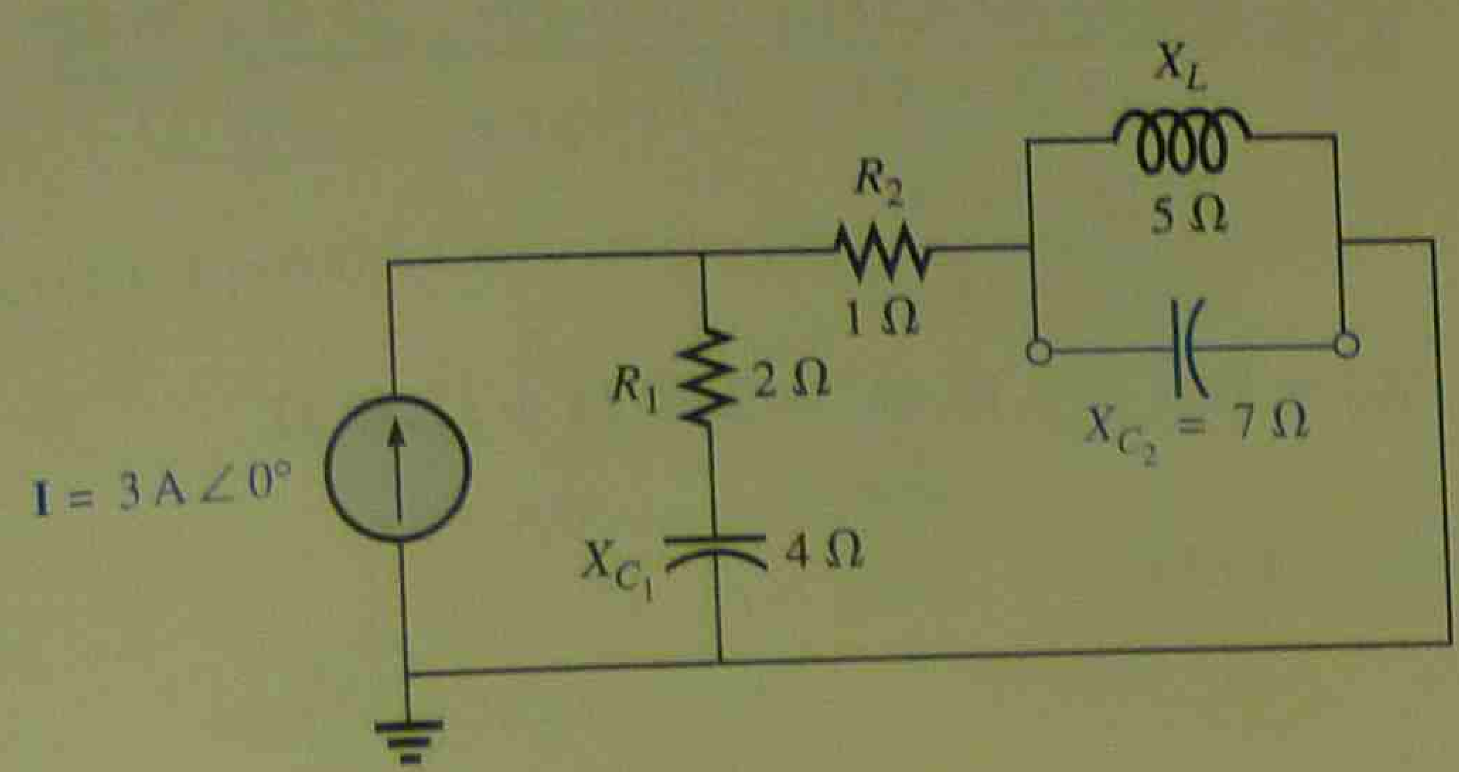


FIG. 18.66

Solution:

Steps 1 and 2 (Fig. 18.67):

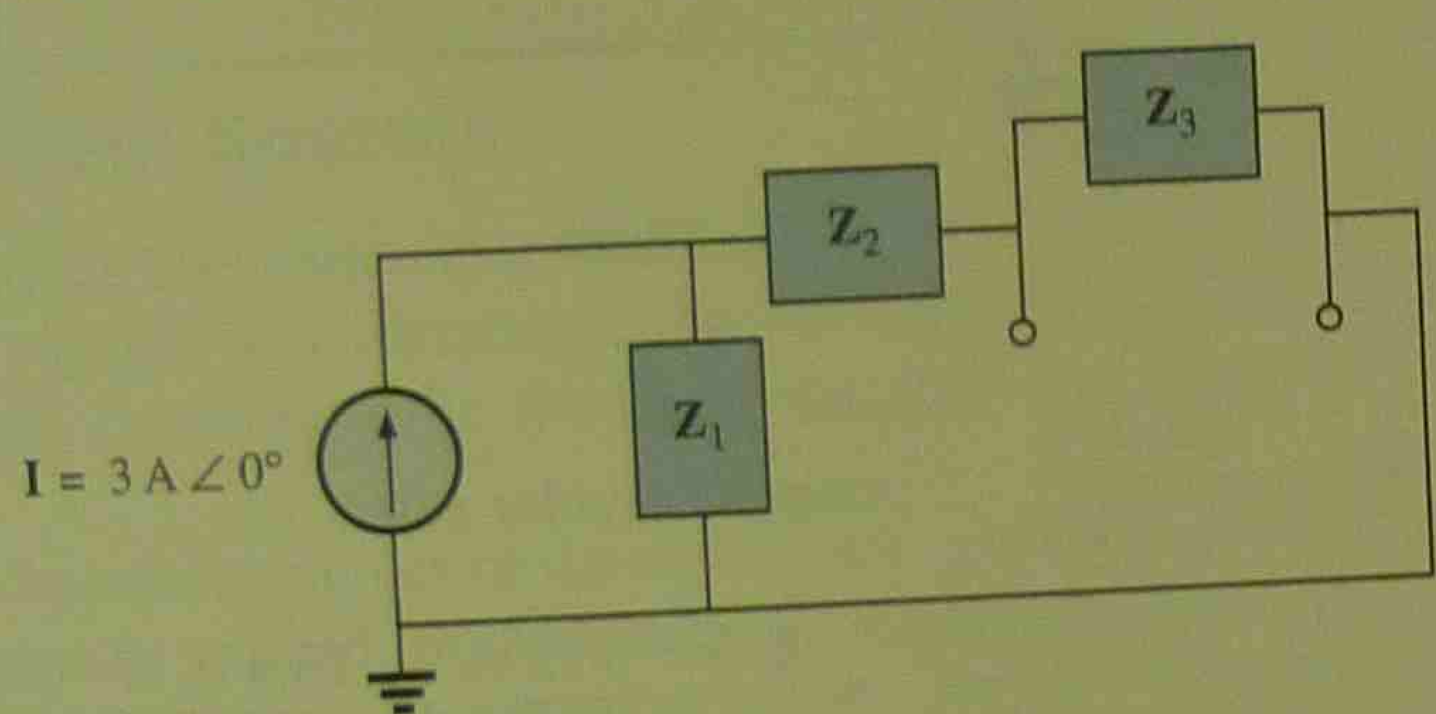


FIG. 18.67

$$\mathbf{Z}_1 = R_1 - jX_{C_1} = 2\ \Omega - j\,4\ \Omega$$
$$\mathbf{Z}_2 = R_2 = 1\ \Omega$$
$$\mathbf{Z}_3 = +jX_L = j\,5\ \Omega$$

Step 3 (Fig. 18.68):

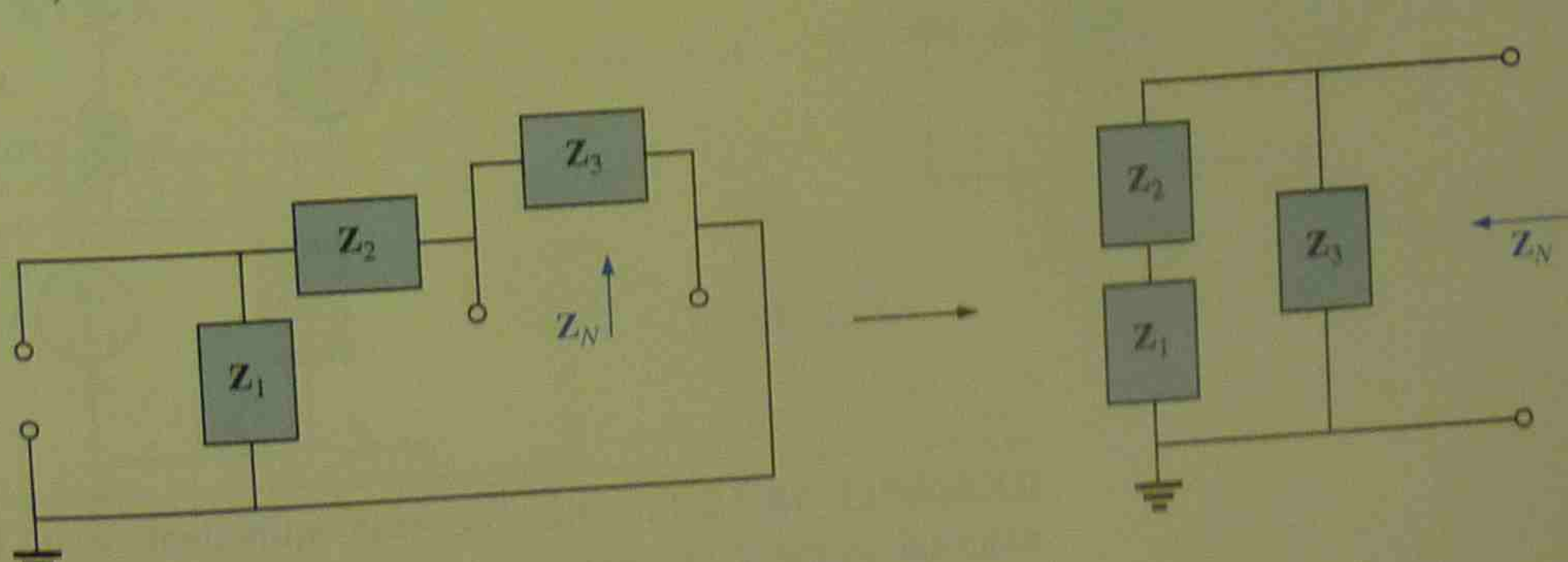


FIG. 18.68

$$\mathbf{Z}_N = \frac{\mathbf{Z}_3(\mathbf{Z}_1 + \mathbf{Z}_2)}{\mathbf{Z}_3 + (\mathbf{Z}_1 + \mathbf{Z}_2)}$$

$$\mathbf{Z}_1 + \mathbf{Z}_2 = 2\,\Omega - j4\,\Omega + 1\,\Omega = 3\,\Omega - j4\,\Omega = 5\,\Omega\,\angle -53.13^\circ$$

$$\mathbf{Z}_N = \frac{(5\,\Omega\,\angle 90^\circ)(5\,\Omega\,\angle -53.13^\circ)}{j5\,\Omega + 3\,\Omega - j4\,\Omega} = \frac{25\,\Omega\,\angle 36.87^\circ}{3 + j1}$$

$$= \frac{25\,\Omega\,\angle 36.87^\circ}{3.16\,\angle +18.43^\circ}$$

$$\mathbf{Z}_N = 7.91\,\Omega\,\angle 18.44^\circ = \mathbf{7.50\,\Omega + j2.50\,\Omega}$$

Step 4 (Fig. 18.69):

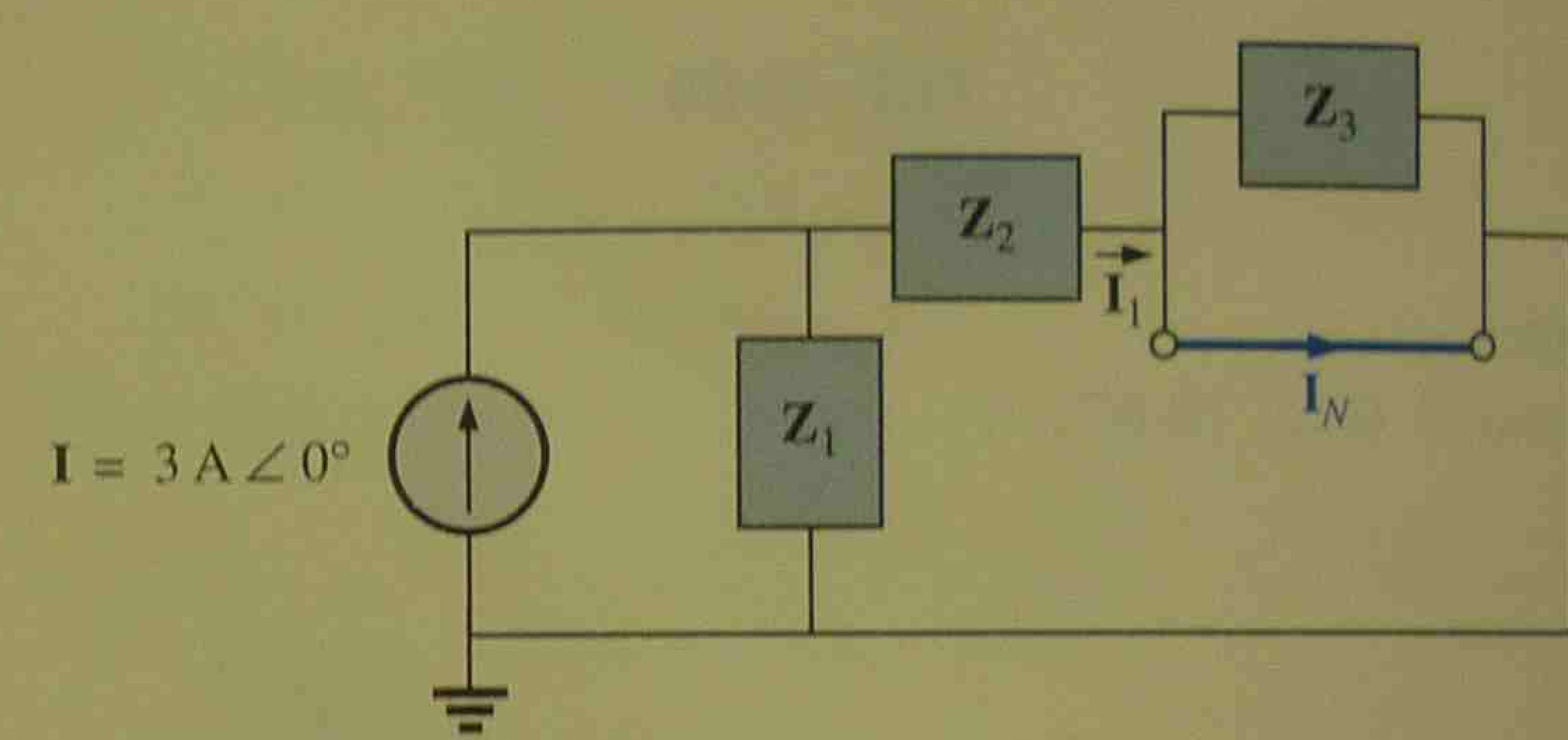


FIG. 18.69

$$\mathbf{I}_N = \mathbf{I}_1 = \frac{\mathbf{Z}_1\mathbf{I}}{\mathbf{Z}_1 + \mathbf{Z}_2} \qquad \text{(current divider rule)}$$

$$= \frac{(2\,\Omega - j4\,\Omega)(3\text{ A})}{3\,\Omega - j4\,\Omega} = \frac{6\text{ A} - j12\text{ A}}{5\,\angle -53.13^\circ} = \frac{13.4\text{ A}\,\angle -63.43^\circ}{5\,\angle -53.13^\circ}$$

$$\mathbf{I}_N = \mathbf{2.68\text{ A}\,\angle -10.3^\circ}$$

Step 5: The Norton equivalent circuit is shown in Fig. 18.70.

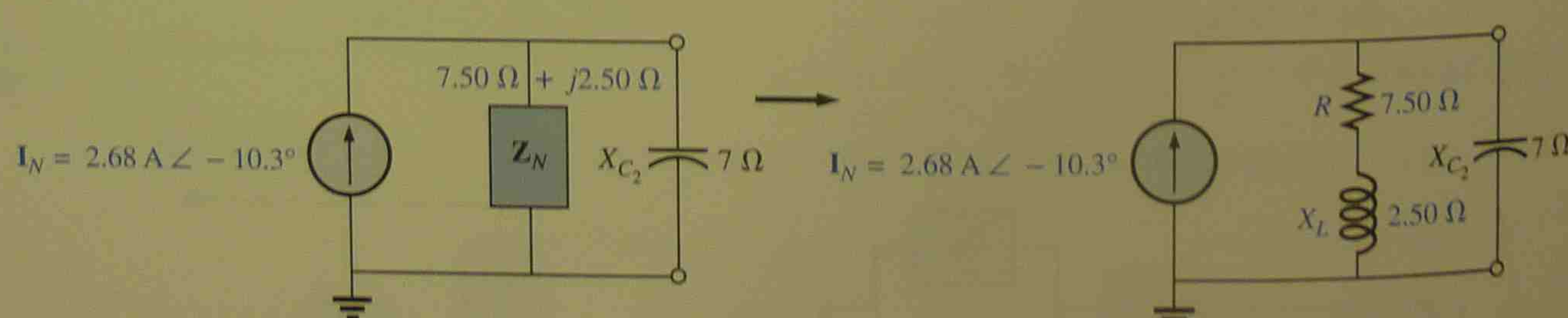


FIG. 18.70

EXAMPLE 18.16 Find the Thevenin equivalent circuit for the network external to the 7-Ω capacitive reactance in Fig. 18.66.

Solution: Using the conversion between sources (Fig. 18.71), we obtain

$$\mathbf{Z}_{Th} = \mathbf{Z}_N = 7.50\ \Omega + j2.50\ \Omega$$
$$\mathbf{E}_{Th} = \mathbf{I}_N\mathbf{Z}_N = (2.68\ \text{A}\ \angle -10.3^\circ)(7.91\ \Omega\ \angle 18.44^\circ)$$
$$= 21.2\ \text{V}\ \angle 8.14^\circ$$

The Thevenin equivalent circuit is shown in Fig. 18.72.

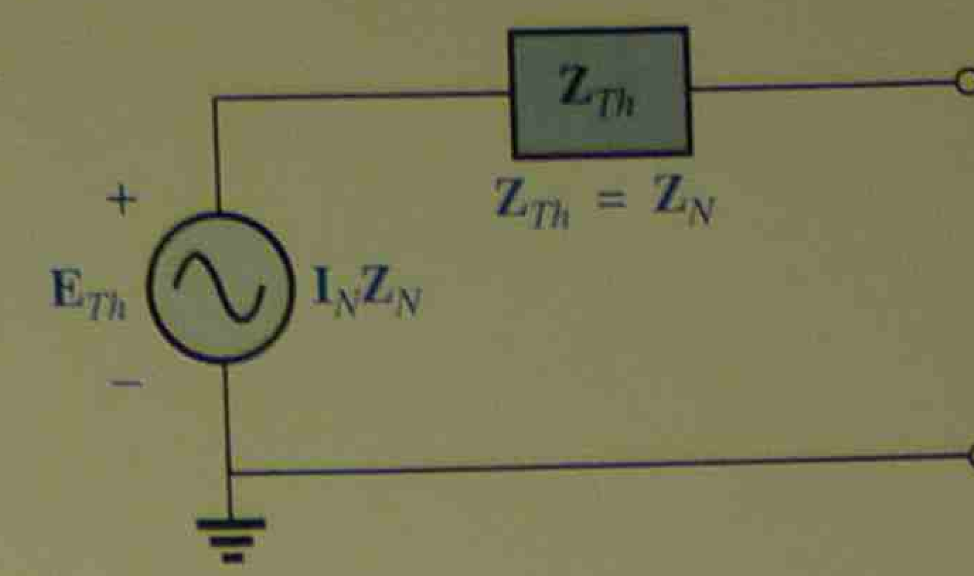


FIG. 18.71

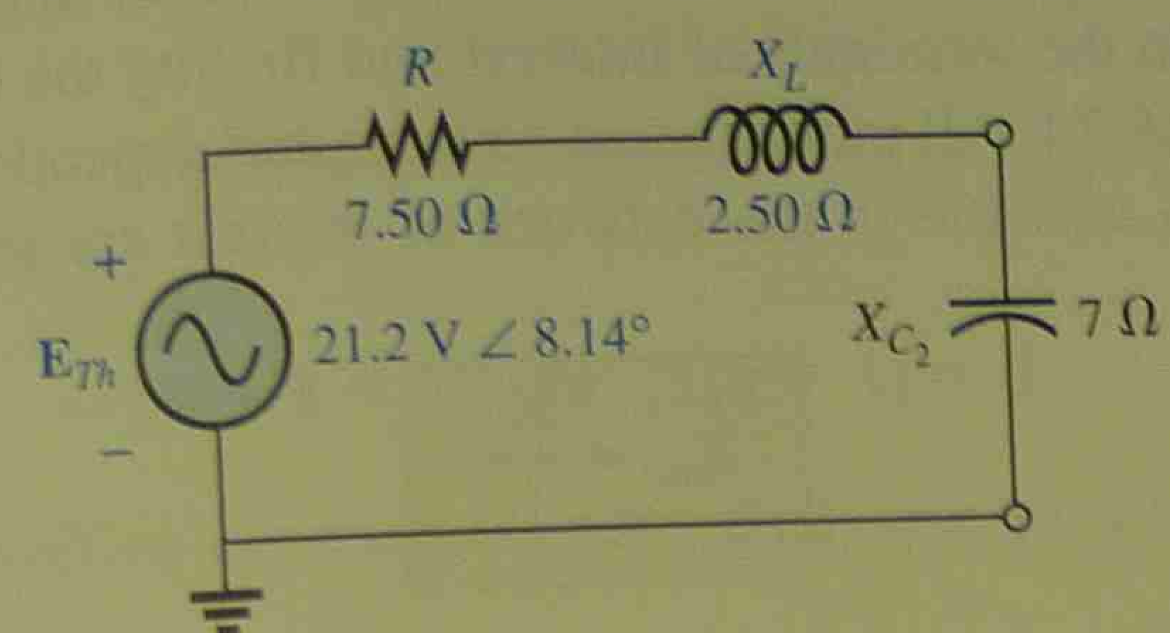


FIG. 18.72

Dependent Sources

As stated for Thevenin's theorem, dependent sources in which the controlling variable is not determined by the network for which the Norton equivalent circuit is to be found do not alter the procedure outlined above.

For dependent sources of the other kind, one of the following procedures must be applied. Both of these procedures can also be applied to networks with any combination of independent sources and dependent sources not controlled by the network under investigation.

The Norton equivalent circuit appears in Fig. 18.73(a). In Fig. 18.73(b), we find that

$$\mathbf{I}_{sc} = \mathbf{I}_N \tag{18.9}$$

and in Fig. 18.73(c) that

$$\mathbf{E}_{oc} = \mathbf{I}_N\mathbf{Z}_N$$

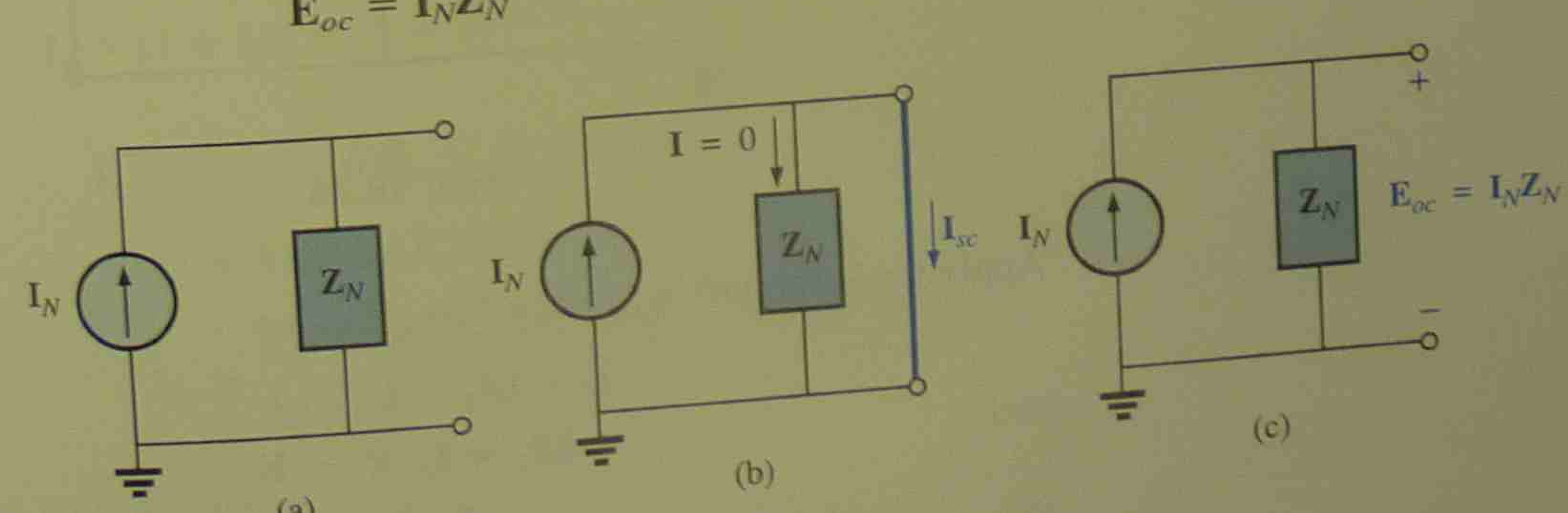


FIG. 18.73

Or, rearranging, we have

$$\mathbf{Z}_N = \frac{\mathbf{E}_{oc}}{\mathbf{I}_N}$$

and

$$\mathbf{Z}_N = \frac{\mathbf{E}_{oc}}{\mathbf{I}_{sc}} \tag{18.10}$$

The Norton impedance can also be determined by applying a source of voltage $\mathbf{E}_g$ to the terminals of interest and finding the resulting $\mathbf{I}_g$, as shown in Fig. 18.74. All independent sources and dependent sources not controlled by a variable in the network of interest are set to zero, and

$$\mathbf{Z}_N = \frac{\mathbf{E}_g}{\mathbf{I}_g} \tag{18.11}$$

For this latter approach, the Norton current is still determined by the short-circuit current.

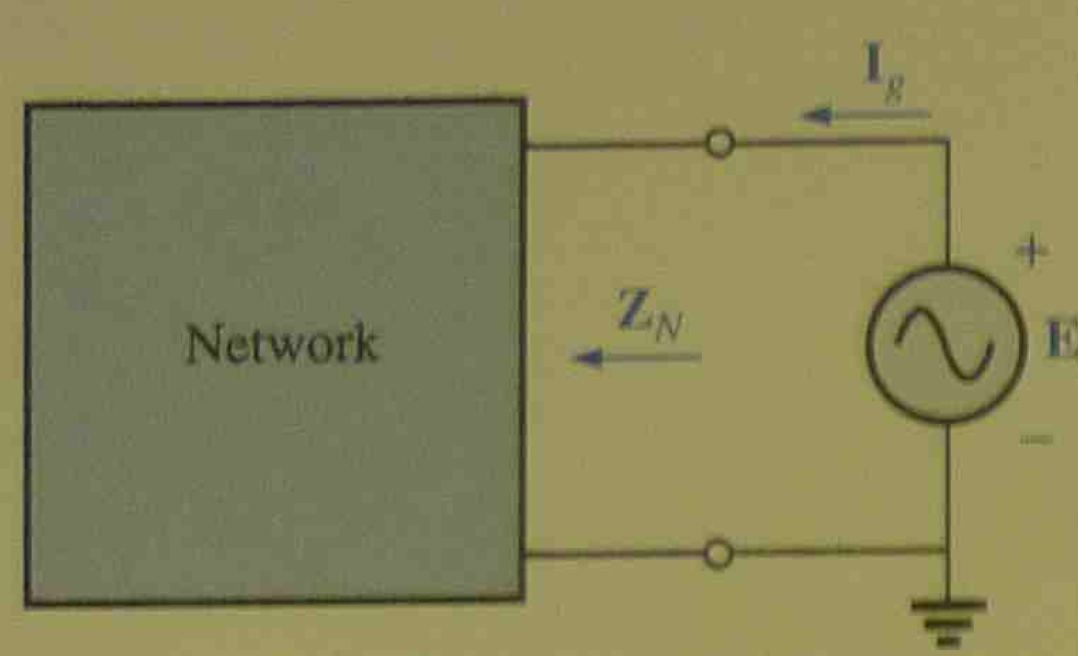


FIG. 18.74

EXAMPLE 18.17 Using each method described for dependent sources, find the Norton equivalent circuit for the network of Fig. 18.75.

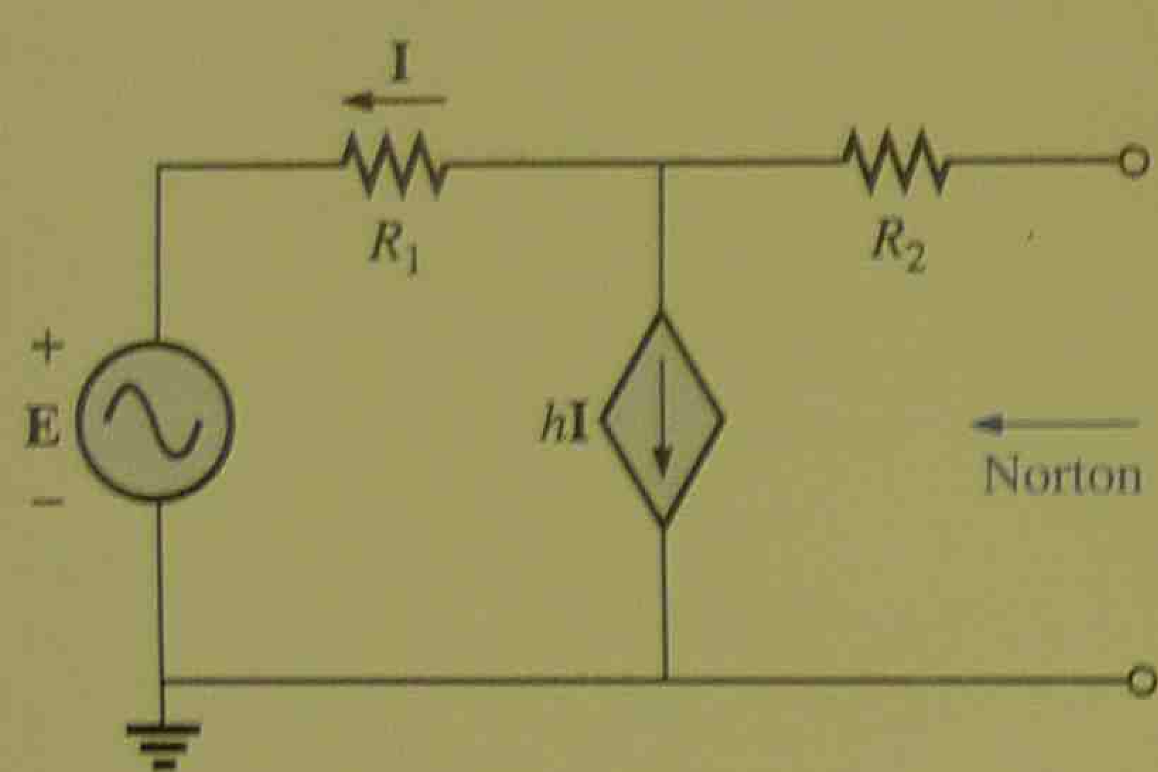


FIG. 18.75

Solution:

$\mathbf{I}_N$:

For each method, $\mathbf{I}_N$ is determined in the same manner. From Fig. 18.76, using Kirchhoff's current law, we have

$$0 = \mathbf{I} + h\mathbf{I} + \mathbf{I}_{sc}$$

or

$$\mathbf{I}_{sc} = -(1 + h)\mathbf{I}$$

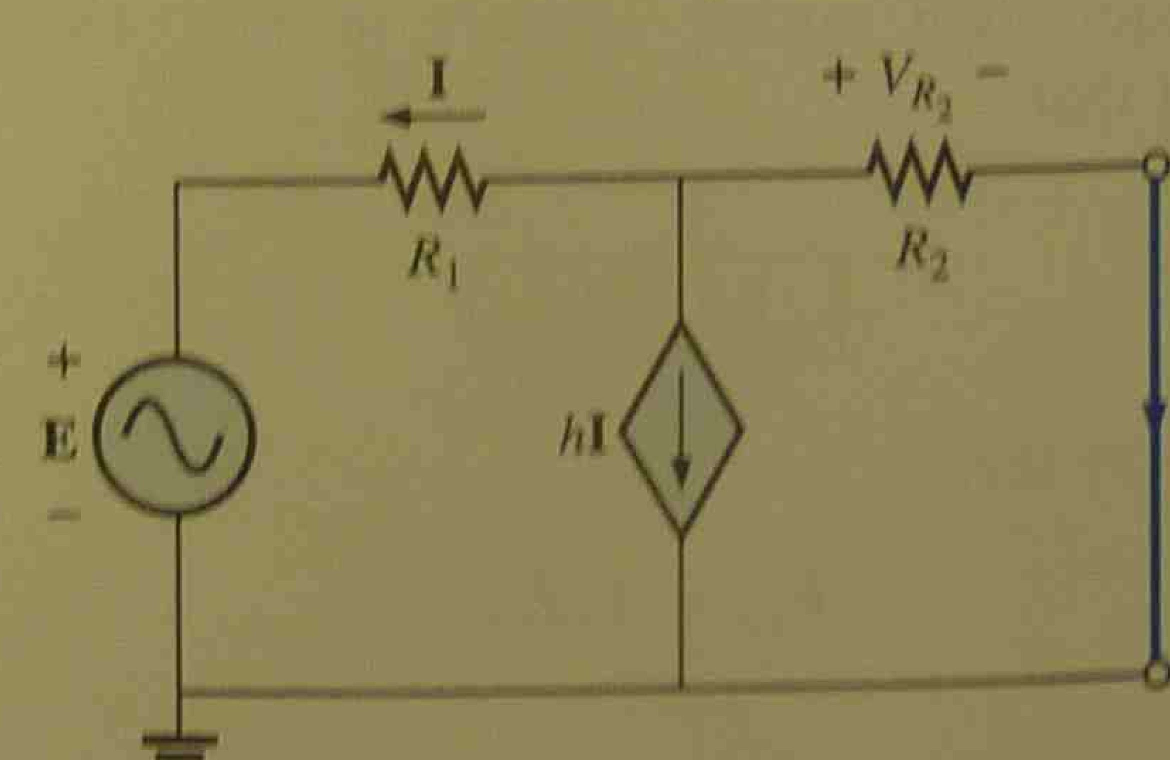


FIG. 18.76

Applying Kirchhoff's voltage law gives us

$$\mathbf{E} + \mathbf{I}R_1 - \mathbf{I}_{sc}R_2 = 0$$

and

$$\mathbf{I}R_1 = \mathbf{I}_{sc}R_2 - \mathbf{E}$$

or

$$\mathbf{I} = \frac{\mathbf{I}_{sc}R_2 - \mathbf{E}}{R_1}$$

$$\mathbf{I}_{sc} = -(1 + h)\mathbf{I} = -(1 + h)\left(\frac{\mathbf{I}_{sc} R_2 - \mathbf{E}}{R_1}\right)$$

so

$$R_1\mathbf{I}_{sc} = -(1 + h)\mathbf{I}_{sc} R_2 + (1 + h)\mathbf{E}$$

or

$$\mathbf{I}_{sc}[R_1 + (1 + h)R_2] = (1 + h)\mathbf{E}$$

$$\mathbf{I}_{sc} = \frac{(1 + h)\mathbf{E}}{R_1 + (1 + h)R_2} = \mathbf{I}_N$$

$\mathbf{Z}_N$:

Method 1: $\mathbf{E}_{oc}$ is determined from the network of Fig. 18.77. By Kirchhoff's current law,

$$0 = \mathbf{I} + h\mathbf{I} \quad \text{or} \quad \mathbf{I}(h + 1) = 0$$

For h, a positive constant $\mathbf{I}$ must equal zero to satisfy the above. Therefore,

$$\mathbf{I} = 0 \quad \text{and} \quad h\mathbf{I} = 0$$

and

$$\mathbf{E}_{oc} = \mathbf{E}$$

with

$$\mathbf{Z}_N = \frac{\mathbf{E}_{oc}}{\mathbf{I}_{sc}} = \frac{\mathbf{E}}{\dfrac{(1 + h)\mathbf{E}}{R_1 + (1 + h)R_2}} = \frac{R_1 + (1 + h)R_2}{(1 + h)}$$

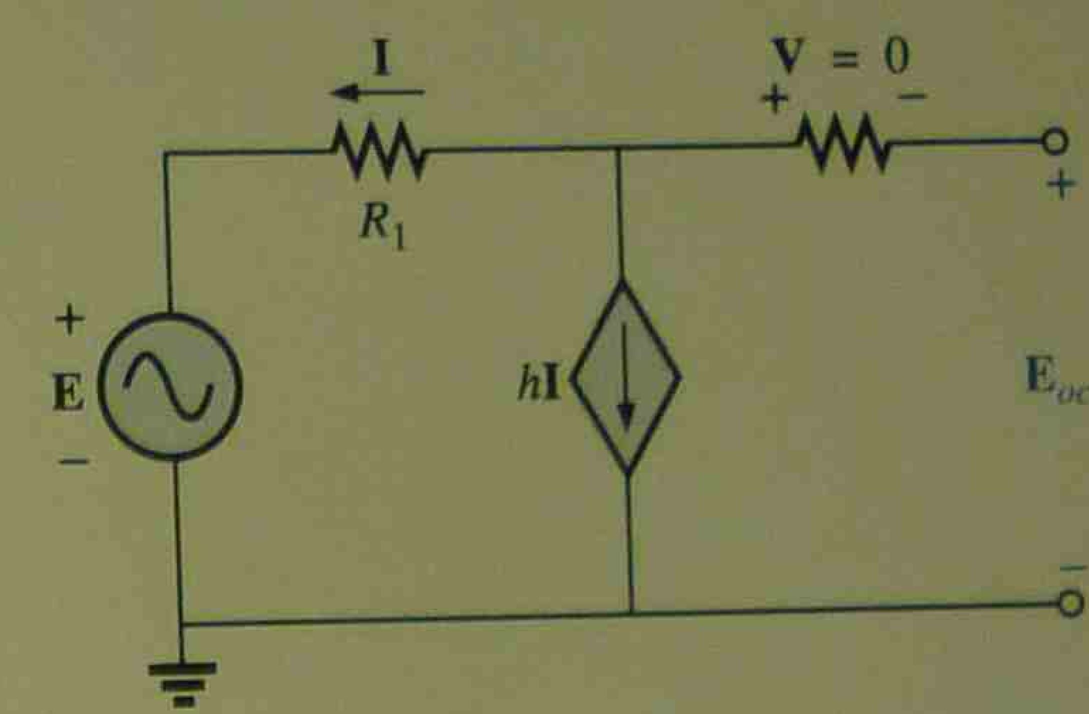


FIG. 18.77

Method 2: Note Fig. 18.78. By Kirchhoff's current law,

$$\mathbf{I}_g = \mathbf{I} + h\mathbf{I} = (1 + h)\mathbf{I}$$

By Kirchhoff's voltage law,

$$\mathbf{E}_g - \mathbf{I}_g R_2 - \mathbf{I}R_1 = 0$$

or

$$\mathbf{I} = \frac{\mathbf{E}_g - \mathbf{I}_g R_2}{R_1}$$

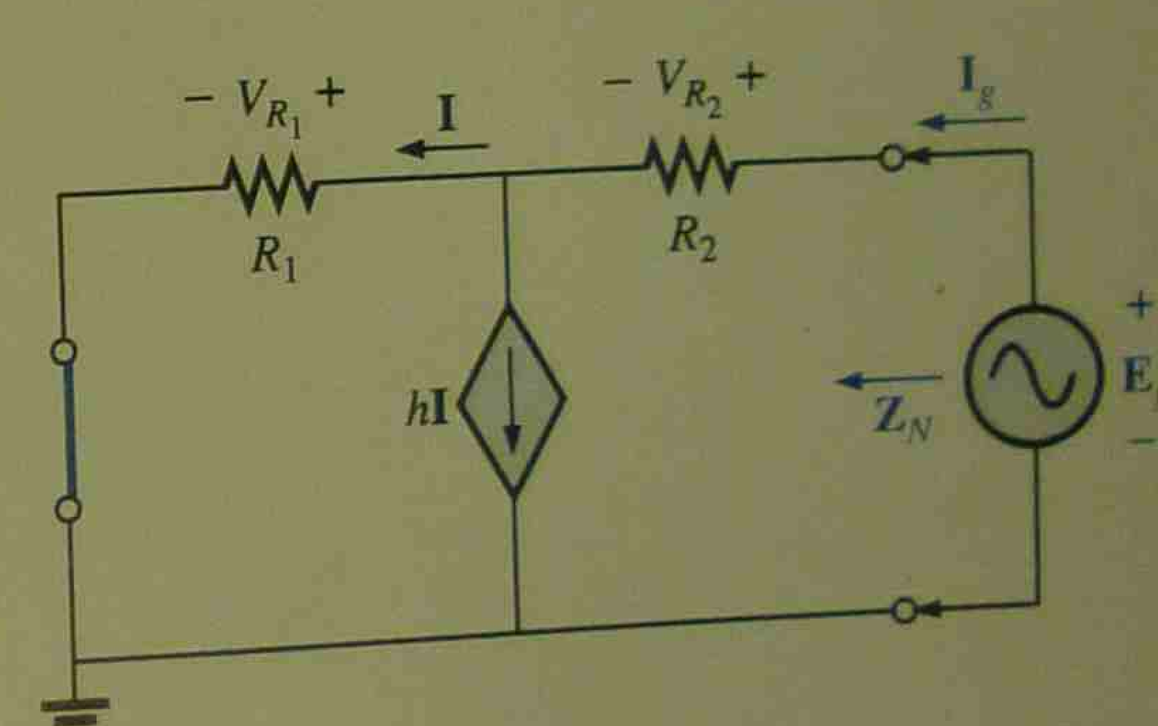


FIG. 18.78

Substituting, we have

$$\mathbf{I}_g = (1 + h)\mathbf{I} = (1 + h)\left(\frac{\mathbf{E}_g - \mathbf{I}_g R_2}{R_1}\right)$$

and

$$\mathbf{I}_g R_1 = (1 + h)\mathbf{E}_g - (1 + h)\mathbf{I}_g R_2$$

so

$$\mathbf{E}_g(1 + h) = \mathbf{I}_g[R_1 + (1 + h)R_2]$$

or

$$\mathbf{Z}_N = \frac{\mathbf{E}_g}{\mathbf{I}_g} = \frac{R_1 + (1 + h)R_2}{1 + h}$$

which agrees with the above.

EXAMPLE 18.18 Find the Norton equivalent circuit for the network configuration of Fig. 18.56.

Solution: By source conversion,

$$\mathbf{I}_N = \frac{\mathbf{E}_{Th}}{\mathbf{Z}_{Th}} = \frac{\dfrac{-k_2R_2\mathbf{V}_i}{R_1 - k_1k_2R_2}}{\dfrac{R_1R_2}{R_1 - k_1k_2R_2}}$$

and

$$\boxed{\mathbf{I}_N = \frac{-k_2\mathbf{V}_i}{R_1}} \tag{18.12}$$

which is $\mathbf{I}_{sc}$ as determined in that example, and

$$\boxed{\mathbf{Z}_N = \mathbf{Z}_{Th} = \frac{R_2}{1 - \dfrac{k_1k_2R_2}{R_1}}} \tag{18.13}$$

For $k_1 \cong 0$, we have

$$\boxed{\mathbf{I}_N = \frac{-k_2\mathbf{V}_i}{R_1}} \qquad k_1 = 0 \tag{18.14}$$

$$\boxed{\mathbf{Z}_N = R_2} \qquad k_1 = 0 \tag{18.15}$$

18.5 MAXIMUM POWER TRANSFER THEOREM

When applied to ac circuits, the maximum power transfer theorem states that

maximum power will be delivered to a load when the load impedance is the conjugate of the Thevenin impedance across its terminals.

That is, for Fig. 18.79, for maximum power transfer to the load,

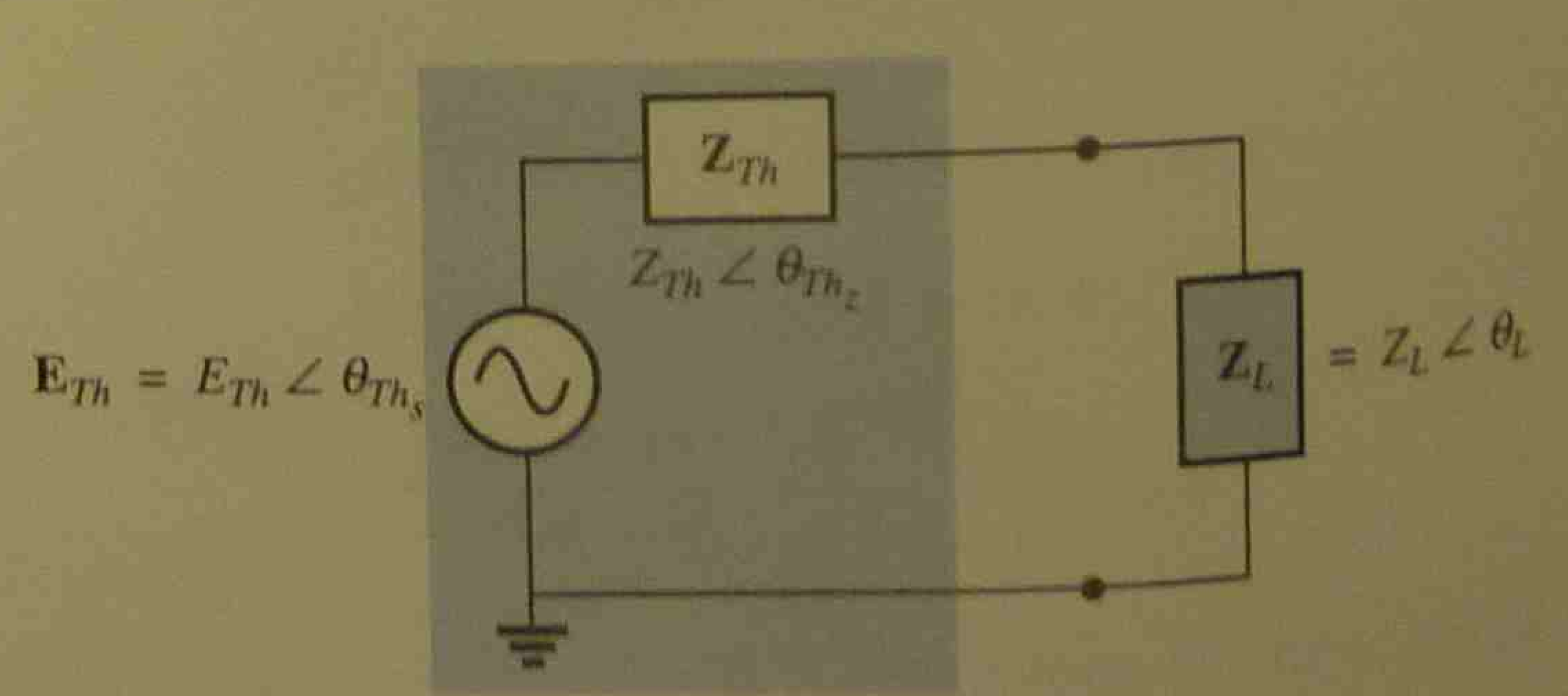


FIG. 18.79

$$Z_L = Z_{Th} \quad \text{and} \quad \theta_L = -\theta_{Th_Z} \tag{18.16}$$

or, in rectangular form,

$$R_L = R_{Th} \quad \text{and} \quad \pm jX_{\text{load}} = \mp jX_{Th} \tag{18.17}$$

The conditions just mentioned will make the total impedance of the circuit appear purely resistive, as indicated in Fig. 18.80:

$$\mathbf{Z}_T = (R \pm jX) + (R \mp jX)$$

and

$$\mathbf{Z}_T = 2R \tag{18.18}$$

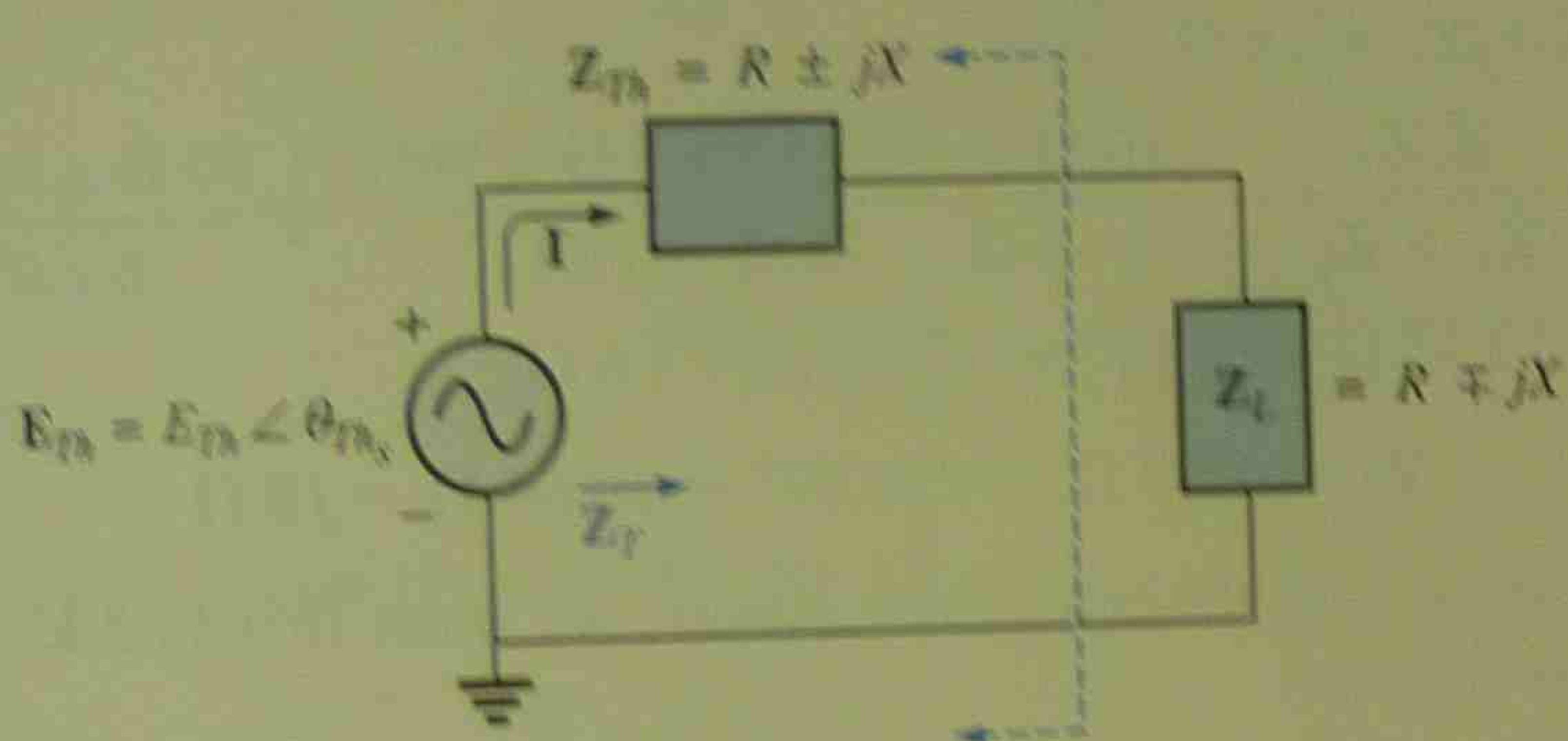


FIG. 18.80

Since the circuit is purely resistive, the power factor of the circuit under maximum power conditions is 1. That is,

$$F_p = 1 \quad \text{(maximum power transfer)} \tag{18.19}$$

The magnitude of the current **I** of Fig. 18.80 is

$$I = \frac{E_{Th}}{Z_T} = \frac{E_{Th}}{2R}$$

The maximum power to the load is

$$P_{\max} = I^2R = \left(\frac{E_{Th}}{2R}\right)^2 R$$

and

$$P_{\max} = \frac{E_{Th}^2}{4R} \tag{18.20}$$

EXAMPLE 18.19 Find the load impedance in Fig. 18.81 for maximum power to the load, and find the maximum power.

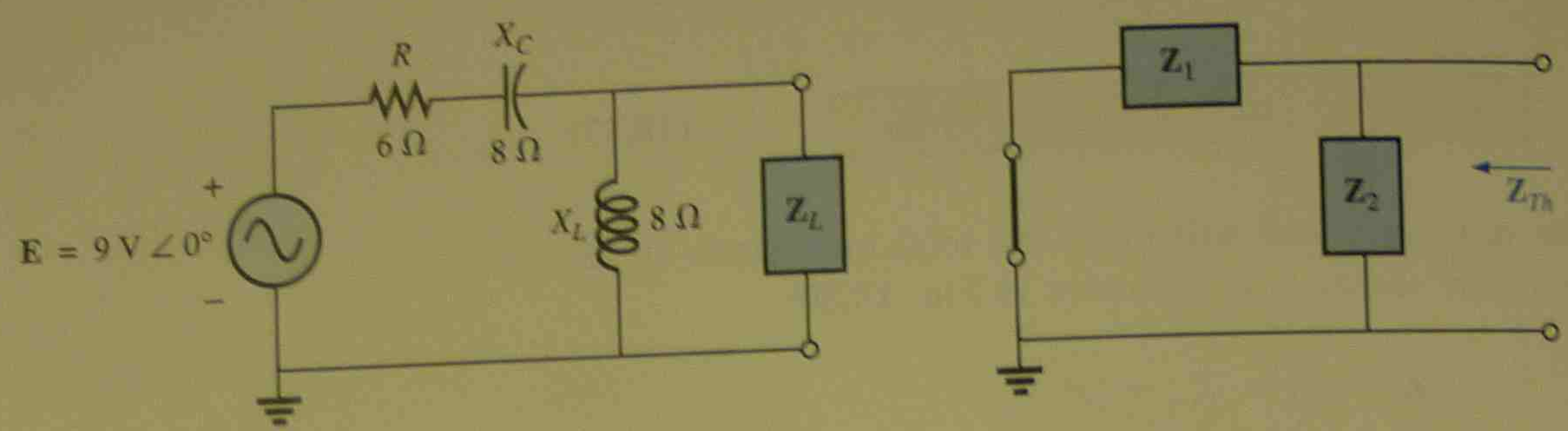


FIG. 18.81

Solution:

$$\mathbf{Z}_1 = R - jX_C = 6\ \Omega - j\,8\ \Omega = 10\ \Omega\ \angle -53.13^\circ$$

$$\mathbf{Z}_2 = +jX_L = j\,8\ \Omega$$

$$\mathbf{Z}_{Th} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2} = \frac{(10\ \Omega\ \angle -53.13^\circ)(8\ \Omega\ \angle 90^\circ)}{6\ \Omega - j\,8\ \Omega + j\,8\ \Omega} = \frac{80\ \Omega\ \angle 36.87^\circ}{6\ \angle 0^\circ}$$

$$= 13.33\ \Omega\ \angle 36.87^\circ = 10.66\ \Omega + j\,8\ \Omega$$

and $$\mathbf{Z}_L = 13.3\ \Omega\ \angle -36.87^\circ = \mathbf{10.66\ \Omega - j\,8\ \Omega}$$

In order to find the maximum power, we must first find $\mathbf{E}_{Th}$ (Fig. 18.82), as follows:

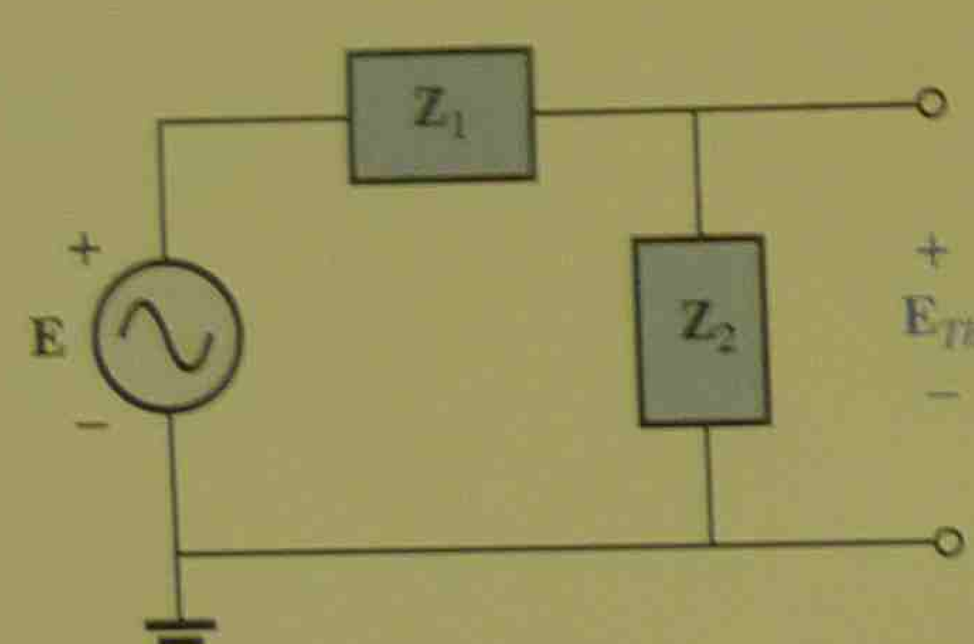


FIG. 18.82

$$\mathbf{E}_{Th} = \frac{\mathbf{Z}_2\mathbf{E}}{\mathbf{Z}_2 + \mathbf{Z}_1} \quad \text{(voltage divider rule)}$$

$$= \frac{(8\ \Omega\ \angle 90^\circ)(9\ \text{V}\ \angle 0^\circ)}{j\,8\ \Omega + 6\ \Omega - j\,8\ \Omega} = \frac{72\ \text{V}\ \angle 90^\circ}{6\ \angle 0^\circ} = 12\ \text{V}\ \angle 90^\circ$$

Then $$P_{max} = \frac{E_{Th}^2}{4R} = \frac{(12\ \text{V})^2}{4(10.66\ \Omega)} = \frac{144}{42.64} = \mathbf{3.38\ W}$$

EXAMPLE 18.20 Find the load impedance in Fig. 18.83 for maximum power to the load, and find the maximum power.

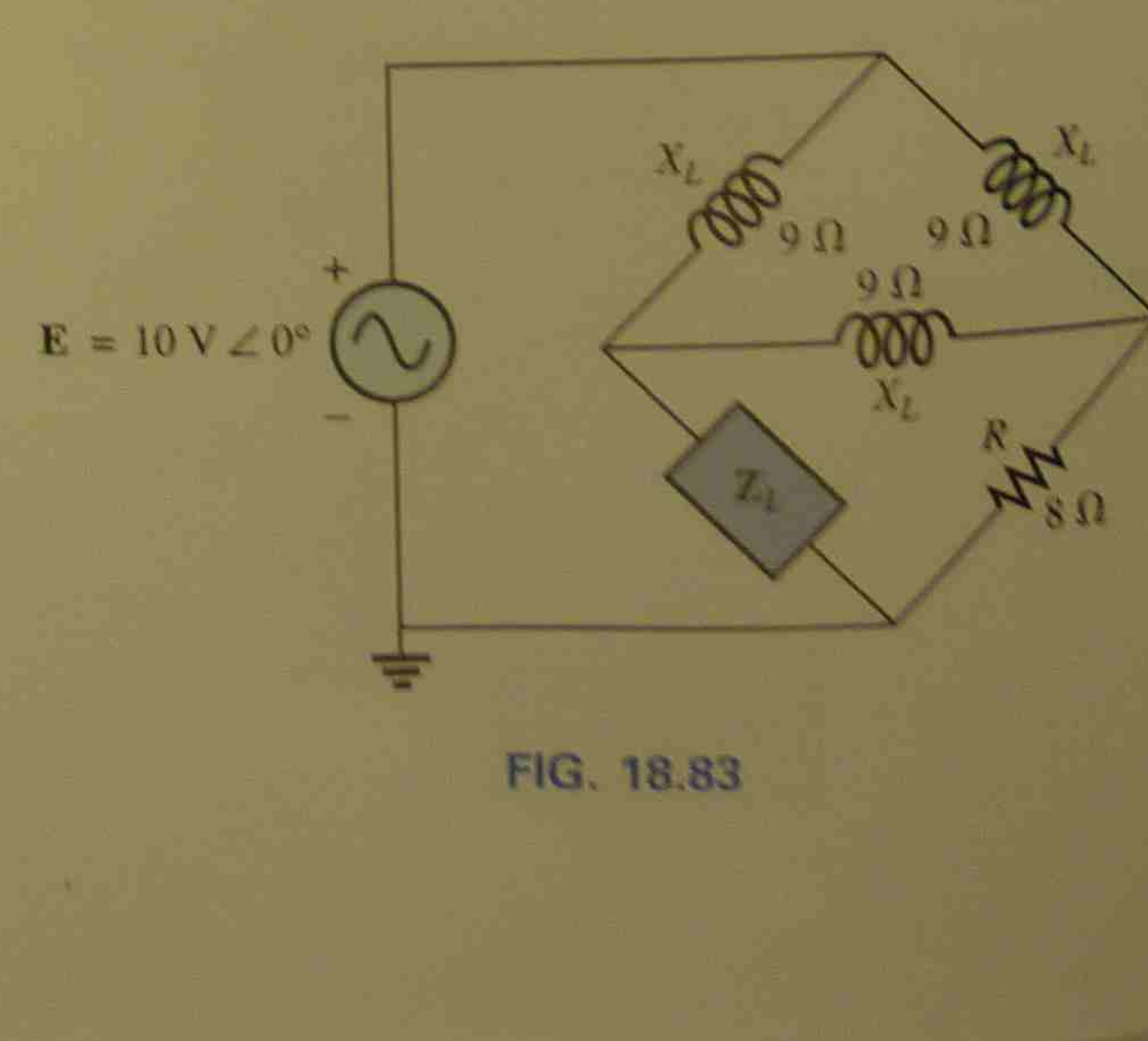


FIG. 18.83

Solution: First we must find $\mathbf{Z}_{Th}$ (Fig. 18.84).

$$\mathbf{Z}_1 = +jX_L = j9\ \Omega \qquad \mathbf{Z}_2 = R = 8\ \Omega$$

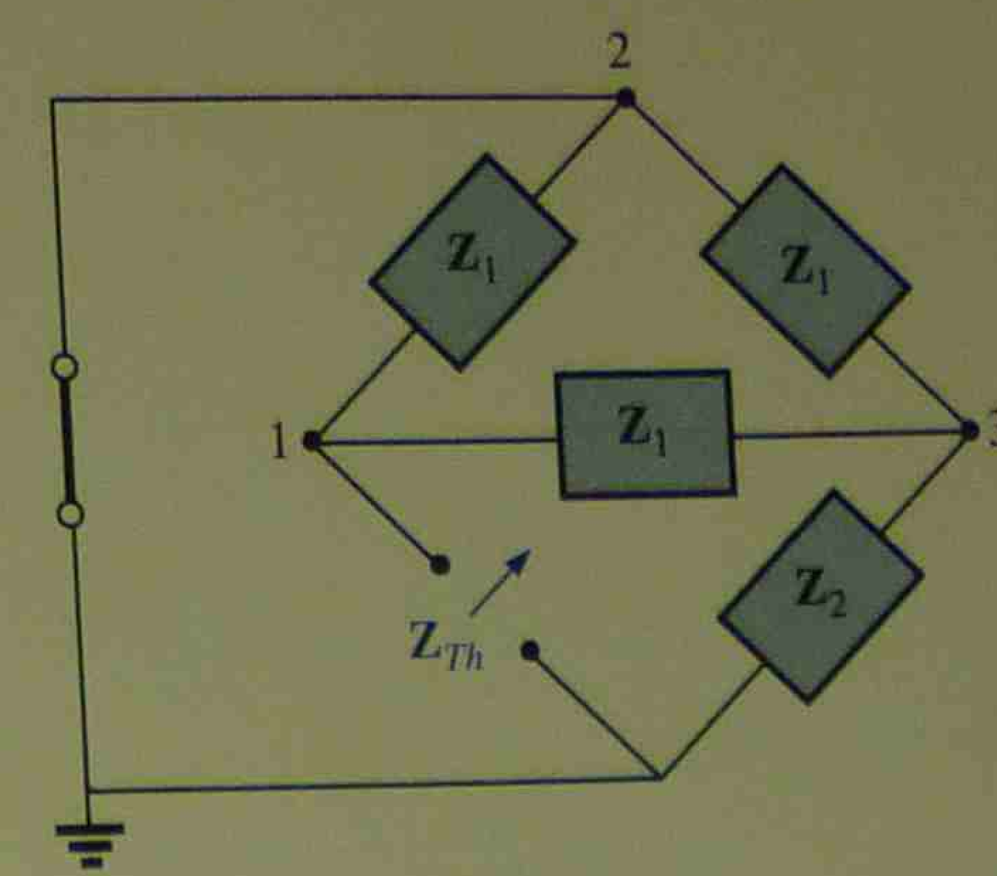


FIG. 18.84

Converting from a Δ to a Y (Fig. 18.85), we have

$$\mathbf{Z'}_1 = \frac{\mathbf{Z}_1}{3} = j3\ \Omega \qquad \mathbf{Z}_2 = 8\ \Omega$$

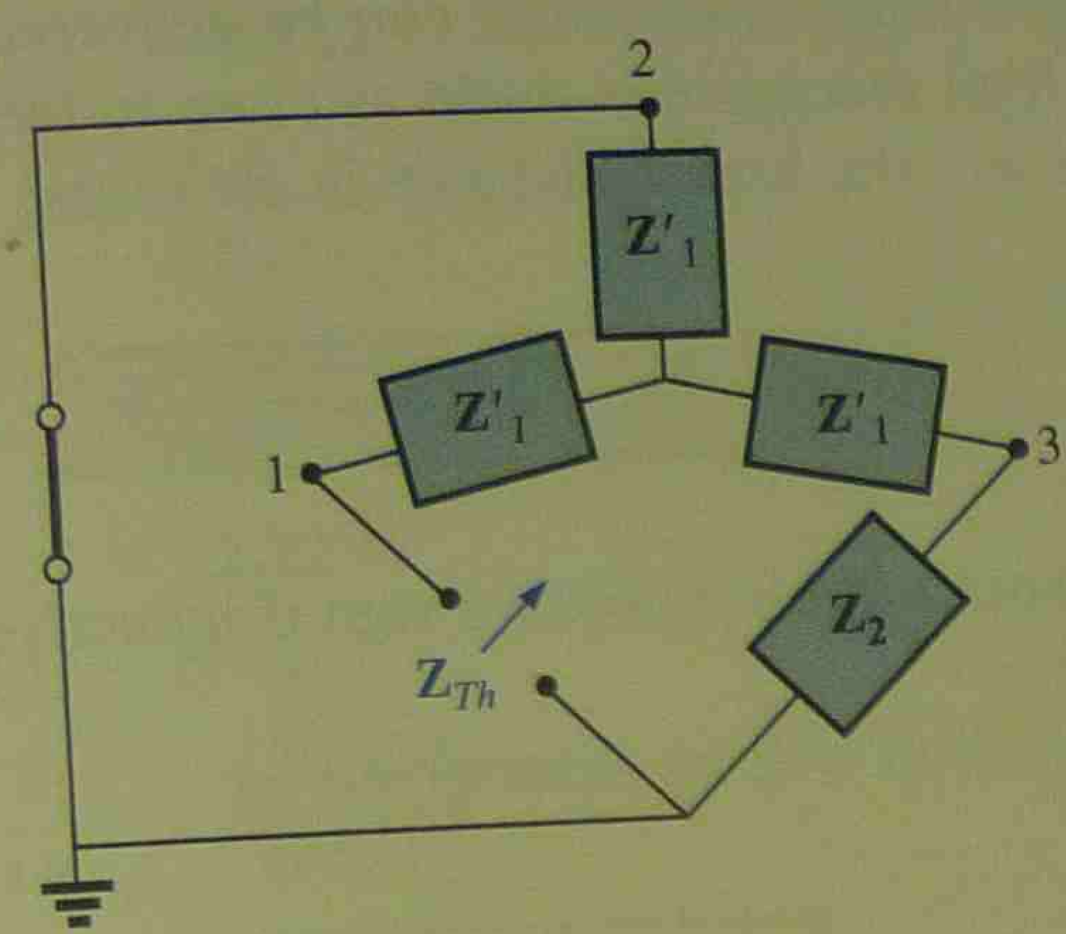


FIG. 18.85

The redrawn circuit (Fig. 18.86) shows

$$\mathbf{Z}_{Th} = \mathbf{Z'}_1 + \frac{\mathbf{Z'}_1(\mathbf{Z'}_1 + \mathbf{Z}_2)}{\mathbf{Z'}_1 + (\mathbf{Z'}_1 + \mathbf{Z}_2)}$$

$$= j3\ \Omega + \frac{3\ \Omega\ \angle 90^\circ (j3\ \Omega + 8\ \Omega)}{j6\ \Omega + 8\ \Omega}$$

$$= j3 + \frac{(3\ \angle 90^\circ)(8.54\ \angle 20.56^\circ)}{10\ \angle 36.87^\circ}$$

$$= j3 + \frac{25.62\ \angle 110.56^\circ}{10\ \angle 36.87^\circ} = j3 + 2.56\ \angle 73.69^\circ$$

$$= j3 + 0.72 + j2.46$$

$$\mathbf{Z}_{Th} = 0.72\ \Omega + j5.46\ \Omega$$

and

$$\mathbf{Z}_L = \mathbf{0.72\ \Omega - j5.46\ \Omega}$$

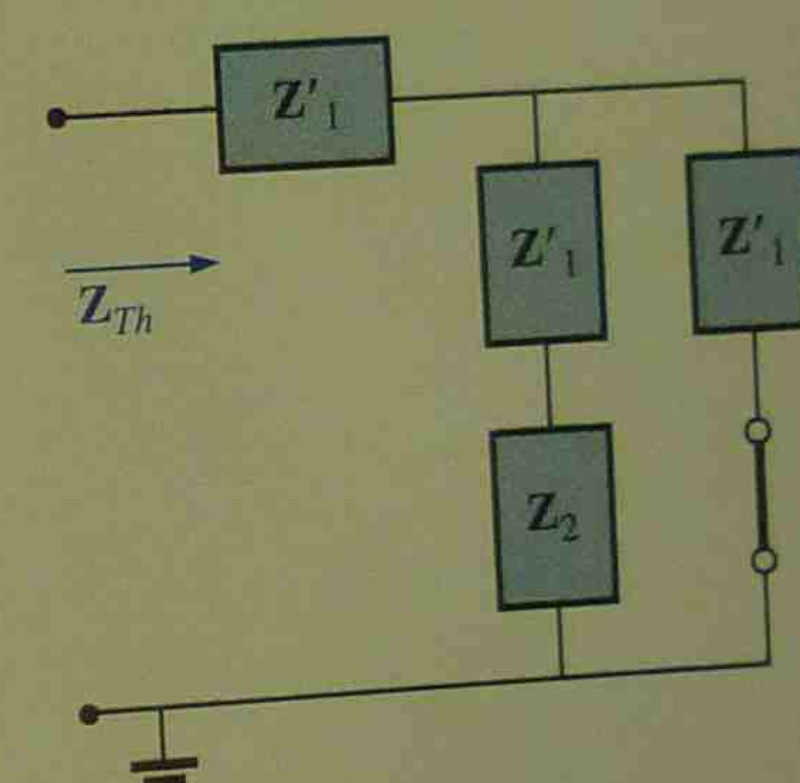


FIG. 18.86

For $\mathbf{E}_{Th}$, use the modified circuit of Fig. 18.87 with the voltage source replaced in its original position. Since $I_1 = 0$, $\mathbf{E}_{Th}$ is the voltage across the series impedance of $\mathbf{Z'}_1$ and $\mathbf{Z}_2$. Using the voltage divider rule gives us

$$\mathbf{E}_{Th} = \frac{(\mathbf{Z'}_1 + \mathbf{Z}_2)\mathbf{E}}{\mathbf{Z'}_1 + \mathbf{Z}_2 + \mathbf{Z'}_1} = \frac{(j3\ \Omega + 8\ \Omega)(10\ \text{V}\ \angle 0^\circ)}{8\ \Omega + j6\ \Omega}$$

$$= \frac{(8.54\ \angle 20.56^\circ)(10\ \text{V}\ \angle 0^\circ)}{10\ \angle 36.87^\circ}$$

$$\mathbf{E}_{Th} = 8.54\ \text{V}\ \angle -16.31^\circ$$

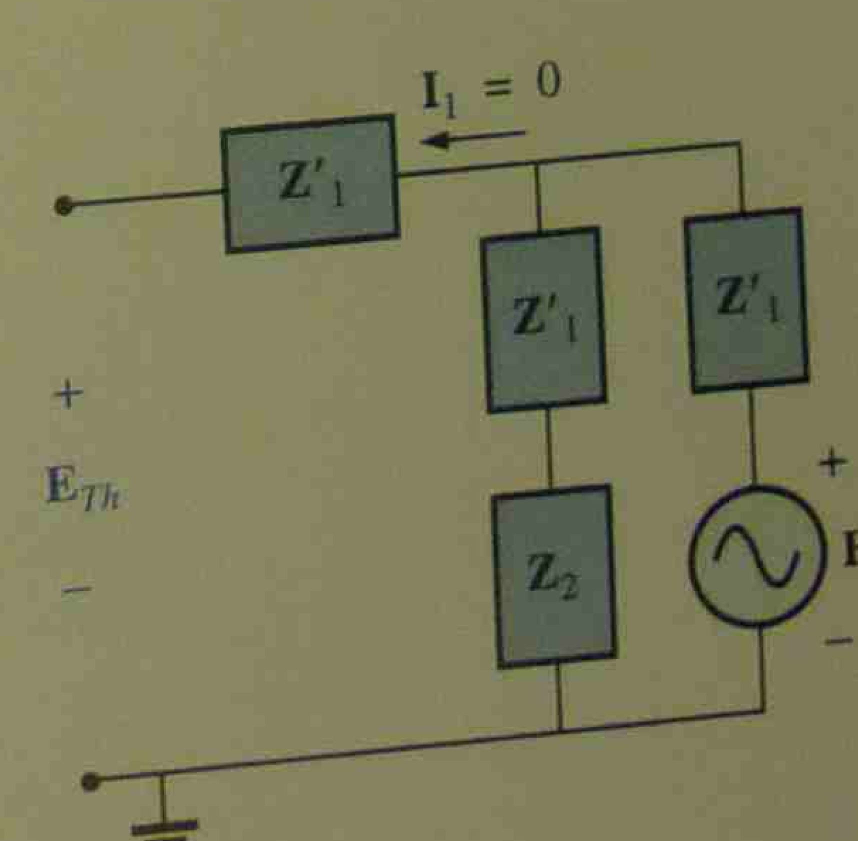


FIG. 18.87

and $$P_{max} = \frac{E_{Th}^2}{4R} = \frac{(8.54\ V)^2}{4(0.72\ \Omega)} = \frac{72.93}{2.88}\ W$$
$$= 25.32\ W$$

If the load resistance is adjustable but the magnitude of the load reactance cannot be set equal to the magnitude of the Thevenin reactance, then the maximum power *that can be delivered* to the load will occur when the load reactance is made as close to the Thevenin reactance as possible and the load resistance is set to the following value:

$$R_L = \sqrt{R_{Th}^2 + (X_{Th} + X_{load})^2} \qquad (18.21)$$

where each reactance carries a positive sign if inductive and a negative sign if capacitive.

The power delivered will be determined by

$$P = E_{Th}^2/4R_{av} \qquad (18.22)$$

where $$R_{av} = \frac{R_{Th} + R_L}{2} \qquad (18.23)$$

The derivation of the above equations is given in Appendix H of the text. The following example demonstrates the use of the above.

EXAMPLE 18.21 For the network of Fig. 18.88:

a. Determine the value of R_L for maximum power to the load if the load reactance is fixed at 4 Ω.
b. Find the power delivered to the load under the conditions of part (a).
c. Find the maximum power to the load if the load reactance is made adjustable to any value and compare the result to part (b) above.

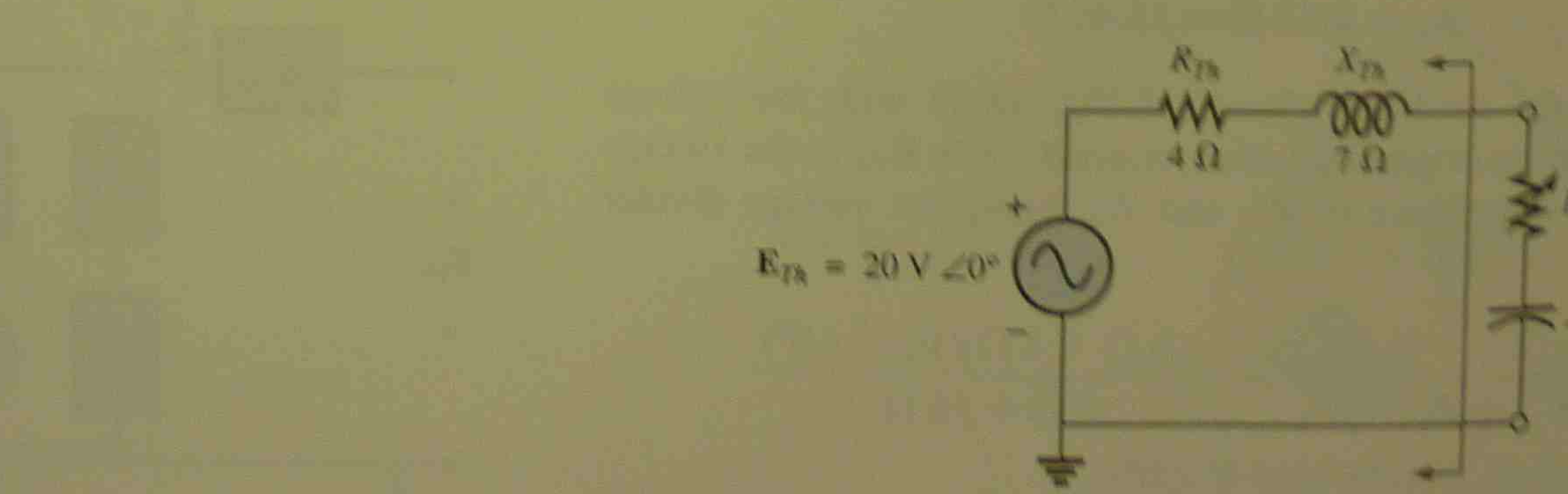


FIG. 18.88

Solutions:

a. Eq. (18.21):
$$R_L = \sqrt{R_{Th}^2 + (X_{Th} + X_{load})^2}$$
$$= \sqrt{(4\ \Omega)^2 + (7\ \Omega - 4\ \Omega)^2}$$
$$= \sqrt{16 + 9} = \sqrt{25}$$
$$R_L = \mathbf{5\ \Omega}$$

b. Eq. (18.23):
$$R_{av} = \frac{R_{Th} + R_L}{2} = \frac{4\ \Omega + 5\ \Omega}{2}$$
$$= \mathbf{4.5\ \Omega}$$

Eq. (18.22):
$$P = \frac{E_{Th}^2}{4R_{av}}$$
$$= \frac{(20\ \text{V})^2}{4(4.5\ \Omega)} = \frac{400}{18}\ \text{W}$$
$$\cong \mathbf{22.22\ W}$$

c. For $\mathbf{Z}_L = 4\ \Omega - j7\ \Omega$
$$P_{max} = \frac{E_{Th}^2}{4R_{Th}} = \frac{(20\ \text{V})^2}{4(4\ \Omega)}$$
$$= \mathbf{25\ W}$$

exceeding the result of part (b) by 2.78 W.

18.6 SUBSTITUTION, RECIPROCITY, AND MILLMAN'S THEOREMS

As indicated in the introduction to this chapter, the substitution and reciprocity theorems and Millman's theorem will not be considered here in detail. A careful review of Chapter 9 will enable you to apply these theorems to sinusoidal ac networks with little difficulty. A number of problems in the use of these theorems appear in the problem section.

18.7 COMPUTER ANALYSIS

The computer analysis of this chapter will be limited to an application of Thevenin's theorem to a network with independent sources and the writing of an input file for a network with controlled sources. The extended use of controlled sources in the analysis of electronic systems places a high priority on the introduction of the subject. There are essentially four types of controlled sources in electronic systems with which a student should become familiar: current-controlled current sources (CCCS), voltage-controlled current sources (VCCS), current-controlled voltage sources (CCVS), and voltage-controlled voltage sources (VCVS). Each has a controlling variable that must be properly listed in the input file to insure the proper magnitude and phase for the controlled source. In BASIC an analysis of the network as appearing in this chapter would have to be applied—simply an extension of the longhand ap-